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RESEARCHING AND USING LEARNING PROGRESSIONS (TRAJECTORIES) IN MATHEMATICS EDUCATION

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The relationship between research and practice has long been an area of interest for researchers, policy makers and practitioners alike. One obvious arena where mathematics education research can make a practical contribution is the design and implementation of school mathematics curricula. This requires research that is fine-grained and focused on individual student learning trajectories as well as large-scale research that explores how student populations engage with the big ideas of mathematics. This research forum brings together work from the United States and Australia on the development and use of evidence-based learning progressions/trajectories in mathematics. In particular, the forum will consider their basis in theory, their focus and scale, and the methods used to identify and validate learning progressions.

INTRODUCTION

Learning progressions, or learning trajectories as they are more commonly referred to in mathematics education, are not new. For instance, it could be said that scope and sequence charts and year level outcome statements represent particular forms of learning progressions/trajectories. While there has been considerable research in particular domains over many years that has contributed to our understanding of how knowledge is constructed and informed practice in those domains, it is only relatively recently that learning progressions/trajectories per se have become the focus of systematic research efforts (e.g., Clements & Sarama, 2004; Confrey, 2008; Daro, Mosher & Corcoran, 2011; Siemon, Izard, Breed & Virgona, 2006).

Ever since Simon’s (1995) introduction of the notion of \textit{Hypothetical Learning Trajectories} (HLT), there has been debate about the meaning and use of learning progressions/trajectories in mathematics education (e.g., see the special edition of \textit{Mathematics Teaching and Learning}, 6(2) in 2004). A common element in the different interpretations and use of the terms is the notion that learning takes place over time and that teaching involves recognising where learners are in their learning journey and providing challenging but achievable learning experiences that support learners progress to the next step in their particular journey. Another common characteristic is that, to varying extents and in different ways, learning progressions/trajectories are based on hypothesised pathways derived from experience and a synthesis of relevant literature, the design and trial of learning activities aimed at progressing learning.
within the hypothesised framework, evaluation methods to assess where learners are in their journey and the efficacy of both the framework and the instructional materials and approaches used.

The focus of a learning progression/trajectory may relate to a particular instructional episode (e.g., Simon, 1995; Tzur, 2007), a specific aspect of the curriculum (e.g., Clements, Wilson & Sarama, 2004) or a much larger field of mathematics learning that encompasses different but related aspects of mathematics (e.g., Confrey & Maloney, 2014; Siemon, Izard, Breed & Virgona, 2006). Their development and use may vary from a reflective practitioner working to understand and support his/her student’s attainment of a specific learning goal over a relatively short time frame through to an extensive network of teachers and researchers working collaboratively to understand how students in general might be supported to progress their learning in a particular domain or field of mathematics over an extended period of time.

Concern with the numbers of students ‘falling behind’ and the considerable range of achievement in any one year level (e.g., OECD, 2014) have prompted educational systems and researchers in a small number of countries to work more closely together to identify evidence-based learning progressions/trajectories that might be used to inform teaching and map student’s progress over time. While these vary considerably in their focus and scale, there is much that we can learn from each other to further the work in this field and to build new knowledge that is likely to make a difference to student learning (e.g., see Daro, Mosher & Corcoran, 2011, p. 13).

The research forum is likely to be of substantial interest to a PME audience as it is concerned with the application and scaling up of research to practice to make a difference in mathematics classrooms. The forum provides an opportunity for a reality check. For example, does this work translate to other settings? Is it a valid use of research conducted for other purposes in other contexts and do the results and affordances outweigh the limitations?

The contributors have been brought together on the basis of their recognised contributions to this field, to consider what is meant by learning progressions/trajectories and explore a range of issues associated with their development and use including theoretical framing, research approaches, implementation and evaluation. It is difficult to succinctly capture the body of work represented here in a way that is both fair and accurate. So for the purposes of building a coherent picture and facilitating discussion, contributors were invited to discuss their work (past, present and future) under three headings: research approaches, starting points and developments, and practical applications and/or implications. These are presented in turn followed by key questions raised by our critical friend, Anne Watson that raise issues concerning the development and use of LT/Ps.
Key questions to be explored in the Research Forum:

What characterises a learning progression/trajectory? What purposes do they/can they serve? How are they different to or compatible with theories of conceptual development?

What is situated and what is universal about learning progressions/trajectories?

What research designs, techniques and evidence are used to develop, evaluate and refine such progressions?

How are learning progressions/trajectories used in practice? How are they related to task sequences used in countries like China and Japan? What impact do they/can they have on teacher knowledge and confidence?

RESEARCH APPROACHES

A variety of research approaches have been used to conceptualise and construct the learning progressions featured here will be discussed in turn.

Tzur

For the past 25 years, my research program consisted of four interrelated components: articulating hypothetical learning trajectories in the areas of multiplicative and fractional reasoning (Tzur 2004, Tzur 2014); explaining mathematics learning as a cognitive change process (Tzur & Simon 2004, Tzur 2011); linking this model to teaching that can promote progression along those trajectories (e.g., Tzur 2008); and identifying shifts in mathematics teacher practices (Jin & Tzur 2011). This four-fold program is rooted in the premise that mathematics teaching is a goal-directed activity, aimed at promoting students’ learning of the intended mathematics. This requires an understanding of how learning of particular mathematics may progress and how teaching may foster such progression.

To strengthen this twofold understanding, my work on articulating HLTs led me to distinguish two kinds of studies on learning trajectories: Marker Studies, which foreground conceptual landmarks that constitute a learning trajectory; and Transition Studies, which foreground the conceptual transformation involved in progressing from less to more advanced landmarks. Because a primary goal of my work on HLT is to contribute to the knowledge base about understanding (learning, teaching), I have conducted mainly transition studies.

Recently I have complemented teaching experiments with two other methods: corroborating empirically grounded models through quantitative methods and elaborating on findings (markers and/or transitions) from previous teaching experiments (Tzur 2014).

Clements and Sarama

Our 30-year work with learning trajectories (LTs) began with the creation and testing of LTs, but has come to span the full range of research and development in education,
contending now that LTs have ramifications for all aspects of curriculum (e.g., ideal, expected, available, adopted, implemented, achieved, or tested, Clements, 2007). This requires a wider range of methods (that we will discuss in subsequent sections), with the focus here being only on the methods we use for the creation, refinement, and validation of LTs.

