The Development of Geometric Reasoning: Middle Years Students’ Understanding of 2-dimensional Shapes

A thesis submitted in fulfilment of the requirements for the degree of Master of Education by Research

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I certify that except where due acknowledgement has been made, the work is that of
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acknowledged; and, ethics procedures and guidelines (RMIT Approval Number:
CHEAN B 0000017502 and DEECD Reference Number: 2015_002882) have been
followed.

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Adrian Berenger
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Both my supervisors have inspired me to keep on researching.

Adrian Berenger
Abstract

Visualisation plays a critical role in geometric reasoning. An ‘image in the mind’ provides students with the necessary structures of a shape in order to define and classify them and then deduce other properties from them. Convincing others through a common mathematical discourse is seen as a necessary component in the meaning-making process of geometry. It is necessary for teachers to have the pedagogical content knowledge necessary to develop and support geometric argument in the classroom.

This study used a design-based research methodology to examine the geometric thinking of students in Years 7 and 8 at two inner-suburban schools in Melbourne, Victoria, Australia. Student written work samples, classroom observations and video-recorded teaching episodes were used to collect data about how students and teachers communicated their understanding of geometric concepts relating to 2-dimensional shapes. A series of geometric reasoning tasks were developed to provide opportunities for students to learn through group work activities.

Students’ use of keywords, visual mediators, narratives and discourse routines were interpreted using Sfard’s (2008) interpretive framework of mathematical discourse. The results of the research from the preliminary tasks show that students do not readily use diagrams to describe shapes, and instead, produce exhaustive lists of known properties of common shapes. This is defined as analysis by the van Hiele levels of geometric thought. The results also demonstrated that students do not engage in mathematical discussion in the absence of clearly defined classroom norms for group work. Student work samples from supplementary tasks revealed a growth in the sophistication of keywords and visual mediators used to describe shapes yet raised questions about the teachers’ readiness to provide appropriate intervention and instruction in geometry.

The main conclusion that can be drawn from this study was that students’ progress in geometric reasoning was hampered by misconceptions by both students and teachers. The role of the teacher was consequential to engendering group work skills in their students because geometric argument necessitates the communication of ideas in constructing endorsed narratives of new knowledge from familiar geometric concepts.
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<td>Australian Association of Mathematics Teachers</td>
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<tr>
<td>ACARA</td>
<td>Australian Curriculum, Assessment and Reporting Authority</td>
</tr>
<tr>
<td>ACM</td>
<td>Australian Curriculum: Mathematics</td>
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<tr>
<td>DBR</td>
<td>Design-Based Research</td>
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<td>DBRC</td>
<td>The Design-Based Research Collective</td>
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<td>DEECD</td>
<td>Department of Education and Early Childhood Development</td>
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<td>DET</td>
<td>Department of Education and Training</td>
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<td>DGE</td>
<td>Dynamic Geometry Environment</td>
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<td>GMK</td>
<td>General Mathematics Knowledge</td>
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<td>Topic-specific Mathematical Knowledge</td>
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Chapter 1

Introduction

Geometry appeals to our visual senses and engages our minds to think in various ways. Children “are naturally intrigued by, and motivated to learn more about, the geometry that defines their world” (Oberdorf & Taylor-Cox, 1999, p. 340). Enjoyment in geometry comes from connecting shapes and pulling them apart in order to solve puzzles. Examples include jigsaws (van Hiele, 1999) and paper-folding activities such as origami (Boakes, 2009; Golan, 2011). These types of activities create a lot of excitement and provide visual engagement for children.

Geometry is significant as a discipline because it can be applied across mathematics (Freudenthal, 1973) and other learning disciplines to help students think and solve problems (Suydam, 1985). Geometry helps develop visual-spatial skills that enhance the ability to solve problems in, for example, the science disciplines (Baker & Talley, 1972; Kozhevnikov, Motes & Hegarty, 2007; Wilder & Brinkerhoff, 2007) and the arts (Akayuure, Asiedu-Addo & Alebna, 2016; Kozhevnikov, Kosslyn & Shephard, 2005, Rosenberg, 1987).

Geometry has a long history, and has played an important part in all cultures (D’Ambrosio, 2003; Massarwe, Verner, Bshouty & Verner, 2010). Therefore, cultural considerations are important in engaging students. By building on students’ mathematical traditions, learning can be enhanced (Sinclair, Bussi, de Villiers, Jones, Kortenkamp, Leung & Owens, 2016). Despite issues around student engagement, there are several other pedagogical hurdles that need to be overcome to address issues of student achievement and attainment in geometry and related fields.

1.1 Background

Geometry stems from several cultural influences developed from practical theories of earth measurement (Clements & Battista, 1992). The etymological root of geometry comes from the Greek (Γεωμετρία), geo for earth and metron meaning to measure. However, the geometric concepts currently taught in schools have little to do with measuring the Earth. The work of Euclid in ‘Elements’ (circa 300BCE) codified the accumulated knowledge of geometry known at the time (Jones, 2002), and recast
geometry as rules that underlie mathematical proof – the theoretical mathematics of definition and deduction (Tall, 2014).

Freudenthal (1973) viewed geometry, at its basic level to be “grasping that space in which the child lives, breathes and moves. The space that the child must learn to know, explore, conquer, in order to live, breathe and move better in it” (p. 403). At its highest level, geometry is built upon stated axioms and definitions which act as starting points from which theorems are developed and proved (Sinclair, Bussi, de Villiers, Jones, Kortenkamp, Leung & Owens, 2016). This high level of geometry is often referred to as deductive reasoning and has always been a teaching and learning challenge.

For many children, the learning of geometry starts before schooling with the recognition of shapes based on their appearance (Kilpatrick, Swafford & Findell, 2001). This then develops throughout primary schooling with the identification of shape properties. Associated measurement concepts also begin quite early when a number is assigned to continuous quantities (Browning, Edson, Kimani & Aslan-Tutak, 2014). Many relationships between lengths or areas that students encounter in their mathematics classroom do not depend upon measuring, but on their knowledge of geometric structures. For example, because rectangles have four right angles, they have two pairs of parallel sides; since parallelograms have two pairs of parallel sides, they have equal opposite angles. These results are established without the aid of rulers and protractors. Establishing geometric relationships requires thinking and reasoning about properties of shapes and their diagrams (Cooke, 2007). Geometry is an excellent vehicle to develop measurement and number concepts, and broader reasoning skills at all levels of learning.

Geometry in school is characterised by spatial concepts involving shapes, transformations, and geometric reasoning (Australian Curriculum, Assessment and Reporting Authority [ACARA], n.d.; Atiyah, 2001). Owens’ (2005) use of the term space mathematics signifies the association of geometry with spatial concepts, and is not meant to imply a discussion about extra-terrestrial space. Spatial reasoning involves visualisation and plays an important part when the stored memories of the brain interact with new sensory information (Owens, 2012). Spatial reasoning is highly related to one’s ability to solve mathematical problems that do not reflect the formal procedures one might be taught in the classroom (Greiffenhagen & Sharrock, 2008), especially
non-routine real world problems with real world applications (Clements & Samara, 2011; Presmeg & Balderas-Cañas, 2001).

A case in point is the use of origami in solving scientific problems. Origami not only appeals to one’s aesthetic appreciation, but also can contribute to improving students’ spatial abilities (Boakes, 2009; Golan, 2011; Tateishi, 2011) and their understanding of geometric concepts (Alperin, 2000; Coad, 2006; Geretschlag, 2002). Origami can help students “to visualise, reason and discover fundamental properties of shapes including their geometrical relations and transformations” (Akayuure, Asiedu-Addo & Alebna, 2016, p. 200). When applied to structural applications, origami has provided geometric solutions to scientific problems (Schenk & Guest, 2011), such as solar sails used in space-based imaging (Furuya, Miyazaki & Takeuchi, 2003), heart stents folded and inserted into a blocked artery (Han, 2013), and emergency shelters consisting of deployable rigid walls designed for military and disaster relief housing (Merali, 2011). When students see the real purpose and real applications for learning mathematical concepts, they are more likely to engage in more challenging and less routine tasks (Barkatsas & Seah, 2015; Herrington, Reeves & Oliver, 2010).

Clements and Battista (1992) emphasised the role of school geometry as mathematising objects, relationships, and transformations, in addition to developing skills to construct visual representations. What is ‘seen’ is often referred to as visualisation. Visualisation involves using imagery to construct vivid detailed pictures of static objects, representing relations among objects, and dynamic transformations of objects (Newcombe & Stieff, 2012). Visual imagery is important if mathematical concepts are to emerge through concrete experiences (Owens & Outhred, 1998). Indeed visualisation is a vital skill in our daily lives, particularly in this information age (Lowrie, 2010), and is an important factor in geometric thinking (Battista, 1990).

Thinking geometrically involves observation, interpretation, visualisation, and logical argumentation that allows “concepts, processes and their uses to be built up, problems to be explored and solved, conjectures to be made and examined, and complex ideas about the world to be communicated in precise, succinct ways” (Booker, 2005, p. 49). For students, thinking geometrically necessitates communication of their conceptions and being able to convince others. If, as Tall (1991) contended, “mathematics is a shared culture” (p. 3), students need to not only parrot facts, but also be able to connect learned facts to construct logical arguments as endorsed mathematical discourse (Battista, 2001; Sfard, 2008).
1.1.1 An Australian Perspective

Historically, Australian educational approaches to mathematics curriculum and teaching were strongly influenced by European, US and UK traditions (Benavot, Cha, Kamens, Meyer & Wong, 1991; Clements & Ellerton, 1996; Kamens & Benavot, 1991; Spyker, 1999). The emphasis of geometry in the UK and the former British dominions of Canada, New Zealand and Australia was on coordinate geometry and transformations, and much less on the Euclidean approach, even though it did persist in the US (Benavot, Cha, Kamens, Meyer & Wong, 1991). Jones (2002) offered an historical perspective on curriculum reforms in the UK, where school mathematics from the 1960s onwards emphasised calculus and linear algebra at the expense of geometry.

There is little doubt that the driving force behind curricular decisions in high school mathematics is the goal of preparing students for the study of calculus. A great deal of the manipulative skill in algebra, trigonometry, and analytic geometry is clearly prized because of its usefulness in calculus, or at least calculus as it has been traditionally conceived. Geometry, in its many guises, gets neglected, with spatial intuition and visualisation being particularly so. A [recent] emphasis on numeracy may only serve to reduce the coverage of geometry in schools just at the time when geometrical education has so much to offer the education of students. (Jones, 2002, p. 129)

Researchers and educators have suggested several challenges in the teaching, learning and assessment of geometry in both primary and secondary schools. One challenge, for example, is the lack of visual and spatial reasoning within the Australian Curriculum: Mathematics [ACM] (Lowrie, Logan & Scriven, 2012). This denotes a geometry curriculum with an emphasis on memorising vocabulary and applying formulae (Seah, 2015b) and a lack of attention to the role of visualisation in developing spatial and geometric reasoning skills. An analysis of the most recent Trends in International Mathematics and Science Study [TIMMS] indicate only 20% of the proportion of Year 8 test items was devoted to geometry (Thomson, Wernert, O’Grady & Rodrigues, 2016) signifying an ongoing lack of emphasis given to geometry. It is not unexpected then that the findings from TIMMS 2007 (Thomson, 2009), TIMMS 2011 (Thomson, Hillman & Wernert, 2012) and TIMMS 2015 (Thomson, Wernert, O’Grady & Rodrigues, 2017) showed that Australian students historically performed poorly in geometry.

A further challenge is the lack of emphasis on geometry in mathematics curricula more generally and the decline of an explicit geometric focus over several decades in Australia and the UK (Hohenwarter & Jones, 2007; Jones, 2000; 2002; Jones & Mooney, 2003). Geometry and spatial thinking were also an ignored or minimal part of early childhood education (Levenson, Tirosh & Tsamir, 2011). This, in turn has
ramifications for the ways that teachers present geometric tasks, and the specific emphases given to geometric reasoning and problem-solving (Clements & Samara, 2011; Herbst & Kosko, 2012).

The predominant mode of teaching and learning in the middle years is typically described as low-level, skill-based, repetitious exercises that can be characterised as ‘example-practice-practice’ (Siemon, Bleckly & Neal, 2012). Teachers rely on procedural rather than conceptual knowledge to make links between geometric ideas. “Repetition and interiorisation of procedures have been seen as an essential part of mathematics learning, for decades it has been known that it has made no improvement in the understanding of relationships” (Gray, Pinto, Pitta & Tall, 1999, p. 2). Although instructionally supporting students’ development of fluency with core ideas and procedures is important (Battista, 2001), procedural practice lacks the flexibility to solve novel problems (Schoenfeld, 1992a). Student success with future problems requires them to make sense of problem situations (Bjuland, 2007), that evolves over time, and is a key component to mathematical thinking (Booker, 2005; Hiebert et al., 2000, Schoenfeld, 1992a).

1.1.2 A Case for Developing Spatial Skills

Recently, there has begun a refocusing of mathematics education, fuelled by technological developments (National Council of Teachers of Mathematics [NCTM], 2013). The current educational interest in science, technology, engineering, and mathematics [STEM] presents opportunities for richer connections of mathematics with other areas of learning. Integration of subject disciplines improves students’ learning experiences and engagement in mathematics, preparing them for the global economy and the 21st century (Becker & Park, 2011; Kuenzi, 2008;). The fastest growing occupations require significant science and mathematics training, and this underscores the need for education and curricular reform (Becker & Park, 2011; Hynes & Dos Santos, 2007). Architects, artists, physicists, designers, doctors and engineers employ skills involving spatial reasoning – an ability to visualise with the mind’s eye (NCTM, 2013). Visualisations can be used to help imagine the unseen (Phillips, Norris & Macnab, 2010). Having good spatial skills is a strong predictor of achievement and attainment in the STEM fields (Obara, 2013; Uttal, Meadow, Tipton, Hand, Alden, Warren & Newcombe, 2013).
Spatial intuition or spatial perception is an enormously powerful tool, and that is why geometry is actually such a powerful part of mathematics – not only for things that are obviously geometrical, but even for things that are not. We try to put them into geometrical form because that enables us to use our intuition. Our intuition is our most powerful tool. (Atiyah, 2001, p. 658)

Several studies examining the role of visual-spatial skills in the science disciplines have shown the impact of visualisation on student outcomes. Baker and Talley (1972) suggested that visualisation skills enhanced the ability of undergraduate students to deal with chemistry concepts, such as chemical reactions requiring physical modelling as an important vehicle for communication and analysis. Kozhevnikov, Motes and Hegarty (2007) found that physics undergraduates’ spatial visualisation abilities directly affected their ability to solve kinematics problems where processing integrated motion problems relied on the availability of spatial working memory resources. Wilder and Brinkerhoff (2007) found that high school students’ understanding of protein structures and functions improved as a result of bimolecular visualisation.

Despite the opportunities that STEM initiatives provide, many teachers are unaware of the benefits of integrative approaches to student learning (Becker & Park, 2011). Research into applications of visualisation in the science disciplines indicates important instructional implications for the development and use of different visualisation techniques that generalise across the science domains (Baker & Talley, 1972; Kozhevnikov, Motes & Hegarty, 2007). This includes incorporating computer-based visualisation instruction (Wilder & Brinkerhoff, 2007).

1.1.3 A Socio-Cultural Perspective
Ethnomathematics is the integration of cultural anthropology and mathematics (Zaslavsky, 1998). It is based on the identification of mathematics developed within different cultural and social groups. Different ways of engaging ethnically diverse student populations provide increased opportunities for student engagement in mathematics (Atweh & Goos, 2011). Ubiratan D’Ambrosio introduced the term Ethnomathematics in the 1970s in order to emphasise the influence of sociocultural factors on the teaching and learning of mathematics. “Geometry is in the heart of every culture and is inherent in every human mind” (Massarwe, Verner, Bshouty & Verner, 2010, p. 2). The connection between ethnicity and mathematics manifests itself strongly in geometry and potentially reduces the gap between geometry studied at school and real world experiences.
Harris (1997) compared women’s work in various cultures and found that geometry and symmetry were foundations for, among others, knitting, weaving, lace making, and dressmaking – displaying geometrical ideas “as a daily visual phenomenon” (Harris, 1988, p. 24). Owens (2012) reported on student-teacher program initiatives in Papua New Guinea that linked culture and mathematics. Owens found that “the teacher’s culture, language, physical and geographic spaces are recognised as having a continuous impact on spatial and other mathematical thinking” (p. 586). Massarwe, Verner, Bshouty and Verner (2010) observed a wide range of geometrical reasoning through activities involving analysis and construction of ornaments among high school Arabic students. Ascher and D’Ambrosio’s (1994) analysis of comparable studies found similar geometric connections made by African students using cultural ornaments and artefacts. These studies indicated the significance of visualisation and spatial reasoning across cultures, and offered useful ways to enhance student engagement in geometry.

Failure to address social and cultural identity through ethnomathematical objectives is a missed opportunity to engage all students in mathematics and can be disempowering to individual students (Mason, 2007). Only recently, the Australian Curriculum has acknowledged cultural aspects in the learning of mathematics and other cross-curricular areas. This includes recognition of Aboriginal and Torres Strait Islander history and culture, and Australia’s engagement with Asia. Year 8 Australian indigenous students’ average mathematics scores are significantly lower than non-indigenous students, and also significantly lower than the international scale average as measured by TIMMS (Thomson, 2009; Thomson, Hillman & Wernert, 2012). These results have not improved over the last 20 years (Thomson, Wernert, O’Grady & Rodrigues, 2016). The average mathematics score for Year 8 indigenous students in 2015 was 438 compared with 509 for non-indigenous students. These results were almost identical to the scores in 2011.

Cultural perspectives addressed by cross-curriculum priorities present several challenges for how well teachers cater for students from different ethnic backgrounds and how well these students engage in a modern mathematics curriculum that is meant to create opportunities and enrich the lives of all Australians (ACARA, n.d.).
1.2 Rationale

Geometry is an undervalued part of mathematics, often seen simply as shape recognition, exploring visual patterns and very little else (Jones, 2002). Given the potential for geometry to develop students’ spatial abilities, reasoning skills, and abilities to solve real-world problems (Jones & Mooney, 2003; Marchis, 2012), there is a strong need for developing specific teaching and learning opportunities. These opportunities should engage all students in activities that enhance visualisation and geometric reasoning. Spatial ability provides us with capacities to perceive the visual world (Gardner & Hatch, 1989). However, there is a basic reluctance of students to use visualisation in ways to communicate geometric concepts (Presmeg, 2006).

There is much literature that highlights the role of visualisation and the role of language in supporting students’ development of geometric reasoning. Owen (1999) stated that students’ use of gestures played an important role in explaining visual images. Ufer, Heinze and Reiss (2009) suggested that construction of mental models was an important predictor of reasoning. Geometric reasoning involves selecting particular properties of a shape and then deducing other properties using mental interpretations of images (Cooke, 2007). The need to communicate what is ‘seen’ in a language that is consistent with a common mathematical dialogue is known as mathematical discourse (Sfard, 2000; Silver & Smith, 1996). Therefore, geometric discourse requires the use of specific terminology and diagrams as norms of exchange from everyday language to a mathematical way of communicating with each other (Moschkovich, 2003).

The purpose of this study was to investigate the ways in which Year 7 and 8 secondary school students and their teachers interacted with geometric tasks designed to develop visualisation and geometric reasoning skills. This study also explored the factors that influence the development of geometric reasoning among middle years students when learning about 2-dimensional (2D) shapes. The data will be examined through a geometrical discourse lens that offers insight into how students communicate their understanding and how they interact with others. There are few studies in geometry education that explore the data in this way. Due to the nature of the research questions, a qualitative research approach was adopted in order to best capture the required data.
Chapter 1

1.3 Research Questions

The main research question of the study is:

*What factors influence the development of geometric reasoning among middle years students when learning about 2-dimensional (2D) shapes?*

The main research question can further be explored by the following sub-questions:

- *How does visualisation underpin the teaching and learning of definitions and classifications of 2D shapes?*
- *How does language support the development of geometric reasoning?*
- *How does instruction influence the learning of geometric concepts?*

1.4 Methodological Approach

In this study, qualitative methods have been employed to generate data for analyses and interpretation. The level of description required to understand how students and teachers communicated geometric concepts necessitated a qualitative approach, allowing for adaptable data collection procedures and interpretations of emerging ideas (Boeije, 2010). A socio-constructivist perspective enabled an understanding of teaching and learning processes as participants interacted with geometric tasks.

A design-based research approach was used to develop and refine geometric tasks. The flexibility of design-based research allowed for an analysis of the nature of learning as it took place in messy classrooms where human variables are deliberately not controlled as in clinical settings (Collins, Joseph & Bielaczyc, 2004). Design-based research takes into account the particular setting and local interpretations (Barab & Squire, 2004; The Design-Based Research Collective [DBRC], 2003). This approach also allows for the active participation of the researcher to interact with students and teachers during the study (Schoenfeld, 1992a).

Participants for this study included Year 7 and 8 male and female students and their mathematics teachers in three multicultural classes across two secondary schools in Melbourne, Victoria. A combination of student-written work samples and associated rubrics, lesson observations, and video-recorded teaching episodes enabled the researcher to make close observations of the ways students engaged with geometric tasks and communicated their thinking. The role of the teacher in providing support and instruction for students during the task phases was also explored.
1.5 Thesis Structure

The role of visualisation in geometric reasoning is at the heart of this thesis. In order to understand how students solve geometric problems involving visualisation, a series of tasks were conducted with students working in groups. As students and teachers engaged in tasks and communicated with each other, elements of mathematical discourse, including the use of keywords and visual mediators, were analysed for patterns and various levels of geometric thinking. The classroom observations revealed that both students and their teachers had several misconceptions and difficulties that impacted on their ability to solve problems using geometric reasoning. Gaps in the design and implementation of tasks also emerged as a research problem requiring further testing in the field to strengthen their quality and impact.

A discussion of relevant historical, Australian, and cultural perspectives was briefly outlined in this chapter. These perspectives identified several barriers to the quality of teaching and learning of geometry in schools, and identified opportunities for teaching considerations to engage students in geometry. The rationale for the study, the research questions, and the methodological approach were also outlined in this chapter.

The literature pertinent to this study is explored and presented in Chapter 2. An overview of important models of geometric thinking and curriculum frameworks will be presented. Findings from the literature concerning the role of visualisation in spatial and geometric reasoning are also discussed. Further, important studies on the role of language and mathematical discourse in developing geometric concepts will be presented. Finally, pedagogical considerations including teacher knowledge for teaching geometry, and their role in managing tasks and facilitating group work will be explored, including significant barriers to developing certain geometric concepts pertaining to this study supported by contemporary research evidence.

The methodology of this study and procedures used for determining the validity and reliability of the data are presented in Chapter 3. A qualitative, constructivist approach has been used that employs design-based research to develop and implement tasks intended to understand the factors that influence students’ geometric reasoning.

The data from this study were summarised and analysed using Sfard’s (2008) interpretive framework for mathematical discourse to identify and develop themes of keywords and narratives for the interpretation. The analysis created a rich and multi-layered picture of the ways students and teachers use mathematical discourse elements
to communicate their reasoning with geometric concepts. The data analyses and interpretation are presented in Chapter 4, using summary tables and figures to illustrate individual and class responses to each task.

Finally, a synthesis of the data with respect to the research questions is provided in Chapter 5. Implications of this study are then outlined before conclusions and recommendations are drawn. Potential future studies are also provided in this final chapter.

1.6 Chapter Summary

An introduction to the study has been presented in this chapter, identifying key perspectives relevant to the teaching and learning of geometry and the significance of developing visual-spatial reasoning to solve real-world problems. The study’s rationale, research questions, and methodological approach were also outlined.

The literature review of important theoretical models for geometric thinking, curriculum frameworks, the role of visualisation, key aspects of language and mathematical discourse, and pedagogical considerations are presented in the next chapter.
Chapter 2

Literature Review

2.1 Introduction

Geometry lends itself to making rich connections between other mathematical domains through the contextual development of mathematical reasoning (Lew, Cho, Koh, Koh & Paek, 2012). These domains include discrete and continuous mathematics, algorithmic thinking, functions, limits and trigonometry (Goldenberg, Cuoco & Mark, 1998). The development of geometric reasoning is also an important auxiliary to solving problems in our daily lives (Couto & Vale, 2014; Marchis, 2012). Despite the benefits of learning geometry, its important links with other aspects of learning through STEM subjects, and geometry’s potential to solve real world problems, the amount of geometry included in high school curricula globally has reduced since the 1960s (Jones & Mooney, 2003). Today, school curricula do not include sufficient opportunities for students to develop spatial abilities (Mann, Mann, Strutz, Duncan & Yoon, 2011; Wai, Lubinski & Benbow, 2009; Webb, Lubinski & Benbow, 2007). Given the increased use of visual media in schools and at home, students need to be provided with explicit instruction in practices of reading, producing and understanding visual representations (Lowrie & Diezmann, 2009).

A review and examination of the literature of relevant theories of geometric thinking are now discussed. Relevant curriculum models were compared and contrasted in relation to their structure and how their content aligned to visualisation and geometric reasoning. Following this, the role of visualisation is discussed, and its implications regarding spatial reasoning and geometric thinking are explained. The role of language and mathematical discourse is given prominence as vital tools for the development and effective communication of geometric conceptual knowledge. Finally, key elements of whole-classroom teaching experiments of pedagogy, tasks designs, and the social setting described by Lamberg and Middleton (2009) are discussed in relation to:

(1) difficulties and misconceptions held by students and teachers as significant barriers to learning geometry;

(2) the role of the teacher in managing instructional practice;
(3) task design elements relevant to this study; and,
(4) the role of group work in the modern geometry classroom.

2.2 Geometric Reasoning Models and Curriculum Frameworks

Several geometric reasoning models underpin a significant volume of the literature and have been widely accepted as normative in this field. The first model, Piaget’s Stages of Cognitive Development developed in 1936, presented a process of the formation of spatial representations that is an important aspect of geometric reasoning (Elia, Gagatsis & van den Heuvel-Panhuizen, 2014; Jones, 2002; Panaoura & Gagatsis, 2010).

The second model, the van Hiele Model of Geometric Thought, is the most significant theoretical model for the development of geometric thinking and was developed in 1957. The van Hiele levels grew from observations Pierre van Hiele made of his own students when they engaged with geometric tasks. This model has had several renumbering and renaming of levels over the decades, and is rarely quoted in curriculum documents today. The inclusion of a preliminary level named prerecognition was added to the original model (Mason, 1998).

Biggs and Collis’ SOLO Taxonomy developed in 1982 attempted to capture the quality of a student’s learning rather than levels of thinking. The SOLO Taxonomy is presented, and a discussion of the developmental theories of Piaget and the SOLO taxonomy is given as a means of comparison and contrast with the van Hiele model.

Finally, The Big Ideas and Essential Understandings in Geometry developed in 2006 by the National Council of Teachers of Mathematics (NCTM) in the US, and the Australian Curriculum: Mathematics (ACM) are included as relevant curriculum frameworks that underpin related research into teacher professional development, preservice teacher training programs, and classroom resources. As significant curriculum models in the context of this study, they are analysed in relation to the structure of the content of geometry and related concepts of visualisation and reasoning.

2.2.1 Piaget’s Stages of Cognitive Development

Piaget believed that geometric thought developed in stages according to experiential order (Levenson, Tirosh & Tsamir, 2011). The model was underpinned by the notion that geometric thinking developed with the age of the child (Huitt & Hummel, 2003; Jones, 2002; Ly & Malone, 2010), and that mental structures developed through the child’s own activity and interactions within the environment (Clements & Battista,
1992). Piaget viewed knowledge as made up of logical structures resulting from actions, and contributing to a total mental structure:

To understand the development of knowledge, we must start with... the idea of an operation. Knowledge is not a copy of reality. To know an object, to know an event, is not simply to look at it and make a mental copy or image of it. To know an object is to act on it. To know is to modify, to transform the object, and to understand the process of this transformation, and as a consequence to understand the way the object is constructed. An operation is thus the essence of knowledge; it is an interiorised action which modifies the object of knowledge. (Piaget, 1964, p. 176)

According to Piaget (2008), the stages of cognitive development are:

- **Sensory-Motor** (from birth): A child reacts to the physical environment. It is this mode in which complex motor skills are acquired. The elements are the objects in the immediate physical environment.

- **Pre-operational Representation** (from about 18 months): Language emerges. A child develops words and images that stand for objects and events. The elements become signifiers.

- **Concrete Operations** (from about 6 years): Symbolic systems develop. Thinking is expressed through the use of symbol systems such as written language and number systems. The elements develop from signifiers to concepts and operations using logic.

- **Hypothetic-deductive Operations** (from about 16 years): Abstract concepts such as principles or theories are considered. Elements are concepts determined by deduced relationships.

The ages provided within each mode are indicative of when each mode typically starts occurring. The pre-operational stage is often referred to as *ikon* in that thinking is associated with images, imagination and language development, and a fifth stage *post formal*, is often included to be associated with challenging and extending the theoretical constructs developed in the formal mode and developing from about 20 years of age (Pegg, 1992).

Piaget conceived of geometry as a study of space, and he differentiated between topological and Euclidean figures (Levenson, Tirosh & Tsamir, 2011). His model has two themes. The first concerns the process of the formation of spatial representations and remains reasonably well supported. The second hypothesis suggests that the progressive organisation of geometric ideas follows a definite order and that this order is more experiential, and has mixed support (Clements & Battista, 1992; Newcombe & Huttenlocher, 2006; Newcombe & Stieff, 2012). For instance, Huitt and Hummel
(2003) found that all individuals do not move to the next cognitive stage as they mature. Jones’ (2002) view was that all types of geometric ideas appear to develop over time, becoming increasingly integrated and synthesised. Bishop (1980) further questioned the sequence of topological, projective, and Euclidean thinking of children learning geometry posited by Piaget.

2.2.2 The van Hiele Model of Geometric Thought

An extension to the ideas of Piaget’s *Stages of Cognitive Development* is the well-documented *van Hiele Model of Geometric Thought* (Levenson, Tirosh & Tsamir, 2011). The work of Pierre and Dina van Hiele provided a framework upon which geometry content may be planned, taught and assessed. According to the van Hiele model, geometric thought develops through five distinguishable levels – *visual, analysis, non-formal, formal deduction*, and *rigour* (Crowley, 1987). Progression is based on the principle that appropriate instruction is needed to move students through several qualitatively different levels of geometric thought and increasingly sophisticated levels of geometric understanding and reasoning (Battista, 2001; Way, 2011).

Van Hiele (1985) did not initially name each level and referred to the first level as a base level or level 0 where geometric thinking is in terms of shape recognition. Several researchers however have redefined van Hiele’s levels over time. According to Mason (1998), the van Hiele levels are:

- **Level 1: Visualisation** (also known as *recognition*). Students recognise features of shapes by appearance alone by comparison with known prototypes. Students perceive shapes but cannot describe shapes based on analysis of their attributes.
- **Level 2: Analysis** (also known as *descriptive*). Students focus on the relationship between parts of a shape and can recognise and name properties of a shape. Students are able to list properties but not discern the properties that are necessary or sufficient to describe shapes.
- **Level 3: Abstraction** (also known as *informal deduction, ordering or relational*). Students perceive relationships between properties and between figures. Logical implications and class inclusions are understood at an informal level.
- **Level 4: Deduction** (also known as *formal deduction*). Students can construct proofs and understand the role of definitions. They understand the meaning of necessary and sufficient conditions by deductive reasoning.
Level 5: Rigour. Students understand formal aspects of deduction as well as non-Euclidean systems at this level.

The van Hiele's model sequence includes five phases of activity to promote the development of higher levels of thinking from one level to the next – Inquiry, Direct Orientation, Explication, Free Orientation and Integration (van Hiele, 1985). Crowley (1987) promoted the van Hiele model as a useful tool for teaching, and emphasised the need for children to be presented with a variety of experiences where carefully chosen materials should ‘set the stage’ for geometric understandings. The van Hiele’s clearly take the final goal of geometry to be the realisation of geometry as a deductive structure (Hershkowitz, Ben-Chaim, Hoyles, Lappan, Mitchelmore & Vinner, 1990).

Pegg (1992) saw the levels as representing a logical hierarchical arrangement with the substance of each level flowing from an analysis or investigation of the previous level, and recommended a need for a level below the base level, as there were identified numbers of students who could not satisfactorily meet the van Hiele's Level 1 criteria. Clements and Battista (1992) also suggested that students first visualise a subset of characteristics of a shape resulting from an inability to distinguish between shapes, and so they proposed the existence of Level 0: Prerecognition. The version of the van Hiele model defined by Mason (1998) is used throughout this study as this contemporary version encapsulates all the levels of the original and preceding models, and includes a numbering and naming scheme.

The van Hiele levels are treated as a hierarchy of geometric thinking and are discontinuous because shifting from one level to another requires shifting between two different paradigms (Ekanayake, Brown & Chinnappan, 2012). When a student is reasoning at a particular van Hiele level, he/she uses particular language and geometric associations within that level (van Hiele, 1985). Essentially, two students who reason at different levels cannot be understood by each other since each uses different linguistic symbols and relationships to communicate their thinking (Suydam, 1985). Gutiérrez and Jaime (1998) found that students may be on different van Hiele levels for different concepts, and may master some abilities but not others within each van Hiele level. However, students can also simulate higher levels by learning rules or definitions by rote or by applying routine algorithms that they do not understand (Pegg, 1992) leading to misjudgment of a student’s level of thinking (van Hiele, 1985).
It is also often the case that teachers and their students “speak a very different language” (van Hiele, 1985, p. 245). In order for a teacher to be able to teach geometry at a student’s current level of geometric thinking, paradigmatic alignment is necessary for each to be understood (Burger & Shaughnessy, 1986; de Villiers, 1987; Houdement & Kuzniak, 2003; Mason, 1998; Panaoura & Gagatsis, 2010). This may mean that the teacher may need to modify the choice of language and images used to align with students’ current levels of thinking.

Language is broadly accepted as one of the main features that characterises one’s current level of geometric thinking (Hershkowitz, Ben-Chaim, Hoyles, Lappan, Mitchelmore & Vinner, 1990), because “each level has its own linguistic symbols” (van Hiele, 1985, p. 246). The visual level begins with non-verbal thinking. At the analysis level, language is used to describe shapes whereas, at the abstraction level, language is used to formulate definitions even though the intrinsic meaning of deduction is not understood (van Hiele, 1999). Lack of mastery at the abstraction level prevents most students from achieving higher levels of geometric thinking as observed by van Hiele. Similarly, several studies of high school students, and primary and secondary pre-service teachers, found that most do not think beyond the abstraction level (Browning, Edson, Kimani & Aslan-Tutak, 2014; Gutiérrez, Jaime & Fortuny, 1991; Halat, 2008; Halat & Sahin, 2008; Mayberry, 1983; Usiskin, 1982).

The van Hiele model has been used in several studies to examine geometric thinking of students, pre-service teachers, and in-service teachers. Mayberry (1983) confirmed the hierarchical nature of the van Hiele model by assessing 19 pre-service teachers’ geometric knowledge of squares, right triangles, isosceles triangles, circles, parallel lines, similarity, and congruence. Results indicated pre-service teachers’ difficulties with deducing relevant facts from a given statement, and difficulties perceiving a proof as a logical chain leading to a conclusion. Learning of geometric concepts in her study seemed to have been by rote. Mayberry found that 48% of pre-service teachers were reasoning at van Hiele level 2: analysis, or lower. Similar studies conducted by Gutiérrez, Jaime and Fortuny (1991) confirmed the low levels of geometric understanding among pre-service teachers.

Lawrie’s (1993) critique of Mayberry’s research suggested that it did not assign appropriate van Hiele levels for some students. Several items of the work did not measure student thinking at the levels indicated, with unequal treatments of concepts across levels, and an unequal distribution of questions across and within levels. This
points to the challenge of designing test items for determining the van Hiele levels of thinking in students. A subsequent study by Lawrie (1999) confirmed the theoretical underpinnings of the van Hiele model – that students speak a different level of language and have different mental organisations associated with each van Hiele level. Lawrie also found that higher performing students had difficulties in interpreting the thrust of lower level questions. This suggested that the van Hiele levels were discontinuous and highlighted the difficulties in using the model to assess student thinking.

Levenson, Tirosh and Tsamir (2011) found that the van Hiele levels might not be discrete and that a learner may display different levels of thinking for different contexts, different concepts, and different tasks, and that the levels may even develop simultaneously. Their study confirmed similar findings by Lawrie, and earlier works by Burger and Shaughnessy (1986) and Mayberry (1983). Nonetheless, Atebe and Schäfer (2008) found that senior high school Nigerian and South African students whose instructional experiences in geometry were aligned to the van Hiele levels showed a better understanding of geometric concepts than those whose experiences were not.

The van Hiele model has strongly influenced research into geometric thinking for 2-dimensional shapes (Owens & Outhred, 2006) and provides a suitable framework for describing students’ geometric concept development. However, the theory defined as discrete level of geometric development is problematic as visualisation, for example, is the first van Hiele level but is also necessary at all other levels of geometric development (Jones, 2002). To understand the cognitive processes underlying geometric thinking and movement from one level to the next, it is necessary to understand how students construct and mentally represent spatial and geometric knowledge beginning with perception and imagery (Battista, 2007).

Similarities between the models of van Hiele and Piaget ascribe student understanding to a series of levels or stages. Clements, Battista and Sarama (2001) used a synthesis of both Piaget’s theory and the van Hiele model as a basis for their research on the Logo Geometry project and found that reasoning can occur at multiple levels at the same time. Nevertheless, there are important differences between the two models. Piaget’s stages relate mainly to geometry as the science of space, whereas the van Hiele model combines geometry as the science of space and geometry as a tool with which to demonstrate a mathematical structure (Hershkowitz, Ben-Chaim, Hoyles, Lappan, Mitchelmore & Vinner, 1990). Further, the van Hiele model suggests that the phases of instruction that help students to progress from one level to the next level (Freudenthal,
1973; Pegg, 1992; van Hiele, 1985; 1999), and concentrates on learning rather than development. This places greater emphasis on language in moving from one level to the next (Pegg, 1992).

### 2.2.3 Structure of Observed Learning Outcome –SOLO Taxonomy

The *Structure of Observed Learning Outcome (SOLO) Taxonomy* evolved as a means of objectively assessing qualitative levels of student learning (Biggs, 1979). It was also developed to address perceived shortcomings in *Piaget’s Stages of Cognitive Development* that placed little emphasis on the role of language, and instead had concentrated on age development rather than learning (Pegg, 1992). The *SOLO Taxonomy* defined two aspects of a student’s response to a question or problem (Biggs & Tang, 2007). The first, *modes of thinking*, are at the functional level of the elements used, and build on the stages outlined by Piaget (Biggs, 1979; Pegg, 1992). While the modes of thinking have much in common with Piaget’s model, newly developed modes do not subsume or replace earlier modes, and latter developed modes could assist growth in earlier developed modes (Pegg, Gutiérrez & Huerta, 1998).

Associated within each mode are *levels of response* which provide a measure of the complexity of how a response relates to information in a question, the amount of memory or attention span required, and the quality of a conclusion or answer (Biggs & Collis, 1982). Jurdak (1991) provided examples of each response level to a task consisting of identifying quadrilaterals from a given sheet of drawn quadrilaterals by putting *S* on each square, *R* on each rectangle, *P* on each parallelogram and *B* on each rhombus; and to define the figure by giving a minimal list for characterising each figure, (the necessary and sufficient conditions to define it). This task was taken from Burger and Shaughnessy’s (1986) earlier work aligning student responses to the van Hiele levels.

The *levels of response* (Biggs, 1979), with examples from Jurdak (1991) are:

- **Pre-structural**: Where the response is based on an irrelevant aspect.
  *Example response:* Did not attempt the task or provided an unrelated response.

- **Uni-structural**: Where the response is based on a single relevant aspect.
  *Example response:* Used one relevant aspect (i.e. the figure).

- **Multi-structural**: Where the response is based on multiple aspects that are seen as independent.
  *Example response:* Used several relevant disjoint aspects (i.e. the quadrilateral and use of its separate properties).
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- Relational: Where the response is based on relationships between several aspects.
  Example response: Used all relevant information and the relationships among them (i.e. the quadrilateral, properties of its components, and sufficient conditions to define the shape).

- Extended abstract: Where the response goes beyond the thinking of the mode in question and exhibits a new way of thinking.
  Example response: Comprehensive use of the given information (i.e. figures, properties, relations) with related hypothetical constructs and abstract principles (i.e. tested on the data).

Serow (2007b) conducted a qualitative research comparison of the complexity of different interview tasks applied to 2D shapes, involving in-depth interviews with 24 Year 8 to 12 students in two secondary schools. Twelve of the students repeated the interview tasks two year later. Analysis of students’ understandings of class inclusion confirmed the hierarchical nature of the SOLO taxonomy. An application of Rasch analysis in Serow’s examination identified hierarchical pathways that were inherent in the development of mathematical concepts (Callingham & Bond, 2006).

Jurdak (1991) suggested that the SOLO taxonomy aligned to the van Hiele model with the exception of a SOLO level of response coinciding with van Hiele level 5: rigour. By aligning the levels within the SOLO taxonomy with Piaget stages of learning, Pegg (1992) proposed that there was a very large range of ability within van Hiele level 3: abstraction, as well as the relational level of the SOLO taxonomy. Namely, three aspects are relevant to the abstraction/relational levels – an ordering of properties; class inclusion of figures; and, simple deductions. The particular depth of understanding of these aspects by students before they are deemed to be thinking at van Hiele level 3: abstraction, is still a research challenge (Pegg, 1992).

Whereas the van Hiele model applies to geometric thinking, the SOLO taxonomy can be applied more broadly (Gray, Pinto, Pitta & Tall, 1999). A statistical application on data tabulation and representation by Reading (1999) found that teachers could use the SOLO taxonomy to assess students’ understanding of statistical concepts. When the SOLO taxonomy is applied to geometry, the application and use of symbols through written language and embedded geometric concepts can be analysed. Collis and Campbell (1987) applied the SOLO taxonomy to 3D tasks and found that children’s ability to conceptualise 3D shapes required a mastery of separate skills of relevant numerical operations and an understanding of the internal structures of solid shapes.
Visualisation in 3-dimensional geometry was a neglected area of research according to Gutiérrez (1996). Lawrie, Pegg and Gutiérrez (2002) advocated for a didactic hierarchy for 3D geometry due to, as they claimed, the lack of attention to ‘solid’ geometry. In their study involving 181 students across all year levels in four secondary schools in New South Wales, SOLO and van Hiele were used to analyse the nature of students’ understanding of 3D cross-section and nets of 3D figures. The range of student responses indicated a hierarchy of difficulties in the understanding of cross-sections of solids using both models. This suggested that a curriculum framework for instruction in understanding the cross-sections of 3D figures could be provided with either the SOLO taxonomy or the van Hiele model, and expanded to include the whole of 3D geometry and spatial reasoning. Levenson, Tirosh and Tsamir (2011) similarly proposed an extension of the van Hiele model to improve reasoning with 3D shapes.

The SOLO taxonomy provides a useful model for the development and assessment of tasks as it allows for a systematic way of describing how a student’s performance might grow in complexity (Jurdak, 1991; Lawrie, Pegg & Gutiérrez, 2002). It has been used to describe where students should be operating, and for evaluating learning outcomes so that the level at which students are actually operating is known (Biggs & Tang, 2007; Pegg & Woolley, 1994). Combining models appears to have research validity, and this confirms earlier work by Pegg. By combining the SOLO taxonomy with the van Hiele model to explain student growth, Pegg (1992) believed that it might place greater emphasis on the role of the teacher to make language an important aspect of teaching to enable student development.

2.2.4 Curriculum Models for Teaching and Learning Geometry

The National Council of Teachers of Mathematics (NCTM)

The NCTM has been a significant mathematics education and curriculum research body in the USA since 1920, and it remains an influential committee providing direction for educational change (McLeod, Stake, Schappelle & Mellissinos, 1995). The Big Ideas in mathematics created by NCTM are key to connecting mathematical concepts with actions (Australian Association of Mathematics Teachers [AAMT], 2009). “A Big Idea is a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole” (Charles & Carmel, 2005, p. 10). This may be seen as ideas that link several mathematical concepts within and across content areas. Understanding of the Big Ideas means that mathematics is not
seen as disconnected concepts, skills, and facts, but rather is seen as a coherent set of ideas (Charles & Carmel, 2005).

The *Big Ideas* framework (NCTM, 2012) for high school geometry underscores the significance of geometric thinking through working with imagery through diagrams. The transition from *Classifying, naming, defining, posing, conjecturing, and justifying as codependent activities in geometric investigations* (Big Idea 4: Year 6 to 8), to *Working with and on definitions* (Big Idea 3: Year 9 to 12), and *A written proof is the endpoint of the process of proving* (Big Idea 4: Year 9 to 12), align with the van Hiele levels.

As a framework for geometry curriculum, the principles of the *Big Ideas* represent mathematics as a coherent and connected enterprise. However, as Siemen, Bleckly and Neal (2012) suggested, the framework makes “no claims about possible learning progressions or developmental priorities beyond what is loosely and perhaps unintentionally implied by the organisation of the list” (p. 21).

**Australian Curriculum: Mathematics [ACM]**

The Australian Curriculum: Mathematics located in ACARA organises curriculum into content strands. Geometry is combined with Measurement signifying the relationship between the two content areas. The content of geometry is organised into sub-strands of shapes, location and transformation, and geometric reasoning, and defined by a set of content descriptors (ACARA, n.d.). Learning of content should be developed through the four proficiency stands of understanding, fluency, problem-solving, and reasoning - adapted from the United States’ report to the National Research Council, *Adding it up: Helping Children Learn Mathematics* by Kilpatrick, Swafford and Findell (2001). The proficiencies are a means by which students can engage with content through experiences and actions (Atweh, Miller & Thornton, 2012).

The Australian Curriculum: Mathematics, however, provides little indication of the role of visualisation in geometry, and any transition toward Euclidean concepts of defining and proving seems to be lacking. Despite Seah’s (2015a) assertion that the Australian Curriculum “emphasises the need to help children develop an increasingly sophisticated understanding of geometric ideas, to be able to define, compare and construct figures and objects, and to develop geometric arguments” (p. 4), and Obara’s (2013) claim that the Australian Curriculum: Mathematics “highlights the importance of teaching spatial reasoning” (p. 21), most learning in primary and secondary contexts
revolves around number and algebra (Lowrie, Logan & Scriven, 2012). The emphasis on geometry and particularly spatial reasoning is not present.

The proficiency strands, however, allow for a “holistic development of numeracy capabilities rather than just a set of essential skills and knowledge” (Day & Hurrell, 2013, p. 52). The emphases on problem-solving and reasoning in the ACM provide opportunities for the building blocks of mathematical reasoning and proof (Richardson, Carter & Berenson, 2010). “In learning the art of mathematical problem-solving, reasoning is required at each decision-making step and in drawing appropriate conclusions” (Wares, 2014, p. 60). Wares (2014) claimed that non-routine problems were a prominent part of any reformed mathematics classroom. His account of problem-solving activities involving paper folding highlighted how several mathematical concepts could be visualised. Angle bisection, congruence of shapes, properties of right triangles, similar triangles, reflection, and rotation become more tangible and vivid, and promote mathematical discussion in the classroom. Richardson, Carter and Berenson (2010) found that through a series of geometric tasks, different solution methods supported students’ ability to make conjecture and to argue, and therefore, enhancing their classroom interactions and increasing their level of sophistication in justifying, explaining, and solving future related mathematical problems. It is through these proficiencies that spatial reasoning and geometric thinking may, therefore, be developed.

The structure of the NCTM’s Big Ideas and ACM are useful articulations of geometric content. The Big Ideas place a greater importance on imagery and the nature of geometric figures, and the ACM proficiencies provide opportunities to develop the Big Ideas in Australian schools (Siemon, Bleckly & Neal, 2012).

2.2.5 Summary

Even though the globality of the van Hiele levels are doubtful (Hershkowitz, Ben-Chaim, Hoyles, Lappan, Mitchelmore & Vinner, 1990; Levenson, Tirosh & Tsamir, 2011), the model provides a useful framework to qualify students’ reasoning and communication of geometric concepts (Burger & Shaughnessy, 1986; Gutiérrez & Jaime, 1998). The model is also useful for describing the ways teachers communicate geometric concepts to their students (Swafford, Jones & Thornton, 1997). Although the legacy of van Hiele pervades a lot of literature in geometry, theories underpinning several other models were also taken into consideration in this study.
An understanding of curriculum structures and content was an important consideration in task selection processes and analyses of student responses to tasks in this study. The NCTM *Big Ideas* provided a coherent curriculum framework for learning geometry, and it highlights the significance of visualisation. The ACM indicated the importance of reasoning and problem-solving through its proficiency stands. The theories of geometric thinking (ie. Piaget, van Hiele, SOLO) and the relevant curriculum models framed visualisation as a necessary component in the development of geometric thinking.

### 2.3 The Role of Visualisation in Geometry

The importance of visualisation can be seen through the explicit attention it is given in *Piaget’s stages of Cognitive Development, the van Hiele Model of Geometric Thought,* and the *SOLO Taxonomy.* While visualisation in the Australian Curriculum: Mathematics is much less formalised, the NCTM makes visualisation explicit – *geometric thinking involves developing, attending to, and learning how to work with imagery* (Big Idea 2: Year 6 to 8), *a geometric object is a mental object that, when constructed, carries with it traces of the tools or tools by which it was constructed* (Big Idea 3: Year 6 to 8), *and working with diagrams is central to geometric thinking* (Big Idea 1: Year 9 to 12).

This section of the literature review describes visualisation as an important process for developing spatial abilities and geometric reasoning. First, visualisation is discussed as external and internal processes for generating and manipulating images – an essential aspect of spatial and geometric reasoning. Secondly, the role of visualisation is presented as a critical element for developing spatial abilities. Finally, the nature of visualisation in geometric reasoning is examined, and its role is argued as a necessary tool for developing and communicating proofs – the ultimate goal of geometry (Serow, 2007b).

#### 2.3.1 Visualisation

Couto and Vale (2014) described geometry as comprising of visualisation and the comprehension of geometrical shapes. Visualisation is also an important aspect of spatial reasoning (Clements, 1982; Liben, 2006; Maier, 1996; Newcombe & Stieff, 2012; Owens, 2015). Van Klinken (2010) saw geometry, as spatial thinking made up of two main skill-sets – spatial orientation and spatial visualisation. As students progress in school, they develop the skills of visualisation, along with intuition, perspective,
problem-solving, conjecturing, deductive reasoning, logical argument and proof (Jones, 2002). Ability to reason about shapes begins visually with non-verbal thinking (van Hiele, 1999) where children judge figures by their appearances without the words necessary for describing what they see (Levenson, Tirosh & Tsamir, 2011).

Both concrete and virtual manipulatives are widely used and have been shown to improve the visual skills of students. Effective use of manipulatives needs to be carefully planned and integral to the lesson (Swan & Marshall, 2009). Concrete experiences that provide tactile-kinesthetic experiences help children develop concepts. Through classroom dialogue, what students know about particular properties of shapes can then be assessed (Clements, 1999; Moyer, 2001). Virtual manipulatives similarly develop an understanding of mathematical concepts by forming dynamic images and by performing actions on objects (Clements, 1999; Moyer, Bolyard & Spikell, 2002). Visualisation, therefore, is not simply an ability to see and describe an object. Visualisation is the ability to represent, transform, generate, document, and reflect on visual information (Hershkowitz, Ben-Chaim, Hoyles, Lappan, Mitchelmore & Vinner, 1990):

[It is] the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings… (Arcavi, 2003, p. 217)

In the research literature, many alternative descriptions are used for the process of visualisation. These include visual reasoning (Dreyfus, 1991), visuospatial ability (Elia, Gagatsis & van den Heuvel-Panhuizen, 2014; Pittalis & Christou, 2010), spatial visualisation (Fennema & Tartrre, 1985; Kosslyn, 1983), imagery (Bishop, 1988; Clements, 1982; Cooke, 2007; Presmeg, 1996), and others. Duval (1998) argued that visualisation was an important cognitive process and described visualisation as the activity of taking an external representation, usually a diagram or picture, and formulating an internal representation or image (Duval, 1999).