Initially we considered a learning trajectory as a device whose purpose is to support the development of a curriculum or a curriculum component. Building on Simon (1995), but emphasizing a cognitive science perspective and a base of empirical research, we conceptualized “learning trajectories as descriptions of children’s thinking and learning in a specific mathematical domain, and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children’s achievement of specific goals in that mathematical domain” (Clements & Sarama, 2004, p. 83). In other words, each learning trajectory has three parts: (a) a goal, (b) a developmental progression, and (c) instructional activities. To attain a certain mathematical competence in a topic or domain (the goal), students learn each successive level (the developmental progression), aided by tasks (instructional activities) and pedagogical moves designed to help students build the mental actions-on-objects that enable thinking at each higher level. We address the determination of the goal in the following section; here we address the other two components.

While others have based their LTs on historical development of mathematics and observations of children’s informal solution strategies (Gravemeijer, 1999), anticipatory thought experiments (that often focus on instructional sequences), or emergent mathematical practices of student groups (Cobb & McClain, 2002 in which instructional design serves as a primary setting for development), our approach is grounded more in cognitive science. We begin by learning from others, conducting comprehensive research reviews (e.g., Barrett, Clements, Sarama, & Cullen, in press; Clements, Wilson, & Sarama, 2004). If details are lacking, we use grounded theory methods and clinical interviews (Clements, 2007; Ginsburg, 1997) to examine students' knowledge and ways of thinking in the content domain, including conceptions, strategies, intuitive ideas, and informal strategies used to solve problems. The researchers set up a situation or task to elicit pertinent concepts and processes. Once a (static) model has been partially developed, it is tested and extended with constructivist teaching experiments, which present limited tasks and adult interaction to individual children with the goal of building models of children’s thinking and learning. Once several iterations of such work reveal no substantive variations, it is accepted as a working model, then subjected to validation and/or refinement through hypo-deductive applications of qualitative methods such as teaching experiments and quantitative methods such as correlational analyses between level scores (Clements, Wilson, & Sarama, 2004) and Rasch modeling (Barrett et al., in press; Szilagyi, Sarama, & Clements, 2013).
Next, sets of activities are taken from successful interventions in the literature or created (or tasks are adapted from previous work) by the developers. In both cases, the key is ensuring that the activities are theoretically valid in engendering or activating the actions-on-objects that mirror the hypothesized mathematical activity of students in the target level (that is, level \( n + 1 \) for students at level \( n \)). Design experiments and microgenetic studies (Siegler & Crowley, 1991) are employed, using a mix of \textit{model (or hypothesis) testing} and \textit{model generation} to understand the meaning that students give to the objects and actions embodied in these activities and to document signs of learning.

\textbf{Confrey and Maloney}

Two major components of our research around learning trajectories over the last twenty years are: developing and validating the Equipartitioning learning trajectory (1995-2011), described here and Confrey’s current research on the LT-based \textit{Math-Mapper 6-8} for middle grades, described in later sections.

We have used a variety of methods in developing LTs. In our original work on the Equipartitioning LT, we began with Confrey’s splitting conjecture (1988; Confrey & Scarano, 1995), namely, that an independent cognitive construct for splitting differs from that of counting. After an extensive literature review on evidence for the independence of this construct, we chose the term “equipartitioning” to clarify that this involved not simply making parts, but making equal-sized parts. Further, we identified two relatively distinct literatures, one for sharing groups fairly and the other for sharing a whole fairly. We integrated these notions of sharing into a single learning trajectory. The new trajectory consists of 16 levels, covering three cases of equipartitioning: Sharing a collection \((na)\) among \( n \) people, sharing a whole among \( n \) people, and finally sharing multiple wholes that did not divide evenly (one with more wholes than sharers and one with fewer wholes than shares, which could be addressed by students in either order, depending on their prior knowledge from instruction and experience) (see Confrey et al., 2014b). To validate the learning trajectory, we undertook two primary research initiatives.

1) Items corresponding to the 16 levels were written and administered to students in grades 1-5. Student item responses were coded, then analysed using item response theory. In general, the items for the LT lower levels were less difficult than the items for the upper levels.

2) In a design study, curriculum units developed to support the LT were used, along with a digital tool we had developed, to collect student data from automated diagnostic tasks that corresponded to the different levels (Confrey & Maloney, 2015). We worked with 12 students, grades 2-4, from high poverty settings, for two summer weeks. We articulated our initial conjectures and conducted a daily debriefing session to revise plans based on each day’s observations (Confrey & Lachance, 2000). We periodically conducted one-to-one interviews with students to understand how their thinking was developing. At the end of the study, we reviewed the data from the diagnostic
assessments, video, and notes, and drew conclusions about how the LT levels, the curriculum, and items might be modified in light of the results. In general, we also described the trajectory in terms of a) the development of the cases, b) the way in which students generated strategies at early stages, c) whether the students developed a sense of properties at the second levels and d) how they showed signs of reasoning in a connected fashion at the higher levels.

**Siemon and Horne**

In 1999, RMIT was commissioned to identify and document what was working in numeracy teaching in Years 5 to 9 where numeracy was seen to involve:

- core mathematical knowledge (in this case, number sense, measurement and data sense and spatial sense as elaborated in the (Australian) National Numeracy Benchmarks for Years 5 and 7;
- the capacity to critically apply what is known in a particular context to achieve a desired purpose; and the
- actual processes and strategies needed to communicate what was done and why (Siemon & Virgona, 2002)

A quasi-experimental design involving a representative sample of 47 Victorian schools was used. In the first phase, data were collected from just under 7000 Year 5 to 9 students using rich assessment tasks and scoring rubrics based on the dimensions of numeracy described above (Siemon & Stevens, 2001). These data were analysed using item response theory, which confirmed that the tasks were appropriate for the cohort tested and that it was possible to measure a complex construct such as numeracy using assessment tasks that incorporate performance measures of content knowledge and process (general thinking skills and strategies) across a range of topic areas using teachers-as-assessors.