Dreyfus (1991) also claimed that the use of diagrams to discover, describe and justify geometric results was necessary for learning about shapes. In contrast, Bishop (1988) and Fischbein (1987) described visualisation as a complex process that involved imagery with or without a diagram, to organise data into meaningful structures that were important in guiding the analytical development of a solution. There was no general agreement about the terminology reflecting the diversity of areas where visualisation was considered to be relevant (Gutiérrez, 1996). However, there was
agreement that visualisation was important in geometry as the types of mental processes involved were necessary for other areas of mathematics. Visualisation, whether internally or externally represented, was also accepted as a central ingredient to mathematical and scientific discourse (Lowrie, 2012; Newcombe & Stieff, 2012, Sfard, 2008).

Bishop (1988) suggested that imagery might be described as either *primitive* (such as imagining a door handle being rotated) or *abstract* (such as an imagined right triangle inscribed in a circle) to imply that the form an image takes is central to how it may be used to communicate visually. According to Kosslyn (1983), the ability to generate and manipulate images involved four processes: *generating an image, inspecting an image, maintaining an image, and transforming and operating on an image*. Mental imagery or internal representations included various forms of concrete and dynamic constructions (Presmeg, 1986) that involved a “picture in the mind” (Clements, 1982). External representations included symbolic systems of mathematics or graphical representations (Lowrie, 2012).

Owens (1999) asserted that the ability to create mental imagery was inherent in us all. Despite this assertion, Presmeg (1986) had earlier found that some individuals do not have a preference for using visual methods for solving problems. Instead, they preferred rote-type learning experiences, while other individuals generated images in particular learning situations. Presmeg’s model defined five types of visual imagery that she observed in her students:

1. *Concrete, pictorial imagery* – pictures-in-the-mind;
2. *Pattern imagery* – pure relationships depicted in a visual-spatial scheme, a schematic image capturing regularities;
3. *Memory images of formulae* – provide quick means of recall of abstract procedures;
4. *Kinesthetic imagery* – involving physical movement; and,
5. *Dynamic imagery* – the image itself is moved or transformed (familiar in the current computer technologies context of learning).

The types of imagery observed by Presmeg (1986) provided a basis for her description of imagery as “the occurrence of mental activity corresponding to the perception of an object” (p. 42). This was deliberately broad enough to include shapes and patterns, and allow for verbal, numerical or mathematical symbols to be arranged in the form of an image.

Lowrie (2012) suggested that a high proportion of tasks have images embedded
in their representation, whereas formerly geometric problems focussed on word-based problems where the teacher taught heuristics of drawing diagrams and imagining a problem scene. Presmeg (2006) found that the use of images and diagrams occurred quite naturally for most but not all students. However, if students have not been taught how to use diagrams, they could not take advantage of the many features that a diagram represents, nor how these features are interpreted and inter-related (Dreyfus, 1991). Fischbein (1987) claimed that diagrams represent the ‘original reality’ via an *intervening conceptual structure* with its own laws and constraints.

In this study, visualisation is understood as a process of generating a mental image, whether static or dynamic (Bishop, 1988). The image generated as a mental construct depicts visual or spatial information (Presmeg, 2006), and is a necessary important tool for reasoning and generating proofs (Dreyfus, 1991). This definition is broad enough to include the types of imagery generated by students as in Presmeg’s Model, and includes processes of constructing and transforming images of a spatial nature that are implicated in geometry. Visualisation is the key to the development of spatial thinking (Owens, 1999), and is at the heart of spatial work (Bishop, 1980).

Essentially, visualisation is vital for communicating, both verbally and non-verbally, conceptual understanding in geometry (Arcavi, 2003; Clements, 1999; Lowrie, 2012; Moyer, 2001; Moyer, Bolyard & Spikell, 2002; Newcombe & Stieff, 2012; and many others). This study examines the ways in which students and teachers utilised visualisation as a communicative tool during tasks dealing with 2D shapes.

### 2.3.2 Spatial Abilities

Spatial ability consists of a set of cognitive processes to construct, represent and manipulate objects (Clements & Battista, 1992; Maier, 1996; Ramful, Lowrie & Logan, 2016). According to Lohman (1996), spatial ability is

> … the ability to generate, retain, retrieve, and transform well-structured visual images. It is not a unitary construct. There are, in fact, several spatial abilities, each emphasising different aspects of the process of image generation, storage, retrieval, and transformation. (p. 3)

According to Yeh (2013), there are three major spatial ability factors. These are:

1. **Spatial visualisation**: the ability to mentally rotate, manipulate, and twist two- and three-dimensional stimulus objects (McGee, 1979). Students with high spatial visualisation ability tend to produce images that are schematic (images that encode the spatial relations described in a problem) in nature, whereas
students with low spatial visualisation ability produce images that are pictorial in nature (Phillips, Norris & Macnab, 2010). This suggests that deficits in spatial visualisation competencies may interfere with the ability to solve word problems in mathematics (van Garderen, 2006).

2. **Spatial orientation**: the comprehension of the arrangement of elements within a visual stimulus pattern; the aptitude for remaining unconfused by the changing orientations in which a figure may be presented; the ability to determine spatial relation with respect to one’s body; is the ability to know where an object is in space and its relationship to the position of other objects such as in mapping (van Klinken, 2010).

3. **Spatial relations**: the ability to mentally transform (i.e. translate or rotate) objects with respect to an environmental frame of reference (i.e. a landmark or cardinal points) while one’s egocentric reference frame does not change. Clements (1999) and Kosslyn (1983) include spatial relations within their broader definitions of spatial orientation and spatial visualisation.

Yeh’s (2013) definition of spatial visualisation is taken from McGee (1979) and is the definition that is also used in this study. This definition links spatial visualisation to the stimulus created by an object or an image. Lohman (1996) also acknowledged the existence of other elements such as spatial perception (the ability to recognise horizontal and vertical lines) and mental rotation (the ability to rotate 2D or 3D figures), as depicted in Maier’s (1996) model, but he labelled them as minor factors (Pittalis & Christou, 2010). Del Grande and Morrow (1993) also claimed that spatial perception included eye-motor coordination, figure-ground perception, perceptual constancy, position-in space, perception of spatial relationships, visual discrimination, and visual memory, involving real and imagined movement or displacement by mentally changing shapes through transformations. Spatial perception skills cannot be considered as minor as they are necessary for students to comprehend tasks involving graphical information (Diezmann & Lowrie, 2012).

Spatial ability, essential for survival is found in many organisms (Newcombe & Huttenlocher, 2006). Human lives require spatial thinking at home, at work, at school, and at play. These all involve space-related actions, perceptions, and representations. Even very young infants perceive spatial qualities like depth, distance, size, shape, and position (Liben & Christensen, 2011). However, according to Liben (2006), the
constructs of space and spatial thinking have no commonly agreed upon set of components or universally shared formalisms.

Despite the variety of spatial factors used to define spatial ability, definitions such as spatial visualisation, appear to make use of visual imagery. For example, McGee (1979) defined spatial visualisation as the “ability to mentally manipulate, rotate, or twist, or invert a pictorially presented stimulus object” (p. 893). Clements (1982) suggested that mental imagery was a type of spatial ability. Conversely, Presmeg (1986) suggested that spatial ability was linked to visual imagery. Prusak, Hershkowitz and Schwarz (2012) also claimed that spatial ability is based on both visual and intuitive considerations.

Mohler (2009) identified that spatial ability improved with age in childhood resulting from increased processing speed, knowledge, and experience, but that it declines with age in adulthood. Further, there are important gender differences at all ages in spatial ability (Bishop, 1980; Halat & Sahin, 2008; Kozhevnikov, Kosslyn & Shephard, 2005; McGee, 1979; Newcombe & Huttenlocher, 2006). Spatial ability favours males across regions, classes, ethnic groups and ages. For example, Boakes (2009) found that males have superior skills in mental rotation when performing origami tasks. Commonly, in spatial tasks, spatial perception, mathematical reasoning, and targeting ability, males outperform females. In verbal fluency, perceptual speed, memory, and certain motor skills, females outperform males. Some of the reasons for this might be a combination of child-rearing, cultural and educational environments (Fennema & Tartre, 1985; Liben & Christensen, 2011). Newcombe and Stieff (2012) argued that gender differences in spatial thinking caused by biological differences were a myth.

Developing spatial concepts through kinesthetic activities with young children has been shown to engage students, and promote mathematical discussion and contextual understandings (Wood, 2008). Duatepe-Paksu and Ubuz (2009) found that visualisation played a critical role in drama-based geometry instruction. This is an example of Presmeg’s (1986) *kinesthetic imagery* – using the human body to model shapes, or geometric concepts. Sfard (2008) also included kinesthetic and gestural forms of communication in her interpretive model of mathematical discourse, and highly relevant to this study.

Owens (1999) provided a *Framework of Imagery for Space Mathematics* that consisted of three aspects of spatial knowledge that become evident in the way students...
behave and respond to tasks through the use of different operational strategies – words, gestures, drawings, or using materials for different aspects of spatial knowledge.

Three aspects of spatial knowledge depicted by Owens (1999) are:

1. **Orientation and Motion**: Changing perspective and orientation are related to motion. Movement is ‘imaged’ by children as they make associations between shapes.

2. **Part-Whole Recognition**: All shapes are made up of parts. Children develop a repertoire of properties of shapes when they notice these parts.

3. **Classification and Language**: Verbal expressions are associated with visual imagery and help define it. Children will associate particular words consistently with particular actions, shapes, and other spatial relationships.

Owen’s (1999) framework highlights the dynamic nature of imagery and links aspects of language as representative of visual imagery inferred from student actions and words. This model portrays the use of imagery to mediate spatial abilities and describes key features of spatial knowledge for young children. However, research into task designs and learning experiences that improve students’ ability more generally to benefit from visualisations that develop their spatial perception and therefore geometric reasoning, seems somewhat scant (Newcombe & Stieff, 2012). There is also a lack of research that considers how well new technologies can serve to exercise, and hence foster, students’ spatial skills (Liben & Christensen, 2011).

There is a strong relationship between spatial ability and the learning of geometry (Gutiérrez, Pegg & Lawrie, 2004). Effective learning involves a range of examples including non-prototypical examples and distractors, including those visually similar to examples, to build valid and strong images of geometric concepts, including dynamic and flexible imagery (Clements & Sarama, 2011). Developing spatial reasoning involves decoding visually represented information and encoding internalised images (Lowrie, 2012). This is a dynamic mental process where the spatial relations between the objects are changed (Maier, 1996).

Spatial ability is an example of higher-order thinking in mathematics and science that helps students solve practical and theoretical problems through the creation of spatial images and mental manipulation of objects (Pittalis & Christou, 2010). Students who develop their spatial abilities are likely to spontaneously generate mental and physical models easily and are likely to succeed in tasks and occupations that require spatial abilities (Lohman, 1996; Mohler, 2008). The significance of spatial skills
lies in its critical importance across the curriculum, particularly within the STEM disciplines (Uttal, Meadow, Tipton, Hand, Alden, Warren & Newcombe, 2013; Wai, Lubinski & Benbow, 2009). Spatial ability enhances educational and occupational outcomes (Lowrie, Logan & Scriven, 2012; Obara, 2013) yet is not explicitly taught nor developed through our current Australian Curriculum Framework.

It is generally accepted that spatial ability comprises several components, of which visualisation (often termed visual-spatial reasoning or spatial visualisation) is one element (Clements & Battista, 1992; Liben, 2006; Maier, 1996; Yeh, 2013). This also refers to the use of combinations of verbal or visual strategies to perform spatial tasks (Liben, 2006). The combination of the verbal and the visual are key aspects of spatial reasoning explored in this study.

2.3.3 Processes of Geometric Reasoning

Reasoning can be defined by five interrelated processes of mathematical thinking. These are categorized as sense-making, conjecturing, convincing, reflecting, and generalising (Bjuland, 2007; Schoenfeld, 1992a). Geometric reasoning involves identifying and selecting particular properties of shapes and then deducing other properties from them (Cooke, 2007). “Any move, any trial and error, any procedure to solve a difficulty is considered as a form of reasoning” (Duval, 1998, p. 37). Reasoning is a mental activity not simply a recitation of a memorised proof (Brousseau & Gibel, 2005). For students to be literate in mathematics they need to be able to decode and construct the language of reasoning:

The language of reasoning is the ultimate goal of mathematics teaching and learning. It refers to the language that the teacher and the students use in mathematics-based problem-solving contexts, and it develops out of the language of reflection and only after descriptive and comparative language is well-established. Causal relationships are central to the language of reasoning, so such language may include comparisons, predictions, inferences, considerations as well as the verbs and adverbs of possibility. (Serow, 2007b, p. 185)

Clements (2014) and Hershkowitz et al. (1990) also saw the ultimate goal of mathematics to be about deductive reasoning. The van Hiele Model of Geometric Thought depicted several preliminary stages of geometric reasoning, preceding level 4: deduction (Mason, 1998). Visual thinking, as in the initial stages of geometric reasoning (van Hiele level 0: prerecognition, and level 1: visualisation) involves thinking tied down by visual ideas. Moving beyond visual stores of images for shapes to analytical knowledge requires connecting with spatial knowledge about shapes (van Hiele level 2: analysis, and level 3: abstraction) – for example, why a shape does or does not belong
to a shape class (Kosslyn, 1983). Geometric reasoning, therefore, is developed through different stages as depicted by the *van Hiele Model of Geometric Thought* (van Hiele, 1985).

Aligned to the *van Hiele Model*, Gutiérrez and Jaime (1998) identified four processes of geometric reasoning:

1. **Recognition** of types and families of geometric figures, and identification of components and properties of the figures.

2. **Definition** of a geometrical concept. This process can be viewed in two ways – as the students *formulate* definitions of a particular concept, and as the students *use* a given definition read in a textbook, or heard from the teacher or another student. This two-way view of *definition* was also noted by Heis (2014).

3. **Classification** of geometric figures or concepts into different families or classes.

4. **Proof** of properties or statements, that is, to explain in some convincing way why such property or statement is true.

Each of the four processes aligns to two or more van Hiele levels of reasoning. For example, the process of *recognition* maps across van Hiele level 1: *visualisation*, where students identify physical attributes of a shape, and van Hiele level 2: *analysis*, where students identify mathematical properties of a shape (Gutiérrez & Jaime, 1998). This model outlines a hierarchy of reasoning processes of which informal and formal deductive methods are the higher levels of abstraction and aligns to both Piaget’s *concrete operational level* and the *relational level of response* in the SOLO Taxonomy (Collis, Romberg & Jurdak, 1986).

Diagrams and other forms of visual representation are an essential and legitimate component of deductive reasoning (Dreyfus, 1991), and are important communicative tools. Computer software provides the advantage of flexibility with images and visual reasoning (such as ‘dragging’ and other changes to shapes) to allow for variance and invariance (NCTM, 2012) of properties to be determined when shapes are transformed (Clements, 1999; Dreyfus, 1991; Moyer, Bolyard & Spikell, 2002; Prusak, Hershkowitz & Schwarz, 2011). The effective use of diagrams as a communicative tool necessitates an understanding of the universal conventions (Diezmann, 1994), also known as mathematical signifiers, to indicate particular properties on a figure (such as a square in a corner \( \square \) for a right angle, or the use of arrowheads \( \iff \) for parallel lines).
A specific geometric diagram embodies the attributes of a class, providing students with prototypes.prototypes in geometry are generalised representations having common visual characteristics and are useful for simple manipulations of orientation. However, they are limited references to geometrical concepts (Gutiérrez & Jaime, 1999) and do not support hierarchical, inclusive definitions (Presmeg, 2006). A prototype has internal constraints of organisation (Duval, 1995). Students need to be able to explore shapes by ‘seeing the parts’ – a notion that Owens (2003) referred to as disembedding. Making sense out of a visual representation involves re-seeing it from a different angle but not mentally moving the object (Tartre, 1990). “The power of seeing geometric properties detach themselves from the figure… might be a potent tool for informing the design of experimental tasks in geometry… so that they build effectively on geometric intuition” (Fujita & Jones, 2003, p. 48). An image is no longer a ‘picture in the head’ but rather images that are more abstract, malleable, less crisp, and often segmented into parts (Kosslyn, 1983).

Moving beyond the reasoning process of recognition defined by Gutiérrez and Jaime (1998) and mediated by diagrams, involves thinking and reasoning about definitions. Ekanayake, Brown and Chinnappan (2003) described definitions as a critical skill of inferring. “Backward inference, forward inference, bi-directional inference, and drawing auxiliary lines are not rules but heuristics that reduce unnecessary inference” (p. 2). The process of looking back on a solution process and reflecting on a solution itself is considered to be a heuristic strategy used for developing an understanding of definitions (Bjuland, 2004; Peterson, 1997). Gray, Pinto, Pitta and Tall (1999) viewed definitions as ‘facing in two ways’ – back to previous experiences and forward to the construction of theorems. The process of defining requires using both known theorems and known properties of shape, deducing a specific result in relation to a figure with given properties, considering alternative definitions of geometrical shapes, and deciding which properties are necessary, sufficient and minimal (Brown, Jones, Taylor & Hirst, 2004).

Deductive reasoning, as a part of geometric reasoning is the way in which relationships are established by thinking about geometric diagrams, exercising powers of mental imagery, and making generalisations (Cooke, 2007; Panaoura & Gagatsis, 2010). It is thinking that progressively passes from where objects and their properties are controlled by perception to where they are controlled by the explication of properties (Panaoura & Gagatsis, 2010). To understand the processes of deductive
reasoning (such as definition and proof) is to understand Euclidean geometry. Euclidean geometry is an axiomatic system enabling a deductive description of the entire body of geometric knowledge that was known at the time of Euclid (Battista, 2001). Euclidean concepts are abstract spatial system concepts because they provide the structures by which locations and objects are represented in reference to an abstract, stable, general system (Liben & Christensen, 2011). The ‘Euclidean approach’ to teaching geometry was often referred to as ‘ruler and compass geometry’ because of its centrality of construction problems in Euclid’s work (Mariotti, 2000). Kilpatrick, Swafford and Findell (2001) suggested returning geometry to its Euclidean roots where students mentally structure and revise their construction of space. It is in the learning of Euclidean geometry where students learned about proofs (Jones & Rodd, 2001).

As students progress to classification and proofs, geometric figures are used to mediate communication of concepts. Proofs are arguments that convince someone who knows the subject, and there is no reason why some of the reasoning cannot be diagrammatic or visual (Dreyfus, 1991). Generalisation is the abstraction of similarities in solutions to problems where steps taken in solutions with similar conditions and goals, result in a rule that may be applied to a larger class of problems (Thompson, 1985).

Deductive reasoning is developed at van Hiele level 3: abstraction (also referred to as informal deduction), where students perceive relationships between properties and between figures, and can create meaningful definitions and justify their reasoning (Mason, 1998). It is not until van Hiele level 4: deduction, where students consolidate their construction of proofs, and understand the role of definitions, knowing the meaning of necessary and sufficient conditions. Deductive reasoning occurs at the Hypothetic-deductive Operations stage of Piaget’s cognitive development levels where elements are concepts determined by deduced relationships (Collis, Romberg & Jurdak, 1986) – a level reached by only 35% of high school graduates in industrialised countries (Huitt & Hummel, 2003). Deduction takes place at the relational response mode of SOLO where students are required to conjecture, prove, justify, generalise and evaluate the validity of geometric statements (Rizvi, 2007). The NCTM curriculum framework also indicated that working with diagrams was central to geometric thinking with and on definitions. Definitions are one of the reasoning processes defined by Gutiérrez and Jaime’s (1998). The three models of van Hiele, Piaget and SOLO, and the NCTM curriculum framework model, all highlight the significance of visualisation on the path
to deductive reasoning. Visualisation is a tool for proofs (Presmeg, 2006), and a necessary process in order to reason deductively (Jones, 2002).

2.3.4 Summary

In order to develop spatial abilities and geometric thinking, visualisation is considered a necessary tool. Spatial ability as described by Yeh (2013) and variations (such as Maier (1996)) point to the key role of visualisation. Owens’ (1999) framework of imagery emphasises the role of imagery in developing spatial relations. Visualisation is also an accepted part of geometric reasoning in all four processes of reasoning as defined by Gutiérrez and Jaime (1998).

Apart from visualisation, language is understood as an important aspect of geometric understanding. Deductive reasoning, for example, requires explanation and validation of conjectures raised in an exploration (Prusak, Hershkowitz & Schwarz, 2012). The development of mathematical arguments from the first years of schooling enables the transition from informal to a more formal thinking method, involving both inductive and deductive processes (Couto & Vale, 2014).

The role of language, from roots of key words, elements of discourse, and the role of definitions are presented in the following section of this literature review.

2.4 The Role of Language in Developing Geometric Concepts

Mathematics is often described as a language with its own rules, convention, symbols and syntax seen by many students to be arbitrary (Austin & Howson, 1979; Morgan, Craig, Schuette & Wagner, 2014). Vygotsky saw thought as being an internalisation of language arising as a means of communication between a child and others (Vygotsky, 1978). Both the van Hiele model and SOLO taxonomy feature language as a critical component of cognitive development. For example, being able to communicate at different levels is one of the underpinning theories of the van Heile model. That is, that as students progress in geometric thought they exhibit a different level of language and communication than previously (Clements & Battista, 1992). The language used by teachers also plays a critical role in the development of understanding 2D shapes and their relations to other shapes. This language provides a level of “representational fluency” (Chinnappan & Lawson, 2005, p. 200) when learning about 2D shapes.

Learning mathematics involves a shift from contextualised speech in everyday life to a more specialised form (Forman, McCormick & Donato, 1997; Moschkovich, 2003). One important shift is from recitation to something resembling ‘real discussion’
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(Cazden, 2001) – that is, moving from answers to questions, to exploring and evaluating another person’s thinking. In order for students to participate in mathematics, it is important for them to become proficient and fluent at being able to communicate their reasoning to other students and their teachers for validation or rejection. The activities they engage in thus must involve communication and language.

Moving from informal discussions, with imprecise descriptions and explanations, to precise, unambiguous descriptions is one of the more challenging tasks for teachers according to Brown, Jones, Taylor and Hirst (2004). Discussion helps students to clarify their thinking and improve their understanding by organising their thoughts in preparation for presenting reasoned arguments. This makes language and communication particularly important in deductive geometry. Roth and Gardener (2012) suggested that encouraging students to define, justify and persuade others in their descriptions and classifications of shapes is important in public accounting and integral to students becoming increasingly knowledgeable. This socio-cultural perspective helps define the collective activity of argumentative talk that transforms individual thought into collective thoughts and actions (Prusak, Hershkowitz & Schwarz, 2012).

Austin and Howson (1979) provided three aspects of language relevant to mathematics – the language of the learner, the language of the teacher, and the language of the subject. Language involves “external operations as aids to the solution of internal problems” (p. 166). Halliday and Matthiessen (2004) also described language in mathematics as a designed semiotic system comprising architecture beyond natural, verbal language. Mathematical cognition, therefore, involves the use of speech, symbols, drawings, gestures and actions as features of language.

Morgan, Craig, Schuette and Wagner’s (2014) review of the literature into communication in mathematics identified language as including verbal and non-verbal modes of communication including symbolism, diagrams and gestures. In contrast, Sfard’s (2008) reference to ‘modalities of mathematical discourse’ included vocal (spoken words), and visual to encapsulate written words and symbols, iconic, concrete, and gestural as means of communication. “Mathematics is a discourse... made distinct by their tools, that is, words and visual means, and by the form and outcomes of their processes, that is, the routines and endorsed narratives that they produce” (p. 161). The discourse used inside the classroom has a significant influence on what and how students learn (Ferreira & Presmeg, 2004).
This section of the literature review expands the notion of mathematics as a language toward a broader definition of mathematics as a discourse inclusive of visual and verbal means pertinent to the learning of geometry. The modalities of mathematical discourse referred to by Sfard (2008) are detailed and presented as a model for interpretation of classroom discourse in geometry. The role of definitions is given prominence as arbitrary constructs of language (Vinner, 1991; Zazkis & Leikin, 2008) that indicate how students and teachers reason about shapes.

2.4.1 Visual and Verbal Tools of Communication

Geometry is an indispensable part of childhood and it involves exploring the environment, describing one’s position, or the position of an object (Sarama & Clements, 2009). Such activities allow children to develop spatial awareness, visualisation, and reasoning abilities (Elia, Gagatsis & van den Heuvel-Panhuizen, 2014).

Gestures are a vital aspect of communicating spatial thinking, and are a natural part of the discourse children and teachers readily use to reason with shapes (Özerem, 2012). “The visuospatial nature of gesture makes it suitable for capturing spatial information. Gestures represent spatial properties and action-based characteristics of concepts. They help speakers activate mental images and maintain these spatial representations in working memory” (Elia, Gagatsis & van den Heuvel-Panhuizen, 2014, p. 740). Hu, Ginns and Bobis (2014) found that ‘pointing’ gestures emerge in the first years of life and play an important role in subsequent vocabulary development. A pointing gesture may indicate a specific part of an object to a peer, enabling him/her to attend to something that, until then, may have remained unnoticed. The gesture of pointing is a process described by Rahim and Olsen (1998) as ‘hand-eye performances’, and it plays a specific role in objectification – to make something apparent (Sabena, Radford & Bardini, 2005). Roth and Gardener (2012) observed that when, for example, students asked to locate cubes in the classroom, young children first use pointing gestures as part of their initial engagement in the task. Gestures linked to words become an important part of child development. The geometrical “definition” is “in hand”—literally in the way the children hold the non-cubes and the cubes. Other communicative resources – words, diagrams, body movements – also accomplish an objectifying purpose. Gestures support student thinking and knowing, and co-emerge with peers’ gestures in interactive situations (Owens, 2015).
Visualisation and representation are at the core of understanding and communicating mathematics. “There is no other way of gaining access to mathematical objects but to produce some semiotic representation” (Duval, 1999, p. 3). Semiotic representations of mathematics require images and descriptions either mentally (internal), or in written form (external) (Larkin & Simon, 1987). Radford (2003) referred to *semiotic means of objectification* as:

… these objects, tools, linguistic devices, and signs that individuals intentionally use in social meaning-making processes to achieve a stable form of awareness, to make apparent their intentions, and to carry out their actions to attain the goal of their activities… (p. 41)

For example, when a teacher discusses a geometry proof problem, it generally involves an oral presentation of a proof and body movements such as pointing at different parts of the figure of the problem (Özerem, 2012). Both verbal and gestural modalities are the frames of reference children use to express their knowledge (Gersmehl & Gersmehl, 2007).

Verbal modalities are also important communicative tools in mathematics (Sfrad, 2008). Halliday (1974) suggested the notion of a *register* to be a set of meanings appropriate to a particular function of language that incorporates existing words (or phases), and creates new words (or phrases). “A register is a functional variety of language – the patterns of instantiation of the overall system associated with a given type of context” (Halliday & Matthiessen, 2004, p. 27). Students do not move spontaneously from their natural language register to the mathematical language register of the classroom or textbooks (Laborde, Conroy, De Corte, Lee & Pimm, 1990). In Duval’s (1999) view, mathematical thinking involves moving between multiple registers of representation. Typically several registers such as symbolic, natural language, and diagrams, and several systems of visualisation “entail a complex cognitive interplay underlying any mathematical activity” (Duval, 1999, p. 6). Mathematisation involves the ability to move flexibly amongst multiple registers incorporating mathematical models to solve problems (Presmeg, 2006), and denotes participation in mathematical discourse (Sfard, 2008).

### 2.4.2 Linguistic Considerations in Geometry

Often the specific terminology used in geometry (and for that matter mathematics) stem from Greek and Latin roots (McIntosh, 1994). The etymology of keywords is the foundation for building bridges between everyday language and mathematical language (Thomson & Rubenstein, 2000). Teaching students to acquire vocabulary is reflected in
research on bilingual learners of mathematics and is important but not sufficient for learning mathematics (Moschkovich, 2002). A focus on making etymological connections with students, presents mathematics from different cultural and historical perspectives, engaging students from diverse backgrounds. Teachers have opportunities to make socio-cultural connections through the language of geometry (Eglash, 1997), an important aspect of student engagement from an ethnomathematical perspective.

The language of mathematics should be concurrently taught with mathematics (Serow, 2007a). Good teachers know and make use of the root meanings of words in their teaching that lead to a better understanding of definitions and the identification of concepts (Mulcrone, 1958). The NCTM standards of mathematics as communication and mathematical connections (http://www.nctm.org/) provides a rationale for developing and implementing opportunities for student learning experiences to involve word roots related to geometry.

The meanings of geometric terms are rooted in their etymological structure and meanings can be deduced without reliance on memory. Conventions tend to dictate how some terms are used above alternatives. For example, the natural etymological extension from a triangle as a three-sided polygon to one with four sides would be quadrangle (Usiskin, Griffin, Witonsky & Willmore, 2008). However, more commonly quadrilateral is used as the preferred term from the Latin quadri for four, and latus, meaning side. In Greek, such a figure was called a tetragon - four angles. The word parallelogram comes from the Greek parallelos for alongside another, and gramm for line. Although only one definition dominates textbooks, other definitions such as a quadrilateral with pairs of equal, opposites sides form an equivalent definition for a parallelogram. This type of definition while etymologically unusual can serve to define hierarchies of quadrilaterals (Usiskin, Griffin, Witonsky & Willmore, 2008). Enabling a transition from informal to formal thinking including inductive and deductive processes (Couto & Vale, 2014), definitions are central to understanding classifications of shapes and hierarchies of shapes.

Roots of keywords are one aspect of geometric language. If, as suggested by Morgan, Craig, Schuette and Wagner (2014), language may be considered to be the “phraseology and vocabulary of a particular domain or group” (p. 844), then another aspect of geometric language is the unique locutions or phrases students are likely to meet. Terms such as regular polygon, right-angle triangle, or square on the hypotenuse qualify particular geometric properties. These technical terms only apply in geometry
(mathematics) and are not part of other linguistic registers. In some cases, meanings are
given to words that only apply in geometry such as *regular* implying *same* rather than *common*.

Knowledge of word roots and locutions are helpful in providing description and
definition of geometric concepts and help students make connections between concepts.
As this study explored the ways students and teachers use language to communicate
geometric concepts, attention to meanings of geometric terms exhibited by participants
are examined in this study.

2.4.3 Mathematical Discourse

Classroom discourse stems from social constructionist views of learning where new
knowledge can be created through interaction with other students and the teacher
(Serow, 2007a). Classrooms are complex social systems where *talk* serves many
purposes of communication. In typical classrooms, there is an asymmetry of rights and
obligations of teachers and students (Cazden, 2001). There are opposing forces in
communication between *authoritative* discourse and *internally persuasive* discourse that
allow opposing viewpoints and norms. Authoritative discourse dominates classrooms
where the teacher tries to create a community of learners. The role of the teacher in
creating a learning environment that is conducive to discussion is vital for learning
mathematics (Forman, McCormick & Donato, 1997; Gillies & Haynes, 2011). Internally persuasive discourse is akin to an individual’s own choice of words or
interpretation used to describe a learning situation or event (Bakhtin, 1981).

Nathan and Knuth (2003) argued that many mathematics teachers had not
participated in professional learning to adequately prepare them to enact *productive
discourse*, that is, “forms of social exchange which provide participants with an avenue
to construct and build on correct conceptions through their interactions with other class
members.” (p. 204). Lampert and Cobb (2003) also found that mathematics classrooms
where mathematical communication and language development are ‘focal’ were not
typical where teachers build the capacity of students to think, reason, solve complex
problems, and communicate mathematically. Typically, the teacher’s discourse
dominates and is characterised by classroom instruction governed by recitation-style
discourse patterns (Prusak, Hershkowitz & Schwarz, 2012). Instead, the balance
between the teacher’s authority and students’ individual voices play a critical role in
promoting the kind of learning and thinking that is valued (Ball, 1991).
Language is transactional and interactional (Brown & Yule, 1983). Discourse concerns the patterns or regularities of language used in context, not only of the language but also of the “rules of human communicative actions” (Sfard, 2000, p. 161). The notion of process is a key ingredient of discourse processes such as “comprehending, producing, reproducing, composing, recalling, summarising, and otherwise creating, accessing, and using discourse representations” (Cazden & Beck, 2003, p. 5). “Thinking is a special case of the activity of communicating… Becoming a participant in mathematical discourse is tantamount to learning to think in a mathematical way” (Sfard, 2001, p. 4). Gee (2015) described discourse as:

... a socially accepted association among ways of using language and other symbolic expressions, of thinking, feeling, believing, valuing, and acting, as well as using various tools, technologies, or props that can be used to identify oneself as a member of a socially meaningful group or “social network,” to signal (that one is playing) a socially meaningful “role,” or to signal that one is filling a social niche in a distinctively recognisable fashion. (Gee, 2015, p. 161)

There are several types of mathematical discourse that are dynamic and ever-changing entities (Sfard, 2000). The discourses that mathematicians use are different from classroom discourses. The objective of student learning is to shift students from everyday talk to talking mathematically using both symbolic, mathematical language and natural language (Sfard et al., 1998). As a communicational approach to learning, the process of changing one’s discursive ways in a certain well-defined manner requires the teacher to modify and exchange the existing student discourse to become able to communicate with members of a mathematical community (Sfard, 2001). In order for students’ reasoning to progress, students should discuss their reasoning with their teacher and peers, and explain the basis for their mathematical reasoning, in their writing and mathematical dialogue (Kramarski & Mevarech, 2003). “The quality of student discourse and learning can be enhanced when students are explicitly taught how to dialogue together and are provided with opportunities to practice these newly acquired skills in partnership with others” (Gillies & Haynes, 2011, p. 363).

Sfard (2008) introduced an interpretive framework for mathematical discourse that was grounded in the assumption that thinking was a form of communication and that learning a school subject such as mathematics, modifies and extends one’s discourse. The framework encapsulated the four interconnected elements of keywords, narratives, visual mediators, and routines. Such discourse analysis with ethnographic approaches is often combined to explore questions of what counts as learning in a local setting (Gee & Green, 1998).
Keywords: The distinctive registers and vocabularies comprise words that students use to describe objects, define properties, and engage in cultural exchanges with other individuals. Learning the mathematical meanings of words and associating those meanings to concepts is an important aspect of learning mathematics (Moschkovich, 2003). Talk, comprising keywords, whether written or spoken, is a tool for reasoning and carrying out collaborative activity (Mercer & Sams, 2006). In mathematics, whole concepts can be represented by an individual keyword, and combinations of specific keywords constitute definitions.

Narrative: is any text framed as a description of objects, of relations between objects, or processes with objects (Sfard, 2007). Narratives are either rejected or substantiated. A substantiated narrative is what Tall and Vinner (1981) referred to as a concept definition. The role of the teacher is to “make implicit judgments about the extent to which students take something as shared and to facilitate communication by explicating the need for further explanation” (Yackel & Cobb, 1996, p. 471). Back-and-forth dialogue in mathematics between students, and with their teacher as part of a disciplined inquiry (Peterson, 1997), is where “student performance demonstrates an understanding of important mathematical ideas that goes beyond application of algorithms by elaborating on definitions, making connections to other mathematical concepts, or making connection to other disciplines” (Newmann & Wehlage, 1995, p. 10). Problem-based inquiry can produce powerful mathematical thinkers who not only perform procedures but also have strong mathematical conceptions and problem-solving skills (Battista, 2001).

Visual Mediators: A picture can be as powerful as a definition (Usiskin, Griffin, Witonsky & Willmore, 2008). An image or diagram used to represent an object can be prototypical or less typical. Prototypical images may differ from individual to individual but serve as metaphors essential to reasoning in solving mathematical problems (Presmeg, 1992). Students typically remember prior experiences with diagrams and attributes associated with a concept, instead of the remembering the concept definition (Cunningham & Roberts, 2010). Visual mediators are aligned to the notion that Tall and Vinner (1981) referred to as the concept image, where a non-verbal representation is evoked in our minds with the concept name. Visual mediators for 2D shapes may be basic and represented by a diagram without any further attention to its properties, or may signify particular properties using personally constructed signifiers or mathematically acceptable conventions.
Routines: Students and teachers follow content-specific rules as they endorse acts or utterances (Lampert, 1990). These patterns of social interaction form part of the routines associated with mathematical discourse. For example, Gray, Pinto, Pitta and Tall (1999) believed these routines to be a necessary component in developing definitions as a process of connecting with previously learned concepts and validation. Routines are a result of an individual’s social experiences (Jungwirth, 1993), and in the classroom “help to cope with the fragility of the mutual understanding between teacher and students” (p. 384).

2.4.4 The Role of Definitions

Language is a powerful method of dealing with geometric complexity. A single word can stand not only for a highly complex structure of concepts and/or processes but also for various levels in a conceptual hierarchy. It takes the power of language to make hierarchical classifications (Gray, Pinto, Pitta & Tall, 1999). A focus on perceived objects leads naturally through the use of language to mental images, culminating in mathematical proof (van Hiele, 1985).

Definitions in geometry help to classify shapes (de Villiers, 1998b; Usiskin, Griffin, Witonsky & Willmore, 2008; Zazkis & Leikin, 2008). It serves the dual role of identifying a category to which a shape belongs, and by indicating how it might be distinguished from other objects in that category (Ndlovu, 2014). Concept definitions are definitions as a form of words used to specify that concept, and a concept image is the total cognitive structure that is associated with the concept (Fujita & Jones, 2006), including all the mental pictures and associated properties and processes (Tall & Vinner in Levenson, Tirosh & Tsamir, 2011). Fischbein (1993) also defined the notion of a figural concept – a square, for example, is a concept as well as a geometric figure. Construction of concepts has a role in definitions, and definitions have a role in constructing concepts. “One cannot think of a shape without knowing its definition, and one cannot know the definition without being able to construct the shape in intuition” (Heis, 2014, p. 608). However, many secondary teachers expect a one-way process for concept formation, that is, that “the concept image will be formed by means of the concept definition and will be completely controlled by it” (Vinner, 1991, p. 71).

Geometric concepts are the result of thinking processes, and they arise from the manipulation of mental objects that should be developed from a very young age (Fischbein, 1993; Mariotti, 1995). There are two major theories for concept formulation.
proposed by Levenson, Tirosh and Tsamir (2011) – the classical view where concepts are represented by a set of defining features, and the prototype view that takes into account characteristic features and not just defining elements. The prototypical examples often have the longest list of attributes. Different shapes may have different numbers of prototypes. For example, triangles have more prototypes than circles. This may be why younger children identify long parallelograms as rectangles, and why students have difficulties defining the angle attributes (Browning, Edson, Kimani & Aslan-Tutak, 2014).

Students prefer to rely on a visual prototype rather than a verbal definition when classifying and identifying shapes (Özerem, 2012; Usiskin, Griffin, Witonsky & Willmore, 2008) as they typically remember prior experiences with diagrams associated with shapes presented by their teachers and in most textbooks (Cunningham & Roberts, 2010), instead of the concept definition. Students’ limited prior experiences result in ‘filtered’, personally constructed definitions and concepts, different from formal definitions and concepts (Chinnappan & Lawson, 2005; Heinze & Ossietsky, 2002; Tall & Vinner, 1981; Turnuklu, Gundogdu Alayli & Akkas, 2013).

Definitions of shapes are often described as a sufficient definition where the minimum amount of properties is used to define a shape (de Villiers, 1998). Definitions should be minimal (Vinner, 1991). For example, a square is a quadrilateral with equal sides and right angles is considered to be a sufficient definition as it contains enough information to exclude all non-examples (de Villiers, 1994; 1998b). Definitions that contain superfluous information such as squares are rectangles because they have four right angles and opposite sides are equal and parallel are often described as uneconomical or inefficient (de Villiers, 1994; 1998b; Usiskin, Griffin, Witonsky & Willmore, 2008).

Squares are rectangles because they have four right angles, is considered to be an inclusive definition. This definition requires a priori knowledge of rectangles to define squares and indicates where a rectangle sits in a hierarchy of quadrilaterals (Usiskin, Griffin, Witonsky & Willmore, 2008). Govender and de Villiers (2003) argued that definitions ought to evolve naturally from previous knowledge, models and real experiences in order to avoid confusion. An exclusive definition of a shape, on the other hand, contains extra information to exclude all other non-examples. Prior to 1930 in the USA, exclusive definitions were commonplace that disallowed squares from being rectangles for example. The National Committee of Mathematical Requirements
in the USA (1923 – 1927) recommended that inclusive definitions be used for geometric figures instead (Usiskin, Griffin, Witonsky & Willmore, 2008).

De Villiers (1994) argued that a serious deficiency in learning experiences can be attributed to a lack of provision for *functional* understanding – students fail to understand the function or value of hierarchical classification of quadrilaterals and prefer a *partitional* classification. Students have difficulties in mastering the application of necessary and sufficient conditions in the form of definitions (Pegg & Woolley, 1994). Students also experience difficulties in interpreting new definitions, in using definitions appropriately, and in fully appreciating the role of definitions in solving problems or creating proofs (Zazkis & Leikin, 2008). These difficulties often arise when students are taught to memorise definitions and rely on limited, prototypical examples (Battista, 2001; Clements, 2004; Cunningham & Roberts, 2010). Hierarchical classifications function can help lead to economical definitions of concepts, and “simplifies the deductive systemisation and derivation of the properties of more special concepts” (de Villiers, 1994, p. 15). Hierarchical definitions also simplify the deductive structure of a set of concepts without having to prove each concept anew (de Villiers, 1987). The challenge facing teachers who may have a limited conceptual understanding is the awareness of the limitations of textbooks and the need to supplement them with adequate definitions and sufficient examples (Cunningham & Roberts, 2010).

### 2.4.5 Summary

A focus on language in mathematics is essential because students need to talk about their ‘linguistic associations’ for words and symbols presented and those discovered in their learning experiences (Crowley, 1987). Students should use their own terms to express their conceptions, and gradually be introduced to standard terminology – one of the principles intrinsic in the van Hiele model, namely that language is developmental (van Hiele, 1985; 1999). Expression of a student’s thinking in his/her own language is important in revealing underdeveloped or misconceived ideas. Language, as used by teachers, should be modeled and encouraged in order for generalisations and exceptions to be developed. A crucial factor in directing student thinking is teacher questioning, because “the nature of a student's geometric explanations reflects that student's level of thinking, questioning is an important assessment tool” (Crowley, 1987, p. 14). Language development through questioning approaches is important in the design of any assessment tasks dealing with geometry. Nason, Chalmers and Yeh (2012) also
identified the need for teachers to replace closed questions with open questions, focus questions, and prediction (what if/what if not) questions requiring students to utilise and extend their own descriptive language to include more appropriate mathematical language and terminology. If teachers do not make language an important aspect of teaching, then they can set up a barrier to student growth (Pegg, 1992). Lack of exposure to the vocabulary of geometry handicaps their efforts to learn concepts (Lee & Herner-Patnod, 2007), and may be the cause of many students’ misconceptions (Oberdorf & Taylor-Cox, 1999).

For students to communicate mathematically, they will need an understanding of the vocabulary that is used routinely in mathematics instruction, textbooks, and word problems (Lee & Herner-Patnod, 2007). Problem-solving involving group work requires the routines of reading mathematical vocabulary in the problems, discussing the meaning of the words, discussing and presenting their solutions, and formulating answers. Language is significantly important in developing these aspects to collaborative problem-solving, exploring phenomena, justifying results, and using conjectures.

Mathematics learning involves procedural competence, solving worded problems, reasoning, presenting mathematical arguments, and participating in mathematical discussion (Moschkovich, 2002). Language, as a communicative tool, is a central part of learning geometry, and is critical to this study. This study used the framework of mathematical discourse proposed by Sfard (2008) to examine students’ geometric discourse as they engaged with geometric tasks. Through this interpretive lens, the means by which teachers and students communicated with each other were analysed for etymological connections to geometric concepts identified by teachers and students, the types of visual and verbal associations used to depict images, describe objects, define shapes, and to explain their thinking. The commognitive development of discourse concerns changes from colloquial to mathematical (Sfard, 2006; 2008; 2012), and is a significant feature of this study.

2.5 Pedagogical Considerations

The teacher’s role in facilitating learning is both complex and challenging. Often mathematics teachers lack sufficient content knowledge (Clements, 2004; Nathan & Knuth, 2003) and expertise to manage the learning of their students and succumb to procedural rather than conceptual learning opportunities (Couto & Vale, 2014).
Important pedagogical considerations include knowing the content of the subject, knowing how to engage students mathematically, and knowing how students are likely to think about mathematics and their likely difficulties with newly presented material (Hill, Blunk, Charalambous Lewis, Phelps, Sleep & Ball, 2008; Morris, Hiebert & Spitzer, 2009).

First, the likely misconceptions and difficulties students have with geometry are described in order to convey the unique complexity associated with learning in this area. Secondly, the impact of similar misconceptions and difficulties observed in pre-service and practising teachers (Cunningham & Roberts, 2010; Gutiérrez & Jaime, 1999; Mayberry, 1983) are discussed as significant barriers to student learning. Thirdly, task designs models and specific tasks are explored as ways of addressing some of the pedagogical challenges associated with teaching geometry and engaging students in non-procedural learning activities (Wares, 2014).

Finally, the role of group work is discussed as a necessary practice in modern classrooms, and as a means of engaging students in the mathematical practice of solving unfamiliar problems and constructing common solutions (Linfield, Coltman, Raban & Margetts; 2012). The specific role of the teacher in effectively managing group work (Doyle, 1983; Good, Reys, Grouws and Mulryan, 1989; Yackel, Cobb & Wood, 1991), and facilitating mathematical discourse (Ball, 1991; Michaels, O’Connor & Resnick, 2008; Sfard, 2001; 2008) is elaborated.

2.5.1 Common Misconceptions of Students and Teachers

There are several misconceptions and difficulties that students might experience when learning about geometry and with associated algebraic relationships across all stages of schooling (Browning, Edson, Kimani & Aslan-Tutak, 2014). These misconceptions are also common amongst primary and secondary pre-service teachers, as well as practising teachers (Cunningham & Roberts, 2010; Gutiérrez & Jaime, 1999; Mayberry, 1983).

Burger and Shaughnessy (1986) found that upper primary school students were not sufficiently grounded in basic geometric concept definitions and students were reasoning at different levels. Erez and Yerushalmi (2006) observed that upper primary school students had a tendency to abide by existing knowledge about shapes and had difficulties changing their beliefs when confronted with unanticipated results when constructing shapes in a dynamic geometry environment. Usiskin (1982) found that many high school students who had difficulties with proofs did not progress through the
van Hiele levels and were not versed with basic levels of geometric terminology and concepts.

Student difficulties with spatial relations are manifest through map reading and navigation activities. The interpretations of maps as flattened geometric representations cause students difficulties from early childhood (Liben & Christensen, 2011; Newcombe & Huttenlocher, 2006). Difficulties with the mapping processes of abstraction, generalisation, and symbolisation reflect an inability to connect concrete and abstract frames of reference (Clements, 2004).

Several misconceptions result from learning geometric concepts by rote (Clements & Battista, 1992; Mayberry, 1983). Students are taught to recognise squares and rectangles differently (Oberdorf & Taylor-Cox, 1999). This can lead them to believe (sometimes into adulthood) that a rectangle cannot ever be a square. Students at all levels have difficulties in recognising shapes in non-standard orientations (Crowley, 1987; Mayberry, 1983). Sometimes, children pay more attention to the non-critical attributes (Levenson, Tirosh & Tsamir, 2011). Some attributes, namely critical attributes, stem from the concept definition while others, such as non-critical attributes, do not. Although the orientation of a figure is non-critical, students may exclude a square rotated 45 degrees from being a square because of its rotated position. Due to an inability to distinguish between critical and non-critical attributes of different quadrilaterals, students experience difficulties learning the hierarchy among quadrilaterals (Erez & Yerushalmy, 2006).

Geometric figures are simultaneously concepts and spatial representations and possess spatial properties like shape, location and magnitude (Panaoura & Gagatsis, 2010). Yet students have difficulties with systematically constructing images (Lohman, 1996), a reluctance to engage with visual modes for reasoning with shapes (Dreyfus, 1991, Presmeg, 2006), and are unable to connect visual and symbolic representations. While students readily recall visual prototypes rather than verbal definitions (Özerem, 2012), they solve geometrical problems by relying on the visual perception of a given geometrical figure rather than on a mathematical deduction based on conceptual knowledge (Panaoura & Gagatsis, 2010). This phenomenon is related to students’ difficulties working with geometrical figures as figural concepts (Fischbein, 1993). Correct and effective geometrical reasoning is characterised by the interaction and the harmony between figural and conceptual aspects of geometrical entities (Mariotti, 1995; Panaoura & Gagatsis, 2010).
When students have difficulties perceiving class inclusions of shape, such as a square not being a rectangle, this is due to an underdeveloped concept image (Marchis, 2012). Ng (2009) also observed that elementary teachers lacked the same understanding of how shapes were related to each other in a study that explored the classification of quadrilaterals. If the figure already has one name, a square, then how can it also be called something else? Believing that all objects have only one name may contribute to the difficulties children have in accepting the hierarchal structure of geometric figures (Levenson, Tirosh & Tsamir, 2011). The use of dynamic geometry software might help develop the concept image of shapes because students can observe how various properties might vary or remain unchanged under digital transformation (Browning, Edson, Kimani & Aslan-Tutak, 2014). Being able to work with variance and invariance is one of the necessary Big Ideas of geometry (NMTM, 2012). However, dynamic geometry environments might reduce any need for proofs as students can ‘see’ mathematical properties too easily (Labrode, 2000; Mariotti, 2000).

Children are often exposed to inaccurate geometric terminology through storybooks and television programs. Searching for regular shapes is a popular activity, but precise language must be used otherwise a search, for example, for squares and circles, may result in a collection of cubes and spheres. To avoid misconceptions in the classroom, teachers should emphasise the properties and characteristics of a concept, provide many examples and counterexamples, pay close attention to the language used, challenge understanding, and broaden generalisations (Oberdorf & Taylor-Cox, 1999). The use of activities involving sorting and classifying, comparing and contrasting are suggested ways for developing geometric understandings in the classroom.

Some textbooks provide inadequate definitions and a restricted number of examples (Cunningham & Roberts, 2010). Usiskin, Griffin, Witonsky and Willmore’s (2008) earlier analysis of high school textbooks claimed that a key issue with student misconceptions came from prototypical representations of quadrilaterals. For example, depictions of rectangles as made up of vertical and horizontal sides contributed to lower van Hiele levels of understanding of geometry, because students “do not realise the generality of geometric figures” (p. 34). Students’ choices of examples of shapes are based on their own prototypes and less on definitions (Hershkowitz, Ben-Chaim, Hoyles, Lappan, Mitchelmore & Vinner, 1990). Teachers often do not have the necessary content knowledge to supplement prototypical representations (Usiskin, Griffin, Witonsky & Willmore, 2008), and many exhibit a limited understanding and an
inability to apply concepts of those that involve prototypical examples (Cunningham & Roberts, 2010).

Students often struggle with understanding associated measurement concepts of perimeter, area and volume (Browning, Edson, Kimani & Aslan-Tutak, 2014). Upper primary students often confuse perimeter and area concepts (Vighi, 2010), and it is common for confusion to exist between formulas and associated units (Owens & Outhred, 2006; Steele, 2013). These issues have also been observed in pre-service and current service teachers, often as a result of a reliance on procedural learning (Couto & Vale, 2014).

Gutiérrez, Pegg and Lawrie (2004) found that students have difficulties visualising 3D representations beyond very familiar shapes. Students also have difficulties perceiving how an object will look if viewed from another perspective (Izard, 1990; Pittalis & Christou, 2010; Serow, 2007b). This lack of 3D visualisation stems from an inability to interpret, visualise or construct nets (Izard, 1990; Maier, 1996; Marchis, 2012; Pittalis & Christou, 2010; Pittalis, Mousoulides & Christou, 2010). It should also be noted that there is an emphasis in the curriculum on 2D-geometry rather than 3D-geometry (Maier, 1996). Latner and Movshovitz-Hadar’s (1999) observations of high school students performing 3D geometry tasks indicated that some student difficulties were due to an inability to create, retain and manipulate a mental image of a solid in 3D space from its 2D representation. Tasks that encouraged students to construct and draw 3D models provided opportunities to develop visualisation skills and an understanding of a solid’s internal structure (Izard, 1990; Pittalis, Mousoulides & Christou, 2010).