In subsequent work on learning progressions HLTs were developed from the research literature related to multiplicative thinking (e.g., see Siemon & Breed, 2006) and later for algebraic reasoning, geometrical reasoning, and reasoning in statistics and probability. The HLT, hereinafter referred to as a draft learning progression (DLP), is used to inform the selection and/or development of rich tasks designed to assess not only the core knowledge associated with the areas of mathematics under consideration but also, students’ ability to apply that knowledge in unfamiliar situations and explain or justify their reasoning. The tasks and scoring rubrics are then trialled with a relatively large number of students in the target population and the data analysed using item response theory (e.g., Bond & Fox, 2015). This allows both students’ performances and item difficulties to be measured using the same log-odds unit (logit), and placed on an interval scale. Items (parts of tasks) that do not fit the model are either rejected or refined and re-triailled. The scale is then interrogated by at least three experts in the field to identify and describe patterns in student performances. This results in the identification of a number of levels or Zones within the progression for which teaching advice is prepared in the form of a learning assessment framework.
THE DEVELOPMENT AND REFINEMENT OF LTS

Piaget’s notion of assimilation, a core constructivist principle, is the starting point for any HLT study I conduct. Assimilation posits that any new learning can only be as good as the goal-directed activities afforded, or constrained, by learners’ available (assimilatory) schemes. To teach and study how learners transform (reorganize) assimilatory schemes into new ones, we thus first engage in articulating fine details of the three parts that might constitute their schemes (von Glasersfeld, 1995). The first part is the mental template (‘situation’) by which learners may make sense of a given ‘input’ (e.g., mathematical task), which triggers the goal(s) they would set to accomplish. This goal calls up the second part of the scheme—a mental activity sequence that the learners have been using to reliably accomplish the goal(s). As the activity ensues, the learners’ goal(s) regulate their noticing of effects that either match or do not match the scheme’s third part—a result they expected to ensue from the activity. Detailing all three parts of learners’ assimilatory schemes is vital, because conceptual change is postulated to commence, and thus possibly be fostered, through their noticing of actual effects that differ from the expected ones.

To articulate learners’ assimilatory schemes that would serve as a starting point for studying HLT, as well as the hypothetical process of change those schemes may undergo, we combine two main sources: task-based interviews with participating learners and scrutiny of previous, relevant research. Using these two sources reflexively, our goal is to detail the precise boundaries between schemes we infer students already have constructed and schemes into which the available schemes could possibly be transformed (yet to be constructed). The notion of precise boundaries includes close attention to one of two stages at which we infer learners’ schemes to have been established (Tzur & Simon, 2004). An anticipatory stage of a scheme is inferred if the learner can use it spontaneously and independently when solving relevant tasks. A participatory stage is inferred if the learner can use it albeit not yet spontaneously and independently (e.g., by somehow being incited for a novel use of an activity).

Our hypotheses of how the intended conceptual transformation (a micro-level learning trajectory) may be fostered differ based on the stage of learners’ assimilatory schemes. If we infer those to be at the anticipatory stage, we identify a relevant participatory stage of a new scheme to serve as the goal for their next learning. Accordingly, we detail ways to proactively promote Reflection Type-I, which is postulated to promote a transition to the participatory stage of the next scheme (Tzur, 2011). In this type of reflection, learners compare between effects they expected and actual effects they noticed to ensue from their activity. Such a comparison provides the mental mechanism for creating a novel, provisional relationship between the goal-directed
activity and its actual effects that can be formed solely on the basis of what has been
previously available to the learner.

If, however, we infer learners’ schemes to be at the participatory stage, we set the goal
for their next learning to be the anticipatory stage of that scheme. Accordingly, we
detail ways to proactively promote Reflection Type-II, which is postulated to promote
that transition. In this second type of reflection, learners compare across mentally
recorded instances in which an activity did or did not ensue particular effects. Such a
comparison provides the mental mechanism for abstracting the regularity (invariant) in
and reasoning for why relationship between the goal-directed activity and its effects
must necessarily be what they are in given, as well as non-routine problem situations.

To illustrate how the above constructs are being used as a starting point, I provide an
example from Tzur and Lambert (2011) that led to identifying 4 sub-stages in first
graders’ shift from counting-all to counting-on, that is, from having no concept of
number as a composite unit to the early onset of that concept. For that study, we
sampled all students who spontaneously and independently used the counting-all
strategy for adding two previously counted collections (e.g., 7 cubes and 4 cubes). Our
inference of the scheme that underlies such a strategy included:

<table>
<thead>
<tr>
<th>Situation + Goal</th>
<th>Activity Sequence</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>Having separately counted all 1s in each of two given collections of tangible items to find their numerosities, set out to find the numerosity of the combined collection</td>
<td>Starting over from 1, count every tangible item in the combined collection by creating 1-to-1 correspondence between those items and number words in the conventional sequence</td>
<td>Reaching the final item to be counted and stating the number word that corresponded to this item to indicate the numerosity</td>
</tr>
</tbody>
</table>

Table 1. Scheme underlying strategy

For a child at the anticipatory stage of this (counting-all) scheme, we set out the goal to
begin constructing a participatory stage of a scheme that would give rise to the concept
of number as composite unit, as indicated by the development of a counting-on strategy. To this end, I created a play activity, called How Far From the Start (HFFS) in which two players step on along large tiles from a marked start, taking turns to roll a die and walk from either the start or the last players position the number of steps implied by the tiles and recording the numeral on a note placed on their endpoint. Then, both learners figure out how far the end tile of the second player is from the start (e.g., 11).

This activity assumes learners will begin finding the total number of steps by assimilating the task into their available scheme, that is, by using counting-all. While they play, the researcher-teacher will begin probing for their reflection on the effect they can notice, namely, always calling out the number on the first note (e.g., 7) when counting to find the combined total. For example, we may ask the players to stop their count while stepping on that tile and tell us if they are surprised to have said this
number word (7) or if they could consider starting at a spot and a number word other than 1. We may also shift from real tiles to a drawn out board game marked with Start and End tiles. This allows us, later, to cover some of the tiles on first path to further orient the learners’ reflection onto the possibility to use the first end tile/numeral as a start. Letting players switch roles and repeating these experiences, enabled them to create a provisional link between their counting-all activity up to the first stopping point and the effect it ensued—starting with the number-after (8) when resuming their count. This new, provisional linkage opens the way not only to starting the count from that stopping point (7) but also to keeping track of the count of 1s in the second walk. That is, a new stage of anticipating where to start is formed at a participatory stage, as the learners replace 1 as the start for finding the combined total by their noticed effect of starting from the first end point.

Conceptual reorganization (accommodation) is another core constructivist principle that, coupled with a corresponding, student-adaptive pedagogy (Tzur, 2013), underlies my development of HLT. Above, I provided a brief description of the two types of reflection and two stages (participatory, anticipatory) that enable reorganization of assimilatory schemes into new ones. By student-adaptive pedagogy, I refer to the cyclic, 7-step process postulated (Tzur, 2008) as an elaboration of Simon’s (1995) seminal introduction of the HLT notion. In a nutshell, these 7 steps include (with pointers to the example of fostering transition from counting-all to counting-on as explained above):

1. Specifying students’ current conceptions;
2. Specifying the intended mathematics;
3. Identifying a mental activity sequence through which the conceptual change may evolve;
4. Selecting and/or adapting tasks to promote the intended learning;
5. Engaging learners in the task while letting them use previously constructed schemes first;
6. Monitoring learners’ progress;
7. Introducing follow-up questions and probes to foster Reflection Type-I and/or Reflection Type-II.