Generalisation is the basis of proof and lies at the heart of mathematics, yet it is an elusive concept for many students (Ekanayake, Brown & Chinnappan, 2003; Hoyle & Jones, 1998). Students have difficulty in transitioning from a computational view to one that conceives geometry as a field of intricately related structures (Jones, 2000). This abstract nature of proofs disengages many students (Jones & Fujita, 2001), and they fail to see the reasons for them (Clements & Battista, 1995; Jones & Rodd, 2001; Senk, 1985). They do not understand that a “proof is a tool for verifying mathematical statements and showing their universality. They allow one to distinguish between fact and fiction” (Hadas, Hershkowitz & Schwarz, 2000, p. 127). Both students and their teachers do not understand why mathematicians place such value on proofs (Chazan, 1993).
According to Tall (2014), students’ emotions play a critical role in mathematical thinking and can have a profound effect on how they make sense of proofs. Lack of success in conquering the difficulty inherent in proofs causes even more difficulties in engagement and can increase mathematical anxiety. Teaching approaches to proofs tend to concentrate on verification and devalue or omit exploration of relationships between properties (Jones, 2000). Lower level students often perceived proofs given by a teacher, as an attempt to verify a result, not understanding that a proof lies in the logical relationship between properties (de Villiers, 1987). “The product of mathematical activity might be justified with a deductive proof, but the product does not represent the process of coming to know. Nor is knowing final or certain” (Lampert, 1990, p. 30).

Proof, through the medium of Euclidean geometry, has been a major part of the senior secondary mathematics curriculum in several international jurisdictions for many years (Tall, 1989), but this has declined over many decades. Fujita and Jones (2006) observed a lack of proving skills in prospective teachers’ in Scotland. They found that only 13% of them could identify squares as belonging to the class of rectangles. Although almost all could draw a square, almost two-thirds could not define it correctly, leaving out any mention of angles or other constraining properties. The decline in geometric proofs as part of a mathematics curriculum was partly due to students finding them extremely difficult and often uninteresting (Howson, 2003; Jones & Fujita, 2001; Senk, 1995). These difficulties resulted from a lack of knowledge of geometric concepts, and lack of knowledge and expertise with basic logical tools (Lew, Cho, Koh, Koh & Paek, 2012).

Students need to be involved in processes that develop reasoning and conceptual understanding of shapes (Battista, 2002). Memorising geometric facts is not as important as constructing meaning using students’ own knowledge and ways of reasoning to perform physical and mental manipulations, abstraction and reflection. Mathematics instruction should be guided by detailed knowledge of the sequence of meanings and mental models that students might construct as they move from intuitive to formal thinking.

Of equal concern to student difficulties and misconceptions are that teachers often retain the same misconceptions and misunderstandings from their own schooling (Cunningham & Roberts, 2010; Gutiérrez & Jaime, 1999; Mayberry, 1983). Kospentaris, Spyrou and Lappas (2011) found that many pre-service teachers had the same poor concept images as primary and secondary students such as identifying right-
angle triangles from triangles whose perpendicular sides are not in the vertical-horizontal (prototype) position (Even & Tirosh, 2002; Gutiérrez & Jaime, 1999). Students may be able to make hierarchical class inclusions but do not understand its function or value (de Villiers, 1994; Heinze & Ossietsky, 2002). Similarly, pre-service primary mathematics teachers rely on personal concept images and personal concept definitions that lead to a ‘partitional’ classification of quadrilaterals (Fujita & Jones, 2006; Turnuklu, Gundogdu Alayli & Akkas, 2013). Several researchers have found that the majority of pre-service teachers do not have the required geometric reasoning levels – van Hiele level 3: abstraction or above, necessary to teach geometry successfully at primary school (Halat & Sahin, 2008; Mayberry, 1983; Ndlovu, 2014).

Gabel and Enoch (1987) suggested that pre-service teachers’ difficulties with solving problems containing the concept of volume might be due to the order in which concepts are presented to them and their lack of spatial skills. Findings suggested that the teaching sequence of length, then area, then volume, creates difficulties for pre-service teachers with low spatial orientation, and due to procedural approaches where one learns a sequence of actions or steps with or without meaning (Even & Tirosh, 2002). Pre-service teachers’ misunderstanding of volume as a concept arises from an adoption of procedural approaches using algorithmic methods that engendered a reproductive approach to teaching (Owens & Outhred, 2006; Pittalis, Mousoulides & Christou, 2010). Zevenbergen (2005) found that “the diversity of responses offered by pre-service teachers… could be comparable to what would be expected from students in an upper primary classroom” (p. 21). Browning, Edson, Kimani and Aslan-Tutak (2014) recommended designing curriculum experiences in geometry that moved beyond procedural and memorisation skills toward developing spatial visualisation and problem-solving skills.

A major issue with pre-service teachers who hold misconceptions in geometrical concepts is that they are then unlikely to provide learning experiences for their students that develop conceptual reasoning. Exactly how pre-service teachers might overcome their misconceptions before entering the classroom presents a significant challenge. Overcoming these issues with in-service teachers in classrooms can only occur through ongoing professional learning programs that are evaluated for their effectiveness in “correcting” teachers’ geometry knowledge for teaching. Students’ acquisition of geometric thought depends greatly upon the teacher’s mathematical content knowledge (Couto & Vale, 2014).
This study identified several misconceptions with geometric concepts and difficulties stemming from imprecise or personal concept images and concept definitions (Tall & Vinner, 1981 in Levenson, Tiros & Tsamir, 2011). These barriers were identified as significant in engaging some students in the series of tasks. The ways some of these misconceptions were spread from one student to other students without appropriate interventions by the teacher was also observed and discussed as barriers to learning.

2.5.2 Challenges in Teaching Geometry

Our daily lives are profoundly affected by pervasive space-related technologies that present ongoing educational challenges (Liben & Christensen, 2011). From a young age, children are exposed to visual forms of communication through their engagement with computer-based activities. Exposure to visual and spatial displays does not necessarily mean they understand the information presented to them (Lowrie, 2012). Because children do not develop strong spatial thinking skills at home, spatial education needs to be explicitly taught (Diezmann & Lowrie, 2012). As most teachers are unlikely to have experienced explicit spatial education themselves, they find it difficult to identify opportunities for explicit instruction in spatial thinking and representations (Diezmann & Lowrie, 2012; Liben, 2006). Geometric problems often require spatial reasoning (Owen & Outhred, 2006), and explicit instruction is critical in its successful application to reasoning and problem-solving (Gillies & Haynes, 2011). Coffland and Strickland (2004) also identified the need for enhanced secondary teacher training in order to integrate technologies for geometric instructional purposes with most teachers who do not recognise their usefulness or necessity.

Many studies in the research literature identify that teachers have little preparation for teaching geometry, and this leads to deficits in their teaching practice (Clements, 2004; Nathan & Knuth, 2003). Deficits in teacher knowledge, in turn, create diagnosed difficulties for students. “In order to be a good professional, capable of teaching maths, it’s crucial to deeply know mathematics and therefore…it’s crucial to have the ability of putting to work the strategies which are capable of making the students learn” (Couto & Vale, 2014, p. 59). Teachers are also confronted with a choice of the role they will assume in the classroom, and what type of instruction they will use to present concepts (Cunningham & Roberts, 2010).
Suydam (1985) pointed out that primary and secondary teachers disagree on what geometric topics are important. For example, educators cannot agree upon the role of transformational geometry involving observations of objects when manipulated by reflection, rotation, and translation. Rarely is school-style geometry, based on Euclidean geometry and transformations, encountered in teacher preparation courses (Kuchemann & Rodd, 2012). This pre-1980 proof-laden curriculum (Battista, 2001) is characterised as rote-type experiences where teachers dispensed information and disengaged students (Chinnappan & Lawson, 2005). Whether teachers have the skills and knowledge to deliver a Euclidean approach to geometry that engages students remains a significant pedagogical question. Professional debates about what is relevant and how it should be taught indicate a clear divide between primary and secondary teachers approach to mathematics education and the emphasis they place on geometry in the curriculum.

Mammana and Villani (1998) suggested these strong disagreements about the aims, contents and methods for teaching geometry from primary school to university exists as there is no “simple, clean, linear hierarchical path from the first beginnings to the more advanced achievements of geometry” (p. 337). Approaches to geometry curricula have been referred to as ‘hodgepodge’ (Battista, 2001; Clements, Battista & Sarama, 2001), consisting of superficially covered concepts with no systematic progression to higher levels of geometric thought and geometric problem-solving skills.

The privileging of number over other areas of mathematics through procedural learning strategies has been at the expense of geometry (Jones, 2002). Jones, Mooney and Harries (2002) reported that, in the United Kingdom, geometry was the area of mathematics that pre-service teachers performed poorly and had the least confidence to teach. Their examination of primary teachers’ knowledge indicated weak levels of geometric vocabulary and low self-confidence to teach geometry. Similar reporting on Australian pre-service teachers’ knowledge of geometry concepts is scant. However, Van Klinken (2010) claimed that a leading cause of Australian primary school students’ underperformance in geometry was due to their teachers’ limited understanding of geometric definitions.

Browning, Edson, Kimani and Aslan-Tutak (2014) found that the struggle pre-service teachers’ had with definitions of geometric shapes was an historical problem. Specifically, many pre-service teachers could not define basic geometrical shapes or solids, and did not know the properties of those shapes (Marchis, 2012). Marchis’ examination into the knowledge of geometry of final year pre-service teachers indicated
that two-thirds did not know how to give a definition of regular 2D shapes, and they also did not know which minimum amount of properties would be sufficient for a definition. Some definitions contain repeated information, and/or missing properties such as *a rectangle is a quadrilateral with four sides*. These findings align to earlier observations of pre-service teachers by Fujita and Jones (2006), where individuals cannot discern which properties are necessary and which are sufficient to describe an object, corresponding to the van Hiele level 2: *analysis* (Mason, 1998).

The qualities of a teacher’s knowledge have a strong influence on how that knowledge is accessed and exploited (Chinnappan & Lawson, 2005; Schoenfeld, 1992a). Teacher expertise is widely acknowledged as being the most significant factor in determining student achievement (Anstey & Clarke, 2010). Kuzniak and Rauscher (2011) acknowledged that the link between teacher subject knowledge and the quality of teaching was not easy to study. Their findings indicated ‘didactical’ obstacles that arose from teachers’ specific choices of methods or because of their limited knowledge of students’ cognitive capacities. Teachers who were presented with student responses to particular geometric problems illustrated a command of basic geometric concepts, but there was a considerable divergence in how teachers interpret student understandings and associated misconceptions in geometry.

The complexities of understanding teacher knowledge as suggested by Shulman (1986) identified the need for a coherent theoretical framework to describe and account for different types of knowledge domains for teaching. Shulman (1986) hypothesised two types of knowledge for teaching – *subject content knowledge* [SCK] representing the organisation of the basic concepts and principles of the discipline in the mind of the teacher, and *pedagogical content knowledge* [PCK] describing particular knowledge of the teachability of subject matter.

Since there are no single most powerful forms of representation, the teacher must have at hand a veritable armamentarium of alternative forms of representation, some of which derive from research whereas others originate in the wisdom of practice. (Shulman, 1986, p. 9)

Hill, Ball and Shilling (2008) extended knowledge definitions particular to mathematics and introduced *mathematics knowledge for teaching* [MKT]. This incorporated, not only the mathematical knowledge common to individuals working in diverse professions, but also the subject matter knowledge that supported that teaching. The MKT framework focussed on *why* and *how* specific mathematical procedures work, how best to define a mathematical term for a particular year level, when to apply
particular generalisations and rules, and the types of errors students were likely to make with particular content (Hill, Blunk, Charalambous Lewis, Phelps, Sleep & Ball, 2008). The four components of MKT described by Morris, Hiebert and Spitzer (2009) are:

1. knowledge of mathematics that most educated people acquire ("common content knowledge");
2. knowledge of mathematics that is unique to, and essential for, teaching mathematics ("specialised content knowledge");
3. knowledge that combines knowledge of content with the knowledge of students; and,
4. knowledge that combines knowledge of content with knowledge of teaching.

The second component, specialised content knowledge "involves unpacking or decompressing mathematical knowledge in order to make particular aspects of it visible for students or to identify the source of students’ difficulties" (Morris, Hiebert & Spitzer, 2009, p. 494) and falls largely outside Shulman’s PCK.

Aligned to the Shulman’s SCK – PCK framework, the AAMT’s Standards for Excellence in Teaching Mathematics in Australian Schools also included knowledge of students in its domain of professional knowledge. This particular standard was an articulation of the knowledge base and appreciation of mathematics appropriate to the students they taught, including knowledge of students’ social and cultural contexts, the mathematics they know and use, their preferred ways of learning, and how confident they felt about learning mathematics (AAMT, 2006).

In developing a framework for analysis of teachers’ geometric content knowledge and geometric knowledge for teaching, Chinnappan and Lawson (2005), defined teachers’ knowledge of geometry [KG] and knowledge of geometry for teaching [KGT] that enabled a detailed and differentiated description of the dimensions of a teacher’s knowledge base for the subject matter of geometry and the teaching of geometry. Levenson, Tirosh and Tsamir (2011) also provided a theoretical framework that combined Shulman’s (1986) and Hill et al.’s (2008) components of knowledge for teaching with the concept image/concept definition theory (Tall & Vinner, 1981 in Levenson, Tirosh & Tsamir, 2011). For geometry, Levenson, Tirosh and Tsamir (2011) suggested that teachers required a comprehensive knowledge of concept image/concept definition theory that “focuses on the image of the concept as well as and opposed to the definition of a concept” (p. 94). Significantly, the model was used to access preschool
teachers’ *topic-specific mathematical knowledge* [TMK] and *general mathematics knowledge* [GMK] regarding concept images, revealing that the framework enhanced teachers’ knowledge for teaching mathematics and could be used to enhance teaching practice more broadly.

Hill, Ball and Schilling (2008) claimed that MKT was necessary for developing instructional practices and resources aligned to student learning. The development, improvement and refinement of pre-service teacher mathematics courses need to focus on mathematical thinking and reasoning, to help pre-service teachers make mathematical connections between content strands. Eli, Mohr-Schroeder and Lee (2013) investigated the MKT for the geometry of both pre-service teachers and practising middle years teachers. They found that the fundamental misconceptions, or lack of connections made by participants could be mediated if pre-service courses for prospective middle years teachers were delivered through the MKT framework. This framework should have a particular focus on making visible and explicit the connections between algebraic and geometric concepts, and creating tasks centred around analysing student work and addressing the types of mathematical connections students make. This study by Eli, Mohr-Schroeder and Lee (2103) identified how teacher preparation for secondary teaching was different to preparation for primary teaching that had a stronger emphasis on pedagogy than on the preparation to teach mathematics.

In a year-long qualitative investigation into pre-service secondary teachers’ SCK and PCK, Canturk-Gunhan and Cetingoz (2013) found that pre-service secondary teachers had difficulties when describing geometric shapes and difficulties with using correct mathematical language effectively. Participants were unaware of the possible student misconceptions in geometry. The study also found that pre-service teachers’ instruction to students had the potential to create misconceptions. These results indicated significant deficits in pre-service courses regarding teacher preparation for teaching geometry, but point to the use of the MKT framework to build a cohesive set of measures, and robust knowledge base, about secondary teachers’ MKT. Very few studies, however, have analysed the MKT for the geometry of practising secondary teachers.

An extensive literature review conducted by Browning, Edson, Kimani and Aslan-Tutak (2014), provided an historical overview of seminal studies into pre-service teachers’ mathematical content knowledge in geometry and measurement, and
identified current gaps in the research literature, as well as necessary improvements in curriculum design. Their review described what was known about pre-service teachers' geometry content knowledge before 1998, between 1998 and 2011, and a view beyond 2011. Across the three timeframes, pre-service teachers’ knowledge was identified as weak and limited, with an over-reliance on procedural processes. Research gaps existed in identifying pre-service teachers’ understanding in geometry in targeted topics, on how they develop their content knowledge using technology (Coffland & Strickland, 2004), and in determining a satisfactory level of geometric understanding for pre-service teachers.

Bjuland (2004) investigated pre-service teachers’ ability to reflect on their learning processes after working on geometry problems in small groups, and found that they had difficulties changing from working on a problem to reflecting on the problem-solving activity. Using student work samples helped to influence professional discourses about teaching and learning, and engaged teachers in a cycle of experimentation and reflection, and presenting opportunities for teachers to engage their students in conversations about their thinking strategies (Kazemi & Franke, 2004). Nathan and Knuth (2003) also advocated the need for teachers to engage in discourse where the teacher was “carefully monitoring the classroom interactions, evaluating and shaping them, and, in so doing, establishing and maintaining norms of classroom discourse” (p. 203). Using students’ work expanded teachers’ opportunities to learn and cultivate professional communities of practice (Little, Gearhart, Curry & Kafka, 2003).

Nason, Chalmers and Yeh’s (2012) notion of professional communities of practice termed knowledge-building communities suggested that shared practice presented opportunities for teachers to be contributors to group efforts to produce and continually improve their conceptual understanding of concrete processes, visual representations, and experimental designs. What teachers want students to learn requires the skills to unpack learning goals in order to inform instruction and plan instructional activities, and this does not occur naturally to pre-service teachers (Morris, Hiebert & Spitzer, 2009). Sullivan (2011) also identified the Japanese lesson study as a model for practising mathematics teachers to yield new ideas about teaching and learning based upon a better understanding of student thinking. In this model, teachers work in groups and are presented with problems that create extensive pre-teaching discussion (neriage), are required plan and discuss approaches to the content (bansho), and observe each other in practice (Takahaski, 2006). The Japanese lesson study model encourages non-
routine ways of thinking suggested by Kuchemann and Rodd (2012) as a central component of geometry – that teachers do not see geometry as a procedure-based subject.

Apart from teachers’ knowledge of geometry impacting on student learning, their beliefs play a critical role in their choice of learning experiences (Ernest, 1989). How teachers question, listen and respond reflects their beliefs about mathematics and how it is taught (Ferreira & Presmeg, 2004; Serow, 2007a). Ertmer (2005) identified teachers’ pedagogical beliefs to be a strong influence on mathematics instruction. Askew, Brown, Rhodes, Johnson and Wiliam (1997) identified two particular teacher orientations as either transmission – emphasising procedural methods and routines of reproduction, or connectionist – emphasising applications of mathematics to new situations. Teachers who believed in the importance of knowledge reproduction exhibited traditional teaching methods, typically textbook-driven, and perceived mathematics as an accumulation of facts and rules (Ferreira & Presmeg, 2004; Liu, 2011). Teachers who believed in knowledge construction emphasised a process of inquiry, and student responsibility for learning induced by working and learning together (Liu, 2011).

There is a significant amount of literature examining pre-service teachers’ knowledge for teaching geometry but very little research into practising teachers’ knowledge. More research is needed to investigate ways of working with teachers as practitioners and collaborators in developing tasks for student engagement in geometry. In this study, teachers were observed in practice and several misconceptions were identified amongst them. These were evidenced by the imprecise use of geometric terminology, and inaccurate concept images and concept definitions. In some cases, the lack of ‘correction’ of imprecise conceptual knowledge exhibited by students in the classroom provided an indication of the teacher’s content knowledge and pedagogical content knowledge.

2.5.3 Task Designs in Geometry
A dynamic view of mathematics learning is one based on active, generative processes by doers and users of mathematics (Schoenfeld, 1992a). This is different from the ways in which much mathematics teaching has been taught and presented to students – where mathematics was more static and structured as systems of facts, procedures, and concepts (Henningsen & Stein, 1997). Geometric problems present challenges to both
teaching and learning as there are often no set procedures or algorithms. Students encounter geometrical situations that require logical thinking and reasoning processes, and teachers need to have the skills to develop tasks that allow students to explore geometry in meaningful and often multiple ways. Steele (2013) suggested that geometry tasks that were grounded in the context of teaching, as in the Japanese Lesson Study Model, should capture nuances of teacher knowledge.

Task designs are often based on different ways of conceptualising mathematics as a complex but stable set of ideas and theories (Liben & Christensen, 2011; Slavin, 1996). Students, however, bring to the classroom a variety of experiences that gives rise to the need for tasks to provide opportunities for the learning of concepts as well as open tasks that promote discovery and challenge. The constructivist notion of understanding suggests that students construct their own understanding by building on their prior knowledge (Brophy, 1998). Therefore, teaching activities that take advantage of students’ prior knowledge will help them make sense of new information. Building on prior experiences, the forms of reasoning expected should be examples of local deduction, where students can utilise any geometrical properties to deduce or explain other facts or results (Gillies & Haynes, 2011; Henningsen & Stein, 1997). Problems suitable for group work include tasks that “illustrate important mathematical concepts, allow for multiple representations, that draw effectively on the collective resources of a group, and have several possible solution paths” (Horn, 2005, p. 219). Boaler and Staples (2008) defined problems of this nature as groupworthy.

Levenson, Tirol and Tsamir (2011) suggested five task types that could be used to encourage concept development: classifying mathematical objects, interpreting multiple representations, evaluating mathematical statements, creating problems, and analysing reasoning and solutions. The implementation of tasks is determined by teachers’ belief, knowledge, teaching experiences, and school norms. Curriculum guidelines offer suggested tasks but it is left up to the teacher to decide on the use of tasks and the emphasis these tasks should take.

In designing and trialing classroom materials, Brown, Jones and Taylor (2003) found that the issue of how much structure to provide in a task was an important factor in maximising the opportunity for geometric reasoning to take place. The role of the teacher is vital in helping students to progress beyond straightforward descriptions of geometrical observations to encompass the reasoning that justifies those observations (Forman, McCormick & Donato, 1997; Gillies & Haynes, 2011; Hiebert et al., 2000).
Guiding principles for developing teaching materials include an expectation to explain, justify, reason, and to provide opportunities for students to be critical of their own and their peers’ explanations (Henningsen & Stein, 1997; Gillies & Haynes, 2011). Effective mathematics teachers believe that students develop mathematically by being challenged to think, explain, and listen (Doig, 2007).

Underlying unsuccessful task implementation is a lack of alignment between tasks and students’ prior knowledge, causing students to fail to engage in high levels of cognitive activity (Doyle, 1983; Henningsen & Stein, 1997). Engaging students in higher levels of reasoning involves more ambiguity and higher levels of personal risk for students than other more routine activities (Henningsen & Stein, 1997). Teachers feel pressured by students’ desire to reduce task complexity and subsequently provide procedural avenues, recipes and algorithms to complete tasks (Doyle, 1983; Hattie & Timperley, 2007), “which strips these problems of their nature of being problems requiring live mathematical thinking” (Brousseau & Gibel, 2005, p. 14). Therefore the role of the teacher in selecting appropriately worthwhile mathematical tasks requires proactive and consistent support of students’ cognitive activity without reducing the complexity and cognitive demand of tasks.

Lawrie and Pegg (1999) explored the difficulties with writing tasks to assess students’ understanding in geometry. Their *Understanding in Geometry Assessment Test* (UGAT) used the van Hiele model as a framework for assessing the geometric thinking levels of Australian secondary school students, where questions had the potential to be answered at several van Hiele levels. The weaknesses of such an assessment, as diagnosed by Lawrie and Pegg, included (1) prompting provided by previous test items (such as diagrams that showed squares in previous questions and then a subsequent question exploring quadrilaterals); (2) the inability for questions to encourage students to respond beyond basic levels of thinking (such as encouraging students to look for relationships between properties); (3) providing prompts in questions that inhibit student thinking; (4) presenting questions that could be answered using recalled facts rather than analysis; (5) test items where students provided too many details (such as listing all known properties for a given shape rather than necessary and sufficient details); and, (6) questions where students provided answers without proper reasoning.

Pittalis, Mousoulides and Christou (2010) described a model for 3D geometric abilities that included recognition and the construction of nets, representation of 3D
objects, structuring of 3D arrays of cubes, and the recognition of the properties of 3D shapes. Boakes (2009) and Huse, Bluemel and Taylor (1994) all claimed that, for example, the act of folding paper in making pop-up books enabled children to learn 3D geometry concepts because it encouraged higher-order thinking skills, problem-solving, and the visualisation of mathematics.

In general, paper-folding tasks can be an effective vehicle for exploring patterns and noting regularities, making conjectures about possible generalisations, and evaluating those conjectures (Wiles, 2013). For example, paper-folding tasks that involve punched holes require students to visualise how a piece of paper can be folded to produce a particular punch-hole pattern (Baker & Talley, 1972). These tasks may be done physically with pieces of paper used to test conjectures (such as in origami activities), or may involve images being folded, refolded and unfolded mentally using simple or compound mental transformations (Milivojevic, Johnson, Hamm & Corballis, 2003). Spatial visualisation tasks involving paper-folding require the “imagining” of an external process (McGee, 1979). “While one is in the course of imagining the external process – one passes through an ordered set of internal states of special relation to or readiness for the successive states of the external process” (Shepard & Feng, 1972, p. 242).

Chinnappan and Lawson’s (2005) interviews with high school mathematics teachers provided a model of teachers’ knowledge of geometry (KG) and knowledge for teaching geometry (KGT). Based on two open-ended questions – What is a Square? and How would you teach square to your students? – their KG and KGT could be assessed by how a teacher defined and used related characteristics of shapes, and how they applied these to real-life examples. Similarly, use of the What is a Square? task by Seah, Horne and Berenger (2016) found that geometric knowledge was developed experientially and not developmentally. While no assertions about the KG or KGT of the teachers involved were made, the authors claimed, “the lack of exposure to geometric shapes in different situations, and the emphasis on visual and concept definitions hinder students’ ability to ‘see’ beyond the obvious physical appearance” (p. 509). Often students provided ‘facts’ linked to personal experiences other than geometric properties.

In a mathematical context, it is important that students filter out those things that may not be mathematical (Gray, Pinto, Pitta & Tall, 1999). Open-ended tasks provide opportunities for students to focus on and extend their own descriptive language
repertoire (Nason, Chalmers & Yeh, 2012). Sfard (1991) suggested that “in order to speak about mathematical objects, we must be able to deal with the products of some processes without bothering about the processes themselves” (p. 10). This ‘structural conception’ allows a student to see a mathematical entity such as a square as a static, unique structure existing somewhere in space and time.

Tasks are often presented with examples of shapes where certain properties are selected to form a definition. For example, as depicted in Quirps (Fox, 2000), students list commonalities and differences from given shapes in order to define an ‘unknown’ shape (aligned to NCTM, Big Idea 4: Year 6 to 8). The ‘spontaneous tendency’ to list properties of shapes corresponds to van Hiele level 2: analysis (de Villiers, 1998b, Mason, 1998). Moving students to van Hiele level 3: abstraction, requires them to understand the connection between facts and procedures, and filter out and order properties where one property precedes or follows another property (Chinnappan & Lawson, 2005; van Hiele, 1985).

Several tasks of the type developed by Cooke (2007) asked students to think in terms of individual and combined properties of shapes without the provision of visual cues (aligned to NCTM, Big Idea 2: Year 6 to 8). Gray, Pinto, Pitta and Tall (1999) proposed that constructing a mental object from ‘known’ properties, instead of constructing properties from ‘known’ objects is an approach for developing advanced thinking for proofs. This “didactical reversal” (p. 6) causes a significant cognitive challenge for students. One particular task by Cooke (2007) asked students to draw a quadrilateral with three right angles (aligned to NCTM, Big Idea 3: Year 9 to 12). Because definitions are arbitrary (Vinner, 1991; Zazkis & Leikin, 2008), a student may decide to define a rectangle by its angle properties as a quadrilateral with three angles. This is a preferred (minimal) definition as definitions should be minimal (Vinner, 1991) and one can prove that the fourth angle is also a right angle.

Much research is needed into effective ways for constructing geometric tasks that help teachers learn how to pose questions and use questioning techniques that will strengthen their mathematics knowledge for teaching (Eli, Mohr-Schroeder & Lee, 2013). Students are not the only ones learning during mathematics lessons (Richardson, Reynolds & Schwartz, 2012; Roth & Gardner, 2012). Primary students in Australia prefer open-ended tasks in geometry (Barkatsas & Seah, 2015), yet the provision of challenging mathematical tasks in middle years’ mathematics remains an issue (Clarke
& Roche, 2009). There appears to be scant research of the types of tasks that secondary students prefer.

This study incorporated task types described by Levenson, Tirosh and Tsamir (2011), along with elements from the task designs proposed by Baker and Talley (1972), Fox (2000), Chinnappan and Lawson (2005) and Cooke (2007). The suite of task designs addressed students’ conceptual development through visualisation, recalling known facts, interpreting representations, classifying shapes, deriving properties of shapes, and analysing reasoning and solutions, through an analysis of keywords, narratives, visual mediators, and routines using a discourse analysis framework proposed by Sfard (2008).

Task designs developed through this study involved students working in groups on geometric problems involving visualisation and explaining their thinking processes to each other and their teachers. The reciprocal relationship between individual thinking and the collective intellectual activities of groups aligns to a socio-cultural perspective on learning (Prusak, Hershkowitz & Schwarz, 2012).

2.5.4 Group Work in the Geometry Classroom

A ‘community of practice’ involves learners participating in the sociocultural practices of a community (Lave & Wenger, 1991) where learners generate and appropriate a ‘shared repertoire of ideas’ (Wenger, 1998). Small collaborative groups communicate and verify these ideas among themselves, and then in whole-class discussion (Sfard, Nesher, Streefland, Cobb & Mason, 1998). Communication as part of a community involves the teacher and students mutually constructing taken-as-shared mathematical interpretations and understandings (Cobb et al., 1991). Communication and endorsed mathematical dialogue require social construction of students into communities. A socio-cultural perspective sees knowledge as an entity that is co-constructed by a ‘communities of learners’ (Brown & Campione, 1994; Osborne, Simon, Christodoulou, Howell-Richardson & Richardson, 2013; Sfard, 2008) and where students construct meaning “from the interplay of what they newly encounter and what they already know, but also from the interaction with others” (Alexander, 2005, p. 11).

Classrooms, as microcosms of mathematical culture (Schoenfeld, 1987), are where acculturation sustains learning, and participation in that culture is how one comes to understand what mathematics is. Linfield, Coltman, Raban and Margetts (2012) also identified this aspect of shared thinking as particularly important for young
students to engage in a common dialogue with peers and teachers in order for ideas to be tested and further consolidated in meaningful contexts of experiences and activities. According to Thompson (1985), Piaget held the view that knowledge can only be constructed with others, reaching a stable state or equilibrium toward which cognition is tended (Slavin, 1996).

Hiebert et al. (2000) offered a useful framework for the dimensions of the classroom and the links between them – the nature of classroom tasks, the role of the teacher, the social culture of the classroom, mathematical tools as learning supports, and equity and accessibility. The shared space of the classroom is what teachers need to manage, and it defines their role. “Teachers represent the community of practice, exemplify valued practices, encourage the development of desired norms, and guide students as they become increasingly competent practitioners” (Even & Tirosh, 2002, p. 214).

Small groups of students working together on problems allow them to construct common solutions to complex problems. These days, technology-enhanced learning opportunities offer innovative ways for students to collaborate as part of a group (Battista, 2001; Coffland & Strickland, 2004). “Collaboration is a coordinated, synchronous activity that is the result of a continued attempt to construct and maintain a shared conception of a problem” (Roschelle & Teasley, 1995, p. 70). Collaboration is then a process by which individuals share meanings relevant to a problem-solving task at hand, and is mediated through shared language.

An implication of social constructivism is that mathematics should be taught through joint problem-solving (Yackel, Cobb & Wood, 1991) that involves multiple players. Cooperative learning has a positive effect on student achievement provided that group goals and individual accountability be established (Slavin, 1990). According to Good, Reys, Grouws and Mulryan (1989), when teachers form work-groups they wish to promote academic outcomes where more students exchange mathematical ideas and develop higher-order thinking. Where teachers take on an active role, pitfalls of student passivity and lack of individual accountability can be addressed (Doyle, 1983). The notion of individual accountability from a discourse perspective was described by Michaels, O’Connor and Resnick (2008) as ‘accountable talk’ where the teacher asks students to restate someone else’s reasoning, or to apply their own reasoning to someone else’s reasoning. Students learn to develop understanding by listening, reflecting, proposing and incorporating alternative views.
For group work to be effective, teachers must prepare students to work cooperatively, be individually accountable, and be open to the ideas of other students (Walmsley & Muniz, 2003). Students who are trained in explicit questioning strategies demonstrate more explanatory behaviour than their untrained peers, and consequently, demonstrate more advanced reasoning and problem-solving skills (Gillies & Haynes, 2011). The teacher’s guiding role includes asking follow-up questions, assisting students to clarify their explanations, and encouraging a variety of interpretations rather than evaluating solutions. These actions become accepted when teachers and students together account for the construction of classroom norms (Yackel, Cobb & Wood, 1991) where substantive learning is understood as an interactive process. Students are motivated to learn when their teachers use language that stresses the strong positive affect about learning, and conveys positive expectations to their students (Turner, Meyer, Midgley & Patrick, 2003).

Effective group work allows students to develop deep understanding through substantive conversations (Grootenboer, 2009; Peterson, 1987). Questioning routines through group work contexts stimulate students to elaborate, clarify, and reorganise their own thinking, and the thinking of their peers. “By expressing ideas and defending them in the face of others’ questions, and by questioning others’ ideas, the students are forced to deal with disagreements” (Bjuland, 2007, p. 5). Teacher-led classroom dialogue requires “questions that provoke thoughtful answers… [and]… answers that provoke further questions” (Mercer & Sams, 2006, p. 509). Peer interactions through collaborative activity are important in mathematical cognition “disequilibrating the student’s egocentric conceptualisations” (Slavin, 1996, p. 49). Students learn from one another because in their discussions of the content, conflicts arise as inadequate reasoning is exposed, and higher quality understandings emerge.

Reflecting on procedures and analysing decisions involves reasoning (Brousseau & Gibel, 2005). The issue of student reflection on learning is difficult to achieve but reconsidering and re-examining solution processes and results is an important step in consolidating knowledge and developing skills to solve problems (Bjuland, 2004; Polya, 1957).

A key aspect of Sfard’s (2008) interpretive framework for mathematical discourse used in this study is that mathematics is a form of communication. Discourses include numerous forms of communication, not just verbal as opposed to Brown and Yule’s (1983) description of ‘languages in action’. This study involved task designs
specifically selected for students to complete in small groups. Under investigation were the ways students interacted with each other and their teachers. Specifically mathematical discourses were examined for both non-verbal and vocal exchanges of reasoning. The teacher’s role as a facilitator of group work and manager of the shared learning space that enabled argumentative talk is critical in ensuring that ground rules encourage students to interact and inter-think (Prusak, Hershkowitz & Schwarz, 2012). The role of the teacher was critical to the study. To determine discourse routines exhibited by students in groups, Bjulund’s (2007) heuristics strategies framework of visualising, monitoring, asking questions, and logical strategies was adapted and used as an analytical tool to track groups of students as they completed tasks.

2.5.5 Summary
The complex role of the teacher requires sufficient content knowledge (Clements, 2004; Nathan & Knuth, 2003) in order to understand the barriers to student learning. Student misconceptions in geometry create significant challenges for teachers in order to engage students in learning activities that require them to think and apply reasoning skills to tasks that are non-procedural in nature. Teachers overcoming their own misconceptions also present an issue without clear pathways to resolving them.

Pedagogical issues also include consideration of task choices and teaching approaches that foster a climate of support and challenge. Group work is seen as a necessary component of modern classroom practice, and as such, teachers need both skills and confidence to manage these practices in ways that ensure all students are able to learn geometry in ways that require shared thinking, mathematical argumentation, and increased student accountability (Michaels, O’Connor & Resnick, 2008).

2.6 Chapter Summary
The review of the literature produced reoccurring themes that emphasised the importance of geometric thinking in mathematics (Battista, 1990; Brown, Jones, Taylor Hirst, 2004; Maier, 1996; Panaoura & Gagatsis, 2009). A seminal theme was the role of visualisation in geometric thinking (Cooke, 2007; de Villiers, 1998a; Duval, 1998; Elia, Gagatsis & van den Heuvel-Panhuizen, 2014; Owens & Outhred, 2006).

The significance of geometry in the mathematics curriculum to develop broader skills of spatial ability and logical reasoning has been articulated in this chapter. Models and frameworks used for describing spatial ability and geometric reasoning were also presented. The blending of curriculum frameworks was useful in emphasising
visualisation through the Big Ideas, and the problem-solving and reasoning proficiencies articulated in the ACM. Both the NCTM and ACM provided an important framework combination in determining the appropriateness of task design choices relevant to the developmental levels of the participants involved in this study.

The role of language in developing and communicating geometric concepts and spatial and geometric reasoning were also examined. Significant pedagogical aspects of teacher knowledge of content and pedagogy, and the role of group work in the geometry classroom were presented and important elements of task design elements applicable to this study were identified.

The complexity and challenges associated with teaching geometry are multifaceted and multi-layered. The role of the teacher is critical in knowing the content knowledge of the subject, and being able to convey that knowledge in meaningful ways that engage and challenge students. Managing aspects of appropriate task selection and understanding how to utilise group work to effectively engineer the learning space that promotes mathematical dialogue requires a level of sophistication and experience.

Several studies involving pre-service teachers indicated poor performance in geometric conceptual knowledge (Cunningham & Roberts, 2010; Gabel & Enochs, 1987; Jones, Mooney & Harries, 2002; Marchis, 2012). Van Der Sandt (2007) suggested that deficits in teacher education had a causal affect on student learning in geometry. However, rarely is school-style geometry, based on Euclidean geometry and transformations, encountered in teacher preparation courses (Kuchemann & Rodd, 2012). Liben (2006) also argued that most teachers in the USA have not experienced explicit instruction in spatial thinking themselves and are therefore unlikely to identify opportunities and resources to build those capacities in their students. Studies such as these raised issues pertinent to this study.

The research design explaining why a qualitative, constructivist approach was a suitable theoretical framework for this study is discussed in Chapter 3. An explication of the design-based research methodology and methods of data generation used in the study are also presented. The steps taken to analyse the data involving an interpretive framework for mathematical discourse are fully explained.
Chapter 3

Research Methodology and Methods

3.1 Introduction

The purpose of this chapter is to provide a detailed explanation of the methodology used to investigate teaching and learning processes in three secondary school classrooms through geometric task designs. Further, the choice of methodology and research methods are explicated as viable approaches for generating data for analysis on how geometric task designs are refined through several phases of implementation. This chapter also provides an explanation of the methods utilised to generate and record mathematical discourse. These data provided the evidence base for the analysis and interpretation of the ways students’ communicate their understanding of geometric concepts, their use of reasoning skills to convince others, and the role of teachers in facilitating learning and developing geometric reasoning skills.

The main research question for this study is: What factors influence the development of geometric reasoning among middle years students when learning about 2-dimensional (2D) shapes?

The sub-questions are re-stated here as the component parts of the methodology are discussed.

- How does visualisation underpin the teaching and learning of definitions and classifications of 2D shapes?
- How does language support the development of geometric reasoning?
- How does instruction influence the learning of geometric concepts?

This study sought to explore how students and teachers interacted with the content of geometric tasks that were designed to enhance spatial and geometric reasoning skills. Multiple data collection methods were used to document the learning and teaching processes in classrooms. An important objective of this research was to refine task designs themselves using a design-based research approach (Barab & Squire, 2004; Collins, Joseph & Bielaczyc, 2004; Herrington, Reeves & Oliver, 2010; Tabak, 2004).
The methodological approach adopted to implement task designs in geometry that develop students’ spatial and geometric reasoning skills is first explained. An argument for the legitimacy of design-based research as a methodology is discussed in relation to “the iterative cycles of testing and refinement” (Reeves, 2006, p. 59) of tasks, and design-based research is defined and defended as a systematic inquiry that embraces the messiness and complexity of the classroom (Cobb & Yackel, 1996; Shavelson, Phillips, Towne & Feuer, 2003). Indeed, design-based research itself is ‘messy’ in that it occurs in naturalistic settings and variables are deliberately not controlled. This makes the context of the classroom central to the particular methodology providing good opportunities to obtain rich data (Yutdana, 2005). However, researching learning and cognition is, in and of itself, not adequate as design-based research (Barab & Squire, 2004). Design-based research necessarily involves interventions of some type (Barab & Squire, 2004; Brown, 1992; Collins, Joseph & Bielaczyc, 2004). This discussion will present design-based research as a feasible methodology and will address what differentiates it from other forms of research; what counts as reasonable and useful warrants for the assertions investigated; the boundaries of a naturalistic context; and how bias was controlled by selecting evidence, in reporting observations, and in developing trustworthy claims.

A theoretical, interpretive framework for mathematical discourse is also presented and explained as part of the methodology. Mathematical discourse involves shifting from everyday discourse to a more precise use of language (Moschkovich, 2003), facilitated by the teacher’s monitoring of classroom interactions (Nathan & Knuth, 2003). Mathematical discourse “is tantamount to learning to think in a mathematical way” (Sfard, 2001, p. 4) and is characterised by four interrelated characteristics – Keywords, Visual mediators, Narratives, and Routines. These characteristics were used to investigate ways students and teachers communicate geometric concepts and reasoning, as well as the role of the teacher in modifying and exchanging existing student discourse in geometry to extend students’ discursive skills.

The next section of this chapter will discuss the research methods used in generating credible, qualitative findings including classroom observations, video-recording teacher instruction, and analysing both instruction and student work samples. A qualitative research design appropriately allows for descriptive and interpretive analysis of teaching and learning through the task design phases. As Creswell (2009) stated, “qualitative approaches allow room to be innovative and to work more within
researcher-designed frameworks” (p. 19). The tradition that is most readily identified as qualitative research is ethnography (T Teddlie & Tashakkori, 2009). In anthropological studies, ethnographical approaches involve a broad range of data collection methods to describe and interpret human cultures. As this research involved watching, listening and asking questions, several methods of data collection were necessary to capture “the meanings that give form and content to social processes” (Hammersley & Atkinson, 2007, p. 22). The social processes of the classroom, exhibited mainly by group work, informed this study.

Issues regarding the four aspects of trustworthiness as proposed by Guba (1981) – truth value, applicability, consistency and neutrality, as well as ethical considerations for the study will be addressed in the final sections of this chapter.

3.2 Methodology – Qualitative Research

This study employed a qualitative research approach. Qualitative research is a situated activity placing the researcher in a natural setting (Denzin & Lincoln, 2005). Patton (2003) stated that “qualitative methods are often used in evaluations because they tell the program story by capturing and communicating the participants’ stories” (p. 2). Moreover, Patton (2002) also described qualitative methods as “ways of finding out what people do, know, think and feel by observing, interviewing, and analysing documents” (p. 145). An outcome of qualitative research can include the evaluation of the effectiveness of programs (Saldaña, 2011). To explore the main and sub-questions for this study, flexible approaches to data generation and analyses were required to support rich descriptions of complex human interactions (Brown, 1992; Collins, Joseph & Bielaczyc, 2004). Qualitative methods allow the researcher to employ adaptable data collection procedures, gain detailed description of participants’ views and experiences, and explanations and interpretations of emerging ideas (Boeije, 2010). This study sought to determine the effectiveness of task designs in developing geometric reasoning in classroom situations as part of a secondary school geometry program in middle years mathematics. A qualitative research approach therefore aligned with this objective of generating rich descriptions of messy classroom experiences.

According to Cobb and Bowers (1999), the socio-constructivist approach provides a framework for an understanding of mathematics learning in classroom communities of practice. As this research also involved understanding how students and teachers construct meaning pertaining to geometric definitions, and the role of
visualisation in the development of geometric reasoning, the type of inquiry necessitated approaches that are aligned with a socio-constructivist theoretical paradigm. The constructivist researcher looks to understand the meanings that constitute actions that deploy a wide range of interconnected interpretive practices that “secure an in-depth understanding of the phenomenon in question” (Denzin & Lincoln, 2005, p. 5). Constructivists believe that all entities are simultaneously shaping each other, and it is impossible to distinguish between cause and effect (Teddlie & Tashakkori, 2009):

A paradigm may be viewed as a set of basic beliefs that deals with ultimates or first principles. It represents a worldview that defines, for its holder, the nature of the "world," the individual's place in it, and the range of possible relationships to that world and its parts (Guba & Lincoln, 1994, p. 107)

The concept of a paradigm “broadens the notion of theory and relate to a community of individuals who share a common theory” (Kuzniak & Rauscher, 2011, p. 133), and includes the entire constellation of beliefs, values, techniques and practices shared by the members of that community. When people share a paradigm, they can communicate in very unambiguous ways. Both the teacher and students attempt to make sense of each other’s verbal and non-verbal activity (Cobb & Steffe, 1983). Knowledge can be “taken-to-be-shared” to the extent that individual constructions function in the same way in given situations (Cobb et al., 1991). However, as students and teachers may adopt different paradigms, these differences may lead to misunderstandings. As Crowley (1987) posited “if the teacher, instructional materials, content, vocabulary, and so on, are at a higher level than the learner, the student will not be able to follow the thought processes being used” (p. 4). Geometric reasoning is at the heart of this research and involves specific ways of communicating logical processes. Effective communication necessitates that both students and teachers have paradigmatic alignment. Verbalisation requires students and teachers to articulate what might otherwise be vague or misconceived ideas (Crowley, 1987).

The constructivist researcher examines the ways in which individuals construct meaning of phenomena or experiences. From a constructivist perspective, social interactions between students, and between the teacher and students, are catalysts for individual students’ cognitive development (Cobb, 2011). “Reality is individually constructed” (Dede, 2004, p. 110), and therefore, multiple meanings are generated as individuals perceive experiences differently. Constructivists do not generally begin with a theory. Rather, they “generate or inductively develop a theory or pattern of meanings” (Creswell, 2009, p. 8) throughout the research process. Characteristically, written
student work samples, participatory observations, and teaching episodes form the basis of the research design methods used to record learning and teaching processes that involve multiple forms of communication within the social construction of the classroom.

Social constructivism places a greater emphasis on learning through social interaction and the value placed on cultural background. For Vygotsky (1978), culture gives the child the cognitive tools needed for development. Adults (teachers, researchers, aides) in the learner’s environment are conduits of culture, which includes language, cultural history, social context, and electronic forms of information access. Those who work in the tradition of Vygotsky, as described by Cobb (1995), employ a sociocultural metaphor to account for children’s development through participation in “cultural activities with the guidance of more skilled partners” (p. 364). These classrooms involve collaborative learning as a process of peer interaction that is mediated and structured by the teacher (Cobb & Yackel, 1996). Discussion can be promoted by the presentation of specific concepts, problems or scenarios, and is guided by means of effectively directed questions, the introduction and clarification of concepts and information, and references to previously learned material (Cobb et al., 1991; Cobb 1995). The teacher supports students’ access to tasks by leading a discussion of task scenarios in the classroom with the goal of ensuring that they become real in the minds of all students (Cobb, 2011).

The current study sought to determine the extent to which students retained new information after completing a series of related tasks involving visualisation. The ways students conceive geometric concepts and communicate their conceptions (or misconceptions) were analysed using an interpretive framework of mathematical discourse defined by Sfrad (2008).

A constructivist researcher captures the individual construction of mathematical understandings and strategies, whereas the social constructivist views the development of socially constructed meaning through negotiation and consensus within a community of learners as a subject of mathematical development (Owens & Outhred, 1998). Social constructivist classrooms typically involve aspects of reciprocal teaching and situated learning that constitutes a community of learners (Brown, 1992; 1994; Brown & Campione, 1994) where students’ participation in activities with the guidance of more skilled partners allow them to internalise the tools for thinking (Cobb, 1995). Social norms are explicit topics of negotiation and include explaining and justifying solutions,
making sense of explanations, mathematical argument, and questioning alternatives to portray the classroom participation structure (Cobb, 1995; Cobb, Confrey, diSessa, Lehrer & Schauble, 2003; Cobb & Yackel, 1996).

Group work was a significant aspect of the research design used in this study. Apart from one preliminary task completed individually, supplementary tasks were conducted in small groups of 3 to 4 students. Students were encouraged to discuss parts of the task and assist each other in developing (what was intended to be) group responses. Tasks were selected from the existing literature and adapted specifically, requiring students to use key geometric terminology, draw diagrams, and develop mathematical arguments relating to 2D shapes.

Qualitative methods generate large amounts of information requiring reorganisation into categories, themes and/or illustrative case examples. These can be analysed through content analysis (Saldaña, 2011), attempting to make sense of, or interpreting phenomena in terms of the meanings people bring to them (Denzin & Lincoln, 2005). According to Patton (2003), the fruits of qualitative inquiry are the themes, patterns, understandings and insights that emerge from the evaluation of fieldwork and subsequent analysis. The use of qualitative methods in this study was appropriate because it enabled a rich collection of data from different perspectives (students, teachers and researcher) within the classroom setting. This research design allowed tasks to be implemented in one school, to be refined as a result of examining student work samples and feedback from teachers, and ‘tested’ in a second school. The qualitative tools, such as student work samples, participatory observations and recorded teaching episodes, provided the best means for an interpretative framework to be applied to understand and describe the role of visualisation in the development of geometric reasoning.

The main research design features (see Figure 1) used to investigate, describe and construct an interpretation of students’ reasoning skills, their social interactions, and the role of the teacher in supporting the learning of geometric concepts, incorporated several research methods. Trustworthiness measures are discussed later in this chapter.
A design-based research approach incorporating Sfard’s (2008) interpretive framework will explain how an interpretation of specific classroom episodes informed the planning and implementation of subsequent classroom episodes. This theoretical approach necessitated qualitative differences between participants in classrooms to be analysed and “the quality of students’ inferred, socially situated mathematical experiences” (Cobb & Bowers, 1999, p. 13) to be interpreted.

The methodological approach and research methods used in this study to generate evidence of how students and teachers used visualisation and geometric reasoning skills to construct new meanings and communicate their thinking through a series of tasks are detailed in the following sections of this chapter.

### 3.2.1 Design-Based Research

Design-based research (DBR) is a relatively new branch of academic pursuit, which places researchers in classrooms in order to develop interventions through task designs with teachers and students. The aspect of *shared thinking* where students work together in an intellectual way to solve problems, clarify concepts, or evaluate activities
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(Linfield, Coltman, Raban & Margetts, 2012) is particularly important. Learning within a DBR framework requires dialogue with colleagues and teachers in order for ideas to be tested and further consolidated through evaluative techniques. Tasks are reviewed and re-tested as an iterative process in classrooms. Task designs are often based on different ways of conceptualising mathematics as a complex but stable set of ideas and theories. Levenson, Tirosh and Tsamir (2011) suggested five task types that could be used to encourage concept development: classifying mathematical objects, interpreting multiple representations, evaluating mathematical statements, creating problems, and analysing reasoning and solutions.

In designing and trialling classroom materials, Brown, Jones, and Taylor (2003) found that the issue of how much structure to provide in a task was an important factor in maximising the opportunity for geometric reasoning to take place. Their findings indicated that the role of the teacher was vital in helping students to progress beyond straightforward descriptions of geometrical observations, and to be able to reason and justify those observations.

A DBR approach was most appropriate for this study because it provided a framework with which to implement and refine geometric tasks that examined student interpretations, and explored the ways in which they communicated their geometric reasoning. This research incorporated unstructured tasks through open-ended questions as well as structured tasks where question items build toward students being able to construct geometric definitions or arguments. Further, this study sought to document how explicit instruction supported student progression toward geometrical conceptual understanding and deductive reasoning skills.

Design-based research methodology has become prominent in educational research in recent years (Cobb & Steffe, 1983). Some confusion in the literature does exist with terminology such as experimental design, design research, teaching experiments, development research and design experimentation, all being used to mean similar things (Tabak, 2004). Confusion also exists with the terminology of research design which is meant to describe the structure upon which data collection and analyses will occur. However, the term design-based research combines design and research, and is particularly suited to educational settings (DBRC, 2003; Gravemeijer & Cobb, 2006; van den Akker, 1999).