When conducting teaching experiments, we develop HLT through two types of analysis—ongoing and retrospective (Tzur et al, 2000). Ongoing analysis focuses on inferring each individual learner’s conceptual progress during the recent teaching episode(s). Inferences are made about changes in the learner’s anticipation, explanation of effects they notice to ensue from their activity, and the extent to which learners can use the newly abstracted anticipation spontaneously. Those tentative inferences constitute Step 1 of the 7-step cycle, which inform Steps 2, 3, and 4 in the design of teaching for the next episode.
After completing all teaching episodes, further development of HLT occurs through retrospective analysis, which focuses on distinguishing and explaining plausible ways in which learners’ mental systems may give rise to their observable behaviours (actions and language). Drawing on the principles of grounded theory methodology (Glaser & Strauss, 1967), retrospective analysis identifies commonalities across different learners’ solutions while striving to specify schemes that, we infer, could serve as conceptual underpinnings of those solutions. Those schemes, for which we detail both the participatory and anticipatory stages, become the markers of HLT. Then, going back to the data, we search for ways in which transition from one scheme (marker) to the participatory and then anticipatory stage of the next one might have took place, along with instructional moves that seemed essential in fostering that learning.

Refinement of HLT is accomplished by further organization of findings from my teams’ work and from other research teams’ studies of similar progressions. (e.g., Clements & Sarama, 2004; Maloney, Confrey, & Nguyen, 2014). While staying close to the data from which the HLT were created, this organization involves sequencing of schemes and transitions between them along a developmental continuum. In collaboration with researchers from other teams, a developmental continuum is linked with more general models, such as the model of units coordination levels (e.g., Hackenberg, 2007), which transcends additive, multiplicative, and fractional reasoning. Further refinement of the HLT is then attained through using the continuum of markers and transitions to teach and study different student populations, such as students identified as having learning disabilities in mathematics (e.g., Hord et al, 2016), teachers (Tzur, Hodkowski, & Uribe, 2016), or across social-cultural settings (e.g., Huang, Miller, & Tzur, 2015). Of course, working with different populations may confirm the HLT we have been developing and/or present challenges that require further refinement.

In the past 25 years, I have worked with several teams that produced two HLT—one focusing on multiplicative schemes (Tzur et al., 2013) and the other on fractional schemes (Tzur, 2014). The markers that constitute each of these are summarised below. Details of transitions from one scheme to the next and the tasks used to accomplish this can be found in previous publications.

The HLT for multiplicative reasoning includes 6 Schemes: (1) Multiplicative double counting (mDC); (2) Same-Unit Coordination (SUC); (3) Unit Differentiation and Selection (UDS); (4) Mixed-Unit Coordination (MUC); (5) Quotiative Division (QD); and (6) Partitive Division (PD). It should be noted that distinguishing UDS was not intended or hypothesized before the teaching experiment, but rather compelled by children who indicated explicit inability to make the conceptual leap from SUC to MUC.

The HLT for fractional reasoning includes 9 schemes (the letter ‘S’ in each acronym stands for ‘Scheme’): (1) Equi-Partitioning (EPS); (2) Partitive Fraction (PFS); (3) Splitting; (4) Iterative Fraction (IFS); (5) Reversible Fraction (RFS); (6) Recursive
Partitioning (RPS); (7) Unit Fraction Composition (UFCS); (8) Distributive Partitioning (DPS); and (9) any Fraction Composition (FCS).

Clements and Sarama

A complete learning trajectory includes an explication of the mental constructions (actions-on-objects) and patterns of thinking that constitute children’s thinking at each level of a developmental progression, how they are incorporated in each subsequent level, and tasks aligned to each level (that promote movement to the succeeding level). The learning trajectories construct differs from instructional design based on task analysis because it is based not on a reduction of the skills of experts but on models of children’s learning, expects unique constructions and input from children, involves self-reflexive constructivism, and involves continuous, detailed, and simultaneous analyses of goals, children’s thinking and learning, and instructional tasks and strategies. Such explication allows the researcher to test the theory by testing the curriculum (Clements & Battista, 2000), usually with design experiments (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003).

When we began, we accepted that the goal of an LT should be determined by standards (ideal or expected curriculum) created by dialectical process among many legitimate stakeholders (e.g., CCSSO/NGA, 2010; NCTM, 2006). When more detail was needed, we used reviews of the research literature to identify objectives that contribute to the mathematical development of students, build from the students’ past and present experiences, and are generative in students’ development of future understanding. We now also believe that LTs should play a more active role in determining, as well as incorporating, goals.

Starting points for LTs differ with different goals. The importance of geometric measurement was well established. However, there was less extant justification for the domain of composing geometric forms. We determined this domain to be significant in that the concepts and actions of creating and then iterating units and higher-order units in the context of constructing patterns, measuring, and computing are established bases for mathematical understanding and analysis (e.g., Mulligan & Mitchelmore, 2013; Park, Chae, & Boyd, 2008; Reynolds & Wheatley, 1996; Steffe & Cobb, 1988).

The shape composition learning trajectory had its genesis in observations made of children using Shapes software to compose shapes. Sarama observed that several children followed a similar progression in choosing and combining shapes to make another shape (Sarama, Clements, & Vukelic, 1996). Sarama re-viewed the behaviors all kindergarten children exhibited and found that children moved from placing shapes separately to considering shapes in combination; from manipulation- and perception-bound strategies to the formation of mental images; from trial and error to intentional and deliberate action and eventually to the prediction of succeeding placements of shapes; and from consideration of visual “whole” to a consideration of side length, and, eventually, angles. We combined these observations with related observations from other researchers (e.g., Mansfield & Scott, 1990) and some elements
of psychological research (e.g., Vurpillot, 1976) to refine this developmental progression. Tasks were designed to elicit each of these hypothesized levels. We conducted clinical interviews using these tasks, validating that the actions-on-objects posited to underlie solutions could be observed. We used quantitative methods, confirming that they formed a reliable and valid sequence (Clements, Wilson, & Sarama, 2004). At that point, we confirmed a developmental progression in which children move levels of thinking—from lack of competence in composing geometric shapes, they gain abilities to combine shapes—initially through trial and error and gradually by attributes—into pictures, and finally synthesize combinations of shapes into new shapes, that is, composite shapes.