Increasingly, groups and individuals of many theoretical perspectives are conducting design-based research, and it draws on a variety of intellectual traditions.
(Bell, 2004). It is different to action research which is typically limited to effecting change in the local setting. Action research according to Kemmis (2008) is a form of “enquiry undertaken by participants in social situations in order to improve…their own social or educational practices” (p. 122). It is a process of self-education for the practitioner generating practical or technical outcomes (Kemmis, 2001). The goal of design-based research, however, is to “develop theories about both the process of learning and the means that are designed to support that learning” (Gravemeijer & Cobb, 2006, p. 75). Theory that informs practice is at the heart of design-based research methodology, and the creation of design principles and guidelines enables research outcomes to be transformed into educational practice (Herrington, Reeves & Oliver, 2010). The development of theory provides an interpretive lens for conveying essential features that “set things apart and pulls thing together” (Brown, 1994, p. 8). Developing theory through design-based research concerns the students’ development of key disciplinary ideas that specify patterns in student reasoning (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003).

Design-based research is described by Bell (2004) as a way of learning important things about the nature and conditions of learning by attempting to engineer and sustain educational innovation in everyday settings. Collins, Joseph and Bielaczyc (2004) viewed it as a methodology for carrying out studies of education interventions. DBR attempts to address the nature of learning in context, not in a laboratory. This intertwining of research and practice aligns with the interventionist nature of education. Brown (1992) stated, “this is intervention research designed to inform practice” (p. 143). The identification of possible explanations and ways for redesigning learning environments or instructional artefacts is an intended outcome of DBR (Shavelson, Phillips, Towne & Feuer, 2003). Brown (1994) referred to an ‘orchestration’ of the environment to foster meaningful and lasting learning in collaboration with students and teachers.

Design-based research is contextualized within educational settings, with a focus on the setting guiding the design process (Dede, 2004). Design-based research methods typically include descriptions of how learning unfolds through classroom life (Tabak, 2004), requiring the researcher to be present. To this respect, many variables are deliberately and appropriately not controlled. “The goal is not to sterilize naturalistic contexts” (Barab & Squire, 2004, p. 11). The design is an evolving process. As design-based research occurs within particular settings, local interpretation plays a critical role
in providing sustainable innovation. Sustainable innovation requires local understandings of how and why an innovation works within a particular setting at a particular time (DBRC, 2003). The researcher is an active participant that reserves the right to stop students at any time and ask them to rationalise what they are doing, and how that action would help them solve the problem at hand (Schoenfeld, 1992b).

Brown (1992) presented the notion of an Intentional Learning Environment that differs from a traditional classroom in that students behave as researchers and teachers monitor progress, where inquiry and discovery are guided, and involve tools for intentional collaboration and reflection. Content goals, as well as social goals, are achieved through design experiments, where students come to value the expertise of other students. Design experiments intend, therefore, to encourage distributed expertise amongst a community of learners. This is what Lave and Wenger (1991) termed a community of practice. The idea that students respect the contributions of others is, what Collins, Joseph and Bielaczyc (2004) called, diverse expertise. However, the mere existence of a community of practice is not sufficient to drive change in adopting innovations (Zaritsky, Kelly, Flowers, Rogers & O’Neill, 2003). Gravemeijer and Cobb’s (2006) metaphor of learning ecologies emphasised that learning communities were interacting systems rather than separate factors that influenced learning:

Elements of a learning ecology typically include the tasks or problems that students are asked to solve, the kinds of discourse that are encouraged, the norms of participation that are established, the tools and related material means provided, and the practical means by which classroom teachers can orchestrate relations among these elements. (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003, p. 9)

Developmental research is synonymous with design-based research (Reeves, 2000). Gravemeijer (1994) also described classroom-based research and instructional design in collaboration with teachers as developmental research where each informs each other – “a cyclic alternation of development and research” (p. 449). The first step involved developing and conducting an anticipatory thought experiment (Cobb & Bowers, 1999; Gravemeijer, 1994) that also predicts how the instructional activities might be comprehended through interaction in the classroom and how students’ interpretations and solutions might evolve as the students participate in them - to envision how learning and teaching are likely to proceed along some hypothetical learning trajectory (Simon, 1995); assuming mathematical activity occurs in predictable ways and that all students can benefit from the same learning task. The learning trajectory is hypothetical in that the actual learning trajectory may differ from what the
The basic research cycle as depicted by Cobb and Bowers (1999), suggests that instructional development requires analysis in classrooms that, in turn, informs ongoing instructional design efforts (see Figure 2).

![Diagram: Instructional Development (guided by domain-specific instructional design theory) Classroom-based Analysis (guided by interpretive framework)]

**Figure 2.** The developmental or transformational research cycle.

The phases of conducting a design experiment involve: preparing an anticipatory instructional design and interpretive framework; trialling in the ‘test bed’ of the classroom; and conducting retrospective analyses (Gravemeijer & Cobb, 2006). In approaching design in this manner, the designer formulates conjectures about both the course of the classroom community’s mathematical development and the means of supporting and organising it (Cobb, 2011; Sandoval, 2004). Student-student and student-teacher interactions are important in developing mathematical concepts (Owens & Outhred, 1998). Typically, video-recordings of student engagement and teacher activity provide an important part of the documentation of learning processes (Shavelson, Phillips, Towne & Feuer, 2003).

Design-based research is directed at creating products and processes for the improvement of student learning and teaching skills (Zaritsky, Kelly, Flowers, Rogers & O’Neill, 2003). Local interpretation plays an important role in terms of sustaining innovation/intervention (DBRC, 2003). The phases of design-based research can, therefore, be lengthy and time-consuming.

An outcome of DBR is the development of prototypical products (DBRC, 2003; Ekanayake, Brown & Chinnappan, 2003) where the value of prototypes need to be evaluated against the background of local instructional theories (Gravemeijer, 1994) that derive evidence-based claims that account for the naturalistic setting (Barab & Squire, 2004). “Prototypically, design experiments entail both ‘engineering’ particular forms of learning and systematically studying those forms of learning within the context defined by the means of supporting them” (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003, p. 9).
Chapter 3

The four connected phases of design-based research as defined by Reeves (2006) include – *problems, solutions, iterative cycles, and reflection*. Underpinning this approach to design-based research requires intensive collaboration with teachers to develop guidelines that enable research outcomes to be transformed into educational practice. An ‘ideal’ intervention or ‘evolutionary prototyping’ is desirable (Herrington, Reeves & Oliver, 2010). The aim of DBR is not to implement complete interventions, but to generate successive prototypes in a cyclic or spiral fashion. “Analysis, design, evaluation and revision activities are iterated until a satisfying balance between ideals and realisation has been achieved” (van den Akker, 1999, p. 7).

Reeves (2006) model (see Figure 3) suggests that DBR requires problems to be determined collaboratively and solutions to be developed and then tested in the field, with the final phase requiring a reflective process to inform future implementation decisions. This study involved an examination of the role of visualisation in geometric reasoning. Through the DBR processes and interpretive framework employed, inferences were drawn about task implementation, the role of group work, and the role of the teachers in developing geometric reasoning skills for 2D shapes.

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<table>
<thead>
<tr>
<th>Analysis of practical problems by researchers and practitioners in collaboration</th>
<th>Development of solutions informed by existing design principles and technological innovations</th>
<th>Iterative cycles of testing and refinement of solutions in practice</th>
<th>Reflection to produce “design principles” and enhance solution implementation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Refinement of problems, solutions, methods, and design principles</strong> Four phases of design research (Reeves, 2006, p. 59)</td>
<td></td>
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</tr>
</tbody>
</table>

- **Problems** – *What factors influence the development of geometric reasoning among middle years students when learning about 2-dimensional (2D) shapes?*

- **Solutions** –
  - The ways students use visualisation and the role of the teacher may best be observed through geometric task designs that require them to communicate their thinking using verbal and non-verbal communication. (Brown, Jones, & Taylor, 2003).
  - Geometric reasoning is best facilitated by task designs where all students to perform physical and mental manipulations (Battista, 2002).
  - Tasks involving sorting and classifying, comparing and contrasting are suggested ways of developing geometric reasoning (Oberdorf & Taylor-Cox, 1999).

- **Iterative cycles (DBR)** – methods used to generate the data to address the research questions.

- **Reflection** – to inform future development and implementation decisions.

*Figure 3. Phases of design-based research.*
Several challenges arise due to the complexity of real-world situations. These challenges include situating teachers’ activities in the schools in which they work and developing an interpretive framework to document the collective learning experiences of both students and teachers (Cobb, Zhao & Dean, 2009). One of the limitations of this type of research as suggested by Sandoval (2004), is that it attempts to develop localised theories. How localised are localised theories? Further, in developing these theories, there is a need to account for the complexities of multiple interactions in complex settings (Sandoval & Bell, 2004). Replicability and generalisability are challenges in DBR that can be overcome by rich descriptions of the ‘treatment’ (Hoadley, 2002). Rich description involves a range of research tools to construct ‘narratives of change’ (Tabak, 2004). These research tools are presented later in this chapter.

3.2.2 Sfard’s Interpretive Framework

Discourse, involving students and their teachers, is an important aspect in understanding students’ initial interpretations of tasks, as well as their ability to communicate responses. The role of discourse (Brown, Jones, Taylor & Hirst, 2004; Crowley, 1987; Nason, Chalmers & Yeh, 2012; Pegg, 1992) and gestures (Roth & Gardener, 2012; Özerem, 2012) is well documented and are critical components for communicating geometric ideas. The role of discourse through whole class discussion organised by the teacher “constitutes an additional indicator for teacher skillfulness” (Leikin & Rota, 2006, p. 46). Also, explored through discourse is “the teacher’s ability to receive and review what has been said and to judge what to offer by way of an individually-tailored response which will take learners’ thinking forward” (Alexander, 2008, p. 17). Teachers are able to see the limitations of tasks and become better able to select and use tasks that enable greater exploration and discussion (Silver & Smith, 1996).

Mathematical discourse is exhibited by four critical properties – Word use, Visual mediators, Narratives, and Routines (Sfard, 2008). Seah, Horne & Berenger (2016) defined these critical properties as:

- **Word use** – Shapes are described and defined in distinctly mathematical ways using *keywords*. Their usage reveals how a student sees and interprets that shape.
- **Visual mediators** – Visual objects that are operated on as part of the communication process are known as *visual mediators*. They help define shapes and their properties in a universal visual format.
• **Narratives** are a sequence of expressions or statements used to frame descriptions of objects. Spoken or written narratives are subject to rejection or acceptance as deductive accounts of an endorsed consensus. Mathematical theories, definitions, proofs, and theorems are examples of discursive constructs resulting from endorsed narratives.

• **Routines** are specific repetitive patterns characteristic of creating and substantiating narratives about shapes.

The keywords and visual mediators give rise to the narratives and routines one might apply to mathematical practices as the “taken-as-shared ways of reasoning, arguing, and symbolising established while discussing particular mathematical ideas” (Cobb, Stephan, McClain & Gravemeijer, 2010, p. 126). Only a few researchers have used this framework to analyse students’ communication of geometric concept knowledge.

Seah, Horne and Berenger (2016) used this framework to analyse how students recorded what they knew about a square. In their study of four secondary classrooms, written work samples were evaluated for the types of keywords, visual mediators and narratives students used in order to communicate personal and formal conceptual knowledge about a square. The results showed a lack of students’ exposure to geometric concepts that, in turn, hindered the development of their geometric reasoning abilities. Specific use of the framework indicated that students had difficulties using mathematical signifiers to communicate the geometric meaning of the properties of a square. As part of the discourse used among mathematicians, mathematical signifiers communicate specific attributes a shape might have, such as equal or parallel sides, or right angles. Students showed an adherence to prototypical representations and had very limited ability to use keywords in formulating accurate and complete narratives such as definitions. Their lack of geometric thinking was reflected by their inability to use mathematical discourse to communicate their conceptual understandings.

Sfard’s work (2000, 2001, 2006, 2007, 2008) largely focussed on the modification and changes in learner discourse. In a study into how a dynamic geometry environment [DGE] changes discourse, Sinclair and Yurita (2008) applied Sfard’s interpretive framework to record changes in discourse engendered by the introduction of DGE. This research focussed mainly on changes in teacher discourse with students and
revealed changes in geometric vocabulary, visual mediators and narratives being used to perceive and reason about mathematical objects.

These studies indicated that the use of Sfard’s interpretive framework was appropriate for interpreting the ways in which students thought about 2D geometric concepts, and how they communicated their conceptions to their teachers and each other. Further, this discourse framework was suitable for analysing how teachers themselves understood and communicated their own conceptions of the same geometric concepts. A socio-constructivist’s view – that all entities (teachers, students, and tasks) are simultaneously shaping each other – assumes “that the discursive features of teachers’ communication in the classroom will be highly influential in student learning since their own ways of thinking and communicating will also change” (Sinclair & Yurita, 2008, p. 6). Leikin and Rota (2006) argued that the roles of teachers and students in dialogue are almost symmetrical, and when discussion is of a heuristic nature (questioning, translating, repeating utterances, stating facts, constructing a logical chain, and providing feedback) the teacher is able to move learning into “new mathematics territory” (p. 55).

The design-based methodological approach provides for rich data to be generated that captures all four critical properties of mathematical discourse as defined by Sfard’s interpretive framework (2008) for mathematical discourse. The range of methods utilised in this study will be explained in the next section of this chapter.

### 3.3 Data Gathering Methods

This section outlines the procedures undertaken for determining the participants for the study, the task selection processes, the specific tools and research methods used to inform the design of the research. Instruments used, including subsequent modifications, and data collection techniques are also explained.

Participant selection and sampling procedures are discussed, and a coding scheme is provided to ensure the anonymity of all schools, teachers, and students who contributed to this study. As this study employed a design-based research approach, the setting for this study took place in non-clinical, ‘messy’ classrooms in order to observe human interactions in their natural setting. Student work samples and the recording of teacher instruction and student interactions were major components of the research design. As identified by Cobb, Zhao and Dean (2009), design-based research focuses on students’ mathematical learning either as they interact one-on-one with a researcher or
as they participate in classroom communal processes. This research methodology is used for understanding *how, when, and why* educational interventions work in practice, as research that is detached from practice may not account for the influence of the context of the setting (DBRC, 2003).

The selection of tasks as well as the specific order in which they were conducted is elaborated so as to provide a rationale for the choices taken to best create opportunities for students to learn new concepts, communicate with each other, and reflect on their own learning experiences. As learning from initial implementation of tasks led to refinements being made, these steps are explained.

The methods used for generating data included student written work samples, participatory observations, teacher notes and commentary, researcher field notes and recorded teaching episodes. These are described in detail in this section of the chapter.

### 3.3.1 Sample Selection and Coding

Any secondary schools in Victoria were eligible for this study in so far as they offered a traditional mathematics program governed by the Australian Curriculum ([www.australiancurriculum.edu.au](http://www.australiancurriculum.edu.au)) and within the broader policy framework of the Department of Education Training [DET] Victoria ([www.education.vic.gov.au](http://www.education.vic.gov.au)). However, in order to advance the project, two particular schools were invited to participate in this research. These co-educational public schools were known to the researcher through his professional network as a former Numeracy Coach for the former Northern Metropolitan Region [NMR] in DET Victoria. The sample selection was, therefore, purposive or judgment sampling (Tongco, 2007). The researcher, through the same professional network, also knew one of the teachers in this study.

Purposive sampling is used when the research design necessitates that the researcher makes decisions about which participants would most likely contribute to the study in terms of both relevance and depth (Jupp, 2006). In this way, purposive sampling is somewhat strategic as qualified experts are chosen as informants who know about the culture (Tongco, 2007). “Qualitative researchers recognise that some informants are richer than others and that these people are more likely to provide insight and understanding for the researcher” (Marshall, 1996, p. 523). Purposive sampling is a valid method for this study because random sampling was neither feasible nor efficient. Nevertheless, this type of sampling procedure does present the potential for data to be selective (Teddlie & Tashakkori, 2009). A random sampling method may have provided
the best opportunity for results to be generalised to a population but it is not the most effective way of developing an understanding of complex issues relating to human behaviour (Marshall, 1996). The interactions between students and their teachers present a very complex set of human behaviours.

The rationale for this type of sampling is, therefore, based on pre-existing professional relationships, prior knowledge about each of the schools, their known teaching approaches, and their known interested cohort of teachers willing to participate in this research. It was the researcher’s belief that having knowledge about these schools would lead to a more rich and in-depth understanding and theorising about task designs, and the teaching and learning of visualisation and geometric reasoning.

The RMIT Human Research Ethics Committee HREC, which abides by the NHMRC ethical clearance requirements (https://www.nhmrc.gov.au/guidelines-publications), approved the research study and DET Victoria gave permission to conduct research in Government schools before contact with any participants was made. These procedures are fully addressed in Section 3.5.

The two public secondary schools involved in this study from Melbourne, Victoria, Australia comprised of two Year 8 classes in the pilot school and one Year 7 class in the second school. Once the first school had been identified for the initial phase of this study, informal email contact was made with the principal and informal phone contact was made with the school mathematics coordinator to gauge initial interest in being involved in the study. The coordinator determined classes that were most suitable for the research based on local decision-making and teacher availability. Teachers were contacted via email and through an initial visit to the school to explain the extent of their role in the study. Formal letters of invitation were provided to interested teachers along with a letter to the parents of their students outlining the study and the level of involvement of their child. A research schedule of mutually agreeable times during Term 4, 2015 was agreed via email.

During this initial phase, and after analysing component parts of the tested task designs and making modifications, a second school was sought in the same way as the first school. All parties at the second school were formally invited to participate. The second school had only one class available to be involved in this study. An agreed schedule of mutually agreeable times to conduct the study during Term 1, 2016 was negotiated directly with the teacher.
All participation was voluntary. The names of the schools were coded and pseudonyms were used for all participants to protect their identities. All students in each class were similarly coded to indicate their school, their class, and their pseudonym. Examples of the codes and pseudonyms are provided in Table 1.

Table 1  

<table>
<thead>
<tr>
<th>School 1 (S1)</th>
<th>School 2 (S2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class A (Year 8)</td>
<td>Class A (Year 7)</td>
</tr>
<tr>
<td>Teacher</td>
<td>Teacher</td>
</tr>
<tr>
<td>S1A.Teacher</td>
<td>S2A.Teacher</td>
</tr>
<tr>
<td>School 1, Class A</td>
<td>School 2, Class A</td>
</tr>
<tr>
<td>Student 1</td>
<td>Student 1</td>
</tr>
<tr>
<td>S1A.James</td>
<td>S1B.James</td>
</tr>
<tr>
<td>School 1, Class A</td>
<td>School 1, Class B</td>
</tr>
<tr>
<td>Student 2</td>
<td>Student 2</td>
</tr>
<tr>
<td>S1A.Mary</td>
<td>S2A.Mary</td>
</tr>
<tr>
<td>School 1, Class A</td>
<td>School 2, Class A</td>
</tr>
</tbody>
</table>

In School 1, Class A comprised 16 male and nine female students. Class B comprised 15 male and nine female students and was part of an accelerated learning program, where students were selected for this program based on their Year 6 primary school reports, their primary school teacher’s evaluation of their general abilities across subject disciplines including mathematics, and results from a select entry assessment. In School 2, the participating class consisted of 15 male and six female students in Year 7.

3.3.2 Geometric Task Selection

As a result of the review of the literature into geometric tasks, five tasks involving spatial and geometric reasoning were identified. These were adapted to include minor modifications of terminology but, in essence, the tasks remained unchanged. The researcher developed two additional tasks by combining several aspects of tasks described by Cooke (2007). A student reflection item was included to allow students to describe their learning experiences and rate each task. A rubric for each task was also developed to anticipate likely student responses for the purpose of data analysis, as well as providing a mechanism for teachers to provide written feedback for the researcher and to students on an individual basis. Tasks were conducted in the two schools at different stages of the study in order for initial results from School 1 to be interpreted and modifications of task designs to be made before being conducted in School 2. A rationale for task selection is provided in the task outlines in this section of the chapter.

The tasks and adjoining rubrics (see Appendices 1 - 7) form the first iterative step of the design-based research process. These tasks are summarised in Table 2.
Table 2  
*Task Summary (in order of implementation)*

<table>
<thead>
<tr>
<th>Preliminary Tasks</th>
<th>Reference</th>
<th>Appendix</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quirps</td>
<td>adapted from Fox (2000)</td>
<td>Appendix 1</td>
</tr>
<tr>
<td>What is a Square?</td>
<td>Seah, Horne &amp; Berenger (2016)</td>
<td>Appendix 2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Supplementary Tasks</th>
<th>Reference</th>
<th>Appendix</th>
</tr>
</thead>
<tbody>
<tr>
<td>Properties of Squares and Rectangles</td>
<td>adapted from Cooke (2007)</td>
<td>Appendix 3</td>
</tr>
<tr>
<td>Properties of Parallelograms and Rectangles</td>
<td>adapted from Cooke (2007)</td>
<td>Appendix 4</td>
</tr>
<tr>
<td>Quadrilaterals by Properties (1)</td>
<td>adapted from Cooke (2007)</td>
<td>Appendix 5</td>
</tr>
<tr>
<td>Quadrilaterals by Properties (2)</td>
<td>adapted from Cooke (2007)</td>
<td>Appendix 6</td>
</tr>
<tr>
<td>Paper-folding</td>
<td>adapted from Cakmak, Isiksal &amp; Koc (2014)</td>
<td>Appendix 7</td>
</tr>
</tbody>
</table>

The schools did not require any initial preparation before the commencement of the study, apart from completing all the necessary ethical steps required by DET Victoria and the RMIT University HREC (refer to Section 3.5).

### 3.3.2.1 Preliminary Tasks

The purpose of the preliminary tasks was fourfold. First, these tasks were conducted to orient students and the teacher toward the research. This involved the researcher and teacher observing and recording students while they completed the preliminary tasks. Secondly, these tasks provided initial understandings on how well students engaged in both structured and open tasks in geometry. Thirdly, the teachers were able to provide feedback to students via a scoring rubric; and fourthly, the use of the preliminary tasks provided the teacher with an opportunity to determine student groups for supplementary tasks.

One structured task, *Quirps*, was selected as an instrument that teachers would use to coordinate small groups. Students were not normally organised into working groups for any of their mathematics instruction although both teachers had indicated that their students had done some group work activities previously. The type of collaborative group work referred to by teachers is unknown. As mathematical discourse was an important part of the research design, it was necessary to observe students working in groups as organised by their teacher. It was essential that a “community of discourse” (Brown & Campione, 1994, p. 237) be established early in the study in which constructive discussion, questioning, and criticism were the mode rather than the exception for student discourse.
**Quirps**

*Quirps* (see Appendix 1) was an “effective way to familiarize students with a concept and to make them aware of its distinguishing characteristics” (Fox, 2000, p. 575). This task was selected as a preliminary task that required students to use more than rote-acquired facts about standard geometric figures. Fictitious shapes, *quirps*, were presented to students requiring them to determine distinguishing characteristics of 2D shapes in order to develop a definition for a *quirp*.

*Quirps* was designed as a structured written task that required students to respond to items by identifying similarities and differences in groups of 2D shapes. A marking rubric was developed for teachers to provide feedback to students about the quality of their responses, and to the researcher about potential task refinements. The task was further intended to provide students with an opportunity to work in small groups. This was a necessary component as this study aimed to record different aspects of student dialogue in order to help build a picture of the types of mathematical discourse used for geometric descriptions and definitions by both students and teachers.

**What is a Square?**

*What is a Square?* (see Appendix 2) (Seah, Horne & Berenger, 2016) was developed as an open task that required students to record what they knew about a square. This task was selected to generate individual student written accounts that were intended to inform the study about how much geometric knowledge students were able to recall from prior experiences, and to understand the ways they might communicate these mathematical constructs. This task had the potential to elucidate different levels of student geometric thinking as described by the van Hiele levels (Mason, 1998). A rubric served to assist with the compilation of teacher feedback to students and the researcher.

This task was utilised to understand the types of mathematical discourse properties used by individual students, and the narratives used by the teachers. As Vinner (1991) stated, “to acquire a concept means to form a concept image” (p. 69). A related task, conducted by Heinze and Ossietzky (2002), with Year 8 students in Germany showed that most students used their personal concept image and ignored concept definitions.

The researcher deemed this task useful to confirm findings from similar studies into students’ use of concept images, concept definitions, and difficulties with mathematical language (de Villiers, 1994; Zazkis & Leikin, 2008).
Supplementary Tasks

Five supplementary tasks were conducted as structured written tasks that were completed in fixed groups. The same groups of students worked on all five supplementary tasks. Keeping the same students in each group was necessary to track student dialogue and geometric concept development as a result of completing the set of tasks. The ways students engaged through group work were also explored, so fixing the groups allowed for a clearer examination and analysis. Written student work samples and recorded teaching episodes were also generated for analyses of these tasks.

Properties of Squares and Rectangles

This task was developed by the researcher as an amalgam of several aspects addressed in Cooke’s (2007) Properties of a rectangle (see Appendix 3). In particular, students were required to produce diagrams, identify common and unique properties of shapes, and produce a definition that would include squares and rectangles but exclude other quadrilaterals by attending to ‘right-angleness’.

The purpose of this task was to direct students toward addressing the relationship between common shapes by attending to common properties and shape restrictions. It was intended to produce rich discourse through student written and verbal responses. This task was used to determine whether students were thinking at the descriptive level or informal deduction level (van Hiele, 1999) based on their written and verbal accounts.

Properties of Parallelograms and Rectangles

The researcher developed Properties of Parallelograms and Rectangles (see Appendix 4) in a similar way to the previous task. It required students to produce diagrams, identify common and unique properties, and produce a definition that would include parallelograms and rectangles by attending to parallelism. Students’ use of diagrams is particularly important in conveying geometric concepts (Chazan, 1993; Hanna, 2000).

As for the previous task, the purpose of this task was also to direct students towards addressing the relationship between common shapes by attending to common properties and shape restrictions. Further, it was intended to produce rich discourse through student written and verbal responses. Further, it was intended to address shape classification and shape hierarchy reflected in the research questions of this study. Classification is closely related to defining (de Villiers, 1994; Jones, 2000). The types of definitions developed by students, whether hierarchical (inclusive) or partitional
could be analysed as an indicator of their geometric thinking abilities.

**Quadrilaterals by Properties**

*Quadrilaterals by Properties* developed by Cooke (2007) as a single task was separated into two tasks (1) (see Appendix 5), and (2) (see Appendix 6) in this study because preliminary communication with teachers in School 1 suggested that this would be more manageable for their students.

**(1)** This task provided students with another opportunity to work in small groups. Essentially, it focussed on students drawing different quadrilaterals with differing numbers of right angles. The intention of this task, as developed by Cooke (2007), was to promote student thinking in terms of properties of particular quadrilaterals. This task challenged students to reason about independent and connected properties, and had the potential to generate rich mathematical discourse between students. At each stage of the task, an extra condition was imposed drawing attention to an “unnecessary assumption of simplification” (p. 144). Students needed to draw the most general quadrilateral possible that satisfied the given constraints. Data from this task directly aligned to several aspects of the research questions, particularly those concerning the use of visual mediators to depict the range of quadrilaterals, and reasoning skills to draw valid conclusions about quadrilaterals with a specific number of right angles.

**(2)** Similarly, this task extended on the previous task and had the same intended purpose of promoting verbal and written discourse and, in turn, addressing shape hierarchies. This particular task required students to produce quadrilaterals with differing side conditions – pairs of *opposite* or *adjacent* sides. This task may have introduced two important keywords often used to describe and define shapes, and so the recording of classroom discourse through student work samples, student conversations and teacher interactions were important in establishing whether or not students could accurately understand these terms and apply them to depict particular quadrilaterals.

**Paper-folding**

*Paper-folding* (see Appendix 7), adapted from Cakmak, Isiksal and Koc (2014), required students to predict punch-hole patterns that resulted from unfolding a square piece of paper. Paper-folding activities, such as punch-hole patterns and origami, have the potential to significantly improve students’ spatial visualisation abilities (Akayuure, Asiedu-Addo & Alebna, 2016; Hershkowitz, Ben-Chaim, Hoyles, Lappan, Mitchelmore...
& Vinner, 1990; Kozhevnikov, Kosslyn & Shephard, 2005; Pope, 2002; Salthouse, Babcock, Skovronek, Mitchell & Palmon, 1990; Seah, 2014) and understanding of geometric concepts such as symmetry, similarity and congruence (Pope & Lam, 2011). This task enabled students to use mental manipulation to predict and draw patterns, and then form generalisations.

3.3.3 Conduct of the Study

Preceding the task implementation phase of the study, and in order to establish rapport with the participants, an initial visit to each school was organised at mutually agreeable times to the teachers and the researcher. The purpose of these visits was to gain insights into each teacher’s instructional mode, observe classroom dynamics, and develop a profile of each classroom vis-à-vis organisation and availability of resources.

3.3.3.1 Initial School Visits

During the initial school visits, the researcher met the students of each class, and observed teaching and learning processes in each classroom. This allowed an individual classroom profile of each class’s established routines for teaching and learning of mathematics to be formulated. These visits occurred two weeks prior to the implementation of the tasks and involved a single 90-minute period of observation per class. Individual classroom profiles including particular routines of classroom organisation, teacher instruction, and student learning behaviours are summarised in Chapter 4.

3.3.3.2 Task Implementation

School 1 was treated as a pilot school where all tasks were conducted and analysed using a combination of field notes of observations, video-recording of student interactions, student written responses, feedback from teachers to students via a marking rubric, and feedback from teachers to the researcher via verbal and written commentary. Tasks were staged in order to allow the teachers time to analyse student work samples, process their responses and feedback in order to inform the design of modifications necessary for future steps of implementation. The classroom teachers organised the students (with parental permission) to be recorded in small groups, and the other students of the class into separate groups. The teacher of each class determined group sizes of three or four students and labelled each group alphabetically.

The phases of task implementation (see Figure 4) shows, for example, that the first phase of this study took place in School 1 during Term 4, 2015 once a week over a
five-week period. There were preliminary discussions with the teacher participants about their role in the study. Specifically, teachers were asked to provide minimum instructions about how task items were to be completed. Teachers were asked to direct students towards working collaboratively with each other in their allocated groups. During the second phase in School 2, which took place over five weeks in Term 1, 2016, adjustments to tasks included a written set of teacher instructions to ensure that the teacher did not prompt students in any preamble before students engaged with tasks.

The next section outlines the methods used to generate data in schools to address the research questions.

3.3.4 Student Written Work Samples

Each task was developed as a paper-based activity for individual students to record their responses to task items that were analysed for evidence of mathematical discourse exhibited primarily by written narratives and diagrams. Tasks included opportunities for students to provide reflective responses about what they had learned, as well as any difficulties they may have experienced while attempting each task. Students were also asked to rate each task on a five-point scale from very difficult (1) to very easy (5).

The use of student work samples helped to influence professional discourse about teaching and learning between the teacher and researcher by ‘engaging teachers in a cycle of experimentation and reflection’ (Kazemi & Franke, 2004, p. 204). In this study, student work samples provided opportunities to engage teachers while students...
completed tasks and, retrospectively as they marked student work against a rubric providing written and verbal feedback to students, as well as for the researcher in regard to potential improvements to task designs.

3.3.5 Participatory Observations
The researcher’s observations were a data gathering technique used to record several types of interactions. Students were observed as they engaged with the tasks individually, with each other, and with their teacher. Some observations were video-recorded on an iPad and/or iPhone at particular stages through the lessons when the researcher was able to detect that students were engaged in substantive dialogue relating to the task. The various levels of student interactions were important in understanding the forms of communication as a means of determining students’ geometric reasoning skills. This type of observation within a design-based research approach is referred to as participatory. The researchers participate in local educational practices in the role of curriculum designers and curriculum theorists directly positioned in the social and educational context of the classroom (Barab & Squire, 2004).

The need to interpret the ways in which students and teachers interacted with task designs were necessary for the context of a design-based research framework and required local interpretations to be understood (DBRC, 2003). Task implementation required dialogue with students in order to understand their thinking and learning processes. As dialogue in small groups was anticipated and encouraged, the research design specifically required students to explain and clarify their understanding and reasoning of geometric concepts as they communicated with peers. One of the key tenets of empirical research includes evidence that is based on direct observation collected in an objective and unbiased way (Patton, 2003; Ritchie, Lewis, Nicholls & Ormston, 2013). In this study, the researcher recorded observations in classrooms using teacher and researcher field notes, transcription of student-researcher dialogue, and video-recorded teaching episodes.

3.3.6 Teacher Notes and Commentary
Anticipating the range of likely student responses, the researcher developed a rubric for each task conducted in the pilot school for teachers to mark student work samples (see Appendices 1 to 7). This initial phase was a necessary step to envision how students’ interpretations and solutions might evolve as students participated in each task. The feedback from practical experience into (new) thought experiments induces an iteration
of development and research (Gravemeijer, 1994). These rubrics also provided teachers with an opportunity to understand the different types of student thinking and ways of communicating geometric concepts. Teachers used these rubrics to correct student work, provide written feedback to students, as well as offer commentary to the researcher about potential task improvements.

Teachers were requested to maintain their own notes of observations made during classroom activities and recorded any significant observations as students completed the tasks. This was necessary for describing and explaining problems associated with task designs to inform future iterations. Teacher observations provided opportunities for them to be reflective in the ways students worked to solve geometric problems and how they communicated their responses. Ad hoc field notes were a useful means of contributing to a reliable analysis of the appropriateness and effectiveness of each task.

Teachers were involved directly in the research by looking for patterns and quality of student responses, recording and reporting to the researcher problems with task questions, and difficulties experienced by the students. Their written commentary acted as feedback to the students. This provided the researcher with insights into each teacher’s own geometric knowledge. Teacher commentary also occurred in dialogue with the researcher as tasks were being conducted.

3.3.7 Researcher Field Notes
The researcher maintained field notes throughout the study. Firstly, notes made during the participatory observations were recorded using a diary. These were ad hoc thoughts, ideas, questions to students and teachers, and ideas for task improvements that came to mind as tasks were being implemented. Secondly, written notes involved one-to-one explanations and discussions with individuals or groups of students. These field notes served to supplement what was recorded as participatory observations, and to compare and contrast observations made by teachers.

3.3.8 Recorded Teaching Episodes
This data gathering technique was introduced as a means of recording and observing instructional practice in School 2. Recorded teaching episodes have been used extensively to improve instructional practice (Knight, 2009; Takahashi, 2006). The data assists in understanding the role of the teacher and students in whole class teaching and learning. It also helps to document students’ thinking, the ways they communicate
mathematical concepts, and to interpret how they interact with each other and their teacher in constructing the meaning of these concepts. Its purpose in this study was to analyse aspects of instructional practices exhibited by the teacher, and to understand to the ways that the teacher used and encouraged mathematical discourse with her students. Further, the ways the teacher used gestures to communicate geometric concepts were also recorded.

While students completed tasks, the teacher and researcher observed the students and determined aspects of the task that needed to be explicitly taught. The focus was to address difficulties experienced by students, drawing together key geometric concepts that students were expected to record in their workbooks. Teaching episodes occurred after students had attempted each task and where the whole class was involved in dialogue that focussed on keywords, narratives and diagrammatic representations of 2D shapes. The teacher used a range of questioning techniques to extract as much detailed information from students and then provided additional information in order to build on students’ geometric knowledge and understanding of new concepts.

Each episode was recorded on an iPhone and/or iPad as a way of capturing the teacher’s instructional approach and knowledge in relation to geometry. It was used primarily as a means of generating evidence of teaching that might lead to geometric reasoning.

3.4 Methodological Challenges – Issues of Reliability and Validity

Research validity is increased when multiple sources of evidence and data converges (Yin, 2009). Data triangulation from a variety of distinct sources provides greater opportunity for accurate inferences (Teddlie & Tashakkori, 2009). Triangulation, as described by Creswell (2009), is a strategy for building coherent justification of themes and adds to the validity of the study. Patton (2003) suggested that triangulation was a strategy to enhance the rigour and credibility of analysis. In this study, triangulation was achieved by combining data from several qualitative methods (see Figure 5) using supporting computer software as appropriate.

![Figure 5. Triangulation.](image)
To ensure the validity of the findings, a number of measures were factored into this inquiry. Validity, as described by Shavelson, Phillips, Towne and Feuer (2003), lies in testing theories that emerge through working collaboratively with teachers in classrooms, co-constructing knowledge, and in capturing the specifics of practice with close attention to contextual factors. The following steps were undertaken to increase the validity of findings in this study:

- Teachers in the pilot school were provided with their preliminary data within one week of it being analysed. This was done to provide an opportunity for teachers to check the validity of their data and its analysis.
- Teachers provided written comments to students and reflections that were examined to extract common themes. These components added meaning and depth to some of the student interpretations of tasks and contributed to the validity of this study through triangulation of these data sets.

Guba (1981) suggested that there are four major concerns relating to trustworthiness – *truth value, applicability, consistency and neutrality*. Consideration of trustworthiness concerns and triangulation of data provide multiple lenses of rigour, and strengthens the research findings.

### 3.4.1 Truth Value

Truth value (akin to credibility) refers to establishing confidence in the truth of the findings of the inquiry (Guba, 1981). In this study, multiple methods of data generation were considered and implemented, enabling the triangulation of data to enhance the credibility of the interpretation. These methods evolved in each phase of the design-based research framework and increasing the internal validity of the reported findings.

An analysis was conducted using an established interpretive framework and included a ‘thick description’ that presented a more accurate interpretation of human experiences (Krefting, 1991). The dual-phase research design, and the multiple tasks and their iterations, provided various layers of testing of credibility.

### 3.4.2 Applicability

Whether or not the findings can be transferred and generalised beyond the confines of this study depends on it being able to be useful in ‘situational variations’ (Krefting, 1991). This qualitative study was unique and less amenable to be generalised as it was designed to suit the particular participants in their naturalistic setting where variables were deliberately not controlled – a feature of design-based research. This study’s
applicability is described by its transferability. As this research took place under comparable conditions in two schools, it increased the study’s applicability to other settings similar in nature. Its ‘rich and vigorous presentation of the findings’ (Graneheim & Lundman, 2004, p. 110) might enhance its transferability to contexts outside of this study.

### 3.4.3 Consistency

An inquiry’s reliability relates to its consistency – that is, the extent to which the study is dependable (Krefting, 1991). It is indicative of whether the range and uniqueness of experiences are analysed and reported. Examining the data through multiple lenses ensures the consistency and accuracy of the findings and interpretation. This study captured and reported a range of responses to tasks across schools and within classes and provided a solid platform from which to defend the dependability of its findings.

### 3.4.4 Neutrality

Objectivity, or neutrality, is achieved if the bias of the research is screened out. Reliability and validity are indicators of objectivity achieved in a qualitative study by decreasing the distance between the researcher and the participants (Krefting, 1991). This study involved repeated visits and observations completed over several weeks. This ensured adequate submersion in the classroom allowing the participants to become accustomed to the researcher. Once again, triangulation of multiple sources of data, the seeking of reoccurring patterns of behaviour and checking perspectives, all contributed to the neutrality of the study.

### 3.5 Ethical Considerations

The RMIT Human Research Ethics Committee [HREC] approved this study (see Appendix 8). In accordance with the requirements of RMIT University:

- Principals were emailed an invitation to participate in this study seeking their permission to conduct the study in their school. This detailed the purpose of the study, its benefits and possible risks (see Appendix 9).
- Teachers were emailed a Participant Information and Consent Form [PICF] that detailed the purpose of the study and sought their permission to be observed and video-recorded as they conducted geometric tasks with students (see Appendix 10).
• Students of those classrooms were provided with a letter to their parents that outlined the purpose of the research and sought their permission for their child to be recorded as they engaged with the tasks in the classroom (see Appendix 11).

As the research took place in government schools, ethics approval was sought by the Department of Education and Training [DET] Victoria [previously called the Department of Education and Early Childhood Development (DEECD)] and outlined the purpose of the study and methodology to be undertaken (see Appendix 12). Approval was granted and a courtesy letter was sent to the North Western Regional Director in Victoria informing her about the study to be undertaken.

Participation in this research was voluntary with participants afforded the right to withdraw at any stage without prejudice. They were also informed that they could remove their data from the study. Data were coded and pseudonyms were used to protect the anonymity of participants and schools. All data was stored securely in a lockable filing cabinet, treated with the utmost level of confidentiality, and data were not used for any other purpose than to inform this study. These measures, together with the methodological approaches undertaken constituted a low level of ethical risk.

3.6 Chapter Summary
By observing the ways in which students and teachers interacted with geometric tasks in the classroom, and by documenting the ways in which groups of students interacted with each other and their teachers, this research identified task designs to promote visualisation, and spatial and geometric reasoning. Designs of such tasks are an important tool as they build on students’ geometric intuition (Fujita & Jones, 2003).

Tasks that demanded high degrees of visualisation were utilised to gain insights into the role of visualisation in the development of geometric reasoning. Developing the geometric eye, as termed by Godfrey (1910), was critical in understanding geometrical properties, and so task designs had a strong visualisation focus. Task designs were sought and refined as a means of mitigating misconceptions and for building functional understanding. Tasks, therefore, addressed some of the documented common misconceptions and likely difficulties with shape classification and hierarchy of quadrilaterals.

Design-based research provided the most appropriate methodological framework for examining task designs through two phases using an established interpretive framework for mathematical discourse by Sfard (2008). The keywords, visual
mediators, narratives, and routines used by students to communicate thinking and reasoning were investigated by employing a range of methods resulting in credible and defendable findings.

A discussion of the data and the findings from the two schools involved in the study is provided in Chapter 4. A comprehensive analysis of the results from each task and outline of modifications made in successive phases of the study will be discussed. The findings and interpretations are presented using the four interrelated characteristics of mathematical discourse as an interpretive framework.
Chapter 4

Data Analysis and Interpretation

4.1 Introduction

This chapter presents the data of the study, its analysis and interpretation. Initially, this chapter describes the context of the study involving Year 7 and 8 students at two schools in Melbourne, Victoria, Australia. Data were collected through written student work samples, through participatory observations of students and teachers in classrooms, and through video-recorded teaching episodes. The data were analysed to understand the role of discourse as it relates to the development of visualisation and geometric reasoning skills.

The next section of this chapter presents an analysis based on the interpretive framework of mathematical discourse as proposed by Sfard (2008) (see Chapter 3, p. 81). The findings for each of the preliminary tasks conducted at both schools are summarised and interpreted in terms of the keywords, visual mediators, and narratives participants used while solving each of the tasks. Misconceptions exhibited by the participants are also discussed in the context of mathematical discourse. Aspects of group work are presented as another component of the study where the role of participants was examined. A summary of each preliminary task is provided before data from supplementary tasks are presented using Sfard’s (2008) interpretive framework. Tables and figures have been used extensively to summarise data and organise themes into a logical order for discussion.

The use of routines as defined by Sfard is discussed separately before the summary of this chapter is provided. Conclusions, recommendations, and limitations of the study, are provided in Chapter 5.

4.2 Context of the Study

In order to situate the study, a profile of each school, each teacher, and each class is presented. Teacher profiles and class profiles were determined from initial school visits (see Chapter 3, p. 91).
4.2.1 School, Teacher and Individual Class Profiles

**School 1 (S1)**

School 1 is a single-campus, coeducational secondary school located in the inner northern suburbs of Melbourne and established in 1915. The school offers a full Year 7 to 10 curriculum and Victorian Certificate of Education (VCE) program. A Select Entry Accelerated Entry Program [SEALP] has been operating since 1999. Students are selected into the program on the basis of ability and achievement testing, teacher reports and interview. There is one class of SEALP students in each level from years 7 to 10. After completing Year 12, 61% of students transition into university. Its enrolment in 2015 comprised 970 students of which 52% of students had a language background other than English. Of the student cohort, 41% of students were girls and 59% were boys.

**Teacher A (S1A.Teacher)**

- Female, graduate teacher with 2 years of full-time teaching experience.
- 2015 was her first year of teaching at School 1.
- Taught mathematics to Year 8, 9 and 12 students.

**Class A (S1A)**

Mathematics classes are conducted in several general-purpose classrooms across the school. In the initial school visit, the teacher briefly introduced the researcher to the class and then provided some examples of algebraic problems on the board. She clarified students’ understanding by asking several questions. She set an exercise from the textbook for students to complete and then roamed the classroom monitoring students and assisting those who raised their hands. Furniture was arranged in rows where students worked individually. Students accessed mathematics exercises via an online textbook that they completed in their workbooks. Some students discussed these exercises with neighbouring classmates.

There was considerable noisy and unrelated conversation mainly from the male students, several of whom completed little work other than copying problem examples from the whiteboard. There were no resources available in the classroom for students to use to assist them. Several students had their laptops and accessed textbook materials online. A few students played online games instead of completing the set mathematics work. The researcher asked students what they thought of mathematics, what they would do when they did not understand the work, and what they thought of this study.
Common responses were that mathematics was boring, that they asked the teacher when they needed help, and they were unperturbed by this study.

**Teacher B (S1B.Teacher)**

- Female, leading teacher with 25 years of fulltime teaching.
- 2015 was her second year of teaching at School 1.
- Taught mathematics to Year 7, 8(SEALP) and 11.
- She was also the Mathematics Coordinator for the school.

**Class B (S1B)**

In the initial school visit, the teacher briefly introduced the researcher to the class and then provided some examples of algebraic problems on the board. She also clarified students’ understanding through several questions to individual students. She set an exercise from the textbook for students to complete and then roamed around the classroom monitoring students and assisting those who raised their hands. Furniture was arranged in rows where students accessing online textbook exercises that they completed in their workbooks. Some students discussed these problems with neighbouring classmates.

The teacher stopped the class halfway through and asked students questions in an attempt to gauge their level of understanding and progress, She then set some worded problems from the text. All students had their laptops and accessed textbook materials online. All students remained on task for the whole lesson except when the researcher asked them what they thought of mathematics, what they did when they did not understand the work, and what they thought of this study. Responses consistently were that mathematics was ‘Ok’, that they asked the teacher or each other when they needed help, and they were generally interested in this study.

**School 2 (S2)**

School 2 is a single-campus coeducational secondary school located in the outer northern suburbs of Melbourne, re-established in 2010. The school offers a full year 7 to 10 curriculum and shares VCE offerings with a similar sized neighbouring secondary school. After completing year 12, 31% of students transition into university. Its enrolment in 2016 comprised 450 students of which 70% of students had a language background other than English. Of the student cohort, 31% of students were girls and 69% were boys.
Teacher A (S2A.Teacher)

- Female, accomplished teacher of 10 years full-time teaching experience.
- 2016 was her sixth year of teaching at School 2.
- She taught mathematics to Year 7, 9 and 11.
- She was also the Mathematics Coordinator for the school.

Class A (S2A)

In the initial school visit, the students lined up at the door and waited until the teacher invited them into the classroom. As students entered the classroom, the teacher directed them to their workspace. The teacher introduced the researcher to the class explaining in detail the purpose of this study. She told the students to turn on their computers and continue working on the exercise given at the beginning of the week. Students worked individually and in silence as the teacher called individual students to show her their workbook. She recorded the progress of online exercises and monitored students from the desk at the front of the room. As students raised their hands with questions, she approached them and provided individual instruction. Furniture was already arranged for students to work in groups as the room was formerly set up as a task centre.

All students remained on task for the whole lesson. The researcher asked students what they thought of mathematics, what they did when they did not understand the work, and what they thought of this study. Responses consistently were that they didn’t like mathematics but doing it on the computer was ‘easy’, that they were only allowed to ask the teacher for help, and they were interested in this study as it gave them an opportunity to do ‘something different’.

4.3 Analysis of Tasks

The following sections detail the analysis that was used to interpret the observations, the written records associated with the tasks, and aspects of student misconceptions and teacher knowledge. Data from two preliminary tasks are analysed and discussed with a range of student responses and an analysis of mathematical discourse exhibited by students. Subsequently, the five supplementary tasks conducted in small groups are analysed and presented. As a result of the findings from School 1, task modifications and their implementation in School 2 are explained. Table 3 provides a summary of the research process used to conduct the analysis and interpretation.
Table 3

Summary of research process

<table>
<thead>
<tr>
<th>School 1 (Pilot School)</th>
<th>School 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class A</td>
<td>Class B</td>
</tr>
<tr>
<td>Year 8</td>
<td>Year 8 (SEALP)</td>
</tr>
</tbody>
</table>

| Students | 16 male students | 15 male students | 15 male students |
|          | 9 female students | 9 female students | 6 female students |

| Preliminary tasks | Quirps (Group task) | What is a Square? (Individual task) |
| Supplementary tasks (All group tasks) | Properties of Squares and Rectangles | Properties of Parallelograms and Rectangles | Quadrilaterals by Properties (1) | Quadrilaterals by Properties (2) | Paper-Folding |

<table>
<thead>
<tr>
<th>Data generation methods</th>
<th>field notes</th>
<th>student written work samples</th>
<th>recorded student dialogue</th>
<th>teachers to students feedback</th>
<th>teachers to the researcher feedback</th>
</tr>
</thead>
</table>

4.3.1 Preliminary Tasks

Quirps and What is a Square? were analysed by classifying the types of keywords, visual mediators, and narratives used by students to communicate their thinking. Misconceptions and difficulties captured by recorded discourses are also reported for each task.

Quirps

In this three-part structured task, students were provided with a set of diagrams and asked to identify quirps based on similarities and differences of properties such as sides and angles, to draw their own quirps, and develop a definition of a quirp. The task involved the concept of angles and an understanding of polygons that examined whether students focus on a figure’s appearance or its parts and properties.

Keywords

A polygon is any shape made up of straight sides. A quirp being a polygon without a right angle is an endorsed narrative. Initial analysis focussed on keywords that students used to describe types of shapes, angle conditions, and boundary lines. As indicated in Table 4, there were variations with particular keyword usage between students in S1A and students S1B. Only 57.2% of students in S1A identified the necessary angle condition for quirps compared with 91.3% of students in S1B. Similar numbers of students used shape as a keyword in the construction of their definitions for quirps. No student used straight to describe the sides of a quirp. However, for 26.1% of students in
S1B who used *polygon*, the word *straight* would have been a redundant word. Similarly the same 26.1% of students that used *polygon* did not use the word *shape*, which would also have been a redundant term. However, 69.6% of students in S1B who used *shape* did not use *straight sides* which was required in order to indicate that they understood *quirps* to be polygons.

Table 4  
**Keywords students used to describe a quirp**

<table>
<thead>
<tr>
<th>Keywords</th>
<th>S1A</th>
<th>S1B</th>
</tr>
</thead>
<tbody>
<tr>
<td>polygon</td>
<td>0</td>
<td>26.1</td>
</tr>
<tr>
<td>2D</td>
<td>0</td>
<td>4.3</td>
</tr>
<tr>
<td>shape</td>
<td>71.4</td>
<td>69.6</td>
</tr>
<tr>
<td>right angles</td>
<td>42.9</td>
<td>65.2</td>
</tr>
<tr>
<td>90°</td>
<td>14.3</td>
<td>26.1</td>
</tr>
<tr>
<td>lines</td>
<td>0</td>
<td>4.3</td>
</tr>
</tbody>
</table>

The analysis of keywords used by students in S1B suggested that they were able to articulate the important angle condition for a *quirp* in greater proportion than students in S1A. Students at van Hiele level 1: *visualisation* would not recognise the types of angles in a figure, whereas students thinking at the higher level of *analysis* are expected to recognise types of angles and key characteristics of straight sides for polygons.

**Narratives**

In constructing a definition of a *quirp*, particular combinations of keywords constitute an endorsed narrative. As seen in Table 5, the types of narratives used to define a *quirp* indicated that a higher proportion of students in S1B were able to properly construct an accurate definition. Only 4 students from this class, however, were able to explicitly define a *quirp* as a polygon without right angles. Given that all the shapes presented to students in the task comprised straight sides, the definition of a *quirp* being a shape without a right angle was accepted in the context of the task.

Table 5  
**Endorsed narratives students used to describe a quirp**

<table>
<thead>
<tr>
<th>Endorsed Narratives</th>
<th>% of responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;A quirp is...&quot;</td>
<td>42.9</td>
</tr>
<tr>
<td>a shape without a right angle (90°)</td>
<td>9 students</td>
</tr>
<tr>
<td>a polygon without a right angle (90°)</td>
<td>4 students</td>
</tr>
<tr>
<td></td>
<td>69.6</td>
</tr>
</tbody>
</table>

The evaluation of endorsed narratives showed that students in S1B were more proficient in constructing an accurate definition than students in S1A. All other responses were
either incomplete, incorrect, or contained definitions that did not focus on both the angle and straight side properties such as the student responses listed in Table 6.