Instructional tasks in which children worked with shapes and composite shapes as objects were designed. We wanted them to create, duplicate, position (with geometric motions), combine, and break apart both individual shapes (units) and composite shapes (units). We designed physical puzzles and software environments that required and supported use of those actions-on-objects. Simultaneously, we documented what elements of the teaching and learning environment, such as specific scaffolding, contributed to student learning—planned a priori or occurring spontaneously. Thus, designs are not determined fully by reasoning. Intuition and the art of teaching play critical roles.

Work with the measurement LT differed in several ways. The larger literature allowed us to use a research synthesis to form the initial LT (Sarama & Clements, 2002). The presence of assessment tasks, empirical results and theory allowed us to validate the first LTs with Item Response Theory, creating an equal-interval scale of scores for both the difficulty of items and the ability of the persons assessed. To measure measurement competence, we sequenced the items, strictly maintaining the order within each measurement domain but intermingling items across domains according to the available developmental evidence, including age specifications from the literature and difficulty indices from our pilot testing. Thus, we posited that items were organized according to increasing order of difficulty across domains, but our theoretical claims that this sequencing represented increasingly sophisticated levels of mathematical thinking were made only for items within a given domain. We submitted the results of administering this revised instrument to the Rasch model, validating the developmental progressions for length, area, and volume in multiple studies (Barrett et al., in press; Szilágyi, Sarama, & Clements, 2013). We similarly used and validated instructional sequences, many again from the extant literature.

We believe that full validation of an LT requires validation of the instructional tasks and their implementation in real classrooms.

Confrey and Maloney

Previous efforts. Our original work on equipartitioning led us to make the knowledge base on learning trajectories more accessible to greater numbers of teachers. Doing so
required us to explore the use of new forms of visual representations for the LTs. Our first version was a “hexagon map” ([www.turnonccmath.net](http://www.turnonccmath.net)) that used the Common Core Standards themselves as a framework for 18 LTs for grades K-8. The research team unpacked the content of each LT into an explanation of the LT and related research (Confrey & Maloney 2014). Ultimately, using the standards as the backbone of the LTs was dissatisfying, due to at least two limitations: 1) it tied us to the standards constraining divergence from them, and 2) for parsimony, each standard was embedded in only one LT, because we used each hexagon only once.

**New LTs and learning map.** Working to improve the visualization for greater usefulness to teachers and students simultaneously, the new work has been to develop a “learning map” for grades 6-8 (the content as framed generally in the Common Core Standards). It is called a “learning map” because it is built on a fundamental re-articulation of underlying learning trajectories, specifying how students’ ideas become increasingly sophisticated as they engage with increasingly complex tasks during instruction. The DLS tool “Math-Mapper 6-8” (MM6-8), comprises 1) the learning map, 2) a diagnostic assessment and reporting system that corresponds directly to the learning trajectories, 3) a means to access curricular resources via the web and a curated library of links, 4) a Sequencing tool and calendar to organize all the foregoing components across the school year, and 5) an analytics system for interpreting various levels of use of the tool by students and teachers.

![Math-Mapper 6-8 learning map (fields and big ideas only)](image)

**Fig. 1. Math-Mapper 6-8 learning map (fields and big ideas only)**

The learning map is hierarchically organized, with four fields of mathematics incorporating nine big ideas (Figure 1). Each big idea comprises 2-5 *relational*...
learning clusters or RLCs (24 in all) of related constructs (64 in all). Each construct is associated with a LT and is also associated with relevant CCSS-M standards. The new learning map was developed to be a foundational organizer for the diagnostic assessment and reporting system. The new LTs are more specifically descriptive of student behaviours than those in the hexagon map.

Developing LTs across all four fields of mathematics has been informative. First of all, the hierarchy sets up three levels of trajectories. Each construct is made up of an LT. Then closely-related constructs are formed into clusters, and each cluster’s shape establishes a progression of constructs that itself proceeds from less to greater sophistication with varying structures (e.g. there may be two constructs at the same level that can be taught in either order or taught in tandem). Finally the clusters within each big idea themselves are formed into another progression of sophistication of reasoning. We regard the overall hierarchical structure of the map to describe an evolution of the idea of an LT—showing how the mathematical landscape of middle grades can be conceptualized with LT structure underlying it at multiple levels of scale.

In our extensive work with LTs, we have learned a great deal about how they can be structured. While acknowledging the importance of teachers’ own negotiating the process of developing (hypothetical) LTs in instruction (Simon 1995), many researchers (e.g. Battista, 2011; Sarama & Clements, 2009; Barrett, et al., 2012; Van den Heuvel-Panhuizen & Buys, 2008) have set about to document likely student behaviors, utterances, and beliefs in order to guide curricular development and aid teachers in leveraging student thinking. This work involves identifying target understandings and likely starting points, and delineating observed likely intermediate events of significance for the respective paths. LTs do not delineate stages as in a Piagetian stage theory (Lehrer & Schauble, 2015; Clements & Sarama, 2014). Instead, they describe meaningful probabilistic states that students are likely to encounter as they work to understand an idea. LTs are not recipes or rules for instruction, but guides, resources, and indicators that can help teachers build on student thinking in moving students toward more sophisticated understandings. These student behaviours, utterances, and beliefs resemble examples of “genetic epistemology,” (Piaget, 1970) episodes with epistemological content drawn from the perspective of the learner and his/her experiences, and which change over time as the results of encountering a series of carefully designed tasks or scaffolded discussions. They also are evidence of the emergent behaviours tied to local instructional theories discussed by Gravemeijer and Cobb (2006).

We have identified several types of epistemological objects that arise repeatedly in middle grades LTs in the Math-Mapper 6-8 learning map (building on earlier recognitions of epistemological objects in student learning research). The first is a naïve or partial conception. An example from equipartitioning is that all equal parts of a whole are congruent. This serves a worthwhile purpose for beginners, and speaks to students’ experience with cut pizza slices, the construct is later constraining, when
students need to discern a variety of shapes of one half of a given whole. A second epistemological object is limited representations, for example, an ordered list of values of data placed into a primitive dot plot that lacks spacing for missing values (known as a case plot). A third type of object that serves an intermediate learning goal is a strategy that may be limited in its efficiency, for instance, forms of skip counting used in repeated addition versions of early multiplication. Other types of objects used to build LTs are cases, as described by variation theory (Marton, 2015) which often are useful in movement up an LT. Typically at higher levels of an LT, one witnesses emergence of properties that then guide the student in how to operate on particular examples, and generalizations that describe how to put strategies and cases together into a structure with varying degrees of justification and proof.