Table 6

<table>
<thead>
<tr>
<th>Samples of definitions of a quirp formulated by students</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1A</td>
</tr>
<tr>
<td>S1A.Henrietta: A <em>quirp</em> is a shape</td>
</tr>
<tr>
<td>S1A.Nancy: A triangular shape</td>
</tr>
<tr>
<td>S1A.Anthony: It is an unusual shape</td>
</tr>
<tr>
<td>S1A.Samuel: A shape with a point</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

These narratives indicated that the students in S1A did not have sufficient knowledge of geometric shapes to articulate much beyond a *quirp* being a shape whereas the students in S1B incorporated several keywords – *polygon, 2D, irregular, right angles, and parallel lines*.

**Visual mediators**

The use of diagrams is critical in communicating a ‘taken-as-shared’ mathematical discourse in geometry. Geometric diagrams represent an essential component of the visual logic required to engage in mathematical argument and aid geometric reasoning. Visual imagery is associated with verbal expressions (Owens, 1999). Formal visual mediation incorporates signifiers that are the universal way mathematicians code diagrams to indicate properties of shapes such as right angles, sides of equal length and pairs of parallel lines.

Only one student, S1B.Michael, produced a diagram with formal visual mediators to show parallelism (see Figure 6).

![Figure 6. Formal visual mediators to show parallelism.](image)

However it did not precisely match his correct definition of a *quirp*: ‘a shape without a right angle (90°)’. The student had associated parallelism with having no right angles. This coupled with his use of a prototypical parallelogram in his narrative showed that his understanding of both constructs was fragmented.

Almost 50% of students in S1B were able to produce correct examples of *quirps*.
compared to only 14.3% of students in S1A, indicating that only these students were able to determine similarities and differences from a set of polygons and then produce appropriate *quirp* diagrams. All the other students included at least one diagram containing a reflex angle (concave *quirp*). Other students had difficulties in producing appropriate diagrams indicating an inability to compare the critical attributes of a set of given shapes. Table 7 summarises the ways students communicated their understanding using visual mediators.

Table 7  
Visual mediators students used to describe a *quirp*  

<table>
<thead>
<tr>
<th>Visual mediators</th>
<th>% of responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>No diagram</td>
<td>S1A 14.3</td>
</tr>
<tr>
<td>Incorrect diagram</td>
<td>S1B 0</td>
</tr>
<tr>
<td>Copied examples on the page. No new diagrams</td>
<td>S1A 14.3</td>
</tr>
<tr>
<td>Correct shape without signifiers</td>
<td>S1B 47.8</td>
</tr>
<tr>
<td>Accurate shape (correct use of signifiers)</td>
<td>S1A 0</td>
</tr>
<tr>
<td></td>
<td>S1B 4.3</td>
</tr>
</tbody>
</table>

The lack of emphasis placed on neatness and accuracy in communicating visual responses indicated that they did not think that diagrams served a clear purpose. As indicated in Table 8, students found it difficult to draw diagrams that did not contain right angles. Most of these diagrams appeared with right angles although without signifiers. These students had correctly identified and defined *quirps* from the provided diagrams indicating difficulties in using diagrams accurately to match their definitions. The analysis of student visual mediators is not conclusive for this task as students may know how to signify right angles, however, as there is no accepted way of indicating angles that are not right angles, this task may have led to confusion for some students.

Table 8  
Samples of diagrams students drew to explain a *quirp*  

<table>
<thead>
<tr>
<th>Diagram</th>
<th>S1A.Henry: A <em>quirp</em> is a shape that does not have a right angle</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>S1A.James: A shape without a right angle</td>
</tr>
</tbody>
</table>

Of the students in S1A, 14.3% did not produce any diagrams. *Pre-structural* responses (Biggs & Collis, 1982) by 27.2% of the class indicated a lack of engagement with the task as a result of difficulties with identifying geometric concepts and being able to communicate them, or because they did not understand the expectations of the teacher.
Student misconceptions and difficulties

The analysis of narratives and visual mediators revealed several misconceptions and difficulties students had with 2D shapes. Apart from not being able to produce diagrams to communicate clear geometric concepts, some students included circles and did not recognise that all quirps consisted of straight sides. For example, S1B.Angus wrote ‘A quirp is a shape with one or more right angles’ but identified quirps correctly as having no right angles in the diagrams provided to students. Some of his diagrams (see Figure 7) could have matched his (incorrect) definition had he used right angle signifiers.

*Figure 7. S1B.Angus’ diagrams of quirps.*

Table 9 shows samples of students’ misconceptions and difficulties with geometric concepts resulting from the quirps task.

<table>
<thead>
<tr>
<th></th>
<th>Student Misconception</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1A.Nancy:</td>
<td>A triangular shape</td>
</tr>
<tr>
<td>S1A.Mia:</td>
<td>A shape with an odd angle</td>
</tr>
<tr>
<td>S1B.Zac:</td>
<td>A shape with one or more right angle</td>
</tr>
<tr>
<td>S1B.Josephine:</td>
<td>A shape with angles which are acute and obtuse but no right angles</td>
</tr>
</tbody>
</table>

Some triangular shapes would fit the definition, but S1A.Nancy included a 4-sided figure. Nancy indicated a conception of triangles as having a vertex at the top of a ‘pointy’ shape. This is often seen in younger children where a prototypical orientation of a triangle has a vertex located at the top regardless of whether it has three sides (Levenson, Tirosh & Tsamir, 2011). It is unclear what S1A.Mia meant by an ‘odd’ angle. S1B.Zac had difficulties using right angle signifiers for his quirps and it was not obvious which of his quirps have ‘one or more right angles’. S1B.Josephine included a circle but also a reflex angle in one of her diagrams illustrating that she does not understand the reflex angle concept because she had stated quirps to consist of acute and obtuse angles only.
Group work

Recorded observations of both classes of students indicated that most students had little experience working in small groups in mathematics despite indications from their teachers that group work was a common feature of mathematical instruction. Some students were observed to copy another student’s work instead of discussing and exploring key ideas themselves. This was confirmed by work samples where each member in some groups produced similar errors. As students raised their hands to ask questions of their teacher or the researcher, they were instructed to ask other students in their group. However, students in these classes were observed to work individually and not discuss the task with each other. Student work samples confirmed that each student operated as if the task was to be completed individually. There was no intervention by teachers when students did not engage with the task. Clear protocols for group work were not observed, nor were the teachers’ expectations for group collaboration communicated to students and for each student to attempt all components of the task.

Summary

The responses indicated that many students could identify similarities and differences between groups of polygons. However, many had difficulties in formulating a complete definition for an unknown polygon named a quirp. Further, many students had not understood the significance of diagrams and the use of signifiers in communicating geometric concepts. Geometric relationships are established by thinking about geometric diagrams, exercising powers of mental imagery, and making generalisations (Cooke, 2007). It is unknown what type of instruction in geometry may have occurred in previous years, but many students may not have had experiences with developing geometric definitions where the role of keywords have a direct relationship to mental images and diagrams.

It was important to incorporate student opinions to suggest possible refinements to task items, as well as providing them with opportunities to reflect on their own learning experiences, and to be able to articulate their knowledge of geometric concepts. Few students were able to indicate that they were dealing with an unknown geometric concept or that the problem was unfamiliar. Students also expressed their difficulties in terms of ‘figuring out the similarities and differences’ and ‘noticing the pattern’. A sample of the ways some students communicated their learning experienced by completing the task (see Figure 8) indicated attention to mathematical concepts:
S1A.Tara: looking at something you don’t know and tryin’ to figure it out.
S1B.Tarquam: I learnt different shapes have distinguishing characteristics which helps to classify them into numerous groups
S1B.Andrew: the more you look at a pattern the easier it becomes to find a pattern
S1B.Stacey: I learned that I can recognize unfamiliar patterns

Figure 8. Student descriptions of their own learning about an unknown concept.

Often students stated that they learned nothing and that their main difficulty was not knowing what a *quirp* was. Some students in S1A (27.2%) believed a *quirp* was something that they should already know. This might explain why they provided blank responses. This was confirmed by the teacher’s feedback (see Appendix 13) on the task that ‘students feel that a *quirp* is something they should know about, or will learn about’.

Students’ rating of the task did not reflect any pattern. An equal number of students rated the task as easy or difficult. This might be explained in several ways. As students completed the task very quickly, they may not have perceived it as challenging or engaging. Students may have known how to respond correctly to the task or were unable to communicate their thinking precisely which may have resulted in them perceiving the task itself as being difficult. Students may also not have understood the purpose for doing the task, and so they may have provided an inaccurate rating of its level of difficulty.

The *Quirps* task was not conducted at School 2 because this analysis from the pilot school (S1) identified several limitations to the task. Firstly, asking students to draw shapes without right angles meant that only approximated shapes could be drawn, as there was no formal way of signifying angles that are not 90°. The task could have been redesigned where *quirps* have right angles that are easier for students to draw and use signifiers. However, several other reasons provided a justification for removing this task from future study. The task did not sustain student interest nor did it draw out rich descriptions from students. Specifically, many students did not engage in the geometric reasoning required by the task. The structure of the task did, however, reveal that students knew little or did not value the importance of formal visual mediation when communicating geometric ideas. To be able to deduce what a *quirp* was required students to coordinate narrative and visual mediation that was beyond the students’ current geometric knowledge.

*What is a Square?*

In this task, students were asked to record as much information as possible about a
square. This was an open task completed individually without any time constraint in order to maximise student responses. Students in School 1 were allowed to ask questions about the task before they commenced. While the focus was on the keywords and on the narratives used to describe properties of a square, the ways in which students used or did not use visual mediators was an important aspect of understanding how students communicate geometric ideas, as well as understanding how they use visual mediators. Student questions such as ‘Can we draw a diagram?’ may have prompted other students to draw diagrams or write responses that they would not necessarily have thought of doing themselves. Most students completed the task within 20 minutes.

Students were required to construct a list of properties, a definition, and use geometric diagrams in order to develop a complete description of a square. Communication in geometric reasoning requires descriptions, explanations and diagrams in order to organise thoughts (Brown, Jones, Taylor, & Hirst, 2004). It is the combination of these aspects that individual students included that showed their ability to order properties leading to an evaluation of their understanding of classification and ultimately, their understanding of a hierarchy of quadrilaterals.

After analysis of the data from the pilot school, School 1, alterations to tasks and their implementation were made before being conducted in School 2. These were:

- The teacher provided very precise instructions and protocols to students at the commencement of the task. The teacher’s instruction to students was “Impress me with everything you know about a square. Be as accurate as you can!” (see Appendix 14). Questions about the task were not allowed so as to avoid prompting by other students that may have occurred at the pilot school.
- After students had submitted their responses, a teaching episode occurred that involved the brainstorming of task items with students. This allowed the researcher to capture more of the student-teacher dialogue necessary for analysing the mathematical discourse at the instructional level. Teacher notes on the whiteboard were captured and video-recorded by the researcher.

**Keywords**

The spontaneous tendency of most students was to make a list of all properties rather than provide an economical description aligned to the van Hiele level 2: *analysis* (de Villiers, 1998b). A square may be sufficiently (economically) defined as a quadrilateral with four equal sides and a right angle. However, a square might be defined as a
rectangle with four equal sides. This type of definition requires the definition of a rectangle primarily. Definitions form an essential part of the knowledge structure that affects the learner’s thinking processes (Tall & Vinner, 1981; Vinner, 1991). Knowing the definition of a concept, however, does not guarantee an understanding of the concept (de Villiers, 1998b; Vinner, 1991). There are at least nine definitions found in high school textbooks as shown in Table 10a. This variety of possible definitions results from squares lying at the bottom of most hierarchies of quadrilaterals (Usiskin, Griffin, Witonsky & Willmore, 2008). Under inclusive definitions, all squares, therefore, are rhombuses, rectangles, parallelograms, kites and trapezoids.

Table 10a  
**High school textbook definitions of a square** (Usiskin, Griffin, Witonsky & Willmore, 2008)  

<table>
<thead>
<tr>
<th>A square is…</th>
<th>Number of texts</th>
</tr>
</thead>
<tbody>
<tr>
<td>a rectangle with four congruent sides</td>
<td>38</td>
</tr>
<tr>
<td>a rectangle with a pair of consecutive congruent sides</td>
<td>14</td>
</tr>
<tr>
<td>a parallelogram that is both a rectangle and a rhombus</td>
<td>10</td>
</tr>
<tr>
<td>a parallelogram with four equal sides and four right angles</td>
<td>7</td>
</tr>
<tr>
<td>a parallelogram with one right angle and two adjacent sides congruent</td>
<td>5</td>
</tr>
<tr>
<td>a quadrilateral that has four equal sides and four right angles</td>
<td>4</td>
</tr>
<tr>
<td>a rhombus with four equal angles</td>
<td>3</td>
</tr>
<tr>
<td>a rhombus with a right angle</td>
<td>2</td>
</tr>
<tr>
<td>a quadrilateral that is both a rhombus and a rectangle</td>
<td>1</td>
</tr>
</tbody>
</table>


Zazkis and Leikin (2008) identified further definitions for a square (Table 10b). Their use of ‘expert sample space’ included definitions used in instruction and found in textbooks, several of which are based on diagonal or symmetrical properties of squares. The first definition using the term *regular* has a specific mathematical meaning that may not have been known or understood by students. No student used this term to describe a square.

Table 10b  
**Definitions of a square from an expert sample space** (Zazkis & Leikin, 2008)  

<table>
<thead>
<tr>
<th>A square is…</th>
</tr>
</thead>
<tbody>
<tr>
<td>a regular quadrilateral</td>
</tr>
<tr>
<td>a rectangle with perpendicular diagonals</td>
</tr>
<tr>
<td>a rhombus with equal diagonals</td>
</tr>
<tr>
<td>a quadrilateral having 4 symmetry axes</td>
</tr>
<tr>
<td>a quadrilateral symmetric under rotation by 90°</td>
</tr>
</tbody>
</table>

Initial analysis of keywords summarised in Table 11, showed that students in S1B used words such as *quadrilateral, shape, right angles, angle sum,* and *parallel* in higher proportion than the other two classes. Of students in S1B, 85% were able to specify the necessary condition of 4 sides of equal length compared with 28.6% and 25% of students in S1A and S2A respectively. The other necessary condition for squares as having right angles was specified by 90%, 61.9% and 10% of students in S1B, S1A and S2A respectively.

Table 11
*Keywords students used to describe a square*

<table>
<thead>
<tr>
<th>Keywords</th>
<th>S1A</th>
<th>S1B</th>
<th>S2A</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 sides</td>
<td>81.0</td>
<td>85.0</td>
<td>55.0</td>
</tr>
<tr>
<td>lines</td>
<td>23.8</td>
<td>0</td>
<td>20.0</td>
</tr>
<tr>
<td>4 edges</td>
<td>0</td>
<td>30.0</td>
<td>10.0</td>
</tr>
<tr>
<td>straight</td>
<td>28.6</td>
<td>15.0</td>
<td>5.0</td>
</tr>
<tr>
<td>equal sides</td>
<td>19.0</td>
<td>50.0</td>
<td>20.0</td>
</tr>
<tr>
<td>even</td>
<td>14.3</td>
<td>15.0</td>
<td>10.0</td>
</tr>
<tr>
<td>same length</td>
<td>9.5</td>
<td>35.0</td>
<td>5.0</td>
</tr>
<tr>
<td>2D</td>
<td>28.6</td>
<td>60.0</td>
<td>70.0</td>
</tr>
<tr>
<td>quadrilateral</td>
<td>14.3</td>
<td>65.0</td>
<td>0</td>
</tr>
<tr>
<td>3D (reference to cubes)</td>
<td>38.1</td>
<td>35.0</td>
<td>45.0</td>
</tr>
<tr>
<td>shape</td>
<td>38.1</td>
<td>80.0</td>
<td>10.0</td>
</tr>
<tr>
<td>I face</td>
<td>14.3</td>
<td>20.0</td>
<td>10.0</td>
</tr>
<tr>
<td>4 corners</td>
<td>61.9</td>
<td>25.0</td>
<td>75.0</td>
</tr>
<tr>
<td>points</td>
<td>4.8</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>vertex</td>
<td>4.8</td>
<td>35.0</td>
<td>0</td>
</tr>
<tr>
<td>4 angles</td>
<td>33.3</td>
<td>70.0</td>
<td>5.0</td>
</tr>
<tr>
<td>right angles</td>
<td>28.6</td>
<td>75.0</td>
<td>0</td>
</tr>
<tr>
<td>90°</td>
<td>38.1</td>
<td>35.0</td>
<td>10.0</td>
</tr>
<tr>
<td>angle sum</td>
<td>0</td>
<td>35.0</td>
<td>0</td>
</tr>
<tr>
<td>parallel</td>
<td>28.6</td>
<td>40.0</td>
<td>5.0</td>
</tr>
</tbody>
</table>

Collectively, 75%, 14.3% and 5% of students in S1B, S1A and S2A respectively were able to provide a combination of keywords that led to a valid definition of a square. Only students in S1B (35%) correctly included that the angle sum of a square was 360°. The students in S1B had a better understanding of the concept of a square exhibited by the range of individual keywords and their correct combinations.

However, in terms of a sufficient definition - the minimum amount of properties needed to describe the concept (Marchis, 2012) – only one student from each class could list key concepts in a sufficient manner and did not provide redundant information when describing a square. This was indicated by Angus:
S1B.Angus: They have four sides. The four sides have to be equal in length. They have four right angles inside the square.

However, S1B.Angus also thought that ‘If you turn the shape it can be turned into a diamond’. High proportions of students in all classes used redundant word combinations such as 4 sides, quadrilateral, straight and 2D. For example, students who stated 2D quadrilaterals or 4-sided quadrilaterals, may not have understood that quadrilaterals are already 2-dimensional which did not need to be stated, and that all quadrilaterals have four sides which again did not need to be stated. Having an understanding of which properties constituted a sufficient definition of a concept indicated that a student’s level of geometric thinking was at van Hiele level 4: deduction. Only S1B.Angus indicated this level of geometric thinking.

**Narratives**

In geometry, narratives are constructed through knowledge of axioms and theorems and substantiated by deduction (Sfard, 2008). Narratives for a square used by 14.3% of students in S1A indicated some understanding of broader concepts such as transformational ideas. Squares were correctly linked to other quadrilaterals by 55% of students in S1B, indicating a certain degree of understanding of a hierarchy for quadrilateral by these students (see Figure 9).

S1A.Molly: …looks the same if you rotate it, flip it
S1B.Michael: …polygon, square is a rectangle, angle sum is 360°
S1B.Enrico: …a square is a rectangle but a rectangle isn't a square, it's a parallelogram
S1B.Stacey: …are rectangles but rectangles are not squares…
S1B.Angelo: …parallelogram…

*Figure 9. Examples of narratives for a square.*

While students provided elaborate lists of square properties in many cases, several also made personal connections to life experiences such as toast, house designs, pixels and puzzles. 42.9% of students in S1A made reference to the letters that constitute the word square, attending to non-geometric aspects. S1A.George’s response (see Figure 10) was similar to the way most students listed concepts. He identified sides, angles and other properties of a square, but did not include a definition, nor any reference to right angles, nor any visual mediation.
Students in S1B often provided more extended descriptions, accompanied by visual mediators. Some students (25%) were also able to include formulae for perimeter and area, and other students (20%) made personal responses to architectural designs and ‘what happens to your eyes from too much TV’ (S1B.Joseph).

**Misconceptions and difficulties**

There were similar types of misconceptions across all three classes as shown in Table 12. The data indicated that a higher proportion of students in S1B (35%) understood that a rotated square standing on a corner was either a diamond or rhombus. A significant proportion of students in both schools made reference to a 3D object (cube or box) despite earlier referring to it as a quadrilateral or 2D shape.

<table>
<thead>
<tr>
<th>Common misconceptions</th>
<th>% of responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>2D version of a cube, box, 6 faces</td>
<td>S1A</td>
</tr>
<tr>
<td>vertical and horizontal</td>
<td>14.3</td>
</tr>
<tr>
<td>rotated becomes a diamond</td>
<td>4.8</td>
</tr>
<tr>
<td>rotated becomes a rhombus</td>
<td>0</td>
</tr>
<tr>
<td>stretched to become a rectangle</td>
<td>0</td>
</tr>
<tr>
<td>made up of 2 triangles</td>
<td>4.8</td>
</tr>
</tbody>
</table>

Unexpectedly, the data indicated that S1B (the accelerated class in Year 8) had very similar misconceptions with their understanding of the concept of a square as students in the other two classes. Four students from S1B, one from S1A and four from S2A recorded multiple misconceptions. Olivia’s narrative is one such example:

**S1B.Olivia:** …two equilateral triangles can be put together, the 3D form of it is a cube, the two vertical lines are parallel as are the horizontal lines, if rotated becomes a diamond…
Misconceptions were common across all classes indicating imprecise thinking about squares. For example, S2A.Khaled’s use of diagrams and description (see Figure 11) was typical of several students who confused squares with cubes.

At the end of the task, the teacher in School 2 conducted a brainstorming activity as a whole class exercise in order to gauge student understanding of concepts and emphasise the significance of accuracy in labelling diagrams and the precision of keywords. During this activity the teacher supported responses from students using questioning techniques to draw out further descriptions from her students (see Figure 12).

Student A: [a square] has a right angle
Teacher: how would I show it here? (invited the student to add to the diagram on the board)
Teacher: what does a right angle mean? (drawing out further meaning)

Figure 12. Dialogue of teaching episode for indicating a right angle.

Further discussion with the students also drew out more accurate descriptions moving students from 4 sides, for example, to 4 equal sides. Another student was invited to the whiteboard to explain how equal sides should be indicated on the diagram (see Appendix 15). The teacher also guided a discussion about parallel lines (see Figure 13) and included them on the diagram. Further key ideas and terminology, such as diagonals and symmetry, were also extracted from the students by the teacher folding a square piece of paper. During this part of the discussion, students were encouraged to use hand gestures to communicate ideas of horizontal and vertical symmetry.

Student B: a square has parallel lines
Teacher: what does that mean?
Student B: go in the same direction (uses his hands to show motion in one direction)

Figure 13. Dialogue of teaching episode for indicating parallel lines.

This teaching session provided an opportunity for the teacher to draw out and build upon several ideas that students collectively already had about the properties of 2D shapes. Not one student produced a list of concepts as elaborate as the collective contribution of the students guided by their teacher. The type of questioning exhibited
by the teacher drew attention to mathematical terminology and the use of diagrams to convey specific geometric properties. She encouraged and supported students to volunteer ideas and challenged them to provide more detailed descriptions.

This type of instruction was an important change to the task implementation as it allowed more knowledgeable students in the class to support other students’ conceptions in a public way. The teacher challenged students to provide more accurate descriptions in their responses thus building a detailed summary of the properties of a square on the whiteboard which students were expected to record in their workbooks.

The teacher later drew the following shape on the board (see Figure 14) and asked the students what it was. The resulting episode indicated a misconception held by the teacher that was transferred to students, that is, a tilted square is a diamond. *Diamond* was written above the drawn shape as well as the words *a rhombus*.

![Diamond](image)

**Teacher:** So, a rhombus. A diamond. What are you telling me?

**Student C:** Same thing.

*Figure 14.* S2A. Teacher’s tilted square and dialogue.

The acceptance of the student’s response indicated that both the teacher and the students believed that the non-critical attributes of the figure such as its orientation are important in its *concept definition* (Levenson, Tirosh & Tsamir, 2011).

S2A. Abdul’s response that a square “…is three-quarters of an A4 paper…” was most interesting to the researcher. However, this comment went unchallenged by the teacher. At the end of the teaching episode, the researcher invited him to explain what he meant. Abdul folded a piece of A4 paper at 45° to create a square and said that the square would be three-quarters of the original A4 page. He said that he could prove it by measuring it (see Appendix 16). The researcher asked him to fold an A4 page to three-quarters of its size and fold it along its diagonal (see Figure 15).

![Paper-folding proof](image)

*Figure 15.* Researcher’s paper-folding proof.
S2A. Abdul was convinced through the paper-folding activity that his conjecture was incorrect. S2A. Abdul’s response to the original task is included (see Figure 16).

![Image](image.png)

**Figure 16.** S2A. Abdul response to *What is a Square?*

*Visual mediators*

Use of visual mediators varied from basic shapes without any signifiers to squares with the correct use of signifiers for both side lengths and right angles. These are summarised in Table 13. Where students in School 1 were permitted to ask questions before they commenced the task, including whether a diagram could be included, more than half of the students in S1A did not provide any diagrams. Not one student provided an accurate diagram of a square with signifiers. In contrast, 60% of students in S1B provided accurate diagrams, whereas, in the case of School 2 where students were not permitted to ask questions beforehand, no student provided an accurate diagram and 65% did not draw anything at all.

Table 13

<table>
<thead>
<tr>
<th>Visual mediators</th>
<th>% of responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>No diagram</td>
<td>S1A</td>
</tr>
<tr>
<td>Incorrect diagram</td>
<td>57.1</td>
</tr>
<tr>
<td>Basic shape (no signifiers)</td>
<td>0</td>
</tr>
<tr>
<td>Correct shape with RA signifiers</td>
<td>33.3</td>
</tr>
<tr>
<td>Correct shape with equal side signifiers</td>
<td>9.5</td>
</tr>
<tr>
<td>Accurate shape (correct use of all signifiers)</td>
<td>0</td>
</tr>
</tbody>
</table>
| S1A. Callum’s example (see Figure 17)    | showed a general outline of a square and was similar to those that provided a basic shape. Sides were not straight; there were neither formal signifiers for right angles nor equal side lengths.
Figure 17. S1A. Callum’s diagram of a square.

S1B. Angelo’s diagram (see Figure 18) was similar to those students in S1.B who used visual mediators to convey equal sides and right angles.

Figure 18. S1B. Angelo’s diagram of a square.

The lack of diagrams by students in both S1A and S2A indicated their lack of understanding of the need to illustrate key geometric concepts. The students in S1A discussed the inclusion of diagrams with the teacher, whereas students in S2A did not. It was not automatic for students to include diagrams nor was it seen as necessary.

Where students used diagrams, these ranged from real life examples to basic shapes and diagrams including signifiers. The variation in ways that students used signifiers showed a lack of consistency in understanding these mathematical conventions used in geometric discourse.

According to Özerem (2012), students relied on the visual prototype instead of applying definitions when identifying shapes. Of the 14.3% of students in S1A and 5% of students in S2A that provided keyword combinations leading to a definition, none drew a square accurately. Of the 9 students in S1A that provided basic square diagrams, 2 students provided keyword combinations that may have led to a definition. In the case of the 5 students in S2A, 1 student provided keyword combinations that may have led to a definition. This indicated that for students in these two classes, the diagrams they provided did not lead to a definition of a square, and the opposite is true as well. The keywords to identify key concepts of equal sides and right angles did not lead to a correct diagram. Samples of student diagrams with accompanying narratives are provided in Table 14.
Table 14

*Visual mediators used to describe a square and accompanying narratives*

<table>
<thead>
<tr>
<th>S1A.David: It’s an equilateral… can be split into triangles</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1A.Andre: It has vertical and horizontal lines…its in 4 square</td>
</tr>
<tr>
<td>S1A.Aiden: 2D versions of cube…of different rotations</td>
</tr>
<tr>
<td>S1A.Tara: All the same…it makes two rectangles</td>
</tr>
<tr>
<td>S1B.Harvey: …even though some of these shapes are ‘different’ they are still squares (eg. diamonds, parallelograms)…a rectangle is a stretch out square’</td>
</tr>
<tr>
<td>S1B.Stacey: A rhombus is a tilted square….if you cut a square diagonally you get a triangle</td>
</tr>
<tr>
<td>S1B.Hannah: 3D version of a square is a cube…strong unseen bases (square based pyramid)</td>
</tr>
<tr>
<td>S2A.Fadi: It has right angles in its corners and it has one face</td>
</tr>
<tr>
<td>S2A.Mary: That a square has 4 even sides. It has 4 corners. It’s a shape, can be 3D or 2D</td>
</tr>
</tbody>
</table>

*Summary*

The responses indicated considerable variation in student knowledge between classes. The interpretive framework enabled the analysis of what students knew about a square to be examined through ways they engaged in mathematical discourse. While a square may be a familiar mathematical shape, students had difficulties accurately representing them in order to communicate their understanding of key concepts. Diagrams were not automatic tools of mediation between what individual students knew about a square and how they would communicate that knowledge. There were significant difficulties with providing sufficient information to define a square even though students often claimed that they had minor difficulties with the task. Misconceptions were common to students and the teacher in School 2 as demonstrated in the documented teaching episode.
There was variation in how students reflected on their learning. Students in S1A often stated that they had learned ‘nothing’ from the task and had experienced no difficulties with some alternative responses (see Figure 19). Students in S1B were able to describe their learning as needing to remember facts or that they learned that they knew more about squares than they had realised. Typically, students expressed that the difficulties they experienced were about remembering facts.

\[\begin{align*}
\text{S1A.Ava:} & \quad \text{I learnt how to describe a square to an unsuspecting person} \\
\text{S1A.Andre:} & \quad \text{I learnt pretty much nothing about a square} \\
\text{S1B.Eric:} & \quad \text{…running out of ideas}
\end{align*}\]

Figure 19. Student reflections on learning.

Most students in S2A indicated that the task was easy but wrote very little information that was accurate. This indicated that they might not have known what information about a square was relevant or accurate. Students also were not accustomed to being challenged to commit their thinking in writing in an open question format.

There were differences with the ways in which the teachers supported student learning. The scoring rubric where teachers corrected student responses in order to provide students with feedback provided insights into the differences in teacher knowledge. S1A.Teacher commented on the use of equilateral as being a triangle reference, and confused students’ descriptions of properties such as 4 parallel lines, but also suggested that vertices was a ‘great use of language’. She also ticked (✓) the use of corners and points for angles, even for equal sides for S1A.Henry’s response ‘rotate a square to make a diamond’, and for S1A.Andre’s response ‘it has vertical and horizontal lines’ (see Appendix 17). This might indicate that this teacher’s own geometric knowledge may be weak and that this might potentially contribute to students having an underdeveloped understanding of geometric concepts over time. The use of corners and points for angles, for example, and the listing of properties indicated the van Hiele level 2: analysis (Mason, 1998), by at least the students. Teachers and their students tended to exhibit similar misconceptions when assessed on their knowledge of basic geometric figures (Hershkowitz & Vinner, 1984; Mayberry, 1983).

In contrast, S1B.Teacher readily corrected keywords and student narratives. She questioned the use of even sides, vertices, and corners. She purposefully did not mention straight sides, equal sides, or pairs of parallel sides when students listed or indicated parallelism on diagrams, nor did she give false definitions. When a teacher responds to students appropriately by addressing their misunderstandings and readily
correcting their errors, it is indicative of a high-level of [geometric] knowledge and high quality instruction (Hill, Blunk, Charalambous, Lewis, Phelps, Sleep & Ball, 2008).

In summary, S2A.Teacher exhibited powerful teaching techniques through her questioning techniques that extracted further precision from students as they volunteered responses during her teaching episode. She modelled mathematical discourse by attending to keywords and definitions mediated by diagrams with signifiers. Geometric concepts were communicated to her students verbally, in written form including diagrams, and through gestures. However, it also shed light on the misconceptions with the square concept held by this teacher and provided an explanation for how students might develop misconceptions in the first instance.

### 4.3.2 Summary of Preliminary Tasks

The analysis of both preliminary tasks indicated differences between the ways students in different classes communicated their understanding of major geometric concepts. The students in S1B more readily produced diagrams to convey geometric concepts and were aware of the implications of using signifiers and the meanings of keywords. Accurate diagrams invariably accompanied descriptions. However, often there was a mismatch between the keywords and narratives used to describe shapes or properties of shapes with the visual mediators provided. The students in S1A may have understood the importance of using diagrams and accurate geometric terminology, but may not have acquired an understanding of geometric concepts or skills to communicate them.

The students in S2A had considerable difficulties with engagement in these types of learning activities where there was a need for thinking and communicating in precise ways. This may stem from what students accepted as a routine approach to learning mathematics in high school – that is, via a computer-based program where responses to mostly multiple-choice questions were completed individually and were immediately rewarded. It was a place where students did not engage in mathematical discourse with each other, nor challenge each other’s thinking, nor collaborate on learning tasks. This was the regular mathematics pedagogy across Years 7 and 8 in this school.

The purpose of the preliminary tasks was to orient students toward geometry. Students in both schools were learning other topics of mathematics when the study commenced. These two preliminary tasks were of a different nature. **Quirps** was conducted as a structured group task and **What is a Square?** was conducted as an
individual, open task. This phase of the study indicated the need for further examination of how students and teachers communicated concepts involving visualisation and geometric reasoning.

Sfard’s (2008) interpretive framework was a useful lens for examining mathematical discourse. It illustrated the frequency of keywords used by students to describe 2D shapes. Analysis of student narratives also provided initial data about how students might readily use definitions, reveal their misconceptions, and communicate their thinking to other students. The summaries of visual mediators also indicated the type of diagrams students used, often as an adjunct, to describe an unknown shape, quirp and a common shape—a square.

The ways students worked in groups in order to develop more accurate use of diagrams, keywords and geometric descriptions was further examined through a series of supplementary tasks. The following section describes the analysis of the supplementary tasks conducted in both schools.

4.3.3 Supplementary Tasks
Five structured supplementary tasks were selected in order to provide further opportunities for students to use visualisation and geometric reasoning to solve problems. The interrelated characteristics of mathematical discourse that students were likely to use to communicate their thinking through group work activities were recorded using individual student-written submissions and video-recording of student interactions. Each student’s submission in School 1 was assessed against a rubric. Teaching episodes conducted in School 2 were video-recorded and analysed using the same discourse framework.

Properties of Squares and Rectangles
This task required students to write down responses to a series of questions in relation to squares and rectangles. It essentially involved students drawing squares and rectangles, listing their properties, and identifying common properties in order to develop an inclusive definition. Oberdorf and Taylor-Cox (1999) suggested that students have difficulties with describing squares as rectangles as this ‘new’ information does not logically connect with what they have learned previously. Students have a tendency to use partitional definitions (Heinz & Ossietzky, 2002).

This task was completed in small groups of 3 or 4 students. This task addressed the commonly held misconception that squares are not rectangles (Marchis, 2012). In
order to capture aspects of students’ dialogue that may either build understanding in their peers, or consolidate misconceptions, the researcher required students to work in groups. The researcher’s intention to capture ‘new’ learning through the task design and implementation was addressed through individual student evaluations requiring students to record any new learning.

The teacher of each class told students to read the questions carefully and complete each section as accurately as possible. The teachers monitored student groups and observed interactions between students. Interventions, by teachers while students completed the task, were limited to behaviour management in School 1. Students attempted the task without further direction or prompting from the teacher.

As a result of analysing the findings from School 1, this task design and its implementation were refined when conducted at School 2. The task modifications are documented below:

• The teacher reminded students about key concepts (right angles and equal sides) from the previous teaching episode in order to promote accuracy and elaboration in the students’ responses. This occurred at the beginning of the class with a brief questioning of students knowledge. The teacher were asked to observe student groups and monitor their work output, recording any significant points for discussion during the subsequent teaching episode.

• A rubric was used to assess individual student responses in School 1. This generated data regarding which sections students completed but it did not offer an efficient nor effective means of providing students with feedback about their thinking. The researcher determined that the most efficient and potent form of feedback would occur during teaching episodes.

• After students had submitted their responses, a teaching episode involved the teacher brainstorming the task items with students. This was written on the whiteboard and video-recorded by the researcher.

Keywords
There are several endorsed narratives for squares and rectangles that would constitute an inclusive definition. For example, squares and rectangles may be defined as quadrilaterals with right angles, or as parallelograms with right angles. Alternatively, they may be defined as polygons with two pairs of parallel sides perpendicular to each other. Essentially, squares are quadrilaterals with sides of equal measure and right
angles, and rectangles are quadrilaterals with right angles. Both definitions of squares and rectangles result in pairs of opposite and equal sides. As Usiskin, Griffin, Witonsky and Willmore (2008) claimed, several inclusive definitions for squares result from them lying at the bottom of the hierarchy for quadrilaterals. This means that squares may be defined by properties of other quadrilaterals. For example, because a square is a rectangle means that its diagonals have the same length and bisect each other; because a square is also a parallelogram, it means that it has two pairs of equal opposite angles.

As indicated in Table 15, the *keywords* used by students to describe squares were comparable to *keywords* that they used to describe rectangles. Students in S1B used keywords to describe these shapes at a higher proportion than the other two classes. In S1B, 94.7% and 84.2% of students used combinations of keywords that constituted an accurate definition of a square and a rectangle respectively, compared with students in S1A – 33.3% (square) and 23.8% (rectangle), and students in S2A – 41.2% (square) and 5.9% (rectangle).

<table>
<thead>
<tr>
<th>Keywords for a square</th>
<th>% of responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>quadrilateral</td>
<td>14.3 68.4 0</td>
</tr>
<tr>
<td>2D</td>
<td>23.8 15.8 29.4</td>
</tr>
<tr>
<td>polygon</td>
<td>0 15.8 0</td>
</tr>
<tr>
<td>4 sides</td>
<td>14.3 94.7 41.2</td>
</tr>
<tr>
<td>4 lines</td>
<td>0 0 17.6</td>
</tr>
<tr>
<td>even sides</td>
<td>33.3 0 11.8</td>
</tr>
<tr>
<td>equal sides</td>
<td>33.3 94.7 29.4</td>
</tr>
<tr>
<td>4 right angles</td>
<td>38.1 31.6 0</td>
</tr>
<tr>
<td>4 90° angles</td>
<td>33.3 68.4 29.4</td>
</tr>
<tr>
<td>4 angles</td>
<td>4.8 0 0</td>
</tr>
<tr>
<td>corners</td>
<td>33.3 5.3 100</td>
</tr>
<tr>
<td>vertices</td>
<td>9.5 15.8 5.9</td>
</tr>
<tr>
<td>edges</td>
<td>19.0 10.5 11.8</td>
</tr>
<tr>
<td>parallelogram</td>
<td>0 26.3 0</td>
</tr>
<tr>
<td>parallel lines</td>
<td>28.6 15.8 29.4</td>
</tr>
<tr>
<td>parallel sides</td>
<td>4.8 21.1 0</td>
</tr>
<tr>
<td>angle sum</td>
<td>14.3 47.4 0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Keywords for a rectangle</th>
<th>% of responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>quadrilateral</td>
<td>14.3 57.9 0</td>
</tr>
<tr>
<td>2D</td>
<td>9.5 15.8 5.9</td>
</tr>
<tr>
<td>polygon</td>
<td>0 0 0</td>
</tr>
<tr>
<td>4 sides</td>
<td>19.0 0 17.6</td>
</tr>
<tr>
<td>4 lines</td>
<td>0 0 17.6</td>
</tr>
<tr>
<td>even sides</td>
<td>14.3 0 35.3</td>
</tr>
<tr>
<td>equal sides</td>
<td>19.0 31.6 17.6</td>
</tr>
<tr>
<td>4 right angles</td>
<td>28.6 26.3 17.6</td>
</tr>
<tr>
<td>4 90° angles</td>
<td>14.3 68.4 0</td>
</tr>
<tr>
<td>4 angles</td>
<td>0 0 0</td>
</tr>
<tr>
<td>corners</td>
<td>33.3 5.3 64.7</td>
</tr>
<tr>
<td>vertices</td>
<td>9.5 10.5 5.9</td>
</tr>
<tr>
<td>edges</td>
<td>19.0 10.5 5.9</td>
</tr>
<tr>
<td>parallelogram</td>
<td>0 26.3 0</td>
</tr>
<tr>
<td>parallel lines</td>
<td>23.8 15.8 0</td>
</tr>
<tr>
<td>parallel sides</td>
<td>4.8 21.1 0</td>
</tr>
<tr>
<td>angle sum</td>
<td>14.3 42.1 0</td>
</tr>
</tbody>
</table>

Both squares and rectangles were understood to also be parallelograms by 26.3% of students in S1B. Students across all classes continued to use *vertices* and *edges* which are keywords used for 3D Euclidean geometry. They also continued to use words, such as *quadrilaterals*, but also used *polygons* or *2D shapes* which were both redundant as
quadrilaterals are already 2D polygons. This occurred in 15.8% of students in S1B but none in the other two classes. Three students in S1B also used *equilateral* when describing a square – a term usually used for triangles with sides of equal length. One of these students also indicated some understanding of diagonal bisection (see Figure 20).

S1B.Harriot: If diagonal cuts are made, all places where the cuts meet = 90°.

*Figure 20. S1B.Harriot’s discourse on diagonal bisection.*

Students in S1B, the accelerated class, may have been aware of more keywords, but had difficulties with using correct terminology and with providing sufficient descriptions for both squares and rectangles. The majority of students in S1B aligned to the van Hiele level 2: *analysis*, where students listed properties but were unable to discern which properties were necessary and which were sufficient to describe an object. The inclusion of redundant keywords occurred in differently to students in the other two classes. For example, 9.5% of students from S1A included *pairs of equal sides, 4 right angles* and *sets of parallel lines* when describing a rectangle, and 17.6% of students in S2A used *2D shapes, 4 right angles and pairs of parallel lines (sides)* when describing a square.

Students in S2A continued to use *corners* for angles of a square (100%) and for a rectangle (64.7%) – a term used by younger children. This was despite having completed the preliminary task *What is a Square?* that involved an extended teaching episode where the angle concept was discussed with the whole class. However, more students in this class specified squares as having *right angles* and *parallel lines* which was not a common response in the preliminary task, increasing from 5% to 29.4% for both terms. Commonly, students included key concepts like *parallelism* and *symmetry* in their description. It could be argued that students have retained these ideas from the teaching episode on *What is a Square?*

*Narratives*

A rectangle is a quadrilateral with 4 right angles, where a square is a special case of a rectangle with 4 sides of equal length. Table 16 illustrates the range of endorsed narratives for squares and rectangles. These indicated stark differences between students in S1B from the other two classes. No student in S2A could provide an inclusive definition for squares and rectangles despite an extended teaching episode about squares
and other quadrilateral conducted by their teacher. More than 50% of students in S1A had the same level of difficulty with deduction implicit in the task.

Table 16
Definitions for squares and rectangles

<table>
<thead>
<tr>
<th>Endorsed Narratives</th>
<th>S1A</th>
<th>S1B</th>
<th>S2A</th>
</tr>
</thead>
<tbody>
<tr>
<td>“Only square and rectangles are quadrilaterals… that have four right angles (90°).”</td>
<td>33.3</td>
<td>73.7</td>
<td>0</td>
</tr>
<tr>
<td>that have four 90° corners.</td>
<td>26.6</td>
<td>57.9</td>
<td></td>
</tr>
<tr>
<td>[that are also] parallelograms with right angles (90°)</td>
<td>5.3</td>
<td>10.5</td>
<td></td>
</tr>
<tr>
<td>where every point is a right angle.</td>
<td>6.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>* previously used corners for angles.</td>
<td>6.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>that have at least one right angle.</td>
<td>6.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>* definition sufficiency</td>
<td>6.6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Of the 73.7% of students in S1B that attempted to provide a definition, the necessary property for squares and rectangles of right angles was included. Some students seemed to have developed an understanding of a hierarchy of quadrilaterals, as indicated by S1B.Barry’s statement connecting and ordering square and rectangles in relation to parallelograms:

S1B.Barry: …because they are parallelograms with right angles

Several inconsistencies with matching accurate definitions with lists of properties and visual mediators were detected in many student written-work samples. Even though 82.4% of students in S1A attempted a definition, common misconceptions about rectangles created significant difficulties with this task. Out of all the classes, only one student in S1A was able to provide an inclusive and sufficient definition of a square and rectangle as being a quadrilateral with at least one right angle, but in a previous section of the task described a rectangle as follows:

S1A.Callum: …two sides are longer than the other

Misconceptions and difficulties

Of the 6 students in S1A that provided an inclusive definition of a square and rectangle, S1A.Henry had also written that rectangles had ‘two different length lines’. Many other students in this class experienced misconceptions with squares and rectangles with an understanding that rectangles must have a pair of opposite sides longer than the other pair. This was the case whether diagrams were included or not. This indicated that these students were not yet at the geometric thinking van Hiele level 2: analysis where,
according to Crowley (1987), students should be able to make generalisations for the class of shapes. Students expressed the same misconception of rectangles being ‘long’ shapes in different ways (see Figure 21):

S1A.Ryan: longer than a square
S1A.Andre: 2 long lines
S1A.Lucas: two short sides exactly the same
S1A.Christopher: uneven length of edges
S1A.Henrietta: two sets of even sides
S2A.Liam: A rectangle is thinner than a square in width

Figure 21. Misconceptions of rectangles.

Some students had difficulties using parallel effectively. Often, students simply wrote the term parallel in reference to squares and rectangle, but in isolation and without elaborated statements or visual mediators with signifiers. This indicated a lack of understanding of this concept and confirmed geometric thinking at van Hiele’s level 2: analysis, where properties were recognised but the relationship between these properties were not yet understood.

In listing the properties of a square, S2A.Liam’s response of ‘\(\frac{3}{4}\) of A4 paper’ was unusual and indicated that he had retained some information from the previous session about \(\frac{3}{4}\) of an A4 sheet of paper was a square. Because this information provided by another student was not discussed or challenged by the teacher at the time, it may have led some students to retain and then repeat this incorrect information. S2A.Abdul, who had previously volunteered this information, did not repeat this in his response. This is likely to have resulted from the researcher providing an individual session with Abdul about his idea and challenging the notion that \(\frac{3}{4}\) of an A4 sheet of paper makes a square.

Toward the end of the lesson, the teacher of S2A summarised student ideas in relation to the task on the board. During this teaching episode, a student drew a rectangular shape on the board. The following conversation (see Figure 22) indicates the teacher’s questioning skill, but also revealed a common misconception held by students.

Teacher: How do we know this is a square?
Student D: We can measure it with a ruler.
Teacher: What happens if I don’t have a ruler?
Student E: It’s too long.
Teacher: What’s too long?
Student E: It’s a rectangle.
Teacher: It’s a rectangle. What makes it a rectangle?
Student E: Rectangles are like long. Squares are short.
Teacher: OK.
Student F: All sides are not equal.

Figure 22. Questioning technique of S2A.Teacher.
The misconception that rectangles are ‘long’ shapes went unchallenged possibly because this was a preferred definition held by the teacher – she may adhere to partitional definitions without any clear purpose. However, the teacher was able to shift the student’s thinking from a geometric argument involving measuring, described as ‘experimental verification’ (Kunimune, Fujita & Jones, 2010), to thinking about critical attributes of shapes but still quite away from an inclusive definition for squares and rectangles based on right angles conditions.

**Visual Mediators**

The range of diagrams and visual mediators used by students are provided in Table 17. Every student in S1B provided accurate diagrams for both squares and rectangles. In comparison, 71.4% and 94% of students in S1A and S2A respectively produced basic or incomplete diagrams of squares after completing the task *What is a Square?*

<table>
<thead>
<tr>
<th>Visual mediators for a square</th>
<th>% of responses</th>
<th>Visual mediators for a rectangle</th>
<th>% of responses</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>S1A</td>
<td>S1B</td>
<td>S2A</td>
</tr>
<tr>
<td>No diagram</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Incorrect diagram</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Basic shape (no signifiers)</td>
<td>52.4</td>
<td>0</td>
<td>88.2</td>
</tr>
<tr>
<td>Correct shape with incomplete signifiers</td>
<td>0</td>
<td>0</td>
<td>5.9</td>
</tr>
<tr>
<td>Correct shape with inaccurate signifiers</td>
<td>19.0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Accurate shape (correct use of signifiers)</td>
<td>28.6</td>
<td>100</td>
<td>5.9</td>
</tr>
</tbody>
</table>

These proportions were similar for their diagrams of rectangles, indicating that students in S1A and S1B continued to have difficulties with understanding the significance of diagrams in communicating geometric concepts. Even though students were able to describe squares and rectangles, they were not always able to draw accurate diagrams.

Almost every student in S2A provided neat diagrams of squares and rectangles where they had obviously used a ruler for their shape. However, only 5.9% had indicated both right angles and equal sides on diagrams of squares suggesting little
retention from the previous session about the importance of using visual mediators with signifiers in communicating geometric properties. Instead, students had retained the importance of neatness also emphasised by the teacher during the teaching episode. Table 18 provides a range of diagrams used by students to represent squares and rectangles, use of keywords and narratives.

Table 18

Visual mediators, keywords and narratives used for squares and rectangles

<table>
<thead>
<tr>
<th></th>
<th>similar properties</th>
<th>inclusive definition</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>S1A. Samuel:</strong></td>
<td>4 right angles</td>
<td>(not given)</td>
</tr>
<tr>
<td><strong>S1A. Max:</strong></td>
<td>(not given)</td>
<td>(not given)</td>
</tr>
<tr>
<td><strong>S1A. Ava:</strong></td>
<td>quadrilateral right angles</td>
<td>that have 4 right angles</td>
</tr>
<tr>
<td><strong>S1B. Angelo:</strong></td>
<td>angles 90° 4 sides</td>
<td>that have all angles at 90°</td>
</tr>
<tr>
<td><strong>S1B. Hannah:</strong></td>
<td>opposite sides are equal quadrilateral 4 right angles 360° A = L X W</td>
<td>shapes with only 4 right angles</td>
</tr>
<tr>
<td><strong>S2A. Abdul:</strong></td>
<td>4 sides 4 corners</td>
<td>(not given)</td>
</tr>
<tr>
<td><strong>S2A. Natalie:</strong></td>
<td>They have 4 sides 4 corners They are 2D shapes</td>
<td>(not given)</td>
</tr>
</tbody>
</table>

These responses showed that many students were able to indicate right angles and equal sides using appropriate signifiers. The students that used signifiers tended to use as many as they could when drawing diagrams. In the case of S1B.Hannah, additional information was illustrated on the diagrams showing further understanding that the diagonals of a square bisect each other at right angles. Several students, like Hannah did not know what was necessary and sufficient and used signifiers in their diagrams to indicate all that they knew. Several students used parallel line signifiers accurately even
though these were redundant given that 4 right angles had already been indicated. In these cases, students had not been able to deduce that the use of some signifiers was unnecessary when drawing diagrams. Students had further difficulties in articulating an inclusive definition for squares and rectangles. Oberdorf and Taylor-Cox (1999) suggested that students were taught that squares and rectangles were different shapes, and thereby could not see squares as a subset of rectangles.

The students that were able to provide an inclusive definition indicated the common property of four right angles. S1B.Angelo’s diagrams of a square and rectangle illustrated a higher-order of geometric thinking defined by van Hiele level 3: abstraction, allowing him to reason that only one right angle needed to be signified for both shapes. This type of thinking depicted an understanding of the sufficient properties of squares. However, defining rectangles requires more than pairs of equal opposite sides and one right angle to be signified.

**Group work**

Even though students were in close proximity to each other, they did not automatically work together to solve problems. The tendency to immediately ask the teacher questions instead of discussing the tasks with each other was observed by the researcher across all classes. Some groups in S1B did appear cohesive, but most of them preferred to work individually than readily engage in dialogue associated with the task such as checking and validating responses with other group members. This indicated that group work was not a normal part of the classroom routine for learning mathematics.

Group problem-solving was clearly not apparent as a pedagogical strategy for teaching mathematics in School 2. Individual work samples did not match other group members’ responses to task items. In this case, students missed the opportunity to work collaboratively to these solve problems. Students’ ability to be able to define, justify and persuade others is integral to developing reasoning (Roth & Gardener, 2012).

**Summary**

The use of appropriate keywords and accurate visual mediators together has led many students in S1B to produce precise descriptions and definitions. Other students’ interpretations of their own diagrams may have lead to the development of incorrect definitions for rectangles. For example, when students indicated on a diagram of a rectangle, two pairs of equal opposite sides, this, in turn, was incorrectly understood that both pairs of sides must be of different lengths.
Students had difficulties in reflecting on their own learning and being able to articulate what they had learned in a precise way. Zimmerman (2002) found that students were rarely asked to evaluate their own learning or estimate their competence with new tasks. Generally, students also believed that they would be ‘told’ maths (Stodolsky, Salk & Glaessner, 1991). If students have limited experiences to learn through experimentation and inquiry then they would have difficulty valuing such learning activities.