Elaboration, Items, and Assessments. An LT elaboration is a design and development tool that is central for developing the LTs and for ensuring coherence of the learning map with the diagnostic assessments. These “living” documents serve to record and support the evolution of the LT. The LT elaborations specify the wording of each LT level, any (partial) conceptions or misconceptions associated with any specific level, and delineation of cases associated with levels (which typically includes the kinds of numbers or values that are particularly germane to illustrating students’ reasoning and behaviours, and which are used in the assessment items.

The assessment items are all newly designed items developed by the research team to focus on conceptual aspects of the constructs, to support deep student reasoning and flexibility, not just skill development. The elaboration documents are used iteratively as a basis for development of the LT level-specific assessment items. Conversely, the team closely analyses student item response data to evaluate the apparent validity of the LT levels in relation to each other.

Each assessment covers an individual RLC (i.e. one or more constructs), contains 8-10 items, and is designed to require about 20-30 minutes. Multiple forms of the same assessment are developed. Most teachers administer assessments about 2/3 of the way through an instructional unit. They are not intended to be graded, but to provide students and teachers actionable feedback on student (and whole class) understanding of the mathematical concepts. Students typically score between 20-70% correct; retests and practice tests are available to allow students to retry, and to improve their depth of understanding.

Real-time assessment reporting. The student reports show the overall percent correct on each construct, an item matrix that displays each construct, the items the student actually took, and whether the responses were right, wrong, or skipped. Students can select the incorrect or skipped items and resubmit them to change their percent correct. Teachers receive whole-class reports for each construct in the form of a set of “heat maps,” each being a matrix with the LT proficiency levels listed vertically and the students ordered from weakest to strongest overall construct performance along the horizontal axis. The teacher can tap on a progress level to display the related item. Cells are coloured differently for incorrect and for relatively more correct responses.
Based on general expectations of less to more difficulty for higher proficiency levels, the response patterns tend to show increasing correct from bottom left to top right.

**LT and assessment validation.** There may be up to five constructs in an RLC, each with 6-8 levels. Therefore, items must be sampled across the LTs. The multiple assessment forms for each RLC share at least 3 common items, to support whole-class discussion. We encourage teachers to use multiple forms in a class. Each time a student takes an assessment, the results add to our knowledge database of student responses, and to their understanding of the LT, and to our confidence in predicting student progress. We use various psychometric models to explore the optimal modelling of LTs and assessments results. When results for an LT seem unidimensional, IRT is used; otherwise we consider structural equation modeling, CDM, or Bayesian models. These are low-stakes assessment for learning, so the diversity of approaches will add to our understanding of the particular LTs and student reasoning about and learning of constructs, without subjecting this work to artificial constraints regarding dimensionality typical of high stakes assessment modelling.

Math-Mapper 6-8 is being field-tested at three different schools, where the learning map is being incorporated in instructional planning, and the assessments are administered regularly to students, enabling us to collect 50-300 responses per item to analyse. As a result of this the items have gone through a rigorous review and validation process.

Ultimately, this is only the first phase of a complete validation argument. We will be studying the use of the tool over longer periods of time, which will allow us to determine how students improve understanding with the use of the tool, if teachers can use the tool to elicit more student thinking and participation, and find ways to improve the performance of various subgroups of students.

**Siemon and Horne**

Our research on learning progressions is premised on a socio-cultural perspective of learning that views learning “as both a process of active individual construction and a process of enculturation into the mathematical practices of wider society” (Cobb, 1994, p. 13). It is aimed at identifying optimal pathways for teaching and learning key aspects of school mathematics based on an assessment of what might be regarded as students’ taken-as-shared knowledge in Australian mathematics classrooms. A valid criticism of this approach is that it does not necessarily reflect what is possible when students are exposed to high quality mathematics teaching over time (e.g., Boaler, 2008). But the reality is that not all teachers have the knowledge, confidence and local support needed to implement high quality effective practices. Nor do they necessarily have the time and resources to identify each student’s particular learning needs in relation to every single aspect of the mathematics curriculum even if this was desirable. The main rationale for working at scale in relation to a small number of really big ideas in mathematics is that this establishes a plausible, probabilistic model for establishing where learners are in their learning journey in relation to those ideas.
critical to student’s progress in school mathematics (Siemon, Bleckly & Neal, 2012) and a framework to support teachers progress learning. The following sections will summarise our work.

**The Middle Years Numeracy Research Project (MYNRP, 1999-2001)**

A detailed analysis of the distribution of item responses provided by just under 7000 students in the initial phase of the MYNRP project revealed that there was as much variation in performance in any one year level as there was in the whole cohort and that this difference in curriculum terms was of the order of 7 years (i.e., approximately Year 2 to Year 8). While there were variations in measurement and data sense and spatial sense, all of the more difficult items were concerned with number sense, in particular anything that involved multiplying and dividing larger whole numbers, proportional reasoning, fractions, decimals and percentages, and situations not easily modelled in terms of a count of equal groups (e.g., combinatoric problems and problems involving rate or ratio). Characterised by Vergnaud (1988) in terms of the multiplicative conceptual field, these results prompted a follow-up project, the aim of which was to develop a more finely grained, evidence-based learning progression for multiplicative thinking that could be used by teachers to identify starting points for teaching and progress student learning.

**Scaffolding Numeracy in the Middle Years (SNMY, 2003-2006)**

At the time, there was a considerable body of literature concerned with particular aspects of multiplicative thinking. However, very little of this was specifically concerned with how these aspects relate to one another and how and when to support new learning both within and between these different aspects of multiplicative thinking (Siemon & Breed, 2006). Given evidence to suggest that where teachers are supported to identify and interpret student learning needs in terms of teacher accessible, evidence-based frameworks, they were more informed about where to start teaching, and better able to scaffold their students’ mathematical learning (e.g., Clarke, 2001), it seemed sensible to produce a similar framework for multiplicative thinking.

For the purposes of the SNMY project, multiplicative thinking was defined by: a capacity to work flexibly and efficiently with an extended range of numbers (e.g., larger whole numbers and rational number); an ability to recognise and solve a range of problems involving multiplication or division; and the means to communicate this effectively in a variety of ways (for example, words, diagrams, symbolic expressions, and written algorithms).

Initially a broad HLT, derived from a synthesis of the research literature on students’ understanding of multiplicative thinking, proportional reasoning, decimal place-value and rational number was developed (see Siemon & Breed, 2006). The HLT was used to select, modify and/or design a range of rich tasks including two extended tasks (e.g., Callingham & Griffin, 2000). The tasks were trialled and either accepted, rejected or further modified on the basis of their accessibility to the cohort, discriminability, and perceived validity in terms of the constructs being assessed. Secondly these rich
assessment tasks and partial credit scoring rubrics were trialled and subsequently used to inform the development of the learning and assessment framework for multiplicative thinking (LAF). Finally an eighteen month action research study involving research school teachers and the research team, progressively explored a range of targeted teaching aimed at scaffolding student learning in terms of the LAF.

The results from the first round of assessment of just over 1500 year 4 to 8 students were analysed using item response theory and the subsequent variable map was used to link different aspects of multiplicative thinking and identify qualitatively different levels of understanding and strategy usage indicated by student responses (Siemon, Izard, Breed & Virgona, 2006). While these levels were largely consistent with the initial HLT, we were able to collapse one level and elaborate on others. Rich text descriptions for each level were derived from the performances on each item at each level to form the basis of the LAF. In acknowledgement that the levels were approximations based on responses identified at similar locations on the scale and in recognition of the fact that the purpose of the LAF was to help teachers scaffold student learning, the levels were referred to as zones. The LAF so derived comprises eight hierarchical zones ranging from additive, count all strategies in Primitive Modelling (Zone 1) through Intuitive Modelling, Sensing, Strategy Exploring, Strategy Refining, Strategy Extending, and Connecting to the sophisticated use of proportional reasoning in Reflective Knowing (Zone 8).

The notion of targeted teaching and the subsequent use of the LAF will be described in a later section but it suffices to say here that the teaching response to student’s identified learning needs tended to be more effective in primary (i.e., Year 5 and 6 classrooms) than in Years 7 to 8 classrooms (Siemon, Breed, Dole, Izard & Virgona, 2006).

Reframing Mathematical Futures Priority Project (RMF, 2013)

Funding was obtained from the Australian Mathematics Science Partnership Programme (AMSPP) Priority Project round to explore the efficacy of and the issues involved in implementing a targeted teaching approach in secondary schools using the SNMY materials. Twenty-eight schools located in lower-socio economic settings across Australia participated in the 10-month study. Nominated ‘specialists’ in each school were provided with professional learning and supported to work with at least two other teachers at their school to implement a targeted teaching approach to multiplicative thinking. The SNMY assessments were conducted in August and November of 2013. Matched data sets were obtained from 1732 students from Years 7 to 10 with the majority (59%) from Year 8 (Siemon, 2016). The overall achievement of students across the 28 schools grew above an adjusted effect size of 0.6 indicating a medium influence beyond what might be expected (Hattie, 2012).

Reframing Mathematical Futures II Project (RMFII, 2014-2017)

The RMFII project is an AMSPP Competitive Grant project that was formulated in direct response to the findings of the initial RMF project. That is, that one of the major
reasons for secondary school teachers’ reluctance to adopt a targeted teaching approach to multiplicative thinking was their perception that this was not related to the curriculum they were expected to teach. Even though an analysis of the Australian mathematics curriculum at the time found that approximately 75% of the Year 8 curriculum required or assumed student access to multiplicative thinking (Siemon, 2013). The project aims to develop, trial and evaluate a learning and teaching resource to support algebraic, statistical and spatial reasoning in Years 7 to 10 that will enable teachers to identify and respond to student learning needs using a targeted teaching approach aimed at improving students’ mathematical reasoning. For this purpose, mathematical reasoning is seen to encompass the core knowledge needed to recognise, interpret, represent and analyse algebraic, spatial, statistical and probabilistic situations and the relationships/connections between them; an ability to apply that knowledge in unfamiliar situations to solve problems, generate and test conjectures, make and defend generalisations; and a capacity to communicate reasoning and solution strategies in multiple ways (i.e. diagrammatically, symbolically, orally and in writing) (Siemon, 2013; 2016).

This is a non-trivial exercise involving an extended research team with recognised expertise in each domain. It requires the identification of Draft Learning Progressions (DLPs) for algebraic, spatial and statistical reasoning from existing research, the development and validation of rich tasks to assess and refine the DLPs using item response theory, the preparation of targeted teaching advice, and the development and trial of a series of online professional learning modules. To date, DLPs have been identified from the literature for algebra, geometry and statistical reasoning and over 250 individual assessment items have been trialled with more than 3600 students in Years 7 to 10. The initial analysis provides ‘proof of concept’, that is, that it is possible to scale the underlying constructs. Further trial work is being undertaken at the time of writing to validate and elaborate the scales.

APPLICATIONS OF LEARNING PROGRESSIONS/TRAJECTORIES

This section differs from the previous two in that it has amalgamated the responses of the four research teams to highlight the ways in which LT/Ps are being used to impact practice and shape further research. Once again only key references will be included here in the interests of space.

Curriculum and Standards

Three of the four bodies of work reported here used national curriculum statements and/or standards as a starting point for their work on learning trajectories/progressions. As this work unfolded, however, it became increasingly clear that researchers needed to go beyond such documents and look to the research literature more generally to inform their investigations. This had the added advantage of not only informing curriculum development and examining the effectiveness of that curriculum but building a better and deeper understanding of what was needed to achieve curriculum goals even to the extent of providing evidence that questioned the appropriateness of
those goals. This changed the role of LT/P’s from serving mainly as the core of curriculum development projects to having implications for all aspects of curriculum. For example, Clements and his colleagues developed a number of LTs for the NSF-funded Building Blocks project and curriculum (Clements & Sarama, 2013a). While this was designed to comprehensively address standards for early mathematics education for all children, evaluations have shown that Building Blocks can be effective, with large effect sizes even when compared to another research-based curriculum not built upon LTs (Clements & Sarama, 2008).

This and other work in this area led Clements and his colleagues to conclude that any comprehensive and valid scientific curriculum development program in education should address two basic issues - effect and conditions - across three domains - practice, policy, and theory. For instance, the question - is the curriculum effective in helping children achieve specific learning goals? examines effects in relation to practice. The question - are the curriculum goals important? – examines effects in relation to policy, and the question – why is the curriculum effective? – invites an exploration of effects in relation to theory. To achieve these goals satisfactorily and scientifically, developers must draw from existing research so that what is already known can be applied to the anticipated curriculum; used to structure and revise curricular components in accordance with models of children’s learning such as research-based learning trajectories; and conduct formative and summative evaluations in a series of progressively expanding social contexts. As an example of this process, Clements and Sarama offer their work on TRIAD (Technology-enhanced, Research-based, Instruction, Assessment, and professional Development model), which has been implemented at scale and evaluated.