Even though students were asked to work in groups, from their submitted work, it often appeared as if they had completed the work separately suggesting that students had limited experiences in actually working on tasks together. Good, Reys, Grouws and Mulryan (1989) observed that students often wanted to work independently because mathematics has always been presented to them in this way. This was supported by observations of table arrangements before tasks commenced as well as conversations about group work with teachers.

Group work was not an obvious instructional routine. Tables were arranged in rows and students sat facing the white board waiting for direct teacher instruction from the front of the classroom. When asked to formulate groups, students took several minutes to arrange furniture in a way that would be conducive to group work. The role of the teacher, while students attempted the task, was reduced to observing from a distance rather than listening in on student conversations, asking questions and challenging students to explain their thinking to other members.

Properties of Parallelograms and Rectangles
In this task, students were required to draw diagrams and list properties of a parallelogram, list similarities with rectangles, and formulate an inclusive definition. In deductive reasoning, what is important is to have a sense that because a shape has certain properties, others must also be true (Cooke, 2007). Students needed to recognise that rectangles are parallelograms because both have two pairs of opposite parallel sides. These are not facts to be remembered but rather it is important for students to be able to deduce these by interpreting the geometric information by what they ‘see’ in their minds (Battista, 2002; Fujita & Jones, 2003; Owens, 2003). Students “should be given opportunities to explore and engage in activities that can bring out the need to name and define these shapes, and comprehend how mathematicians make decisions about how to organise mathematical knowledge” (Seah, 2015a. p. 4).
Subsequent to the analysis of the results in the pilot school, the task was conducted in School 2 with several alterations to the way it was implemented. The task modifications were:

- The teacher reminded students about key concepts (right angles and equal sides) from the previous teaching episode. This occurred at the beginning of the class.
- Where a rubric was used previously for student responses in School 1, it was determined that the most efficient and potent form of feedback would occur during teaching episodes as had occurred with the previous tasks.
- After students had submitted their responses, a teaching episode involved the teacher brainstorming the task with students. This was written on the whiteboard and video-recorded.

**Keywords**

The keyword *parallel* is embedded in the word parallelogram, providing students with an essential property for this shape, that is, it is a shape made up of pairs of parallel lines. The endorsed narrative, therefore, is a parallelogram is a quadrilateral containing two pairs of parallel sides. This narrative forms an inclusive definition because it includes rectangles and rhombuses. An analysis of keywords used by students indicated that the students in S1B and S2A included *parallel* in their lists of properties in larger proportion to the students in S1A.

Students in S1A and S2A continued to use words such as *corners* and *faces*, but to a lesser extent than in previous tasks. Students in S2A often used *lines* and *corners* for sides and angles when listing properties of parallelograms and rectangles. However, *parallel* and *straight* were also often incorporated into descriptions about their similarities. This indicated that some of these students had retained key information from the previous teaching sessions. Individual students in S1B included *co-interior angles, congruency,* and *diagonal bisection*. Table 19 shows the keywords used by students across classes for describing properties of parallelograms and rectangles.
Table 19
*Keywords used to describe parallelograms and rectangles*

<table>
<thead>
<tr>
<th>Keywords</th>
<th>S1A</th>
<th>S1B</th>
<th>S2A</th>
</tr>
</thead>
<tbody>
<tr>
<td>quadrilateral</td>
<td>13.6</td>
<td>31.6</td>
<td>0</td>
</tr>
<tr>
<td>2D</td>
<td>0</td>
<td>10.5</td>
<td>11.1</td>
</tr>
<tr>
<td>polygon</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4 sides</td>
<td>22.7</td>
<td>15.8</td>
<td>11.1</td>
</tr>
<tr>
<td>lines</td>
<td>13.6</td>
<td>5.3</td>
<td>61.1</td>
</tr>
<tr>
<td>equal</td>
<td>13.6</td>
<td>63.2</td>
<td>5.6</td>
</tr>
<tr>
<td>4 angles</td>
<td>22.7</td>
<td>0</td>
<td>22.2</td>
</tr>
<tr>
<td>corners</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>vertices</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>edges</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>faces</td>
<td>9.1</td>
<td>36.8</td>
<td>0</td>
</tr>
<tr>
<td>opposite</td>
<td>0</td>
<td>42.1</td>
<td>0</td>
</tr>
<tr>
<td>diagonal</td>
<td>27.3</td>
<td>84.2</td>
<td>61.1</td>
</tr>
<tr>
<td>parallel</td>
<td>0</td>
<td>21.1</td>
<td>0</td>
</tr>
<tr>
<td>cointerior</td>
<td>0</td>
<td>26.3</td>
<td>0</td>
</tr>
</tbody>
</table>

Few students in S1B (36.8%) and S1A (9.1%) identified that parallelograms and rectangles have opposite sides of equal length. One student in each of these classes also stated that both shapes have equal opposite angles. In developing an inclusive definition, these properties did not constitute an endorsed narrative for these students.

*Narrative*

Fourteen students developed inclusive definitions for parallelograms and rectangles, as shown in Table 20. These students provided the necessary and sufficient condition for all parallelograms as having two pairs of parallel sides that resulted in a parallelogram having opposite sides being equal. Only S1B.Joanne included this redundant information. Being able to know the meaning of necessary and sufficient conditions was an indication that these students were thinking at van Hiele level 3: *abstraction*, and also suggested they were able to reason and understand the hierarchical nature of quadrilaterals.

Table 20
*Definitions for parallelograms and rectangles*

<table>
<thead>
<tr>
<th>Endorsed Narrative</th>
<th>S1A</th>
<th>S1B</th>
<th>S2A</th>
</tr>
</thead>
<tbody>
<tr>
<td>“All parallelograms and rectangles… [are quadrilaterals that] have two pairs of parallel sides.”</td>
<td>13.6</td>
<td>52.6</td>
<td>5.6</td>
</tr>
</tbody>
</table>

(3 students) (10 students) (1 student)
All other students that had difficulties developing an inclusive definition did so as they considered shapes to be unique – in that they held a *prototypical view* of shapes that excludes all others, rather than a *classical view* where concepts are represented by a set of defining features (Levenson, Tirosh & Tsamir, 2011). Due to an underdeveloped *concept image*, students had difficulties in perceiving class inclusions of shape believing that for example, a rectangle was not a parallelogram (Marchis, 2012).

All the students in S1B provided accurate lists of properties for parallelograms in addition to side lengths, angles and parallel lines (see Figure 23):

- **S1B.Zac:** diagonal creates 2 congruent triangles
- **S1B.Andrew:** two pairs of cointerior angles which add up to 180°

*Figure 23. Additional properties of a parallelogram by students in S1B.*

These properties indicated a logical implication of parallel lines for a parallelogram and thus indicated that these students were thinking at van Hiele level 3: *abstraction*.

**Misconceptions and difficulties**

As detected in the previous tasks, students often described rectangles and parallelograms as having two longer sides. Another misconception was detected when students depicted parallelograms as ‘tilted’ rectangles. Students in S1A indicated a higher rate of this misconception than the other two classes, as shown in Table 21.

**Table 21**

<table>
<thead>
<tr>
<th>Misconceptions</th>
<th>% of responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>parallelograms are tilted rectangles</td>
<td>22.7 4.8 5.6</td>
</tr>
<tr>
<td>parallelograms and rectangle have longer sides</td>
<td>18.2 0 5.6</td>
</tr>
</tbody>
</table>

Even though students in S2A did not express the same misconceptions at the same rate as students in S1A, this might be explained by the correct use of visual mediation of parallelograms. Many drew them with signifiers for parallel lines that did not lead them to conclude that parallelograms have two longer sides and two shorter sides. Their attention was not drawn to the side lengths by their correct use of visual mediators. However, 27.3% of students in S1B drew basic shapes of what appeared to be squares or rectangles that also did not imply the ‘tilted’ aspect of the misconception. In other words, these students may have understood parallelograms to include squares and rectangles, which they then used as a visual mediator for parallelograms – the prototypical parallelogram was not provided.
Other less common misconceptions were observed in student work samples indicating that some students in S2A were unable to retain specific information from previous teaching episodes. In the case of the pilot school, students in the same group repeated these types of misconceptions (see Figure 24):

S1A.Lucas: …two acute angles, two obtuse angles  
S1A.Anthony: …4 angles, two bigger and two smaller  
S2A.Abdul: …two parallel lines at the top  
S2A.Mary: …have equal length of sides

*Figure 24. Student misconceptions of parallelograms.*

Further difficulties that students had with this task were depicted in the type of diagrams they provided, and their use of signifiers. For example, some students interpreted *parallel* as horizontal or vertical lines that could be attributed to prototypical views of these shapes.

*Visual mediators*

Every student in S1B included a diagram of a parallelogram with signifiers. These, however, were often diagrams that indicated pairs of equal sides only, and further, several of these diagrams contained inaccuracies such as diagonals intersecting at right angles. These students did not know or realise that for rhombuses, diagonals intersect at right angles but not for all parallelograms.

For students in S1A, 13.5% provided diagrams that indicated parallelism for parallelograms. The other 18.3% provided diagrams that indicated pairs of equal sides. The rest of the class continued to provide basic shapes without any signifiers. For students in S2A, 16.7% were able to provide diagrams that indicated parallelism, and one student used signifiers for pairs of equal opposite sides. Of this class, 33.3% of students provided diagrams with inaccurate signifiers such as drawing the prototypical shape of a parallelogram but marking all sides as equal. Table 22 illustrates the range of diagrams used by students for parallelograms.

<table>
<thead>
<tr>
<th>Visual mediators</th>
<th>% of responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>No diagram</td>
<td>9.1</td>
</tr>
<tr>
<td>Incorrect diagram</td>
<td>9.1</td>
</tr>
<tr>
<td>Basic shape (no signifiers)</td>
<td>50.0</td>
</tr>
<tr>
<td>Correct shape with incomplete signifiers</td>
<td>0</td>
</tr>
<tr>
<td>Correct shape with inaccurate signifiers</td>
<td>0</td>
</tr>
<tr>
<td>Accurate shape (correct use of signifiers)</td>
<td>31.8</td>
</tr>
</tbody>
</table>

Table 22 *Visual mediators used by students for a parallelogram*
This analysis showed that significant proportions of S1A and S2A did understand the significance of diagrams in mediating their understanding about shapes. There continued to be an emphasis on listing facts through memorisation rather than using diagrams to indicate reasoning and this was indicated by inclusive and sufficient definitions.

An understanding of how diagrams connected with narratives was also examined. Table 23 shows a sample of diagrams and narratives used by students to express similarities of parallelograms and rectangles in order to articulate an inclusive definition.

Table 23

<table>
<thead>
<tr>
<th>similar properties keywords</th>
<th>inclusive definition narrative</th>
</tr>
</thead>
<tbody>
<tr>
<td>they have equal sides</td>
<td>[they] have equal sides</td>
</tr>
<tr>
<td>tilted rectangle</td>
<td>[have] two short and two long – exactly the same length</td>
</tr>
<tr>
<td>lines are even</td>
<td>[have] 2 equal sides</td>
</tr>
<tr>
<td>A rectangle is a parallelogram because it has 2 sets of parallel lines</td>
<td>[contain] at least 2 pairs of parallel lines</td>
</tr>
<tr>
<td>360°…2 sets of parallel sides</td>
<td>[have] at least 2 equal angles</td>
</tr>
<tr>
<td>A parallelograms’ only difference to a rectangle is that it’s corners aren’t 90°. In all other ways, it is the same.</td>
<td>only have 2 pairs of equal AND parallel sides. No more, &amp; no less. They also do not have any additional sides</td>
</tr>
<tr>
<td>four lines or more</td>
<td>are the same</td>
</tr>
<tr>
<td>4 sides, their 2D shape, their parallel</td>
<td>have parallel lines, and they belong in a big family</td>
</tr>
</tbody>
</table>

S1B.Louie’s depiction of a parallelogram included right angles making it a rectangle with the narrative that ‘a rectangle is a parallelogram’. This might indicate that his thinking was the opposite to ‘that a parallelogram is a rectangle’. Louie recognised the similarities in the two shapes but his use of right angles might suggest an inability to
order properties necessary to understanding hierarchies of quadrilaterals. It is otherwise unclear why Louie would choose to draw a rectangle – a specific parallelogram. S2A.Alex’s use of vertical and horizontal lines suggested an understanding of parallel as lines drawn in the same direction but he was unable to provide a diagram of a parallelogram despite knowing it has four lines.

During the teaching episode at the end of this task at School 2, several students contributed to the diagram of a parallelogram on the board. Each student successively and successfully added further detail to the class’s diagram indicating pairs of parallel sides and pairs of equal sides. There was a discussion about opposite angles but this needed to be supported by the teacher. This teaching episode evolved into a focussed lesson on the (interior) angle sum of quadrilaterals.

When the teacher drew the following shape (see Figure 25) on the board, students immediately referred to it as ‘a diamond’. Even after further elaboration about equal side lengths and non-right angles, students continued to refer to the shape as a diamond, yet some knew about a rhombus.

Figure 25. S2A.Teacher’s diamond and classroom dialogue.

This episode indicated that students had recalled incorrect information from a previous lesson, and it also indicated confusion in the teacher’s mind about her inclusion of ‘diamond’ as a quadrilateral. Teachers and their students tend to exhibit similar patterns of misconceptions (Swafford, Jones & Thornton, 1997; Yeo, 2008).

The teacher summarised properties of a parallelogram and corrected students who used corners or vertices for angles. Collectively as a class, students were able to
develop a definition that all parallelograms and rectangles were quadrilaterals with two pairs of parallel lines, and each pair of parallel lines having the same length.

When discussing what was different about parallelograms and rectangles (see Figure 26), students had difficulties with articulating correct responses.

Student J: Rectangles do not have parallel lines.
Student K: You know how parallelograms have all equal lines? A rectangle doesn’t.

*Figure 26. Two students in S2A discussing parallelograms.*

When the teacher drew a rectangle on the board, immediately students were able to identify that a rectangle must have right angles. The diagram was vitally important in mediating students’ understanding.

*Groups*

Students in all classes continued to sit together but worked independently. They already knew that the teacher or the researcher would not immediately respond to individual questions. This did not automatically lead to much group dialogue or debate about rectangle and parallelogram properties. It also did not lead to any collaboration about an inclusive definition. In the case of S1A.Lucas and S1A.Anthony, their teamwork led to the sharing of a misconception that parallelograms ‘have two long sides and two short sides’. All of the teachers did not manage their students’ group work effectively because they tended to leave students working by themselves while sitting together, and they did not pose any questions to students.

*Summary*

Visual mediators together with narratives used by students indicated a clear difference between the classes. Generally, students in S1B already had a more sophisticated repertoire of geometric language that they were prepared to use in comparison with the other two classes. Students in this class used keywords, descriptions and definitions more accurately and were able to order properties, moving toward a sufficient definition. Their definitions tended to address parallelism rather than side lengths and this suggested that many of these students were thinking at van Hiele level 3: *abstraction* (where students logically order properties and can distinguish between the necessity and sufficiency of a set of properties (Burger & Shaughnessy, 1986)). There was an alignment between visual mediators and narratives used by these students and this indicated that their concept images of certain shapes coincide with their concept definitions, and resembled a more ‘formal’ concept definition – one accepted by the
Students found it difficult to articulate their own learning. Communicating what was known or learned is an important demonstration in how and why students think mathematically, and are able to provide a logical argument. Students in S1A rated the task as ‘easy’ despite large sections of the task being incomplete or inaccurate. Many students were unable to articulate their learning through this task and to describe their difficulties with particular aspects of the task. Self-reflection was also clearly not a significant part of student learning for this class. For students in S1A, only two groups responded to what they learned using definitions and new facts, such as ‘a 4-sided shape will always have 4 corners’. This might be evidence of their inexperience with working in small groups. It also indicated an inability to critically reflect on and articulate individual learning through performing tasks. The inexperience of the teacher to set expectations for group performance and output may also have been a contributing factor to students not completing particular tasks and being unable to reflect upon them.

In contrast, the majority of groups in S1B expressed that they had learned about similarities between a parallelogram and a rectangle, and about parallel sides. Others mentioned that they had learned about diagonal lines bisecting in the centre and ‘that many shapes were also parallelograms’. This indicated a higher level of understanding of hierarchical aspects of quadrilaterals than students in S1A.

Students in S2A rated the task as ‘easy’ even though their responses indicated a significant degree of difficulty. One student claimed that he had learned that parallel lines were straight – a reasonable conclusion in terms of the design of the task. The task did not involve a definition of parallelism itself. An indication of student understanding of a hierarchy was not apparent because an inclusive definition for all parallelograms and rectangles was required but did not emerge from the task. Usiskin, Griffin, Witonsky and Willmore, (2008) identified several inclusive definitions for rectangles (Table 24) that required an understanding of parallelograms. Neither version of these hierarchical definitions were stated by students in S2A, this confirming difficulties with the hierarchical classification of quadrilaterals.

Table 24

<table>
<thead>
<tr>
<th>Definitions of a rectangle as a parallelogram found in high school textbooks</th>
<th>Number of texts</th>
</tr>
</thead>
<tbody>
<tr>
<td>a parallelogram with four right angles</td>
<td>35</td>
</tr>
<tr>
<td>a parallelogram in which at least one angle is a right angle</td>
<td>30</td>
</tr>
<tr>
<td>an equiangular parallelogram</td>
<td>7</td>
</tr>
</tbody>
</table>
Student thinking about these shapes as exhibited through their use of keywords, diagrams and narratives appeared inconsistent. This may be as a result of prototypical examples of shapes limiting students’ abilities to identify relevant definitional properties (Heinze & Ossietsky, 2002).

**Quadrilaterals by Properties (I)**

Students were required to define a quadrilateral (a concept already examined in previous tasks) and to draw quadrilaterals with different numbers of right angles, and then to construct a conclusion about quadrilaterals that had three right angles. At each stage of the task, students were required not to repeat previously drawn versions of quadrilaterals.

After analysis of the results from School 1, task modifications were made to address key aspects of teaching. These included formal teacher instructions about how students should work together and how feedback to students would occur. The task modifications for School 2 were:

- The teacher reminded students about key concepts from the previous teaching episode. This occurred at the beginning of the class through verbal description/definition of parallel lines and right angles along with diagrams using formal visual mediation.
- No rubric was used to provide feedback to students. The teacher gave verbal feedback during the teaching episode as students contributed answers to task questions. According to Roth and Gardner (2012), students who are asked to define, justify, and persuade others, is integral to developing a better understanding of geometric concepts.
- An extended teaching episode involved the teacher and students writing on the whiteboard and was video-recorded by the researcher.

**Keywords**

An endorsed narrative for a quadrilateral is a 4-sided polygon. As all polygons are 2D and have straight sides with a matching number of angles, these keywords would be superfluous. Students in S1B (81.3%), S1A (88.2%) and S2A (61.1%) identified 4 sides of which only 43.8% of students in S1B included polygon. To this respect, all other students listed known facts about quadrilaterals but these facts were not incorporated into a definition. At this stage, an economical use of keywords had taken
place with classes documenting the essential terms rather than providing an exhaustive list of facts.

This progression from listing facts to selecting necessary properties to communicate the concept of a quadrilateral indicated that some students were beginning to notice the interrelationships between the properties of a figure. This is described by van Hiele level 3: abstraction. However, at this van Hiele level, students do not know the purpose definitions play to ensure that somebody else knows exactly what figure one is talking about (de Villiers, 1998b). Table 25 shows the types of keywords used by students to describe quadrilaterals.

Table 25
Keywords used by students to describe a quadrilateral

<table>
<thead>
<tr>
<th>What is a quadrilateral?</th>
<th>S1A</th>
<th>S1B</th>
<th>S2A</th>
</tr>
</thead>
<tbody>
<tr>
<td>2D</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>polygon</td>
<td>0</td>
<td>43.8</td>
<td>0</td>
</tr>
<tr>
<td>closed shape</td>
<td>0</td>
<td>0</td>
<td>33.3</td>
</tr>
<tr>
<td>4 sides</td>
<td>88.2</td>
<td>81.3</td>
<td>61.1</td>
</tr>
<tr>
<td>4 lines</td>
<td>0</td>
<td>18.8</td>
<td>0</td>
</tr>
<tr>
<td>straight</td>
<td>0</td>
<td>43.8</td>
<td>22.2</td>
</tr>
<tr>
<td>connected</td>
<td>5.9</td>
<td>18.8</td>
<td>0</td>
</tr>
<tr>
<td>4 angles</td>
<td>17.6</td>
<td>6.3</td>
<td>0</td>
</tr>
<tr>
<td>corners</td>
<td>0</td>
<td>0</td>
<td>33.3</td>
</tr>
<tr>
<td>angles sum</td>
<td>0</td>
<td>31.3</td>
<td>0</td>
</tr>
</tbody>
</table>

Students in S1A and S2A were economical with their use of keywords to describe a quadrilateral. They stated a reduced amount of information instead of extensive lists of properties to define a quadrilateral and assumed but did not state 2D shape or polygon. Students in S1B used a wider range of keywords by providing compound statements that connected several properties some of which were redundant. All students in S1B defined a quadrilateral as four-sided even though 3 students from different groups used lines instead of sides. Many went further (see Figure 27) by including additional information such as connected lines, straight sides, a polygon, and 31.1% specified the angle sum of 360°.

S1B.Andrew: A quadrilateral is a four straight sided polygon, which the interior angles add up to 360°.
S1B.Joseph: A shape with 4 straight, connected lines & 4 interior angles that add up to 360°.
S1B.Seth: A quadrilateral is a polygon with 4 connecting straight sides. They also have 4 interior angles.

Figure 27. Students’ definitions of a quadrilateral.
Narrative

There are several endorsed narratives for quadrilaterals. For example, a quadrilateral is a (2D) shape with four straight sides or it is a 4-sided polygon. The term *quadrangle* also appears in the literature to name 4-sided polygons (Heinze & Ossietsky, 2002; Usiskin, Griffin, Witonsky & Willmore, 2008). This term includes the necessary conditions of four *(quad)* angles. Table 26 shows the types of endorsed narratives used by students to define a quadrilateral.

Table 26

<table>
<thead>
<tr>
<th>Endorsed Narratives</th>
<th>% of responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>“A quadrilateral is a…</td>
<td>S1A</td>
</tr>
<tr>
<td>four-sided polygon</td>
<td>0</td>
</tr>
<tr>
<td>(2D) shape with four straight sides</td>
<td>0</td>
</tr>
</tbody>
</table>

The majority of students in S1A and S2A stated ‘four sides’ which is insufficient information to define a quadrilateral. Of students in S1B, 18.8% included both *polygon* and *4 sides*. S1A.David questioned ‘Why don’t they call it a quadrangle?’ indicating attention being given to the etymological meaning of mathematical terminology by this student. Some students from S2A added a further description that the shape needed to be *closed* (33.3%) and referenced *four corners* (33.3%). This may have resulted because students retained information from previous teaching episodes where the teacher had modelled the drawing of quadrilaterals using a ruler and emphasised that all polygons were closed shapes with straight sides.

Visual mediators

Almost all students provided basic diagrams for quadrilaterals with no right angles. In many cases however, students provided squares and rectangles with unmarked angles. This occurred with 70.6%, 75% and 83.3% of students in S1A, S1B, and S2A respectively. These shapes were then repeated for the series of diagrams where students indicated one, two or three right angles as the question item required. Table 27 shows a decrease in accurate shapes being depicted by all classes from one task item to the next. In the case of S2A, students were unable to manage this task by ensuring that they did not repeat shapes as they progressed through each section and as required by the task.
Table 27

**Diagrams of quadrilaterals with varying numbers of right angles**

<table>
<thead>
<tr>
<th>Quadrilateral with…</th>
<th>% of responses</th>
<th>% of responses</th>
<th>% of responses</th>
<th>% of responses</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>any</td>
<td>one right angle</td>
<td>two right angles</td>
<td>three right angles</td>
</tr>
<tr>
<td><strong>Visual mediators</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>No diagram</td>
<td>S1A 0</td>
<td>S1B 0</td>
<td>S2A 11.1</td>
<td>S1A 0</td>
</tr>
<tr>
<td></td>
<td>S1B 0</td>
<td>S1B 0</td>
<td>S2A 27.8</td>
<td>S1A 23.5</td>
</tr>
<tr>
<td></td>
<td>S2A 61.1</td>
<td>S1A 6.3</td>
<td>S2A 6.3</td>
<td>S1A 61.1</td>
</tr>
<tr>
<td>Incorrect diagram</td>
<td>S1A 0</td>
<td>S1B 0</td>
<td>S2A 11.1</td>
<td>S1A 0</td>
</tr>
<tr>
<td></td>
<td>S1B 0</td>
<td>S1B 0</td>
<td>S2A 27.8</td>
<td>S1A 23.5</td>
</tr>
<tr>
<td></td>
<td>S2A 61.1</td>
<td>S1A 6.3</td>
<td>S2A 6.3</td>
<td>S1A 61.1</td>
</tr>
<tr>
<td>Basic shape (no signifiers)</td>
<td>S1A 0</td>
<td>S1B 0</td>
<td>S2A 11.1</td>
<td>S1A 0</td>
</tr>
<tr>
<td></td>
<td>S1B 0</td>
<td>S1B 0</td>
<td>S2A 27.8</td>
<td>S1A 23.5</td>
</tr>
<tr>
<td></td>
<td>S2A 61.1</td>
<td>S1A 6.3</td>
<td>S2A 6.3</td>
<td>S1A 61.1</td>
</tr>
<tr>
<td>Inaccurate signifiers</td>
<td>S1A 0</td>
<td>S1B 0</td>
<td>S2A 11.1</td>
<td>S1A 0</td>
</tr>
<tr>
<td></td>
<td>S1B 0</td>
<td>S1B 0</td>
<td>S2A 27.8</td>
<td>S1A 23.5</td>
</tr>
<tr>
<td></td>
<td>S2A 61.1</td>
<td>S1A 6.3</td>
<td>S2A 6.3</td>
<td>S1A 61.1</td>
</tr>
<tr>
<td>Accurate shape with signifiers</td>
<td>S1A 100</td>
<td>S1B 100</td>
<td>S2A 88.9</td>
<td>S1A 64.7</td>
</tr>
<tr>
<td></td>
<td>S1B 100</td>
<td>S2A 93.8</td>
<td>S1A 87.5</td>
<td>S2A 35.3</td>
</tr>
<tr>
<td></td>
<td>S2A 0</td>
<td>S1A 0</td>
<td>S1B 50.0</td>
<td>S2A 0</td>
</tr>
</tbody>
</table>

These responses further indicated that students thought it was satisfactory to repeat the same shape, often squares or rectangles, and to indicate the number of right angles required by the task item instead of reasoning that squares and rectangles, for example, must have 4 right angles.

**Narratives for quadrilaterals with three right angles**

Table 28 provided a summary of narratives used by students when required to draw a conclusion about quadrilaterals with three right angles. This indicated that many students had difficulties with providing a valid justification based on deductive reasoning.

Table 28

**Endorsed narratives for quadrilaterals with three right angles**

<table>
<thead>
<tr>
<th>Narratives</th>
<th>% of responses</th>
<th>S1A</th>
<th>S1B</th>
<th>S2A</th>
</tr>
</thead>
<tbody>
<tr>
<td>“A quadrilateral with three right angles then…”</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>it must have 4 right angles… as the angle sum is 360°. (correct justification)</td>
<td>17.6</td>
<td>37.5</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>it must have 4 right angles… (without justification)</td>
<td>35.3</td>
<td>50.0</td>
<td>11.1</td>
<td></td>
</tr>
</tbody>
</table>

Many students stated it was an impossible task – to produce a quadrilateral with three right angles. Some students commented that the task needed to state ‘only three right angles’ which indicated that there was a level of deduction being exhibited by these students to pose this comment. Student reasoning in relation to quadrilaterals with 3 right angles showed difficulties with the concept of quadrilaterals (see Figure 28).

S1A.Henry: you can draw it but only if the q says nothing about joining the lines
S1A.Daniel: there are many with 3RA as the q doesn't say only 3 sides
S1A.Mia: It has to have a curved side
S1B.Stacey: can have a polygon but not a quadrilateral

*Figure 28. Samples of student reasoning for quadrilaterals with three right angles.*
Student narratives often did not match their diagrams and vice versa. This indicated that many students did not see both components as working together to mediate understanding and reasoning. After defining a quadrilateral, students were required to draw *any* quadrilateral. If students had a meaning of *any*, it was different from the intention of the task item. Many students began with a square or rectangle for their version of *any* quadrilateral not knowing that *any* quadrilateral implied the whole class of quadrilaterals – the general and not the specific.

Table 29 provides a sample of student narratives for defining quadrilaterals, diagrams of quadrilaterals, and narratives for quadrilaterals with 3 right angles.
<table>
<thead>
<tr>
<th>Draw...</th>
<th>Any quadrilateral</th>
<th>A quadrilateral with one right angle</th>
<th>A quadrilateral with two right angles</th>
<th>A quadrilateral with three right angles</th>
<th>A quadrilateral with three right angles...</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1A.Ava</td>
<td><img src="image1.png" alt="Image" /></td>
<td><img src="image2.png" alt="Image" /></td>
<td><img src="image3.png" alt="Image" /></td>
<td><img src="image4.png" alt="Image" /></td>
<td>will always have four</td>
</tr>
<tr>
<td>S1A.Tara</td>
<td>a shape that has four sides or four right angles</td>
<td><img src="image5.png" alt="Image" /></td>
<td><img src="image6.png" alt="Image" /></td>
<td><img src="image7.png" alt="Image" /></td>
<td>has 4 sides, but it isn’t working so I do not think it is possible</td>
</tr>
<tr>
<td>S1A.Andre</td>
<td>a shape with 4 sides</td>
<td><img src="image8.png" alt="Image" /></td>
<td><img src="image9.png" alt="Image" /></td>
<td><img src="image10.png" alt="Image" /></td>
<td>we could not find out how to make one</td>
</tr>
<tr>
<td>S1A.Henry</td>
<td><img src="image11.png" alt="Image" /></td>
<td><img src="image12.png" alt="Image" /></td>
<td><img src="image13.png" alt="Image" /></td>
<td><img src="image14.png" alt="Image" /></td>
<td>you can draw it but only if the question says nothing about joining the lines</td>
</tr>
<tr>
<td>S1B.Angelo</td>
<td>is a polygon with 4 sides</td>
<td><img src="image15.png" alt="Image" /></td>
<td><img src="image16.png" alt="Image" /></td>
<td><img src="image17.png" alt="Image" /></td>
<td>You can’t find a quadrilateral with just 3 right angles</td>
</tr>
<tr>
<td>S1B.Brian</td>
<td>a shape with 4 sides</td>
<td><img src="image18.png" alt="Image" /></td>
<td><img src="image19.png" alt="Image" /></td>
<td><img src="image20.png" alt="Image" /></td>
<td>It is impossible to have a quadrilateral with only three right angles</td>
</tr>
<tr>
<td>Name</td>
<td>Description</td>
<td>Diagram</td>
<td>Diagram</td>
<td>Diagram</td>
<td>Diagram</td>
</tr>
<tr>
<td>----------</td>
<td>--------------------------------------------------</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
</tr>
<tr>
<td>S2A.Ali</td>
<td><em>a four sided shape</em></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S2A.Fadi</td>
<td><em>4 sides</em></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S2A.Ian</td>
<td><em>is a closed shape with 4 sides</em></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S2A.Albert</td>
<td><em>a four sided straight shape</em></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Students in S1A and S2A indicated the right angles as each task item required. Most students in S1B indicated equal side lengths, parallel lines and right angles showing an appreciation of the necessity for narratives to match diagrams with the use of signifiers. This, in turn, also indicated a higher level of deductive reasoning, such as two adjacent right angles would create parallel lines. This level of thinking is defined as van Hiele level 3: *abstraction* where students perceive relationships between properties.
Some students in S1A indicated a fourth right angle when asked to draw a quadrilateral with three right angles and but they did know whether this response was permissible. Other students did not want to indicate what type of angle the fourth angle must be, or in the case of S1A. Henry (see Table 29, p. 146) left the diagram unfinished to avoid creating a fourth right angle.

Students in School 2 had significant difficulties completing most sections of this task although most provided simple diagrams of quadrilaterals corresponding to the right angle conditions specified in each task item. They could not reason that three right angles necessitated a fourth for a quadrilateral. It appeared that this level of abstraction was beyond the level of geometric thinking of the students in this class, and this was confirmed by the majority of students who provided incomplete submissions. The teacher suggested that students attempt the next task on quadrilaterals anticipating that students would experience similar levels of difficulty and would complete it quickly. This allowed her to provide an extended teaching episode where she would address student difficulties and provide definitions and examples of new or unfamiliar keywords.

Misconceptions and difficulties
Students often had difficulties with understanding the mathematical meaning of any where this implies a general case. Not knowing or being taught the meaning of any in a geometrical context may have inadvertently led students to draw the same quadrilateral – squares and rectangles in most cases – and then used the number of right angle signifiers as specified in each question. Many students also expressed that the task needed to contain the word only, especially when it called for a quadrilateral with three right angles. Students were able to reason that a fourth right angle must exist but did not explain their reasoning nor represent it on a diagram because they might have believed that they had used incorrect reasoning. Students interpreted the question as asking them to draw a quadrilateral with only three right angles to which most concluded that it could not be done.

Some of the common difficulties exhibited by students resulted from a continued lack of understanding of the importance of using signifiers with diagrams, and being able to connect geometric information to produce one diagram. For example, drawing a quadrilateral with three angles and being unable to conclude that the fourth must be signified as a right angle, and then concluding that it must be a rectangle. Only
S1A. George made the correct conclusion ‘it is a rectangle as a quadrilateral can only have 1, 2 or 4 right angles’, albeit without any justification. All other students that attempted the task indicated that they operated at van Hiele level 2: analysis, where students did not see relationships between these properties.

Group work
If replicating mistakes was an indication that students have copied from each other (rather than collaborated) to produce a common response then several groups across classes had difficulties with group work and merely focussed on completing work rather than understanding it. Student work samples, including errors, were identical in some cases with no record of any real dialogue supporting collaborative learning practices. The norms of group work and expectations were therefore not previously negotiated as a whole class. While group work might be an ideal pedagogical method of teachers, many students do not understand exactly what their role should be during collaborative tasks, presumably since these processes have not been established in normal classroom routines.

Summary
Students in S1A asserted that they had learned ‘nothing’ from the task and rated it as ‘medium’ in terms of its level of difficulty. Students in S1B claimed they had learned that a quadrilateral could not have only three right angles. They also rated the task as ‘medium’ matching their level of success and difficulty with the task.

For S2A students, even though the majority of sections for the majority of students were incomplete, many of them rated the task as ‘easy’, indicating a superficial rating given their difficulties. Student diagrams were neater than in previous tasks as a result of using rulers for accuracy, and they included key features, such as right angles, as required by the task items. However, an understanding of a relationship between right angles, parallel lines and equal sides was not well understood. Teaching episodes for this class had resulted in enhanced diagrammatic representations of geometric concepts but this had led to minimal improvements in students’ geometric thinking levels.

Several factors led more students to use the keywords of 4 sides for the narrative to help define a quadrilateral. Feedback from the teachers via assessment rubrics of preliminary tasks, some aspects of group work, and formative qualities of the tasks themselves have all influenced the ways in which students in this study communicated geometric concepts.
Quadrilaterals by Properties (2)

Students were required to draw a range of quadrilaterals with different side and angle conditions. Students were not required to use keywords or define anything as such. This task was conducted at School 2 immediately after students had completed the previous task as determined by the teacher and the researcher. The following modifications to the original task conducted in School 2 included:

- No rubric was used to provide feedback to students. Instead, feedback would be given during the teaching episode verbally and publicly with students as they contributed answers to task questions.
- The extended teaching episode involved the teachers and students writing on the whiteboard and was video-recorded by the researcher.

Keywords

Even though keywords and narratives were not part of the written component of the task, students were asked to record what they had learned. Students in S1B did not indicate difficulties with keywords. Of the students in S1A, 50% commented that they did not know the meaning of adjacent indicating that this was a new concept for them. This was also an obstacle for students in S2A. Several students in School 2 did not understand the keywords in the questions – adjacent sides and opposite sides, nor were they able to combine these concepts to produce accurate diagrams based on these conditions. Of this class, 38.1% made no attempt to complete questions and submitted blank responses. It was apparent that this task was beyond the geometric level of understanding for most students in S1A and S2A.

Visual mediators

Several students in S1A had difficulties with interpreting the questions and, therefore, did not attempt sections of the task or instead they drew very basic diagrams. However, every student in S1B attempted all components of the task, and this indicated a level of intrinsic motivation to learn new concepts or be challenged by activities, or higher levels of geometric understanding, thus giving students better access to problems than students in other classes. Even though students in both S1A and S1B produced some diagrams, the majority indicated equal sides but did not know how to use signifiers to indicate equal angles.

All students in S2A had difficulties completing most sections of this task. This indicated that the task was beyond the level of student geometric thinking. Only
S2A. Michelle provided a valid diagram when asked to draw a quadrilateral with one pair of equal sides (see Figure 29). However, Michelle was unable to attempt any additional task items clearly indicating a lack of understanding of keywords, such as opposite and adjacent.

![Figure 29. S2A.Michelle’s diagram of a quadrilateral with one pair of equal sides.](image)

Table 30 summarises the types of diagrams provided by students for all sections of the task. Significant proportions of students in S1A (47%) and S2A (90.5%) had difficulties drawing quadrilaterals using signifiers, and this deteriorated for each successive part of the task down to 70.6% and 100% respectively for the final task item. In contrast, larger proportions of students in S1B were able to provide accurate diagrams although some students misused signifiers.

Table 30
Types of visual mediators used by students used by students for quadrilaterals with different side and angle restrictions

<table>
<thead>
<tr>
<th>Quadrilateral with…</th>
<th>% of responses one pair of equal sides</th>
<th>% of responses one pair of adjacent sides equal and one pair of opposite angles equal</th>
<th>% of responses two pair of adjacent sides equal and one pair of opposite angles equal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Visual mediators</td>
<td>S1A</td>
<td>S1B</td>
<td>S2A</td>
</tr>
<tr>
<td>No diagram</td>
<td>0</td>
<td>0</td>
<td>38.1</td>
</tr>
<tr>
<td>Incorrect diagram</td>
<td>29.4</td>
<td>6.3</td>
<td>38.1</td>
</tr>
<tr>
<td>Basic shape (no signifiers)</td>
<td>17.6</td>
<td>6.3</td>
<td>14.3</td>
</tr>
<tr>
<td>Correct shape with incomplete signifiers</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Correct shape with inaccurate signifiers</td>
<td>5.9</td>
<td>6.3</td>
<td>4.8</td>
</tr>
<tr>
<td>Accurate shape (correct use of signifiers)</td>
<td>47.1</td>
<td>81.3</td>
<td>4.8</td>
</tr>
</tbody>
</table>

Students in S1B demonstrated a consistently higher level of geometric reasoning at each stage of the task, and were able to articulate valid conclusions by successfully combining and representing concepts of adjacent sides and opposite angles through diagrams. Table 31 shows a sample of student diagrams for attempted task components.
Table 31
Samples of diagrams used for depicting quadrilaterals with side and angle restrictions

<table>
<thead>
<tr>
<th>Quadrilaterals with…</th>
<th>one pair of equal sides</th>
<th>one pair of adjacent sides equal and one pair of opposite angles equal</th>
<th>Quadrilateral with two pairs of adjacent sides equal and one pair of opposite angles equal</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1A.Daniel</td>
<td>not provided</td>
<td></td>
<td></td>
</tr>
<tr>
<td>S1A.John</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S1B.Joanne</td>
<td>not provided</td>
<td></td>
<td></td>
</tr>
<tr>
<td>S1B.Angelo</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S1B.Joseph</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S1B.Andrew</td>
<td>not provided</td>
<td></td>
<td></td>
</tr>
<tr>
<td>S1B.Harriot</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Students in S1A continued to have difficulties with using signifiers adequately. Students in S1B out-performed students in S1A by being able to combine key concepts of angle and side conditions effectively, thus demonstrating a higher level of geometric reasoning. Students in S2A had significant difficulties with attempting all sections of this task most likely due to an inability to interpret opposite, adjacent, and to effectively combine those two concepts. Many abandoned the task and this resulted in an extended teaching episode.

Students in S2A provided basic shapes of squares or rectangles, if they drew anything at all. Where students rated the task, most rated it as hard to very hard. Only two students provided similar reflective comments that indicated that they did not know what adjacent had meant. This indicated that the task was beyond their level of student
geometric knowledge. The teacher and researcher decided to end the task and the teacher conducted an extended teaching episode, addressing this and the previous task.

Clearly, students were unable to combine the meaning of various mathematical terms and properties and were also unable to produce an accurate diagram. The teacher generated a word list with definitions - *opposite* and *adjacent* sides and angles - and students recorded these in their workbooks. The teacher scaffolded these definitions using everyday examples and then progressed to geometric definitions applying to quadrilaterals. Finally the teacher produced several diagrams that addressed the condition of quadrilaterals with two pairs of equal sides and included a more challenging example of a concave quadrilateral or chevron (see Figure 30).

*Figure 30. S2A. Teacher’s quadrilateral with two pairs of equal sides.*

This particular teaching episode engaged students, provided new knowledge or reinforced previously known geometric concepts, and focussed strongly on geometric language, definitions and diagrams. Other geometric concepts, such as types of angles and sides, were included in the whole class discussion. The teacher proceeded to emphasise language and challenged her students’ thinking.

The teacher drew a right triangle on the board (see Figure 31), and an excerpt of the discussion provided:

<table>
<thead>
<tr>
<th>Teacher:</th>
<th>What does ‘triangle’ mean? (Waited) Tri? (Waited) Tricycle?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Student L:</td>
<td>Tri means triple.</td>
</tr>
<tr>
<td>Student M:</td>
<td>Three angles….</td>
</tr>
<tr>
<td>Student N:</td>
<td>The other two angles are the same.</td>
</tr>
<tr>
<td>Teacher:</td>
<td>Are they? Are you sure? How do you know?</td>
</tr>
</tbody>
</table>

*Figure 31. S2A. Teacher’s right triangle (a).*

The teacher drew a second triangle (see Figure 32).

<table>
<thead>
<tr>
<th>Teacher:</th>
<th>Are they the same size?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Student O:</td>
<td>They look the same.</td>
</tr>
</tbody>
</table>

*Figure 32. S2A. Teacher’s right triangle (b).*
The teacher proceeded to use her arms to make different size angles in order to convince students that the angles were different. As Owens (1999) stated, gestures play an important role in explaining visual images.

*Group work*

Group work was not a strong feature of learning in this task. Students continued to sit together but rarely engaged in discussion about the task or its challenges. A lack of success or motivation in previous tasks may have contributed to this. Many students in S1A were disengaged by the task probably because it represented a higher level of challenge than previous tasks. Students completed the task as quickly as they could before letting themselves be distracted. The expectations set by teacher needed to ensure that students worked in a collaborative way to achieve a level of success and that they knew strategies when they needed support from within their group. There were a number of blank submissions. Students spent a significant amount of time being distracted without intervention from the teacher. In contrast, the teacher of students in S2A determined quite quickly that the task was beyond the level that most students could manage, and instead she provided a teaching episode on new concepts.

*Summary*

This particular task generated a significant level of challenges for students because it required students to use geometric reasoning skills to combine concepts than simply recalling previously known concepts in a rote fashion. The teaching episode provided an important opportunity for one class to learn new concepts. S1B.Zac claimed that he learned ‘to have one pair of equal opposite angles in a quadrilateral, you have to have another’. Seldom did students provide this level of articulation of geometric reasoning required by this task.

*Paper-folding*

This task required students to work in small groups and record their predicted outcomes to punch-hole patterns on folded squares and then make a general conclusion about patterns based on the way the paper was folded, the number of times the paper was folded, and the number of holes punched. This required students to use visualisation in order to interpret diagrams and predict punch-hole pattern outcomes. This final task was conducted at School 1 and School 2 without any modifications.
The first task item described the way a square piece of paper was folded and diagrams were provided to mediate the verbal description. Students were given multiple-choice answers to select their response. The other two task items required students to provide an interpretation using both written and diagrammatic descriptions of punch-holes and to record the resulting pattern – either vertically or diagonally.

**Visual Mediators**

The interpretation of the listed visual mediators (which were multiple-choice task items) produced differences between the two schools as shown in Table 32. Almost every student in School 1 was able to predict the resulting punch-hole arrangement compared with students in School 2 where only 26.7% could interpret the given written and visual description. Table 33 shows the distribution of choices made by students in predicting a punch-hole pattern.

**Table 32**

*Student responses to given punch-hole patterns*

<table>
<thead>
<tr>
<th>Which one of the arrangements of holes will appear?</th>
<th>S1A</th>
<th>S1B</th>
<th>S2A</th>
</tr>
</thead>
<tbody>
<tr>
<td>No response</td>
<td>0</td>
<td>0</td>
<td>6.7</td>
</tr>
<tr>
<td>Incorrect response</td>
<td>5.9</td>
<td>0</td>
<td>66.7</td>
</tr>
<tr>
<td>Multiple responses</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Correct response</td>
<td>94.1</td>
<td>100</td>
<td>26.7</td>
</tr>
</tbody>
</table>

**Table 33**

*Distribution of choices of punch-hole patterns*

<table>
<thead>
<tr>
<th>Which arrangement of holes will appear?</th>
<th>% of responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1A</td>
<td>No response: 0, A: 5.9, B: 0, C: 0, D: 94.1, E: 0</td>
</tr>
<tr>
<td>S1B</td>
<td>No response: 0, A: 0, B: 0, C: 0, D: 100, E: 0</td>
</tr>
<tr>
<td>S2A</td>
<td>No response: 6.7, A: 46.7, B: 20, C: 0, D: 26.7, E: 0</td>
</tr>
</tbody>
</table>

Students who selected option B visualised a resulting horizontal pattern but did not interpret the effect of the double-fold. The large proportion of students in S2A who selected option A visualised the effect of the double-fold but were unable to predict the resulting pattern.

The task items requiring students to draw punch-hole patterns are shown in Table 34. Students in S1B were able to interpret written instructions with given diagrams to predict punch-hole patterns with almost every student providing correct
responses to both task items – although 62.5% and 31.3% drew inaccurate responses for
the two tasks item respectively. These inaccurate responses were most likely due to the
size of the punch-holes and their spacing rather than confusion with orientation.

Table 34
Student diagrams of punch-hole patterns

<table>
<thead>
<tr>
<th>Visual mediators</th>
<th>S1A</th>
<th>S1B</th>
<th>S2A</th>
<th>S1A</th>
<th>S1B</th>
<th>S2A</th>
</tr>
</thead>
<tbody>
<tr>
<td>No response</td>
<td>6.7</td>
<td>0</td>
<td>65.0</td>
<td>13.3</td>
<td>0</td>
<td>65.0</td>
</tr>
<tr>
<td>Incorrect response</td>
<td>60.0</td>
<td>6.3</td>
<td>10.0</td>
<td>60.0</td>
<td>0</td>
<td>10.0</td>
</tr>
<tr>
<td>Inaccurate response</td>
<td>26.7</td>
<td>62.5</td>
<td>25.0</td>
<td>20.0</td>
<td>31.3</td>
<td>25.0</td>
</tr>
<tr>
<td>Correct response</td>
<td>6.7</td>
<td>31.3</td>
<td>0</td>
<td>6.7</td>
<td>68.8</td>
<td>0</td>
</tr>
</tbody>
</table>

Students in S1A and S2A had difficulty in visualising, predicting and drawing resulting
punch-hole patterns after unfolding, with 65% of students in S2A not providing
responses to either of the two task items. Table 35 shows the types of diagrams used by
students to indicate their interpretation of the instructions and diagrams provided.

Table 35
Responses to punch-hole patterns

<table>
<thead>
<tr>
<th>Punch-hole pattern</th>
<th>expected response</th>
<th>expected response</th>
</tr>
</thead>
<tbody>
<tr>
<td>(vertical/horizontal pattern)</td>
<td>(diagonal pattern)</td>
<td></td>
</tr>
</tbody>
</table>

S1A.Ryan:

S1A.Henry:

S1A.Samuel:

S1B.Enrico: (general pattern is evident but little attention to spacing)

S2A.Ali:

S2A.Noah:

S2A.Michelle:

Students displayed problems with visualisation. Reflecting a pattern mentally and
reasoning spatially created challenges for these students. Some drew lines in an attempt
to help mediate the reflection that was necessary to achieve a resulting pattern.
Keywords

Several keywords are important in describing visualisation. Table 36 shows the types of keywords students used to formulate a general conclusion about resulting punch-hole patterns and to describe their learning through the task. Some students knew that reflection was a significant part of this task, and that mental formation of patterns was important in producing an accurate representation of the resultant pattern.

Table 36
Keywords used by students to describe resulting punch-hole patterns

<table>
<thead>
<tr>
<th>General conclusion for punch-hole patterns</th>
<th>% of responses</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>S1A</td>
</tr>
<tr>
<td>reflection</td>
<td>17.6</td>
</tr>
<tr>
<td>pattern</td>
<td>0</td>
</tr>
<tr>
<td>mirror image</td>
<td>5.9</td>
</tr>
<tr>
<td>symmetrical</td>
<td>11.8</td>
</tr>
<tr>
<td>visualise</td>
<td>0</td>
</tr>
<tr>
<td>imagining</td>
<td>11.8</td>
</tr>
<tr>
<td>thinking</td>
<td>5.9</td>
</tr>
<tr>
<td>prediction</td>
<td>5.9</td>
</tr>
</tbody>
</table>

Narratives

The resulting narratives (see Figure 33) illustrate the mathematical discourse used by students to describe mental patterns:

S1A.Daniel: If a hole is on a side then it will stay on a side, same with holes on corners and near the centre
S1A.John: its all opposites / X2 / Reflections
S1A.Tara: that if you fold a shape and put a hole, there will be more holes in other spots after you unfold
S1A.Mia: The dots are multiplied by 4 and mirrored in both x and y
S1B.Joseph: that you need to multiply the dots for each fold
S1B.Andrew: Two holes punched close together will still be close on a folded piece of paper when unfolded
S1B.Sally: The holes will always reach from one end of the shape to the other
S1B.Hannah: Reflection occurs when paper is unfolded

Figure 33. Statements defining punch-hole patterns.

Students were also asked what they had learned by completing the task (see Figure 34):

S1A.Mia: [I learned]...the dots are multiplied by 4 and mirrored in both x and y
S1B.Joseph: [Difficulties]...visualising the shapes unfolded
S1B.Brian: [Difficulties]...thinking about the pattern in my head
S1B.Kathy: [Difficulties]...trying to imagine the paper being folded
S2A.Khaled: [I learned]...how to visualise things

Figure 34. Statements describing student learning about mental pattern formation.

Of the students in S1A, 52.9% had difficulties in drawing a conclusion based on the number of folds – that for each fold, the number of punched holes doubled. Another
35.3% provided no response at all. Most students in S1B gave a concluding comment with varying levels of sophistication. S1B. Joseph’s concluding comment ‘that you need to multiply the dots for each fold’ was close to an accurate statement about patterns. No student in S2A could construct a conclusion based on the number of folds. This, coupled with the students’ use of visual mediation, reinforced the observations made previously about some students’ inability to visualise, predict and thereby describe patterns.

**Summary**

Students were very closely monitored ensuring that no student folded paper to verify their prediction or to assist with visualising the problem (although one student in S2A asked for permission to fold the paper but this was not granted. This indicated a problem-solving strategy that would be effective in solving the problem but would have provided little insights into the student’s ability to make a prediction based on visualisation).

Students in all groups discussed this unfamiliar task with each other and were observed to use hand movements to communicate aspects of reflection. Students in S1A rated the task as ‘very difficult’ stating that they had learned nothing. Students in S1B responded differently indicating that they had learned about reflection, patterns and symmetry. They produced several inaccurate responses but were able to interpret the task as requiring thinking, visualising, imagining, and possibly indicating the use of visualisation or mental imagery in solving these types of problems.

Generally, students in S2A were unable to reflect on what they may have learned. The majority rated the task as ‘easy’ despite several inaccurate responses. This illustrated that this class were unable to interpret a combination of written and visual instructions and formulate responses about their thinking. This also showed that student reflection on learning was not a routine practice for this class.

**4.3.4 Summary of Supplementary Tasks**

Supplementary tasks completed in small groups were selected by the researcher in order to encourage students to produce diagrams as well as descriptions to determine how well students were able to communicate geometric concepts. The descriptions often required students to produce inclusive definitions using keywords that formulated an endorsed narrative. Definitions supplemented by visual mediators provided insights into students’ understanding of classes of shapes.
The analysis indicated significant differences between the ways students in different classes communicated geometric concepts. For S1A, most students that completed task items indicated that they held prototypical views of shapes. This was evidenced by both diagrams and written narratives. Rectangles were deemed to have a pair of two long sides and two short sides by the majority of students. In contrast, students in S1B were able to identify the necessary condition of four right angles as being the determining property for inclusively describing and defining squares and rectangles.

Only two students in S1A could accurately define rectangles and parallelograms in terms of parallelism. Most used side lengths as the defining property, whereas for most students in S1B, parallelism was used as the defining property. However, misconceptions existed across the two classes with several students suggesting that parallelograms were tilted or slated shapes. This probably originated from holding a prototypical view of shapes (Oberdorf & Taylor-Cox, 1999).

Often, there was a mismatch between the visual mediators used to represent different quadrilaterals and differences in the narratives provided. No student in S1A was able to produce a kite which matched the instruction of drawing a quadrilateral with two pairs of equal adjacent sides and one pair of equal opposite angles. This indicated that many students did not understand the meaning of opposite or adjacent.

Students’ lack of understanding of geometric concepts and inability to readily solve geometric problems was observed in School 2. Students in S2A demonstrated limited knowledge about shape properties evidenced by the frequent use of keywords that were characteristic of younger children, such as corners, vertices, and faces. They had difficulties constructing narratives for definitions, as well as the formulation of reflective comments about their own learning. However, their geometric knowledge did progress mainly as a result of teaching episodes. Students completed more parts of tasks and attempted to complete diagrams using signifiers as the study progressed. They were more readily engaged with tasks at later phases of the study due to the interventions of the teacher mainly through the teaching episodes.

As the analysis indicated, there was a range of difficulties that students experienced using combinations of concepts to interpret a meaning and to produce suitable diagrams or valid definitions. This was more pronounced in School 2 where some students replicated misconceptions communicated by other students and their teacher.
4.4 Routines

*Routines* are structural regularities and well-defined repetitive patterns in student and teacher actions, characteristic of a given discourse (Sfard, 2007). Structural regularities include the structure of a mathematics lesson, as well as the classroom social interaction patterns that form the rituals of mathematics (Cobb, Wood & Yackel, 1991).

Considering the structural regularities of the two classes in School 1, commonalities were that students entered each classroom and sat wherever they chose, often with friends. They took out their notebooks – if they had brought them. They faced the whiteboard and waited for the teacher’s instructions. These routines were automatic. The differences between the two classes at School 1 were that that students in S1A either engaged in loud conversation or played games on their notebooks without direct intervention from the teacher who began her instructions without calling students to attention. Students in S1B, on the other hand, engaged in a discussion about their work or were completing homework problems before the teacher began her instruction.

The structural regularities of the class in School 2 were obvious and predictable. From the beginning of each session, students lined up at the door and waited to be invited into the classroom. Students knew their place to sit as they entered the classroom from a pre-arranged seating plan, and opened their notebooks and immediately, individually began working on an exercise from a computer-based lesson program. This was the established routine for mathematics classes in this particular classroom. As students completed work, they kept written records in their workbooks. The teacher monitored student work as she roamed around the classroom and assisted individual students.

The patterns of discourse routines can be analysed using the four groups of heuristic strategies of *visualising*, *monitoring*, *asking questions*, and *logical strategy* as proposed by Bjuland (2007). The use of visual representations, evidence of monitoring their solution processes by looking back and making adjustments to narratives and visual mediators, or correcting aspects of their work, and attempts to build logical arguments (such as using *if-then structures*) were analysed across five tasks. *Questioning* between students required group work norms to be established. As students completed tasks, mostly individually, examination of students *asking questions* of each other was not possible.

Table 37 summarises the heuristic strategies of *visualising*, *monitoring*, and *logical strategy* exhibited by students. This indicates the percentages of students that used
visual mediators correctly throughout each of the tasks. In the case of *What is a Square?* where no diagram was called for, 70% of students in S1B automatically used a diagram as part of their discourse, compared with 42% and 35% of students in S1A and S2A respectively. For the other four tasks, the majority of students in S1B used visual mediators as required by the task questions. This was not the case for the other two classes, possibly indicating their difficulties with using visual mediators to communicate various levels of understanding of geometric concepts, and perhaps also indicating that the level of difficulty of each subsequent task required knowledge of more advanced concepts like parallel lines, and opposite and adjacent sides.

Table 37

<table>
<thead>
<tr>
<th>Task</th>
<th>Class</th>
<th>% of responses</th>
<th>Monitoring</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>What is a Square?</em></td>
<td>S1A</td>
<td>42.8</td>
<td>23.8</td>
</tr>
<tr>
<td></td>
<td>S1B</td>
<td>70.0</td>
<td>35</td>
</tr>
<tr>
<td></td>
<td>S2A</td>
<td>35.0</td>
<td>45</td>
</tr>
<tr>
<td><em>Squares and Rectangles</em></td>
<td>S1A</td>
<td>100</td>
<td>28.6</td>
</tr>
<tr>
<td></td>
<td>S1B</td>
<td>100</td>
<td>42.1</td>
</tr>
<tr>
<td></td>
<td>S2A</td>
<td>94.1</td>
<td>29.4</td>
</tr>
<tr>
<td><em>Rectangles and Parallelograms</em></td>
<td>S1A</td>
<td>81.8</td>
<td>31.8</td>
</tr>
<tr>
<td></td>
<td>S1B</td>
<td>100</td>
<td>47.4</td>
</tr>
<tr>
<td></td>
<td>S2A</td>
<td>88.9</td>
<td>11.1</td>
</tr>
<tr>
<td><em>Properties of Quadrilaterals (1)</em></td>
<td>S1A</td>
<td>64.7</td>
<td>82.4</td>
</tr>
<tr>
<td></td>
<td>S1B</td>
<td>87.4</td>
<td>62.5</td>
</tr>
<tr>
<td></td>
<td>S2A</td>
<td>38.9</td>
<td>33.3</td>
</tr>
<tr>
<td><em>Properties of Quadrilaterals (2)</em></td>
<td>S1A</td>
<td>35.3</td>
<td>58.8</td>
</tr>
<tr>
<td></td>
<td>S1B</td>
<td>93.7</td>
<td>52.4</td>
</tr>
<tr>
<td></td>
<td>S2A</td>
<td>0</td>
<td>9.5</td>
</tr>
</tbody>
</table>

Evidence of monitoring strategies by alterations to diagrams or corrections of narratives did not indicate any particular pattern. In general terms, some students from all classes revisited their work and made alterations to their responses.

The alterations to student responses were made to either the visual mediators they initially recorded or to the way they used narratives to describe geometric concepts or to deduce definition-type statements. Table 38 provides a sample of the modifications to visual mediators used by students when drawing shapes. Modifications to visual mediators indicated that these adjustments led to more accurate versions of shapes, except S1B.Seth who was trying to draw a tessellation, and S2A.Ian’s depictions of squares indicated confusion with cubes.

Table 39 shows the modifications to the narratives used by students when describing geometric concepts. These modifications indicated either refinements to the initial narrative provided by students, or evidence of self- or peer-correction.
Table 38
Sample of modification to visual mediators

<table>
<thead>
<tr>
<th>Task</th>
<th>Modification to visual mediators</th>
</tr>
</thead>
<tbody>
<tr>
<td>What is a Square?</td>
<td></td>
</tr>
<tr>
<td>S1B.Seth:</td>
<td></td>
</tr>
<tr>
<td>S2A.Ian:</td>
<td></td>
</tr>
<tr>
<td>S1A.Anthony</td>
<td></td>
</tr>
<tr>
<td>S1B.Angus</td>
<td></td>
</tr>
<tr>
<td>S2A.Fadi</td>
<td></td>
</tr>
<tr>
<td>Squares and Rectangles</td>
<td></td>
</tr>
<tr>
<td>S1A.Nancy</td>
<td></td>
</tr>
<tr>
<td>S1B.Eric:</td>
<td></td>
</tr>
<tr>
<td>S2A.Stephanie:</td>
<td></td>
</tr>
<tr>
<td>Rectangles and Parallelograms</td>
<td></td>
</tr>
<tr>
<td>S1A.Ava:</td>
<td></td>
</tr>
<tr>
<td>S1B.Kathy:</td>
<td></td>
</tr>
<tr>
<td>S2A.Fadi:</td>
<td></td>
</tr>
<tr>
<td>Properties of Quadrilaterals (1)</td>
<td></td>
</tr>
<tr>
<td>S1A.Daniel:</td>
<td></td>
</tr>
<tr>
<td>S1B.Harriot:</td>
<td></td>
</tr>
<tr>
<td>Properties of Quadrilaterals (2)</td>
<td></td>
</tr>
<tr>
<td>S1A.Albert:</td>
<td></td>
</tr>
</tbody>
</table>
Table 39
Sample of modification to narratives

<table>
<thead>
<tr>
<th>Task</th>
<th>Modification to narratives</th>
</tr>
</thead>
<tbody>
<tr>
<td>What is a Square?</td>
<td></td>
</tr>
<tr>
<td>S1A.Molly:</td>
<td></td>
</tr>
<tr>
<td>S1B.Enrico:</td>
<td></td>
</tr>
<tr>
<td>S2A.Fadi:</td>
<td></td>
</tr>
<tr>
<td>Squares and Rectangles</td>
<td></td>
</tr>
<tr>
<td>S1A.Nancy:</td>
<td></td>
</tr>
<tr>
<td>S1B.Hannah:</td>
<td></td>
</tr>
<tr>
<td>S2A.Fadi:</td>
<td></td>
</tr>
<tr>
<td>Rectangles and Parallelograms</td>
<td></td>
</tr>
<tr>
<td>S1A.Mia:</td>
<td></td>
</tr>
<tr>
<td>S1B.Barry:</td>
<td></td>
</tr>
</tbody>
</table>

Logical statements are another strategy of discourse routines that might be indicators of geometric reasoning (see Figure 35):

Task: What is a Square?
S1B.Enrico: a square is a rectangle but a rectangle isn’t a square

Task: Squares and Rectangles
S1A.Callum: have at least 1 right angle
S1B.Stacey: that are parallelograms with right angles

Rectangles and Parallelograms
S1A.Ava: have two sets of parallel lines/equal sides
S1B.Angus: have diagonals that will meet in the middle
S2A.Sharee: a square is also a parallelogram

Properties of Quadrilaterals (1)
S1B.Andrew:

Figure 35. Samples of logical strategies.
The ways students used narratives that led to logical arguments or definitions were limited. All the logical statements provided by students indicated an ordering of properties that were consistent with van Hiele level 3: abstraction where “logical implications and class inclusions are understood” (Mason, 1998, p. 4). S1B. Andrew’s modified narrative indicated an attention to precision because his modifications allowed him to draw a better conclusion about quadrilaterals with three right angles. This showed that Andrew was able to provide a logical argument based on the angle sum.

The use of strategies of visualising, monitoring, and logical strategy proposed by Bjuland (2007) was a useful framework for analysing actions relating to discourse routines. However, taken together, these strategies were underutilised across the three classes, indicating that these discourse-specific, heuristic strategies had not been established in these classes as well-defined patterns that constituted routines.

Further analysis of group work across four of the five supplementary tasks was conducted by tracking one group of students’ responses for each class. Table 40 shows the use of discourse by one sample group in S1A.

Table 40
Use of discourse by sample group in S1A

<table>
<thead>
<tr>
<th>Group</th>
<th>Task</th>
<th>Visual Mediators</th>
<th>Keywords</th>
<th>Narratives</th>
<th>Misconceptions</th>
<th>Difficulties</th>
<th>Evaluation Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1A</td>
<td>Square and Rectangles</td>
<td>drew the same basic shapes and shaded them</td>
<td>same use of ‘even’ sides and 90° angles</td>
<td>no definition provided</td>
<td>same incomplete responses for a definition, not specifying four 90° angles</td>
<td>same ‘easy’ rating of task, no reflective comments provided</td>
<td></td>
</tr>
<tr>
<td>Judith</td>
<td>Rectangles and Parallelograms</td>
<td>different diagrams provided by Judith and Nancy only</td>
<td>Judith: ‘parallel lines’ Nancy: ‘even sides’</td>
<td>Nancy: ‘two equal sides’ No responses from other students</td>
<td>consistent use of visual mediators with signifiers, formulation of complete descriptions</td>
<td>variations in rating of task, no reflective comments provided</td>
<td></td>
</tr>
<tr>
<td>Nancy</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Henrietta</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Properties of Quadrilaterals (1)</td>
<td>identical shapes with right angles signifiers</td>
<td>‘four sides’ and ‘four angles’</td>
<td>no definition provided</td>
<td>consistent use of visual mediators with signifiers, formulation of complete descriptions</td>
<td>same ‘medium’ rating of task, same comment of learning ‘nothing’</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Properties of Quadrilaterals (2)</td>
<td>same basic trapezium without signifiers</td>
<td>(not required) (not required)</td>
<td>consistent use of visual mediators with signifiers</td>
<td>consistent use of visual mediators with signifiers</td>
<td>same ‘too hard’ rating of task, Henrietta: ‘I don’t know what adjacent sides are’</td>
<td></td>
</tr>
</tbody>
</table>
Samples of student work in S1A indicated that some mirroring of responses occurred in three tasks where students provided similar shapes and repeated keywords. Evaluations of the tasks reflected that students had worked together in order to provide similar responses.

Examination of responses to *Rectangles and Parallelograms* indicated that students had worked individually using different diagrams for a parallelogram, different keywords of ‘parallel lines’ and ‘even sides’ by two of the students, and variations in how they evaluated the task. Nancy indicated that she understood the concepts of rectangles and parallelograms by providing diagrams with the correct use of signifiers for right angles and opposite equal sides instead of a narrative using keywords (see Figure 36). For this task, students worked as separate entities because, for example, her group did not share Nancy’s response.

![Figure 36. Nancy’s use of visual mediators for a rectangle and parallelogram.](image_url)

The same tracking of one group of students’ responses was conducted in S1B (Table 41).

**Table 41**

*Use of discourse by sample group in S1B*

<table>
<thead>
<tr>
<th>Group</th>
<th>Task</th>
<th>Visual Mediators</th>
<th>Keywords</th>
<th>Narratives</th>
<th>Misconceptions Difficulties</th>
<th>Evaluation Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><em>Square and Rectangles</em></td>
<td>drew accurate shapes with same use of signifiers</td>
<td>same accurate use of ‘interior angles’</td>
<td>same accurate definitions based on 4 right angles</td>
<td>over-use of signifiers ie. used both equal side measure and parallel line signifiers</td>
<td>same rating of task as very easy, range of reflective comments</td>
</tr>
<tr>
<td>S1B</td>
<td><em>Rectangles and Parallelograms</em></td>
<td>different shapes with signifiers</td>
<td>same word list but in different orders</td>
<td>variations – ‘at least two pairs’, ‘two or more pairs’</td>
<td>over-use of signifiers ie. used both equal side measure and parallel line signifiers</td>
<td>variations in rating of task and reflective comments provided</td>
</tr>
<tr>
<td>Louie</td>
<td><em>Properties of Quadrilaterals (1)</em></td>
<td>similar shapes with signifiers</td>
<td>same use of ‘polygon’ ‘straight’ ‘interior angles’</td>
<td>same correct conclusion based on angle sum</td>
<td><em>(none recorded)</em></td>
<td>one evaluation indicting ‘easy’ and learning based on angle sum</td>
</tr>
<tr>
<td>Enrico</td>
<td><em>Properties of Quadrilaterals (2)</em></td>
<td>same revised shape with signifiers</td>
<td><em>(not required)</em></td>
<td><em>(not required)</em></td>
<td><em>(none recorded)</em></td>
<td>medium rating and ‘different shapes… same properties’</td>
</tr>
</tbody>
</table>
Samples of student work in this group indicated that some mirroring of responses occurred in three tasks where students provided similar shapes and repeated keywords, suggesting that students had assisted each other and validated their responses as a group. This is substantiated by similar reflective comments and task evaluations.

Examination of responses to *Rectangles and Parallelograms* indicated that students had worked individually using different diagrams for a parallelogram, different orders of keywords, variations in narratives for how rectangles and parallelograms are similar – ‘at least’ and ‘two or more’ – parallel lines, and variations in how each student evaluated the task. Louie’s depiction of a parallelogram (see Figure 37) indicated that he understood that condition of opposite pairs of parallel sides, but his use of a non-prototypical parallelogram with right angles suggested that he also understood that a square belongs to the class of parallelograms indicating van Hiele level 3: *abstraction* where class inclusions are understood (Mason, 1998). The rest of his group provided prototypical versions of parallelograms with correct parallel side signifiers.

![Figure 37. Louie’s depiction of a parallelogram.](image)

The same tracking of one group of students’ responses was conducted in S2A (Table 42).

<table>
<thead>
<tr>
<th>S2A</th>
<th>Task</th>
<th>Visual Mediators</th>
<th>Keywords</th>
<th>Narratives</th>
<th>Misconceptions</th>
<th>Difficulties</th>
<th>Evaluation Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stephanie</td>
<td>Square and Rectangles</td>
<td>drew the same basic shapes</td>
<td>same use of ‘lines’ and ‘corners’</td>
<td>definitions included ‘sides’, ‘lines’ and ‘corners’</td>
<td>longer and shorter sides for rectangles</td>
<td>same rating as very easy. No reflective comments.</td>
<td></td>
</tr>
<tr>
<td>Abdul</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Raymond</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Rectangles and Parallelograms</td>
<td>Neat and accurate diagrams with mediators</td>
<td>same use of ‘lines’ and ‘corners’</td>
<td>Correct definition based of pairs of parallel lines</td>
<td>(none recorded)</td>
<td>same rating as very easy. Stated recalling facts as a difficulty.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Properties of Quadrilaterals (1)</td>
<td>Used rulers. Squares or rectangles with signifiers</td>
<td>same use ‘corners’ and ‘closed shape’</td>
<td>(none recorded)</td>
<td>Only square and rectangle were provided with number of right angles are required</td>
<td>variations in rating of task and reflective comments provided</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Properties of Quadrilaterals (2)</td>
<td>Incomplete diagrams</td>
<td>(not required)</td>
<td>(not required)</td>
<td>(none recorded)</td>
<td>(none recorded)</td>
<td></td>
</tr>
</tbody>
</table>

Table 42  
*Use of discourse by sample group in S2A*
Samples of student work in this group indicated that mirroring of responses occurred in all tasks where students provided similar shapes and repeated keywords. Evaluations of the tasks reflected that students had worked together in order to provide similar responses.

Examination of the use of visual mediators, confirmed by researcher’s observations, indicated that rulers were used to draw shapes (see Figure 38). This attention to neatness and accuracy suggested that students had recalled instructions from earlier teaching episodes where their teacher had emphasised the need for neatness and accuracy when modelling the use of a straight edge when drawing diagrams on the whiteboard during teaching episodes.

Figure 38. Raymond’s diagram of a quadrilateral with three angles.

An examination of how groups used visual mediators, keywords, and narratives provided further evidence of discourse practices. Similar, correct responses were an indication that collaboration occurred since responses were validated within groups. Similar, incorrect responses were an indication that copying had occurred. To the extent that students mirrored each other’s responses to task items, indicted by repeating the same misconceptions, and evaluating tasks as being ‘easy’ despite obvious difficulties, suggested that students had worked in groups in order to provide answers but did not indicate evidence of collaborative group practices.

The assembly of students in the same proximity to work on problems does not constitute group work where students were engaged in substantive conversations (Grootenboer, 2009; Yackel, Cobb & Wood, 1991). It may lead to students learning new concepts from more knowledgeable students, but it may also result in the consolidation of misconceptions in the absence of teacher intervention. The teacher has a guiding role in group work through questioning and assisting students to clarify their explanations (Good, Reys, Grouws & Mulryan, 1989; Hiebert, 2000; Walmsley & Muniz, 2003; Yackel, Cobb & Wood, 1991). The teacher’s role in coordinating group work to ensure high-level student engagement and achievement requires communicating expectations of individual accountability (Turner, Meyer, Midgley & Patrick, 2003; Walmsley & Muniz, 2003).
4.5 Chapter Summary

In summary, Sfard’s (2008) interpretive framework for mathematical discourse (see Chapter 3, p. 81) was used to analyse students’ use of visual mediators, keywords and narratives to solve geometric tasks. Initially in most cases, students described shapes, such as squares, without diagrams. Visual images were not understood to be a powerful aspect of the discourse for communicating knowledge of geometric concepts. Many students did not understand the role of diagrams to communicate visual information. However, students in all classes progressed in their usage of visual mediators from basic shapes to formal representations, including signifiers for right angles, equal sides and parallel lines.

Secondly, the use of keywords to build accurate narratives also progressed over the series of tasks beginning with very imprecise terminology such as equal sides for even sides, and vertices, corners, edges, faces as used in the preliminary tasks. However, students were often unable to connect visual mediators and keywords to produce accurate arguments or provide convincing responses to several tasks items. This indicated that they were thinking at van Hiele level 2: analysis, listing all the properties the students knew but not discerning which properties were necessary and which were sufficient to describe the shape.

The analysis of data from both schools established several common themes in building geometric knowledge. Where students had a basic knowledge of geometric concepts, demonstrated through their use of keywords and accurate visual mediators, they were able to apply their knowledge to develop sound geometric arguments. Where students did not have a sufficient knowledge of geometric concepts, and in the absence of teacher intervention, little progress was made through the tasks to reach a stage of mathematically acceptable reasoning as substantiated through conventional mathematical discourse. Finally, where students did not have a sufficient knowledge of geometric concepts but did have appropriate teacher intervention, they were able to progress through the tasks. This progress was, in the case of this study, hampered by misconceptions disseminated by both students and the teacher of one of the participating classes.

The effectiveness of group work in building knowledge and developing shared geometric understandings produced mixed results. It highlighted a pedagogical need for teachers to engender group work skills in their students as geometric thinking
necessitates the communication of ideas in constructing endorsed narratives of new knowledge from familiar geometric concepts.

These findings have answered questions about the methodological approach required to produce effective outcomes from the tasks designs and their implementation – namely that subsequent phases require classes to be organised for effective group work and that the role of the teacher is fundamental in driving learning forward. The classroom culture needs to support teaching and learning of geometry where student thinking is sustained and challenged. Specifically, as the analysis has shown, students needed to understand the conditions and expectations of group work. As the analysis of discourse routines using heuristic strategies of visualising, monitoring, and logical strategy proposed by Bjuland (2007) also indicated, students needed to know these specific strategies to improve their opportunities to benefit from collaborative practices by learning what it is to think and communicate mathematically with each other.

A discussion and concluding statement of the main themes of the research questions of this study are discussed in Chapter 5. The discussion synthesises the interpretations of the data made, and explains the meaning of the findings, evaluates its importance, and outlines implications for the factors influencing students’ geometric thinking.
Chapter 5

Discussion and Conclusion

5.1 Introduction

If, as Goldenberg, Cuoco and Mark (1998) suggested, geometry “lends itself to making rich connections to the rest of mathematics” (p. 23), it is essential that students are successful in a geometry curriculum. Ongoing challenges in mathematics education include a lack of emphasis on geometry as a significant branch of mathematics (Hohenwarter & Jones, 2007; Jones, 2000; 2002; Jones & Mooney, 2003). The current Visualisation and spatial reasoning are also deficient in the current Australian curriculum (Seah, 2015b). The findings from this study identify the factors that influenced students’ geometric reasoning skills when learning about 2D shapes.

In this chapter, a discussion of the key findings will be presented in relation to the themes of the research questions, by drawing on the literature to explain and evaluate the results, and by identifying the implications for further study. First, the role of visualisation in defining and classifying shapes is discussed. Definitions and classifications, described as a transition from SOLO multi-structural to relational responses (Biggs & Collis, 1982), or van Hiele level 2: analysis through to level 4: deduction (Mason, 1998), necessitates the use of visualisation in order to communicate geometric reasoning with 2D shapes (Clements, 1982; Clements & Battista, 1992; Jones, 2002; Kosslyn, 1983; Lohman, 1996; Maier, 1996). Secondly, an evaluation of the use of language in developing an understanding of geometric concepts and communicating geometric reasoning is provided. Thirdly, an explanation is given as to the nature of the components of instructional practice necessary for the teaching and learning of geometry in modern classrooms. Fourthly, the main question of this study that explored the factors that influence the development of geometric reasoning among middle years students when learning about 2-dimensional (2D) shapes, is addressed.

Finally, key elements of the study are summarised in the concluding sections of this chapter. Limitations and recommendations of the study are presented, and future research options are considered and suggested in order to advance the findings.
5.2 The Role of Visualisation

The first sub-question examined in this study was:

- *How does visualisation underpin the teaching and learning of definitions and classifications of 2D shapes?*

Visualisation involves the ability to generate and manipulate images (Kosslyn, 1983) and is a necessary component for all levels of geometric reasoning (Jones, 2002). Visualising shapes and their spatial relations is an important part of geometric reasoning (Clements, 1982; Clements & Battista, 1992; Lohman, 1996; Maier, 1996; Ramful, Lowrie & Logan, 2016; Yeh, 2013). Defining and classifying shapes are considered to be important in geometric reasoning processes (Gutiérrez & Jaime, 1998), and are key stages of student thinking in the van Hiele model of geometric thought.

In this study, students and teachers used visualisation skills to interpret images, represent shapes, and communicate their reasoning through gestural, verbal and written means. They described and defined shapes based on known and derived facts. The different types of visual mediators used by students provided a strong indication of their ability to participate in a common mathematical discourse. Teaching episodes used visualisation to introduce new concepts to students and reinforce essential elements of geometric communication, that is, highlighting the significance of geometric diagrams coupled with the accurate use of geometric terminology to develop reasoning processes.

Gestures play a specific role in making concepts apparent (Sabena, Radford, & Bardini, 2005). The use of gestures to build an understanding of new concepts were evident during teaching episodes in School 2, where the teacher explained the meaning of *angles, right angles, parallel lines,* and *opposite and adjacent sides* using hand-eye coordinated movements (Rahim & Olsen, 1998). The teacher also encouraged students to show the meaning of concepts such as vertical and horizontal symmetry using hand gestures. Newly developed concepts involved discourse elements of keywords linked to gestures, and mediated by diagrams of 2D shapes on the whiteboard.

Students’ use of visualisation in *What is a Square?* was largely confined to either images that were drawn without signifiers or with no representation of the shape at all. This indicated *uni-structural or pre-structural* responses (Biggs & Tang, 2007). Students failed to understand that images were part of a geometric description that organises geometric data into meaningful structures (Bishop, 1988; Fischbein, 1987). Diagrams of shapes did not match students’ use of keywords such as *right angles, same*
length and parallel. Some students invented their own personal signifiers if they did not know how to use mathematical signifiers to indicate properties on their diagrams. In *What is a Square?* where students were not asked specifically to draw a shape, 65% of students in S1B drew a square with right-angle signifiers, compared to only 9.5% of students in S1A, and 0% of students in S2A. When asked to draw a square in *Squares and Rectangles*, 100% of students in S1B drew a diagram with right-angle signifiers compared with only 28.6% of students in S1A and 5.9% of students in S2A. Students’ preference for describing shapes using keywords without visual mediators indicated a lack of understanding the formal nature or purpose of geometric diagrams.

The students who did not provide images used similar keywords to those students who did provide images. This suggested that students had learned these concepts by rote (Clements & Battista, 1992; Mayberry, 1983) or may have used an internalised semiotic representation (Larkin & Simon, 1987) – picture-in-the-mind (Arcavi, 2003; Clements, 1982; Presmeg, 1986) – to facilitate their descriptions of a square. Students who constructed these shapes ‘in intuition’ (Heis, 2014) may have known how to represent them as a diagram complete with signifiers for right angles and equal side measures but assumed diagrams were not permissible in relation to the task. The effective use of internalised images indicated a visual reasoning ability based on visual perception rather than on deduction (Panaoura & Gagatsis, 2010) but limited a student’s ability to communicate their thinking. Students had difficulties working with *figural concepts* (Fischbein, 1993) by linking the figure to the concept. Nonetheless, it is conjectured in this thesis that for the basic van Hiele level of recognition where properties of figures are identified (Gutiérrez & Jaime, 1998), diagrams were not necessary for many students to generate the required keywords of *equal sides* and *right angles* used to define a square.

The effective use of images to communicate reasoning with shapes necessitated the use of a diagram and an understanding of geometric conventions (Diezman, 1994). Reasoning with diagrams at higher levels than basic recognition required spatial visualisation and orientation skills in order to disembled properties of shapes (Owens, 2003). In this study, students tended to draw prototypical diagrams of shapes such as vertical-horizontal examples of squares and rectangles. Reliance on visual prototypes restricted students’ reasoning to listing known properties (Özerem, 2012). Dreyfus, (1991) found that while students might generate visual images, they were reluctant to use them for analytical reasoning.
The use of prototype images for 2D shapes was common across all tasks for all classes and explained many students’ difficulties with definitions and classifications of 2D shapes (Presmeg, 2006). Prototypical examples of shapes suggested they had memorised definitions and, therefore, had limited their development of hierarchical classifications (Battista, 2001; Chinnappan & Lawson, 2005; Clements, 2004; Cunningham & Roberts, 2010; Tall & Vinner, 1981; Turnuklu, Gundogdu Alayli & Akkas, 2013). In general, students in this study who produced accurate diagrams with signifiers, whether prototypical or not, performed tasks at a higher level than those who did not produce diagrams. Students who knew more facts tended to over-signify their diagrams with the superfluous use of signifiers, such as indicating four right angles and pairs of parallel sides for rectangles. Where this occurred, lengthy, cumbersome and uneconomical definitions of shapes also followed from over-signified diagrams and extensive lists of shape properties.

In order to be successful in the Paper-folding task, students required visualisation to perform imagined movements (Clements, 1999) of folding or unfolding paper to interpret and predict a resulting spatial pattern formation. This task was an example of how pattern imagery (Presmeg, 1986) required students to use visualisation to anticipate patterns of configurations of punch-holes, in much the same way as some expert chess players can remember configurations of chess pieces. Students in S1B were more successful with this task than the other two classes indicating an attention to precision in recorded responses as well as individual strategies to aid spatial reasoning (such as drawing folding lines to assist with imagined reflections inherent in the task). Students in School 2 had difficulties with producing accurate patterns and describing these mental pattern formations through visualisation processes.

5.2.1 Implications

There are several critical implications for task designs and teaching interventions concerning the learning of visualisation for defining and classifying 2D shapes.

Geometric tasks designs

Tasks allowed students to make choices about how they would communicate their level of reasoning about shapes when trying to define and classify them. Initially, students did not perceive the significance of using visual images to describe properties of a square. Preference for prototypical examples restricted students from reasoning at higher levels. However, in successive tasks where students were asked to draw
diagrams, depictions of shapes showed an improved response to using signifiers by students (see Table 27, Chapter 4, p. 144) – although weaker students in School 2 continued to depict shapes without signifiers despite the teacher’s modelling of diagrams with signifiers during teaching episodes. Students were unable to progress in their communication of mathematical conceptual knowledge that involved visual mediators, thus indicating a uni-structural level response (Biggs, 1979) at most. It could be conjectured that students’ prior experiences of reasoning with 2D shapes were restricted to basic recognition activities. The correct listing of geometric facts without the use of mathematically acceptable visual mediators suggests memorisation might characterise prior learning experiences.

Classification as exhibited through logical argument (Gutiérrez & Jaime, 1998) was rarely observed in this study. Students who compared diagrams demonstrated the extent that visualisation played in classifying shapes. It could be conjectured that students learned some links to a hierarchical classification by means of memory and recitation. The process of proving, however, requires argumentation in order to convince others (Bjuland, 2007; Dreyfus, 1991; Gutiérrez & Jaime, 1998; Schoenfeld, 1992a). There was insufficient evidence in this study to conjecture about students’ ability to use proving processes. There is evidence, however, to suggest that task designs need to focus more on students depicting several versions of a shape in order to isolate the necessary properties to define it. Tasks also need to involve students using definitions to generate geometric concepts.

Further implications on future task designs need to address the paucity of non-prototypical images being used to complete geometric reasoning tasks. Students need experiences in seeing shapes being transformed dynamically. This implies that paper and pen geometric tasks need to be supplemented by the use of dynamic geometry software where objects and relations can be visualised and physically manipulated. Such learning environments offer the possibility for students to “construct knowledge in action and not only by having recourse to language” (Laborde, 2002, p. 15).

Teacher interventions:
The analysis of teaching approaches conducted by one teacher at School 2 indicated a limited understanding of pedagogical approaches to the teaching of visualisation to help students define and classify shapes. The impact of the teaching episodes improved some students’ ability to represent shapes using signifiers. However, the teacher’s own
preference for prototypical examples, coupled with a preference for partitional
definitions, restricted significant improvements in her students’ reasoning ability. Often
teaching episodes involving visualisation, either through diagrams on the whiteboard or
through the use of gestures to describe concepts, resulted in a consolidation of
misconceptions about shapes, for example, as ‘having horizontal and vertical lines’.
This then impacted on students’ abilities to reason with 2D shapes. Instruction needs to
incorporate non-prototypical examples and counter-examples to build geometric
reasoning skills through classroom questioning and dialogue.

5.3 The Role of Language

The second sub-question examined in this study was:

- How does language support the development of geometric reasoning?

Language, in its multiple written and verbal forms, is of critical importance to the
learning of geometric concepts. It provides the means by which teachers can assess
levels of student thinking by the keywords they use to describe geometric concepts.
Progression in geometric reasoning is facilitated by appropriate instruction (Battista,
2001) and involves assigning keywords to visual mediators and developing narratives to
define shapes. The framework for mathematical discourse developed by Sfard (2008)
defined four interrelated elements of keywords, visual mediators, narratives and
routines. This was the primary analytical tool of students’ mathematical discourse used
in this study. Analysis of students’ use of each of these elements revealed certain
aspects of students’ geometric reasoning.

The tasks in this study required students to use keywords to construct narratives
to list, describe or define shapes or geometric patterns. Economical descriptions are
critical in developing hierarchical definitions of shapes (de Villiers, 1994) because they
attend to common properties of shapes and are, therefore, inclusive. Describing shapes
economically helps students to focus on the properties that define a class of shapes and
indicates a higher level of geometric reasoning (Gutiérrez & Jaime, 1998).

Students in this study showed a clear tendency to develop exhaustive lists of
properties rather than being economical with keywords in describing shapes. This
resulted in students providing long-winded statements in the development of
definitions. For example, S1B.Joseph defined a quadrilateral as ‘a shape with 4 straight,
connected lines & 4 interior angles that add up to 360°’. He also defined parallelograms
and rectangles as having ‘2 pairs of equal and parallel sides, have 4 angles that add up
to $360^\circ$ and at least 2 equal angles’ demonstrating that Joseph may have known more facts about 2D shapes but his attempts to define them led to cumbersome narratives. It would have been sufficient to define both parallelograms and rectangles as quadrilaterals with 2 pairs of parallel sides.

In *Quadrilateral by Properties (1)* students ‘pruned’ their descriptions of quadrilaterals to two essential keywords of *sides* and *angles*, whereas a range of unnecessary keywords such as *vertices, edges, faces, lines* and *points* were included in the preliminary tasks. This did not lead to any student in S1A being able to correctly define the shape, in comparison to 94% of students in S1B, and 22% of students in S2A. These differences could be accounted for when considering the effect of the homogeneous structure of classes in School 1 and teacher instruction in School 2. As an accelerated learning class, students in S1B engaged in tasks more readily and reasoned with 2D shapes at a higher level. They worked individually and exchanged information with each other. In the case of School 2, teaching episodes drew attention to keyword usage and clearly improved the precision of student lists of shape properties, resulting in some being able to develop an endorsed narrative for a quadrilateral.

A discussion about the use of language by students cannot be separated from their use of visual mediators, as both are a necessary part of the mathematical discourse in geometry. Language and visual imagery are both internal and external operations (Austin & Howson, 1979; Larkin & Simon, 1987). How students used internalised images of shapes to generate written lists of properties is unknown. Rote learning of concepts can simulate higher levels of geometric thinking (Pegg, 1992). It is likely that rote learning had played a role in these tasks because the level of mismatching between keywords and visual mediators was palpable. Clements (2003) pointed out that in the absence of thoughtful instruction, students could learn geometric concepts by rote. Mayberry (1983) also found that pre-service teachers used similar rote-learning strategies when reasoning about 2D shapes. For students with higher levels of thinking, there was a clear preference to use a combination of keywords and diagrams to describe shapes as defined by van Hiele Level 3: *abstraction* (Mason, 1998) or a SOLO relational response (Jurdak, 1991).

The unique locutions in some tasks, such as *only* and *any* in *Quadrilaterals by Properties (1)*, and *adjacent* and *opposite* in *Quadrilaterals by Properties (2)*, generated a degree of confusion among students. Students did not know that the mathematical use of *any* implied the general and not the specific case, so when asked to draw *any*
quadrilateral, at least 70% of students in each class produced prototypical squares and rectangles without signifiers. Students commented about the absence of the word only in the task item that asked them to ‘draw a quadrilateral with three right angles’. In some cases, this resulted in them drawing the fourth right angle and quoting the angle sum for quadrilaterals as a justification, but in many other cases they simply drew a rectangle indicating three right angles. The majority of students in S1A did not understand the meaning of adjacent and opposite and did not provide a suitable diagram. This revealed a lack of understanding of how particular terms were used in geometry, a lack of compensatory strategies available to students, and a lack of teacher intervention during group tasks.

In School 2, this prompted the teacher to define these terms using everyday examples before discussing their use in relation to quadrilaterals. While etymology was not specifically under examination, this emerged as an additional aspect of geometric language exhibited on occasions by students and teachers, and is worthy of discussion. S2A.David’s unsolicited use of the word quadrangle for quadrilateral pointed to at least one instance where a student recognised particular attributes of a shape and made connections to its 2D name as a concept – the concept definition where words specify the concept (Fujita & Jones, 2006). This occurred after several teaching episodes where the teacher had drawn attention to the meaning of specific keywords such as parallel, diagonal, and symmetry as an important part of learning about 2D concepts.

Language is a key determinant of the levels of geometric thinking exhibited by students (Hershkowitz, Ben-Chaim, Hoyles, Lappan, Mitchelmore & Vinner, 1990; van Hiele, 1985), and the application of Sfard’s framework to the use of students’ mathematical discourse revealed student thinking levels during each of the tasks. While the questions of this study did not focus on the assessment of individual student levels of geometric thinking based on the analysis of mathematical discourse, it can be argued that the levels of thinking of most students were largely confined to van Hiele level 2: analysis. This ‘confinement’ to level 2 for most students could be attributed to a lack of appropriate teacher intervention during activities, a consolidation of misconceptions about 2D shapes, and reliance on rote-learned concepts. Rarely students in S1B exhibited thinking at a range of levels including the higher level of abstraction. In the case of S1B.Andrew, his inclusion of ‘two pairs of cointerior angles which add up to 180°’ for a parallelogram, his correctly drawn visual mediators for all required shapes throughout the study, his identification of spatial patterns of punch-holes, and his use of
routine strategies to self-correct his own narrative statements, were strong indications that his level of geometric reasoning was at a higher level than his cohorts, and demonstrated an ability to apply geometric thinking to define shapes inclusively.

The findings from this study generally support the view that geometric reasoning is more dynamic rather than static, and that progression is best described as continuous rather than at specific stages as implied by the van Hiele model (Burger & Shaughnessy, 1986; Gutiérrez, Jaime & Fortuny, 1991; Usiskin, 1982).

5.3.1 Implications

These findings have implications on tasks designs and appropriate teacher interventions on how language such can support the development of geometric reasoning.

Geometric tasks designs

Students’ use of geometric language revealed that when tasks call for descriptions of shapes, students were able to use prior knowledge of 2D shapes to generate a list of attributes rather than recalling a concept definition. This aligns with Burger and Shaughnessy’s (1986) observations of secondary students, and Cunningham and Roberts’ (2010) observations of pre-service teachers that provided evidence of students’ levels of thinking. Students that readily provided descriptions with diagrams indicated advanced levels of thinking through a multi-structural response (Biggs, 1979).

As tasks progressed, students reduced their lists of keywords from everything they knew about shapes to providing only words essential to defining them. The reduction of keywords revealed a progression in geometric reasoning where students began to discern the properties of shapes that were necessary or sufficient for defining them. Descriptions focusing on critical attributes help to describe some shapes differently to others (Levenson, Tirosh & Tsamir, 2011; Tall & Vinner, 1981). Students in S1B articulated sufficient information about 2D shapes in order to describe them. This level of description aligned to visual mediators with the correct use of signifiers and this indicates the levels of thinking for some students at van Hiele Level 3: abstraction, where they perceive relationships between properties and between figures (Mason, 1998).

In this study, Sfard’s elements of mathematical discourse successfully depicted a range of students’ geometric thinking. The use of the framework also indicated several barriers to student progression as exhibited through diagnosed misconceptions and difficulties when learning about 2D shapes, and implying a specific instruction role.
Teacher Interventions
Progress through the van Hiele levels is dependent on the type of instruction provided to students (Crowley, 1987). The instructional phases of Inquiry, Direct Orientation, Explication, Free Orientation and Integration place language as an important aspect in the development and assessment of geometric reasoning (van Hiele, 1985). Verbal modalities are a necessary part of mathematisation (Presmeg, 2006) and denote participation in mathematical discourse (Sfard, 2008). In this study, when students in School 2 did not know the meaning of particular keywords used in the tasks, explicit teaching episodes provided an important opportunity for the teacher to model these terms and to build up a deeper level of understanding. The use of geometric terminology, coupled with the use of gestures and diagrams constructed on the whiteboard, led to discernible improvements in the quality of classroom dialogue, deeper engagement in successive tasks, and accurate use of visual mediators.

Real discussion involves a shift to a more specialised form (Cazden, 2001; Forman, McCormick & Donato, 1997; Moschkovich, 2003; Sfard et al., 1998). Teaching episodes challenged students to think, explain, and listen, and in doing so provided opportunities for students to extend their own descriptive language repertoire. In contrast, where there was an absence of teacher intervention in School 1, students showed minor improvements in their use of geometric language and overall engagement in the tasks. The observations in this study align with the view that instruction is needed to move students through increasingly sophisticated levels of geometric understanding and reasoning (Battista, 2001; Freudenthal, 1973; Pegg, 1992; van Hiele, 1985; van Hiele, 1999), and where the technical language registers (Halliday & Matthiessen, 2004) and meaning of particular locutions (Morgan, Craig, Schuette & Wagner, 2014) need to be taught.

5.4 The Teacher’s Role in Developing Geometric Concepts
The third and final sub-question examined in this study was:

• How does instruction influence the learning of geometric concepts?

The Australian Curriculum (ACARA, n.d.) not only defines the content of geometry but also the knowledge base required by teachers to be effective in teaching mathematics. “Excellent teachers of mathematics have a sound, coherent knowledge of the mathematics appropriate to the student level they teach, and which is situated in their knowledge and understanding of the broader mathematics curriculum” (AAMT, 2006).
The teachers’ pedagogical content knowledge for geometry includes a blend of curriculum content and pedagogical knowledge of useful ways to conceptualise and represent a geometric concept, and to understand why some students experience difficulties when learning a particular geometric concept (Chinnappan & Lawson, 2005). Teachers have an instructional role in knowing how difficulties may be ameliorated through quality teaching practices (Morris, Hiebert & Spitzer, 2009). Oberdorf and Taylor-Cox (1999) also specified that a teacher’s lack of pedagogical content knowledge might be the cause of many students’ misconceptions. A teacher’s content knowledge has a strong influence on students’ progression in geometric reasoning (Couto & Vale, 2014).

Students need to engage as a community of learners (Lave & Wenger, 1991; Wenger, 1998) in order to generate ideas and to verify their mathematical interpretations and understandings (Sfard, Nesher, Streefland, Cobb & Mason, 1998). Student engagement in shared practice necessitates co-construction of knowledge (Brown & Campione, 1994; Osborne, Simon, Christodoulou, Howell-Richardson & Richardson, 2013), where endorsement of emergent thinking occurs through their interactions with others (Alexander, 2005; Sfard, 2008). The teacher’s role is pivotal in developing norms of collaboration and in guiding students toward end goals (Even & Tirosh, 2002). Individual student accountability is an important factor in effective group work (Walmsley & Muniz, 2003; Yackel, Cobb & Wood, 1991), requiring the teacher to convey expectations of participation and collaboration to students (Turner, Meyer, Midgley & Patrick, 2003) in order to engage them and motivate them to learn. Knowing when to listen, when to question, and when to intervene, are necessary for promoting the learning of geometric reasoning beyond straightforward descriptions (Forman, McCormick & Donato, 1997; Gillies & Haynes, 2011; Hiebert et al., 2000).

Teachers need to identify and provide opportunities for explicit instruction in spatial thinking and geometric representations (Diezmann & Lowrie, 2012; Liben, 2006), and these opportunities were prevalent during task implementation at School 1. However, it did not motivate teachers to provide instructional interventions – a critical aspect of students being successful in reasoning and problem-solving (Gillies & Haynes, 2011). Observations of students in S1A while working on tasks confirmed a high degree of student passivity and lack of accountability. Student work samples showed incomplete or superficial responses, and in several case non-attempts, and this is indicative of the teacher’s lack of understanding of her role in guiding students during
group work activities. Students’ desire to work independently comes about because mathematics has often been presented to them in this way (Good, Reys, Grouws & Mulryan, 1989). It could be conjectured that teacher inaction led to a consolidation, rather than mitigation, of misconceptions that impacted on student performance for future tasks.

In contrast, the ways in which the teacher of School 2 used language to describe and define concepts illustrated the critical role of teachers in building ‘representational fluency’ to the learning of 2D shapes (Chinnappan & Lawson, 2005), often requiring visual mediation. As she observed her students, the teacher made notes of difficulties they experienced while working on each task but without immediate intervention. In each task, whole-class discussion to assess student progress and to summarise newly learned concepts became an integral part of her explicit teaching episodes. The teaching episodes provided opportunities for the teacher to consolidate students’ understanding of geometric concepts through explicit instruction, to increase student participation through questioning, and through conveying her expectations of work quality. Even though students may have ‘copied’ each other in the absence of group work norms being established, the teaching episode allowed for follow-up questions, modelling of solution processes, drawing attention to mathematical language, ensuring accountability of student explanations, and encouraging a variety of interpretations and responses as part of an interactive process of substantive learning (Yackel, Cobb & Wood, 1991).

Teaching episodes provided insights into instructional practices for developing geometric concepts. The explicit instruction of the teacher drew out accurate verbal definitions of geometric concepts mediated by diagrams modelled on the whiteboard by the teacher and students. Nason, Chalmers and Yeh (2012) found that open questions extended students’ descriptive language repertoire and geometric reasoning. Similarly, modelling of open questioning techniques by the teacher in School 2 challenged students to communicate their understanding of concepts, and to learn the skills of listening, reflecting, proposing, and incorporating alternative views – something that Michaels, O’Connor and Resnick (2008) refer to as ‘accountable talk’. In communicating how visual mediators are used in geometry in specific ways to transmit understanding about shape attributes, the students progressed in their use of visual mediators from basic 2D shapes without signifiers, to neat diagrams with signifiers usually constructed with the use of rulers. This shift was a result of the teacher’s use of questioning routines and challenging students to think about particular shape attributes.
and their use of keywords. This modelling shows how collaboration develops shared meanings and shared language (Battista, 2001; Cobb et al., 1991; Coffland & Strickland, 2004; Sfard, 2008) where visual mediation is taken as part of the shared mathematical discourse.

Further, communication of geometric concepts occurred through a combination of verbal, diagrammatic and gestural means. Kinesthetic and gestural forms of communication are relevant to the discourse narrative (Ferreira & Presmeg, 2004; Gersmehl & Gersmehl, 2007; Owens, 1999; Sfard, 2008). For example, the teacher of S2A used two palms and two index fingers to indicate parallel lines in Rectangles and Parallelograms, and used scissor motions with her arm to show the angle concept. Gestures are a natural part of communicative modalities (Owens, 1999; Sfard, 2008), exhibited by this teacher, modelled for her students, and exhibited by her students during subsequent whole-class discussion.

Mathematicians prefer hierarchical definitions of quadrilaterals to partitional definitions because these show the deductive relationship between shapes (de Villiers, 1987; Fujita & Jones, 2006; Turnuklu, Gundogdu Alayli & Akkas, 2013). However, when one student suggested that ‘rectangles are like long. Squares are short’ and the teacher said ‘Ok’ (see Figure 22, Chapter 4, p. 128), it could be conjectured that this ‘partitioning’ of rectangles and squares became a reinforced concept in her students’ minds, thus presenting a barrier to her students from being able to progress in their ability to reason at a higher level. Other misconceptions of the teacher were also exhibited through her teaching episodes. For example, the teacher communicated that a tilted square was a diamond and also a rhombus (see Figure 14, Chapter 4, p. 117). The teacher’s confusion with the naming of shapes was exacerbated in Parallelograms and Rectangles where her questioning and tone indicated to her students a level of uncertainty about ‘diamonds’ (see Figure 25, Chapter 4, p. 138). In another instance, the teacher linked the word triangle to other words with similar roots, such as tricycle – a standard literacy strategy. She also linked the concept of ‘tri’ to diagrams on the whiteboard, however, in each case she presented a prototypical orientation of a right angle triangle and in doing so inadvertently defined right-angle triangles as having perpendicular sides in the vertical-horizontal (prototype) position (Even & Tirosh, 2002; Gutiérrez & Jaime, 1999), potentially misleading her students.

The data revealed several student misconceptions and difficulties across all three classes. Significantly, these misconceptions presented several opportunities for teachers
to engage in whole-class dialogue as exhibited during scheduled teaching episodes in School 2. For example, in *Squares and Rectangles*, 42.9% of students in S1A stated that rectangles have two long sides and two short sides, as well as other misconceptions of the shapes such as having vertical and horizontal sides. Only 27.2% of the class based their definition of rectangles as having four right angles compared with 77.8% of students in S1A. No student in S2A could provide an inclusive definition for rectangles, and more than half of the class made no attempt. In the case of S2A.Abdul who claimed that a square was three-quarters of an A4 sheet of paper (see Figure 16, Chapter 4, p. 118), the researcher corrected this misconception through a private consultation. In a subsequent task, however, S2A.Liam used the same description for a square indicating how when left uncorrected by the teacher, misconceptions can be spread to the minds of other students.

The series of task designs used in this study were intended to promote mathematical discourse in small groups. A key indicator of student readiness to participate in collaborative learning practices was demonstrated by their ability to engage in substantive conversations (Grootenboer, 2009; Peterson, 1987). Other indicators included the quality of their questioning and sustained dialogue leading to common interactions with tasks and joint thinking about solutions (Prusak, Hershkowitz & Schwarz, 2012). Roth and Gardener (2012) claimed that persuading other classroom members was integral to developing reasoning skills.

The results of this study indicated that ‘friendship group’ arrangement of students did not automatically achieve productive group work (as was the case in School 1), nor should it be applied as a behaviour management strategy (as was the case of School 2). Most students had limited prior experiences with working in small groups in mathematics in ways that would support and challenge their thinking. This view was supported by their inability to express their learning after completing tasks. Further, individual students’ submission of blank responses, even though they were assigned to a small group and had peer support available, indicated that they had not engaged in tasks as a group, and that the norms of group work (Yackel, Cobb & Wood, 1991) and norms of classroom discourse (Nathan & Knuth, 2003) had not been fully established prior to the commencement of this study.

The series of supplementary tasks replicated the same learning approaches of individuals unable to discuss task items in a group and engage in distinct routines of mathematical discourse (Bjuland, 2007; Sfard, 2008). Students often replicated each
other’s errors by copying, rating tasks similarly, and had difficulties expressing what they had learned using accurate mathematical terminology. For example, Judith, Nancy and Henrietta from S1A drew the same basic shapes in *Squares and Rectangles*, used the same keywords, provided no definitions and did not record the common property of right angles yet rated the task as ‘easy’ (see Table 40, Chapter 4, p. 164). Students in some groups in S1B repeated the same reflective comments about each task using the same geometric terminology in their written responses thus indicating some level of collaboration may have taken place. For example, Louie, Enrico and Andrew from S1B were able to conclude that if a quadrilateral had three right angles, then the fourth must be a right angle and they quoted the angle sum of 360° as a justification. A higher register of geometric language (Duval, 1999, Presmeg, 2006; Sfard, 2008) was available to these students in S1B in order to construct their logical arguments.

In general, mathematical discussion was not an automatic condition for learning in this study across all three classes. Questioning routines of a nature that provoked thoughtful answers and further questions (Mercer & Sams, 2006) were underutilised by students working in groups. While students were willing to ask questions of their teachers, these were specific to understanding what types of answers were possible, such as ‘can we draw a diagram?’ rather than questions resulting in promoting group or whole class dialogue about geometric concepts.

The social culture of the classroom (Hiebert et al, 2000) defines the role of the teacher in managing the shared space where collaboration is coordinated (Roschelle & Teasley, 1995; Yackel, Cobb & Wood, 1991). If, as Even and Tirosh (2002) suggested, teachers represent the community of practice, then in this study, the development of desired and shared norms of group work, conducive to the mathematical discourse by teachers were not apparent. Their role, while students were assembled in groups, was somewhat confused and limited to watching from a distance. Jones and Herbst (2012) identified poor instructional choices made by teachers to engage in dialogue, placed responsibility for reasoning on their students, and prevented higher levels of reasoning. This pattern has also been observed in this study.

These classroom observations and analyses point to a problem of teaching practice where clear protocols of participation and collaboration as part of group work were not observed nor established despite teachers having self-reported that group work was a frequent teaching strategy. This is further supported by the lack of communication of each teacher’s expectations to their students at the beginning and
during each task, as well as the teacher’s acceptance of many incomplete responses to task items, and the untidy and almost indecipherable use of diagrams and written text by students. Lack of individual student accountability can be addressed when teachers take on an active role (Doyle, 1983), guiding students to discuss content and deal with alternative views – an important part of mathematical cognition (Slavin, 1996).

5.4.1 Implications

These findings have implications for the instructional choices that influence the learning of geometric concepts through the implementation of tasks. The teacher’s instructional role is critical in knowing what and how to teach geometric concepts to higher levels of student thinking and problem-solving.

*Groupworthy* tasks (Boaler & Staples, 2008) allow for the development of new mathematical concepts and rely on the collective resources of a group (Horn, 2005). The teacher needs to make instructional choices that align tasks and students’ prior knowledge in order for them to engage in high levels of cognitive activity (Doyle, 1983; Henningsen & Stein, 1997). This requires the proactive and consistent support of students’ cognitive activity without reducing the complexity and cognitive demand of the tasks (Brousseau & Gibel, 2005).

This study highlights the importance of teacher interventions as students work on geometric tasks. The observations and data analyses raise important issues in terms of group task implementation requirements. Teaching episodes are a necessary component for the learning of geometric concepts, where the teacher explains new concepts, models the use of diagrams, clarifies expectations of student written responses, presents challenging problems, and asks questions. To be effective, teachers need to communicate their expectations of student learning behaviours and work quality clearly from the beginning of each task. The teacher also needed to play an active role during group work activities beyond monitoring students. This involves asking students to explain concepts to each other and challenging students to think about other students’ reasoning. These actions lead to improvements in student engagement and work quality (Michaels, O’Connor & Resnick, 2008). Task implementation also requires the appropriate allocation of students to particular groups. Whether naturally or from prior group learning experiences elsewhere, limited student collaboration was observed with students in S1B.
Beyond task implementation considerations, teachers also need to make choices about their instructional role in presenting concepts (Cunningham & Roberts, 2010; Jones & Herbst, 2012). Teachers can transmit information to students through explicit instruction, or they can be facilitators and teach concepts through problem-solving approaches (Ernest, 1989). The particular teaching orientations of either *transmission* or *connectionist* (Askew, Brown, Rhodes, Johnson & Wiliam, 1997) are influenced by teachers’ pedagogical beliefs (Ertmer, 2005). Teaching episodes in School 2 clearly indicated the influence of the teacher in impacting positively on the learning of new concepts, but also the negative impact in regard to the reinforcement of misconceptions. As tasks phases, supplemented by explicit teaching episodes continued, students demonstrated improvement in their use of visual mediators and keywords. Teachers who are more closely involved in the selection of tasks could have strengthened this aspect of the study. Where teachers were more passive, as in School 1, leaving the learning to the assembly of students in groups, led to limited engagement by many students, and this was one of the consequences of teacher inaction (Doyle, 1993).

Explicit instruction, as a bridge between what is known and what needs to be known, was a feature of the findings in School 2. This showed that this type of intervention was contingent on teachers having a strong pedagogical content knowledge. The quality of intervention provided by a teacher is dependent on their conceptual understanding to teach and promote reasoning in their students. One of the implications of this study is that all students require some level of teacher intervention in order to progress in their geometric thinking. This involves questioning and challenging students to think at a higher level. As the results of this study indicated, even accelerated learning classes who may have achieved higher mathematics scores and may have learning behaviours conducive to collaborative learning, still require teacher interventions in order to progress in their geometric thinking.

5.5 The Development of Geometric Reasoning: Middle Years Students’ Understanding of 2-dimensional Shapes

The main question examined in this study was:

- *What factors influence the development of geometric reasoning among middle years students when learning about 2-dimensional (2D) shapes?*
The sub-questions in this study concerned the role of visualisation, language and instruction in developing geometric reasoning. Reasoning, an important mathematical proficiency (ACARA, n.d.) requires actions of listening and reflecting on someone else’s thinking (Michaels, O’Connor & Resnick, 2008). This two-way process is important for geometric concept formulation (Ekanayake, Brown & Chinnappan, 2003; Gray, Pinto, Pitta & Tall, 1999; Heis, 2014).

Geometric reasoning is developed through written and verbal exchanges (Forman, McCormick & Donato, 1997; Moschkovich, 2003). Sfard’s (2008) framework for mathematical discourse also places visual mediators as “taken-as-shared” ways of reasoning, arguing, and symbolising geometric ideas. The tasks used in this study reflected the elements of task designs identified in the literature for effective learning of visualisation and geometric reasoning (Baker & Talley, 1972; Chinnappan & Lawson, 2005; Cooke, 2007; Fox, 2000; Levenson, Tirosh & Tsamir, 2011). Defining and classifying 2D shapes corresponds to the graduation from van Hiele level 2, where students describe what they ‘see’, to level 3 where students base descriptions of shapes on reasoning about the properties associated with shapes, and then toward level 4 where deduction involves understanding the role of definitions (Mason, 1998). However, in order to be effective, students need a supportive learning framework for new conceptual constructions to become consolidated.

The data revealed that when students used visualisation to produce diagrams they were more successful at formulating their descriptions and definitions of 2D shapes than students who did not produce diagrams. Students needed to know or be shown how to use mathematical signifiers to accurately describe and define shapes. This was largely hampered by a preference for partitional definitions by students. Some students in the accelerated group were successful at producing inclusive definitions, and therefore, were able to classify shapes when they produced diagrams with signifiers. Where students used accurate diagrams for 2D shapes, they were able to reason about the critical attributes that helped define them. In this study, visualisation was critical for describing shapes and moving students to higher levels of reasoning involved in defining them.

The analysis of language through Sfard’s (2008) interpretive framework allowed an interpretation of students’ geometric conceptual knowledge and reasoning abilities. Written language, in the forms of keywords and narratives, cannot be separated from visual mediators when communicating geometric concepts. This study found that the
tendency to list as many properties about a given shape was common among students. As suggested by de Villiers (1998b), students needed to be directed toward making descriptions as short as possible in order to describe the shape economically.

Implications of the teacher’s role are limited to the teacher’s actions in one classroom only. Interventions through teaching episodes resulted in students’ learning of new geometric concepts and ways of communication. Explicit instruction was necessary for the teacher to convey her expectations and access student levels of understanding of concepts. This study found that the consequences of teacher inaction consolidated misconceptions and lead to poor performances by many students.

5.6 Limitations of the Study
Several methodological considerations placed limitations on the quality of evidence generated by this study and the quality of student learning experiences. This study involved an examination of students’ geometric reasoning in three coeducational classrooms across two schools located in the inner, northern suburbs of Melbourne, Victoria, Australia. With such a small sample, it is difficult to draw generalisations about geometric thinking of students in Years 7 and 8. Further, the inclusion of only suburban public schools may have raised issues different to rural and independent schools. The analyses did not look for differences between male and female students and this may have provided further insights into the ways boys and girls learn geometry. Moreover, as this study involved tasks conducted at different year levels and at different stages of the study meant that only two iterations of the DBR process had occurred. Further iterations may have produced more refined tasks that engendered more collaboration among students and suitable for the range of ability levels across Years 7 and 8.

Teachers’ self-reporting of group work as a typical feature of their mathematics lessons should have been examined before the commencement of the study to establish their understanding of group and classroom norms conducive to the learning of geometry and the teacher’s instructional role within the classroom. The lack of clarity of the teachers’ role was a barrier to understanding student learning of new geometric concepts. The lack of group work in all classes in this study impinged on the quality of student learning experiences. Also, the willingness of teachers allowing students to submit blank responses was an obstacle to student learning. Observed student passivity
and lack of accountability by the teachers in this study supported Doyle’s (1993) claims about the pitfalls of teacher inaction and lack of group work norms.

Explicit instruction and teaching episodes was analysed in one school only. In order to fully capture how teachers build student capacity to develop geometric concepts, explicit instruction should have been explored in both schools. The success of meaningful teaching episodes was hampered by the teacher’s own misconceptions and this created significant barriers to student learning, and demonstrated a lack of sufficient geometric content knowledge of the teacher (Clements, 2004; Nathan & Knuth, 2003).

Student misconceptions were manifest throughout this study. The absence of teacher interventions resulted in some misconceptions becoming reinforced and consolidated with other students. For example, when a teacher did not correct student descriptions of rectangles as having long sides and short sides, this reinforced the prototypical image of rectangles in students, indicating the teacher’s preference for partitional definitions, and thus created barriers for student to reason correctly about hierarchical classifications.

*Routines*, an element of Sfard’s (2008) discourse framework, were difficult to analyse due to the lack of group work norms being established in the first instance. This resulted in limited student dialogue during tasks and, therefore, limited data being generated to provide conclusive evidence in this regard. Routines are repetitive patterns characteristic of the given discourse. Specifically, mathematical routines can be noticed whether one is observing the use of mathematical words and mediators, or following the process of creating and substantiating narratives about geometrical shapes. These repetitive patterns of discourse behaviour characterised as *visualising, monitoring, asking questions*, and *logical strategies* (Bjuland, 2007) can only be observed when students actively engage in mathematical tasks.

### 5.7 Recommendations

As a result of this study, the following recommendations addressing the teaching and learning of geometric concepts in middle years classrooms should encapsulate the following:

- Preliminary tasks involving other shapes, such as triangles, would allow a teacher to discuss key aspects of the use of visual mediators, especially how signifiers convey mathematical properties of particular shapes. The use of keywords for describing shape properties and the development of definitions
need to be modelled for students. Preliminary tasks provide an opportunity to develop questioning routines with students and this promotes mathematical discourse and mathematical argument. In order to deal with student passivity and blank responses, the teacher’s expectations of work quality needs to be conveyed from the outset to the students. These include neatness and accuracy with diagrams, as well as students checking over their work. The use of heuristic strategies of visualising, monitoring, questioning and logical statements, as defined by Bjuland (2007), need to be developed within the context of mathematical discourse routines to provide more detailed information about students’ geometric thinking.

- Teachers need to play a vital role in identifying the source of students’ difficulties (Morris, Hiebert & Spitzer, 2009). One such difficulty is the preference for prototypical examples of shapes to be used by students and teachers, as observed in this study. It is fundamental that prototypes are considered and presented as one form of depiction of a particular shape. Other versions of diagrams, such as changed orientation and counter-examples need to be demonstrated in order to build geometric reasoning.

- Modelling of open questioning and problem-solving through group work activities encourages the use of verbal descriptions with visual forms of reasoning (such as gestures). Through focussed attention to the meaning and derivations of keywords, it will help students to develop effective ways of communicating their thinking. Etymological and mathematical terminology considerations can play an intrinsic role in developing geometric concepts (Usiskin, Griffin, Witonsky & Willmore, 2008).

- A summary or whole class discussion at the end of each task would provide further opportunities to assess student levels of understanding as well as promote individual student accountability. Group work and classroom norms need to be constantly evaluated, as does how the teacher facilitates dialogue between students.

- Students need to be aware that listing properties of shapes requires those properties that are necessary and sufficient to describe them and that they need to focus on critical attributes. Describing a shape is not the same as defining the shape. Definitions need to be understood and taught as a powerful linguistic tool
that is used to include or exclude particular shapes, and it is the economical use of keywords that eventually become substantiated as a definition. Communication of descriptions and definitions of geometric concepts necessitates visual mediation.

The role of visualisation is a significant factor that can influence the development of spatial and geometric reasoning skills. Other factors borne out of this study include the role of language and the role of the teacher in facilitating learning opportunities for students. Nevertheless, some improvements are necessary to build on these findings and these implications. The following recommendations signify the directions for further research in order to refine the findings from this study and to improve the implementation of geometric tasks in middle years schooling.

Several issues regarding the preparation of students and the role of the teacher in facilitating and maximising the outcomes of group work were raised. In order to capture key aspects of student collaboration, a culture of dialogue and questioning in the classroom is required. As student learning is socially constructed, future research needs to examine aspects of spatial and geometric reasoning by students working in collaborative groups. Whole group responses to tasks, including recorded dialogue within groups over extended periods, will provide deeper insights into how students are able to construct knowledge with each other.

This study did not shed any light on the retention of geometric knowledge over time. While claims of student thinking and learning progression were made, how much information could be retained and applied to future problems remains unanswered. A longitudinal study could be conducted to examine retention of knowledge gained by students through task designs, and providing another avenue of future research where, in the case of an accelerated learning program, whole cohorts of students remain as a class for several school years. This would allow for tasks to be administered to the same class periodically to track learning progression.

In this study, questions were raised concerning the participant teachers’ pedagogical knowledge and content knowledge of geometry. It is understood that the pedagogical content knowledge of teachers is a vital ingredient in student learning. Teacher expertise in teaching mathematics strongly relates to student achievement (Anstey & Clarke, 2010; Hill, Ball & Schilling, 2008; Kazemi & Franke, 2004). Further research is needed to examine the effect of teaching episodes to facilitate student
thinking beyond van Hiele *visualisation* and *analysis* levels, and which do not consolidate pre-existing misconceptions that students and their teachers hold in geometry.

Gender differences in learning geometric concepts are not new. Newcombe and Huttenlocher (2006) stated that these differences could be detected by the age of 4 years. Liben and Christensen (2011) indicated that both biological and experiential factors contribute to gender differences in spatial development. Kozhevnikov, Kosslyn and Shephard (2005) reported that men tend to use dynamic imagery whereas women use static imagery when solving spatial orientation and mental rotation tasks. Conversely, Halat (2008) found that there were no statistically significant differences between male and female mathematics teachers for geometric thinking. The difference between the ways males and females apply visualisation to solve problems and communicate their thinking needs closer examination. Further evidence is needed into the differences between boys’ and girls’ ability to learn and apply geometric concepts to construct mathematical narratives of logical argument.

### 5.8 Concluding Statement

The collection of tasks used in this study successfully elucidated a range of levels of geometric reasoning exhibited by students. These tasks required students to construct mental objects from known properties of shapes, allowing students to demonstrate levels of geometric reasoning, consistent with Gray, Pinto, Pitta and Tall’s (1999) approach to developing tasks for advanced geometric thinking.

In several ways, the results of this study were expected. It is well documented that geometry has been a discounted part of the mathematics curriculum for many decades with emphasis given to number and algebra. This has resulted in teachers not having the necessary content knowledge and pedagogical skills to teach geometry beyond basic levels and relying on rote-type teaching methods. It is also well known that students have strong personal conceptions of shapes having a preference for prototypes. They also find geometry to be a disengaging part of mathematics with no clear purpose. This manifested itself in this study with inaccurate definitions for shapes and an inability to use diagrams to communicate what is very precise geometric information about shapes. Surprisingly, some results revealed the creativity of students to name a quadrilateral as a *quadrangle* for instance, and to describe a square as *three-quarters of an A4 sheet of paper*. The poor quality and range of misconceptions with 2D
shapes by students in the accelerated learning class also surprised the researcher and suggested that the graduation of students into this class may have been as a result of other subject considerations than their geometrical mathematical abilities.

An important factor of design-based research is that teachers are shaped by the design process (Roth & Gardner, 2012). Ongoing improvements to task designs in geometry and their strategic implementation supplemented by good teaching practices and consolidated group work approaches can lead to improved levels of students’ geometric reasoning. A classroom culture that challenges students to be collaborators with their peers is an essential pedagogical approach for supporting good teaching and learning practices in mathematics classrooms.
List of References


de Villiers, M. (1998a). An alternative approach to proof in dynamic geometry. In L.R. Lehrer & D. Chazan (Eds.), Designing learning environments for developing understanding of geometry and space (pp. 369-394).


Van Klinken, E. (2010). Utilising Year Three NAPLAN Results to Improve Queensland Teachers' Mathematical Pedagogical Content Knowledge (pp. 297-304). In L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education: Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia*. Fremantle: MERGA.


Appendix 1

Task: Quirps

Please write your responses in the space provided below. Answers should be:
- neatly drawn and labelled appropriately
- address all sections of the task

The following are quirps.

The following are not quirps.

Which of the following are quirps?

a. Draw some quirps.

b. What is a quirp?

Student Reflection:
1. Describe what you learned by completing this task.

2. Describe the difficulties you may have encountered.

3. Please rate this task.

<table>
<thead>
<tr>
<th>Very difficult</th>
<th>Quite difficult</th>
<th>Medium level</th>
<th>Quite easy</th>
<th>Very easy</th>
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## Scoring Rubric

### Part a

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<th>Comment</th>
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<tr>
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<td>No shapes drawn.</td>
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</tr>
<tr>
<td>1</td>
<td>One shape without a right angle.</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Two or more shapes without right angles.</td>
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### Part b

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<th>Indicator</th>
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<td>No definition.</td>
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</tr>
<tr>
<td>1</td>
<td>(Sufficient definitions)</td>
<td>A quirp is a shape without right angles.</td>
</tr>
<tr>
<td>2</td>
<td>(Advanced definitions)</td>
<td>A quirp is a polygon without any right angles.</td>
</tr>
</tbody>
</table>

**Comments: Were there any further properties or explanations provided?**

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Appendix 2

Task: What do you know about a square? Name:______________________

Please write your responses in the space provided below.

a. List what you know about a square.

Student Reflection:
1. Describe what you learned by completing this task.

2. Describe the difficulties you may have encountered.

3. Please rate this task.

<table>
<thead>
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<th>Quite difficult</th>
<th>Medium level</th>
<th>Quite easy</th>
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Scoring Rubric

### Definitional properties

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<tr>
<td>1</td>
<td>Four straight sides.</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>It is a quadrilateral.</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Four right angles.</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>It is planar.</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>It is a closed.</td>
<td></td>
</tr>
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### Transformation

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<td>Lines of symmetry</td>
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<td>1</td>
<td>Rotational symmetry</td>
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<tr>
<td>1</td>
<td>Tessellation</td>
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### Formal property-based reasoning

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<td>Parallel lines.</td>
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</tr>
<tr>
<td>1</td>
<td>Perpendicular lines</td>
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<tr>
<td>1</td>
<td>Diagonals.</td>
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### Other

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<th>Indicator</th>
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<tr>
<td></td>
<td>3D. eg. each face of a cube.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Connect to other things. eg. two squares can</td>
<td></td>
</tr>
<tr>
<td></td>
<td>make a rectangle</td>
<td></td>
</tr>
<tr>
<td></td>
<td>In situ. eg. My book is a square.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Non-sequitur (false conclusion). eg. A rectangle</td>
<td></td>
</tr>
<tr>
<td></td>
<td>is a square because it has 4 sides.</td>
<td></td>
</tr>
</tbody>
</table>

Comments: Were there any further properties or explanations provided?

____________________________________________________________________
____________________________________________________________________
____________________________________________________________________
____________________________________________________________________
____________________________________________________________________
____________________________________________________________________
____________________________________________________________________
____________________________________________________________________

230
Appendix 3

Task: Properties of Squares and Rectangles

Name: ____________________________

Please write your responses in the space provided below. Answers should be:
- neatly drawn and labelled appropriately
- address all sections of the task

a. Draw a square.

b. List all the properties of a square.

c. Draw a rectangle.

d. List all the properties of a rectangle.

e. What properties are common?

f. Can you write one definition using one of the properties above that will include all squares and rectangles and no other quadrilateral?

“Only squares and rectangles are quadrilaterals...”

Student Reflection:
1. Describe what you learned by completing this task.

2. Describe the difficulties you may have encountered.

3. Please rate this task.

<table>
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<th>Very difficult</th>
<th>Quite difficult</th>
<th>Medium level</th>
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231
## Scoring Rubric

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<td>No diagram.</td>
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<tr>
<td>1</td>
<td>The basic shape is shown.</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>The diagram contains markings showing equal lengths only. The diagram contains markings showing right angles only.</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>The diagram contains markings showing equal lengths and right angles.</td>
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### Part b

<table>
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<th>Score</th>
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<tbody>
<tr>
<td>0</td>
<td>No properties included.</td>
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</tr>
<tr>
<td>1</td>
<td>Only equal sides mentioned.</td>
<td>Only equal angles mentioned. Angle sum is 360 degrees.</td>
</tr>
<tr>
<td>2</td>
<td>Both equal sides and angles mentioned.</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Additional properties correctly identified. eg. Diagonals, symmetry or parallelism.</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Advanced properties correctly included. eg. Diagonals bisect each other at right angles, correct use of formula for perimeter and/or area.</td>
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### Part c

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<tr>
<td>1</td>
<td>The basic shape is shown.</td>
<td></td>
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<tr>
<td>2</td>
<td>The diagram contains markings showing equal lengths only. The diagram contains markings showing right angles only.</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>The diagram contains markings showing equal lengths and right angles.</td>
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### Part d

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<tr>
<td>1</td>
<td>Only equal sides mentioned.</td>
<td>Only equal angles mentioned. Angle sum is 360 degrees.</td>
</tr>
<tr>
<td>2</td>
<td>Both equal sides and angles mentioned.</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Additional properties correctly identified. eg. Diagonals, symmetry or parallelism.</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Advanced properties correctly included. eg. Diagonals bisect each other at right angles, correct use of formula for perimeter and/or area.</td>
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</tbody>
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### Part e

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<td>Both have four sides (quadrilaterals). Right angles mentioned. Angle sum is 360 degrees.</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Opposite sides being equal also mentioned.</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Additional properties correctly identified. eg. vertical and horizontal symmetry or parallel opposite sides.</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Advanced properties correctly included. eg. Diagonals bisect each other, common rule for perimeter and/or area.</td>
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<tr>
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<tr>
<td>0</td>
<td>No common definition provided.</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>(Incomplete or incorrect response) Only squares and rectangles are quadrilaterals that have four sides. Only squares and rectangles are quadrilaterals that have opposite sides equal and/or parallel.</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(Sufficient definition) Only squares and rectangles are quadrilaterals with four right angles.</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(Additional information) Only squares and rectangles are quadrilaterals with four right angles. Their diagonals bisect each other and are of equal length.</td>
<td></td>
</tr>
</tbody>
</table>

**Comments: Were there any further properties or explanations provided?**

______________________________________________________________________________
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Appendix 4

Task: Properties of Parallelograms and Rectangles

Please write your responses in the space provided below. Answers should be:
• neatly drawn and labelled appropriately
• address all sections of the task

a. Draw a parallelogram.

b. List all the properties of a parallelogram.

c. In what ways is a parallelogram similar to a rectangle?

d. Can you write one definition using one of the properties above that will include all parallelograms and rectangles?

“All parallelograms and rectangles...”

Student Reflection:
1. Describe what you learned by completing this task.

2. Describe the difficulties you may have encountered.

3. Please rate this task.

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<tr>
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234
### Scoring Rubric

#### Part a

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<td>No diagram.</td>
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</tr>
<tr>
<td>1</td>
<td>The basic shape is shown.</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>The diagram contains markings showing equal lengths only.</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>The diagram contains markings showing parallelism only. The diagram contains markings showing both equal lengths and parallelism.</td>
<td></td>
</tr>
</tbody>
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#### Part b

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<td></td>
</tr>
<tr>
<td>1</td>
<td>Only equal opposite sides mentioned. Angle sum is 360 degrees.</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Both equal opposite sides and equal opposite angles mentioned. Both equal opposite sides and two pairs of opposite sides parallel.</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Additional properties correctly identified. eg. non-symmetrical.</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Advanced properties correctly included. eg. Diagonals bisect each other, correct use of formula for perimeter and/or area.</td>
<td></td>
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#### Part c

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<td>Common properties not included or trivial.</td>
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<tr>
<td>1</td>
<td>Both have four sides (quadrilaterals). Angle sum is 360 degrees.</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Opposite sides being equal also mentioned.</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Additional properties correctly identified. eg. parallel opposite sides, equal opposite angles</td>
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</tr>
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<td>4</td>
<td>Advanced properties correctly included. eg. Diagonals bisect each other, common rule for perimeter and/or area.</td>
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#### Part d

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<td>No common definition provided. Parallelograms and rectangles have two long sides and two short sides.</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>(Incomplete or incorrect response) Parallelograms and are quadrilaterals that have four sides. Parallelograms and rectangles have two long sides and two short sides.</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(Sufficient definitions) Parallelograms and rectangles are quadrilaterals with two pairs of opposite sides parallel. Parallelograms and rectangles are quadrilaterals with two pairs of opposite angles equal.</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(Additional information) Parallelograms and rectangles have any two adjacent angles supplementary. Their diagonals bisect each other.</td>
<td></td>
</tr>
</tbody>
</table>

**Comments:** Were there any further properties or explanations provided?
Appendix 5

Task: Quadrilaterals by Properties (1)  Name:___________ Group:_______

Please write your responses in the space provided below. Answers should be:
• neatly drawn and labelled appropriately
• address all sections of the task

a. What is a quadrilateral?

b. Draw any quadrilateral.

c. Draw a quadrilateral with one right angle.

d. Draw a quadrilateral with two right angles.

e. Draw a quadrilateral with three right angles.

Now go back and make sure that the example at each stage is not an example at the previous stage.

f. What can you conclude about a quadrilateral with three right angles?

Student Reflection:
1. Describe what you learned by completing this task.

2. Describe the difficulties you may have encountered.

3. Please rate this task.

<table>
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<th>Quite difficult</th>
<th>Medium level</th>
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<td>-------</td>
<td>-----------</td>
<td>---------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>No definition.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>A quadrilateral is a shape with 4 sides. A quadrilateral is a shape with 4 angles.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(Sufficient definitions) A quadrilateral is a closed shape with 4 straight sides. A quadrilateral is a closed shape with 4 angles. Angle sum is 360 degrees.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(Advanced definitions) A quadrilateral is a planar (2D) shape with straight sides. A quadrilateral is a planar (2D) shape with straight sides joining 4 points such that no 3 points are collinear.</td>
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<tr>
<td>2</td>
<td>The diagram contains 4 intersecting straight lines with or without angles marked.</td>
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<td>The basic shape is shown.</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>The diagram contains 4 intersecting straight lines with more than one right angle marked.</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>The diagram contains 4 intersecting straight lines with exactly one right angle marked.</td>
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<td>2</td>
<td>The diagram contains 4 intersecting straight lines with more than two right angles marked. eg. a rectangle</td>
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<tr>
<td>3</td>
<td>The diagram contains 4 intersecting straight lines with exactly two right angles marked. eg. a trapezium</td>
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<td>The basic shape is shown.</td>
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<td>2</td>
<td>The diagram contains 4 intersecting straight lines with four right angles marked. eg. a rectangle</td>
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<td>No conclusion provided or incorrect response.</td>
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<tr>
<td>1</td>
<td>It is not possible to draw a quadrilateral with exactly 3 right angles.</td>
<td></td>
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<tr>
<td>2</td>
<td>Any quadrilateral with three right angles must have a fourth right angle as the angle sum must be 360 degrees. eg. a rectangle</td>
<td></td>
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Comments: Were there any further properties or explanations provided?
Appendix 6

Task: Quadrilaterals by Properties (2) Name: ___________________ Group: _______

Please write your responses in the space provided below. Answers should be:
• neatly drawn and labelled appropriately
• address all sections of the task

a. Draw a quadrilateral with one pair of equal sides.

b. Draw a quadrilateral with one pair of adjacent sides equal and one pair of opposite angles equal.

c. Draw a quadrilateral with two pair of adjacent sides equal and one pair of opposite angles equal.

Now go back and make sure that the example at each stage is not an example at the previous stage.

Student Reflection:
1. Describe what you learned by completing this task.

2. Describe the difficulties you may have encountered.

3. Please rate this task.

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238
### Scoring Rubric

**Name:** __________________  **Group:** ________

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<td>The basic shape is shown.</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>The diagram contains 4 intersecting straight lines with two pairs of equal sides marked. eg. a parallelogram. The diagram contains 4 intersecting straight lines with three equal sides. eg. a trapezium. The diagram contains 4 intersecting straight lines with four equal sides. eg. a rhombus.</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>The diagram contains 4 intersecting straight lines with exactly one pair of equal sides marked.</td>
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#### Part b

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<td>No diagram.</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>The basic shape is shown.</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>The diagram contains 4 intersecting straight lines with two pairs of adjacent sides equal. eg. a kite. The diagram contains 4 intersecting straight lines of equal length. eg. a rhombus.</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>The diagram contains 4 intersecting straight lines with exactly one pair of adjacent sides equal and exactly one pair of opposite angles equal.</td>
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#### Part c

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<td>The basic shape is shown.</td>
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</tr>
<tr>
<td>2</td>
<td>The diagram contains 4 intersecting straight lines with two pair of adjacent sides equal and two pair of opposite angles equal. eg. a rhombus.</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>The diagram contains 4 intersecting straight lines with two pair of adjacent sides equal and exactly one pair of opposite angles equal. eg. a kite or chevron.</td>
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**Comments:** Were there any further properties or explanations provided?

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Appendix 7

Task: Paper-folding  Name: ____________________  Group: __________

Please write your responses in the space provided below. Answers should be:
• neatly drawn and labelled appropriately
• address all sections of the task

a. A square piece of paper is folded and a hole is punched as shown. After unfolding the paper completely, which one of the arrangements of holes will appear?

   ![Diagram of paper folding](image)

   A  B  C  D  E

b. In a similar way, a square piece of paper is folded and holes punched as shown. Draw the arrangements of holes after the paper is unfolded.

   ![Diagram of paper folding](image)

c. In a similar way, a square piece of paper is folded and holes punched as shown. Draw the arrangements of holes after the paper is unfolded.

   ![Diagram of paper folding](image)

d. What general conclusion can you make from parts b and c above?

Student Reflection:
1. Describe what you learned by completing this task.

2. Describe the difficulties you may have encountered.

3. Please rate this task.

   ![Rating Scale](image)
### Scoring Rubric

#### Part a

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<td>D or E.</td>
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#### Part c

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<td>Correct response (8 dots).</td>
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#### Part d

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<tbody>
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<td>Incorrect response.</td>
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</tr>
<tr>
<td>1</td>
<td>There will be 8 holes.</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>If the paper is folded twice and two holes punched, then 8 holes will result in the final pattern.</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>If the paper is folded twice and two holes punched, then 4 times as many holes will result in the final pattern.</td>
<td></td>
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</table>

**Comments: Were there any further properties or explanations provided?**

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Appendix 8

Notice of Approval

Date: 21 October 2015
Project number: CHEAN 8 0000017502-02/14
Project title: An Investigation of Visualization in the Development of Spatial and Deductive Reasoning of 2 – Dimensional Shapes.
Risk classification: Low Risk
Investigator: Dr Tasos Barkatas and Mr Adrian Berenger
Approved: From: 21 October 2015 To: 30 November 2016

I am pleased to advise the change in project title to "An Investigation of Visualization in the Development of Spatial and Deductive Reasoning of 2 – Dimensional Shapes", change in supervisors to Dr Tasos Barkatas and Dr Rebecca Soah, recording students during the learning process and extension of ethics approval until 30 November 2016 has been granted ethics approval by the Design and Social Context College Human Ethics Advisory Network as a sub-committee of the RMIT Human Research Ethics Committee (HREC).

Terms of approval:

1. Responsibilities of investigator
   It is the responsibility of the above investigator/s to ensure that all other investigators and staff on a project are aware of the terms of approval and to ensure that the project is conducted as approved by the CHEAN. Approval is only valid whilst the investigator/s holds a position at RMIT University.

2. Amendments
   Approval must be sought from the CHEAN to amend any aspect of a project including approved documents. To apply for an amendment please use the 'Request for Amendment Form' that is available on the RMIT website. Amendments must not be implemented without first gaining approval from CHEAN.

3. Adverse events
   You should notify HREC immediately of any serious or unexpected adverse effects on participants or unforeseen events affecting the ethical acceptability of the project.

4. Participant Information and Consent Form (PICF)
   The PICF and any other material used to recruit and inform participants of the project must include the RMIT University logo. The PICF must contain a complaints clause including the project number.

5. Annual reports
   Continued approval of this project is dependent on the submission of an annual report. This form can be located online on the human research ethics web page on the RMIT website.

6. Final report
   A final report must be provided at the conclusion of the project. CHEAN must be notified if the project is discontinued before the expected date of completion.

7. Monitoring
   Projects may be subject to an audit or any other form of monitoring by HREC at any time.

8. Retention and storage of data
   The investigator is responsible for the storage and retention of original data pertaining to a project for a minimum period of five years.

In any future correspondence please quote the project number and project title.

On behalf of the DSC College Human Ethics Advisory Network I wish you well in your research.

Suzana Kowcevic
Research and Ethics Officer
College of Design and Social Context
RMIT University
Ph: 03 9925 2974
Email: suzana.kowcevic@rmit.edu.au
Website: www.rmit.edu.au/dsc
Appendix 9

INVITATION TO PARTICIPATE IN A RESEARCH PROJECT

PORTFOLIO OF SCHOOL OF
Design and Social Context
Education

Principal:

Project Title: An Investigation of Visualization in the Development of Spatial and Deductive Reasoning of 2 – Dimensional Shapes.

Name of Investigators: Adrian Berenger Phone: +61
Dr Tasos Barkatsas Phone: +61

Dear [Name],

I am seeking your permission for a research project being conducted by RMIT University. Please read this sheet carefully and be confident that you understand its contents before deciding whether to participate. If you have any questions about the project, please ask one of the investigators.

About the research
This study is part of a Master of Education at RMIT University, in the School of Education, College of Design & Social Context. It is being conducted by me, Adrian Berenger, under the supervision of Dr. Tasos Barkatsas. The research project has been approved by the RMIT Design and Social Context Human Research Ethics Advisory Network (Project Number: CHEAN-B 0000517562-02/14), and by the Department of Education and Early Childhood Development (Reference Number: 2015_002882).

The purpose of this study is to examine approaches to teaching geometry, the challenges faced by secondary school teachers in the field (culture of classrooms), and to develop tasks that promote deductive reasoning and problem-solving skills. Further, the methodological approach adopted presents design-based research (DBR) as a legitimate method for gaining insights into how students and their teacher interact with specifically chosen tasks and with each other during the teaching and learning process. This study will contribute to the area of deductive reasoning by developing task designs that are able to enhance both student knowledge and teacher knowledge through a design-based research methodology. In doing so, several misconceptions held by students and teachers will be addressed through deductive reasoning and problem solving approaches.

I believe you will be able to advance this research having knowledge of your setting and unique pedagogical approaches. I am personally aware of the work your school has done in relation to mathematics reform.

What we are asking you to do
As the Principal, I am asking you to provide your written permission for the research project to proceed during Term 1, 2016 which will involve geometric task development in mathematics classrooms with students and teachers. This will also involve teacher interviews and classroom observations at times suitable to the school. Teacher interviews will be tape-recorded to assist in developing accurate transcripts subject to their permission. The focus of this part of the research is on gaining an understanding of how students use visual and deductive reasoning in solving geometric problems.

Mathematics lesson observations of the teaching and learning process are critical to this research in order to deepen understandings of new or enhanced pedagogical practices. This will involve recording mainly through a written and video format but may involve photographs of student work on occasions. Some dialogue with students may be required to determine their understanding and interpretation of teacher instructions, as well as an understanding of how students interact with each other in solving geometric problems. Parental consent will be sought through the school. There should be minimal impact of the teaching and learning program of those involved.

Possible risks or disadvantages
There are no anticipated risks or disadvantages if you participate in this project. However, if you are unduly concerned about any matter in relation to this project, you should contact one of the researchers as soon as convenient. We will discuss your concerns with you confidentially and suggest appropriate follow-up, if necessary.

Benefits of the research
Knowledge gained from the research will inform on the types of tasks necessary to engage students in geometry promoting visual and deductive reasoning skills through problem solving approaches. The research will also define teaching and learning approaches necessary to engage students in geometric thinking.
Further information you need to know

All schools and participants will be assigned code names and will remain anonymous. Any data which could identify you or your school will not be used. The data collected during the study may be published in possible journal articles and conference presentations. A thesis will also be provided to RMIT. However, once again, all participants will remain anonymous. All data collected will remain secure on a password-protected computer and be securely stored in a locked filing cabinet. To ensure confidentiality, the data will not be made available to anyone.

Any information that you provide can be disclosed only if (1) it is to protect you or others from harm, (2) if specifically required or allowed by law, or (3) you provide the researchers with written permission. Results will be written up in a Masters thesis and I also intend to publish in academic journals and books.

Your rights as a participant

Participation in the study is voluntary and you may withdraw at any time without prejudice. In this event, any information already obtained will not be used. Any unprocessed data may also be withdrawn and you may access your data at any time. The tape recordings will not be made available to anyone and will be destroyed once the transcripts have been created and checked for accuracy. The transcripts and all other data collected will be destroyed five years after completion of the thesis by RMIT. If you are interested in the final report, you are welcome to request the executive summary which will be forwarded to you.

If you require further information about the research or your role, please contact me on mobile +61 email adrian.berenger@rmit.edu.au. You could also contact my principal supervisor, Dr Tasos Barkatsas on or email tasos.barkatsas@rmit.edu.au.

Thank you for considering this request. I do hope that you will be able to help with this important research and look forward to your participation in the study.

Yours sincerely

Adrian Berenger
BSc. DipEd

Dr Tasos Barkatsas

Security of the website

Users should be aware that the World Wide Web is an insecure public network that gives rise to the potential risk that a user’s transactions are being viewed, intercepted or modified by third parties or that data which the user downloads may contain computer viruses or other defects.

Security of the data

This project will use an external site to create, collect and analyse data collected in a survey format. The site we are using is Nynx. If you agree to participate in this survey, the responses you provide to the survey will be stored on a host server that is used by http://www.qualtrics.com. No personal information will be collected in the survey so none will be stored as data. Once we have completed our data collection and analysis, we will import the data we collect to the RMIT server where it will be stored securely for five (5) years. The data on the http://www.qualtrics.com host server will then be deleted and expunged.

If you have any concerns about your participation in this project, which you do not wish to discuss with the researchers, then you can contact the Ethics Officer, Research Integrity, Governance and Systems, RMIT University, GPO Box 2476V, VIC. 3001. Tel: (03) 9925 2231 or email human.ethics@rmit.edu.au
INVITATION TO PARTICIPATE IN A RESEARCH PROJECT

PORTFOLIO OF
Design and Social Context
SCHOOL OF
Education

Name of Participant: ____________________________

Project Title: An Investigation of Visualisation in the Development of Spatial and Deductive Reasoning of 2 – dimensional Shapes.

Name of investigators: Adrian Berenger  Phone:
Dr Tasos Barkatsas  Phone:

Dear _________,  
You are invited to participate in a research project being conducted by RMIT University. Please read this sheet carefully and be confident that you understand its contents before deciding whether to participate. If you have any questions about the project, please ask one of the investigators.

About the research  
This study is part of a Master of Education at RMIT University, in the School of Education, College of Design & Social Context. It is being conducted by me, Adrian Berenger, under the supervision of Dr Tasos Barkatsas. The research project has been approved by the RMIT Design and Social Context Human Research Ethics Advisory Network (Project Number: CHEAN B 000017502-02/14), and by the Department of Education and Early Childhood Development (Reference Number: 2015_002882).

The purpose of this study is to examine approaches to teaching geometry, the challenges faced by secondary school teachers in the field (culture of classrooms), and to develop tasks that promote deductive reasoning and problem-solving skills. Further, the methodological approach adopted presents design-based research (DBR) as a legitimate method for gaining insights into how students and their teacher interact with specifically chosen tasks and with each other during the teaching and learning process. This study will contribute to the area of deductive reasoning by developing task designs that are able to enhance both student knowledge and teacher knowledge through a design-based research methodology. In doing so, several misconceptions held by students and teachers will be addressed through deductive reasoning and problem-solving approaches.

You are being invited to participate in the study as teachers as I believe you will be able to advance this research having knowledge of your setting and unique pedagogical approaches. I am personally aware of the work your school has done in relation to task development in mathematics.

What we are asking you to do  
As a participant, you are invited to conduct an initial set of task questions with your students in Term 1, 2016 which I will correct and provide feedback, and then participate in follow-up interviews with students at times suitable to you and the school. The completion of the initial task questions will take approximately one hour for students to complete. Any subsequent interviews will be tape-recorded to assist in developing accurate transcripts. However, should you prefer interviews not to be tape-recorded, this will be arranged. The focus of this part of the research is on gaining an understanding of how students use visual and deductive reasoning in solving geometric problems.

It would also assist this research if you could make available any resources which will contribute to developing an understanding of geometric task development and teaching approaches at your school. Any documentation such as planning documents, journals, meeting logs, professional learning products, etc would be valuable.

You are also invited to participate in a series of Mathematics lesson observations at suitable
times. Classroom observations of the teaching and learning process are critical to this research in order to deepen understandings of pedagogical practices. This will involve recording mainly through a written and video format but may involve photographs of student work on occasions. Some dialogue with students may be required to determine their understanding and interpretation of teacher instructions, as well as an understanding of how students interact with each other in solving geometric problems. Parental consent will be sought through the school. There should be minimal impact of the teaching and learning program of those involved.

**Possible risks or disadvantages**

There are no anticipated risks or disadvantages if you participate in this project. However, if you are unduly concerned about your involvement in any aspect of this research or if you find participation in the project distressing in any way, you should contact one of the researchers as soon as convenient. We will discuss your concerns with you confidentially and suggest appropriate follow-up, if necessary.

**Benefits of the research**

Knowledge gained from the research will inform on the types of tasks necessary to engage students in geometry promoting visual and deductive reasoning skills through problem-solving approaches. The research will also define teaching and learning approaches necessary to engage students in geometric thinking.

**Further information you need to know**

The school and all participants will be assigned code names and will remain anonymous. Any data which could identify you or your school will not be used. The data collected during the study may be published in possible journal articles and conference presentations. A thesis will also be provided to RMIT. However, once again, all participants will remain anonymous. All data collected will remain secure on a password-protected computer and be securely stored in a locked filing cabinet. To ensure confidentiality, the data will not be made available to anyone.

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**Your rights as a participant**

Participation in the study is voluntary and you may withdraw at any time without prejudice. In this event, any information already obtained will not be used. Any unprocessed data may also be withdrawn and you may access your data at any time. The tape recordings will not be made available to anyone and will be destroyed once the transcripts have been created and checked for accuracy. The transcripts and all other data collected will be destroyed five years after completion of the thesis by RMIT. If you are interested in the final report, you are welcome to request the executive summary which will be forwarded to you.

If you require further information about the research or your role, please contact me on mobile ………………… or email adrian.berenger@rmit.edu.au. You could also contact my principal supervisor, Dr Tasos Barkatsas on ………………… or email tasos.barkatsas@rmit.edu.au.

Thank you for considering this request. I do hope that you will be able to help with this important research and look forward to your participation in the study.

Yours sincerely

____________________

Adrian Berenger
BSc. DipEd

____________________

Dr Tasos Barkatsas
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Consent Form

Project Title: An Investigation of Visualisation in the Development of Spatial and Deductive Reasoning of 2 – dimensional Shapes.

1. I have had the project explained to me, and I have read the information sheet

2. I agree to participate in the research project as described

3. I agree:
   - to conducting an initial set of tasks questions with students
   - that my voice will be audio recorded
   - to my classroom instruction being observed and video recorded

4. I acknowledge that:
   (a) I understand that my participation is voluntary and that I am free to withdraw from the project at any time and to withdraw any unprocessed data previously supplied (unless follow-up is needed for safety).
   (b) The project is for the purpose of research. It may not be of direct benefit to me.
   (c) The privacy of the personal information I provide will be safeguarded and only disclosed where I have consented to the disclosure or as required by law.
   (d) The security of the research data will be protected during and after completion of the study. The data collected during the study may be published, and a report of the project outcomes will be provided as a thesis to RMIT and for possible journal publications and conference presentations. Any information which will identify me will not be used.

Participant’s Consent

Name: ___________________________ Date: ______________

(Participant)

Participants should be given a photocopy of this PICF after it has been signed.
Appendix 11

Letter to Parents

Dear Parent/Guardian,

I am Adrian Berenger from RMIT University, in the School of Education, Design & Social Context. I am conducting a research project on An Investigation of Visualisation in the Development of Spatial and Deductive Reasoning of 2-dimensional Shapes. A significant component of this research involves lesson observations conducted during some of your child’s numeracy classes during Term 1, 2016. The purpose of these observations is to document student reasoning skills and teaching practice through the development of geometric tasks. The research findings have potential to inform the development of task designs to improve teaching and learning in middle years mathematics classes.

The focus of lesson observations will be on student and teacher interactions as they engage in geometric tasks. This will involve recording students and teachers during lessons mainly through written and video formats but may involve photographs of student work on occasions. Some dialogue with students may be required to determine their understanding of teacher instructions in geometry.

Further information you need to know

Any data which could identify your child will not be used. Your child’s participation in this project is completely voluntary. In addition to your permission, your child will also be asked if he or she would like to take part in this project. Only those children who have parental permission and who want to participate will do so, and any child may stop taking part at any time. You are free to withdraw your permission for your child’s participation at any time and for any reason without penalty. In this event, any information already obtained will not be used.

If you require further information about the research or your role, please contact me on mobile ………………. or email adrian.berenger@rmit.edu.au. You could also contact my supervisor, Dr Tasos Barkatsas on ……………….. or email tasos.barkatsas@rmit.edu.au.

Thank you for considering this request. Please keep the attached copy of this letter for your records.

Yours sincerely

Adrian Berenger

Any complaints about your participation in this project may be directed to
The Secretary, RMIT Human Research Ethics Committee, University Secretariat,
RMIT University, GPO Box 2476V, Melbourne, 3001.
The telephone number is (03) 9925 1745.
Details of the complaints procedure are available from: www.rmit.edu.au/council/hrec

I do/do not (circle one) give permission for my child __________________________ (name of child) to participate in the research project described above.

(Print) Parent’s name

Parent’s signature Date
Appendix 12

Mr Adrian Berenger
79 Rennie Street
THORNBURY 3071

Dear Mr Berenger

Thank you for your application of 5 October 2015 in which you request permission to conduct research in Victorian government schools titled An investigation of Visualization in the Development of Spatial and Deductive Reasoning of 2-Dimensional Shapes.

I am pleased to advise that on the basis of the information you have provided your research proposal is approved in principle subject to the conditions detailed below.

1. The research is conducted in accordance with the final documentation you provided to the Department of Education and Training.

2. Separate approval for the research needs to be sought from school principals. This is to be supported by the Department of Education and Training approved documentation and, if applicable, the letter of approval from a relevant and formally constituted Human Research Ethics Committee.

3. The project is commenced within 12 months of this approval letter and any extensions or variations to your study, including those requested by an ethics committee must be submitted to the Department of Education and Training for its consideration before you proceed.

4. As a matter of courtesy, you advise the relevant Regional Director of the schools or governing body of the early childhood settings that you intend to approach. An outline of your research and a copy of this letter should be provided to the Regional Director or governing body.

5. You acknowledge the support of the Department of Education Training in any publications arising from the research.

6. The Research Agreement conditions, which include the reporting requirements at the conclusion of your study, are upheld. A reminder will be sent for reports not submitted by the study's indicative completion date.
I wish you well with your research. Should you have further questions on this matter, please contact Youla Michaels, Project Support Officer, Insights and Evidence Branch, by telephone on (03) 9637 2707 or by email at michaels.youla.y@edumail.vic.gov.au.

Yours sincerely

[Signature]
Director
Insights and Evidence

6/11/2015
Appendix 13

S1A. Teacher Comments on Quirps task

Thoughts on making this:

- Need a place to record student responses to 'Which of the following are quirps?'
- Students feel that a 'quirp' is something they should know or will learn about. Need to clarify this.
- Rubric needs 'incorrect definition', 'insufficient definition', 'incorrect shape drawn', i.e. 'a shape'.
Appendix 14

Guideline for teachers:

What do you know about a square?

This should be conducted individually (almost like a test). It is a great task to use as baseline data for individual students. It is important that they do not prompt each other so the instruction to students should be as below.

I suggest not allowing questions. Rationale: When students ask questions such as “can we draw a picture?” – these prompt other students to do things that they would not normally do.

Allow a fixed time period say 20 minutes.

Instruction for Students:

“Impress me with everything you know about a square”. Be as accurate as you can!

After the task:

There is a rubric to mark individual student work. This should be marked as if you were providing them with written feedback. So include as many comments for students as practicable.
Save all student written responses and rubrics.
Appendix 15

S2A. Researcher’s recording of Teaching episode

Teacher: Are these straight sides? How?

S: No. I demonstrated that the sides are equal length.

T: What about the parallel sides?

Teacher demonstrates.

New knowledge:

All the angles are equal.

Adjustment of diagram: “revolve.”

Teacher provides.

If this is a rectangle, are the lines the same height? Which one?

S: Horizontal line is on the same height.

T: They’re all the same lines.
Appendix 16

Research notes of discussions with student (S2A)

[Handwritten text]

folded this page and said

3 at Ms. it's a square.

I asked him to prove it
and he said we can
measure — he learnt that
it was a square. I
also should have told
we can simply fold along
the diagonal and it should
be symmetrical.

He suggested this at the end of the
class task when he rrnrnn was
transforming student responses.
## Appendix 17

### S1A. Teacher’s Feedback to Student via Rubric

#### Scoring Rubric

**Definitional properties**

<table>
<thead>
<tr>
<th>Score</th>
<th>Indicator</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Four straight sides.</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>It is a quadrilateral.</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Four right angles.</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>It is planar.</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>It is a closed.</td>
<td></td>
</tr>
</tbody>
</table>

**Transformation**

<table>
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<th>Score</th>
<th>Indicator</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Lines of symmetry</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Rotational symmetry</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Tessellation</td>
<td></td>
</tr>
</tbody>
</table>

**Formal property-based reasoning**

<table>
<thead>
<tr>
<th>Score</th>
<th>Indicator</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Parallel lines.</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Perpendicular lines.</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Diagonals.</td>
<td></td>
</tr>
</tbody>
</table>

**Other**

<table>
<thead>
<tr>
<th>Score</th>
<th>Indicator</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3D eg. Each face of a cube. Connect to other things. eg. two squares can make a rectangle</td>
<td></td>
</tr>
<tr>
<td></td>
<td>In situ eg. My book is a square.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Non-sequitur (false conclusion) eg. A rectangle is a square because it has 4 sides.</td>
<td></td>
</tr>
</tbody>
</table>

**Comments:** Were there any further properties or explanations provided?

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