TRIAD is based on research and enhanced by the use of trajectories and technology. TRIAD places learning trajectories at the core of the teacher/child/curriculum triad to ensure that curriculum, materials, instructional strategies, and assessments are aligned. When implemented with fidelity, TRIAD has shown moderate to strong effects including transfer to other domains (e.g., Sarama, Clements, Wolfe & Spitler, 2012).

As with many researchers in the area Confrey and Maloney started with a specific LT (equipartitioning), then expanded their efforts to examine and analyse K-8 learning in all subfields. They did this first by analysing the U. S. Common Core Standards from a perspective of learning trajectories, but subsequently by building a new tool that uses learning trajectories for guiding instruction and scaffolding digital curriculum. The example they offer is the collaborative work on the Common Core Standards where a group of learning trajectory researchers participated in a joint meeting with the Common Core sponsors and writers, and subsequently provided the writers with summaries of the research to guide their grade-by-grade analysis (Confrey & Maloney 2014). A member of the National Validation Committee, Confrey mapped several early versions of the standards for consistency with the results of that overall research, and made recommendations for strengthening those connections. As with any document subject to competing perspectives, the final CCSS-M seemed consistent in
many areas, and weaker in others. However, this points to the growing recognition of the value that research-based LT/P might play in determining goals for national standards, assessments, curricula, and pedagogy.

In Australia, the National mathematics curriculum is represented by a set of content descriptors (approximately 28 per Year Level) and schools have more control over the instructional materials and pedagogical approaches they use to address the content descriptors. Effect sizes in excess of 0.65 across a number of secondary schools as a result of using the Learning Assessment Framework for Multiplicative thinking (LAF) in 2013 has prompted schools to modify their curriculum offerings in order to accommodate a targeted teaching approach to multiplicative thinking across multiple year levels (Siemon, 2016).

Students and Learning

LT research began with a clear focus on children’s thinking and learning in specific content domains. Initially the focus was on individual student developing schemas in particular mathematical areas (e.g., children’s increasingly sophisticated counting schema, Tzur et al, 2013). While that work continues, there has also been an expansion in the focus of LT work to whole classes and multiple year level cohorts with a particular emphasis on the development and use of formative assessment tools to identify where learners are in their learning journey and better equip teachers to progress that learning (e.g., Sarama, Clements, Wolfe & Spitler, 2012; Siemon, 2016).

Confrey and her colleagues are currently working with multiple schools in multiple school districts with Maths Mapper, an LT-based digital learning system that, among other things, is designed to support the creation of continuity across grades and promote the surfacing of student thinking and strengthening of student agency (Confrey & Maloney, 2015).

Most LT/Ps have been developed and refined with school student populations. However, their application in adult settings has recently been explored by Tzur with both teacher and non-teacher adult learners, many of whom lack foundational schemes for multiplicative and/or fractional reasoning. He has found that applying these LT/Ps has been helpful for these adult learners as well as for children identified by their school systems as students with learning disabilities in mathematics.

An important question arises about LTs developed through studies in western cultures, namely, do they apply to or represent the learning of learners in other cultures. Are these learning frameworks universal or are they a consequence of what learners have had the opportunity to learn?

Teachers and Teaching

As many before, LT/P researchers recognise the importance of looking at domains of knowledge as a means of supporting teachers to better understand the connections between different aspects of mathematics and how that learning might be progressed. A consistent finding of this research is that a major way in which this occurs is through
teachers observing their children’s learning. The value of using assessment data to inform and improve teaching is widely recognised but the difference here is that the observations can be tied to evidence-based frameworks that provide guidance on where to go next in relation to a range of interconnected ideas. This lead Siemon et al (2006) to conclude that a different term, targeted teaching, was needed to distinguish the long-term, multi-faceted nature of the interventions needed to scaffold student’s multiplicative thinking from the equally valid but short-term or spontaneous teaching decisions that might be informed by a pre-test on subtraction or an informal observation of student thinking in the course of a classroom discussion. Targeted teaching is characterized by an unrelenting focus on big ideas framed by evidence-based LT/Ps. It is not a prescribed program, schools and teachers need to appropriate it to their circumstances and capabilities. It is a very organic process that is not in anyway equivalent to systematic streaming/tracking and it is most effective where it has evolved over time with the support of key individuals and the leadership group (Siemon, 2016).

Another way in which LT/Ps support teachers is by providing a shared language around a set of activities and tasks that point to the underlying conceptual structure of the mathematics that is the focus of the LT/P. For example, strengthening teacher community is an important focus of the LT-based Math-Mapper resource. Confrey and Maloney (2015) report that teams of teachers are planning their curriculum using the learning map instead of a set of standards elicit a different kind of conversation about topics. In one school, a teacher described the prior curriculum as “chaotic” and the new one as “calm.” The teachers at the other district found that discussing clusters instead of individual standards helped them ensure that the ideas meant the same thing to them all. They often appealed to the LTs to clarify their thinking (Confrey & Maloney, 2015).

Teacher professional learning has been an element in the trialling, validating and scaling up of LT/Ps across all bodies of work reported here but more recently this has become the focus of research in this area. An example of this is Tzur’s current study of the impact of job-embedded professional development on upper-elementary teachers’ transition toward student-adaptive pedagogy. A substantial part of which engages teachers in learning to notice, infer, and use the two HLT about students’ multiplicative and fractional schemes.

The power of LT/Ps to impact teaching practice and sustain quality approaches over time is evidenced by the follow up work on the Building Blocks project. Clements, Sarama and colleagues expected teachers to decrease in the fidelity in which they taught with learning trajectories after project support was discontinued. However, after two years, they found that the teachers increased the quality of their teaching and the these results were even more positive six years later with the largest predictor of higher fidelity years out was child gain—teachers sustain and increase the quality of teaching when they observe their children learning.
Informing and Extending Research

LT/Ps and the research around them are being used to inform new research. For example Tzur and his colleagues use them (a) to identify participants for a study based on their available, assimilatory schemes and (b) as a suggestive, developmental framework for determining what to teach next. On a much larger scale and with more of an eye to impacting practice at scale, the work of Siemon and her colleagues on mathematical reasoning, Sarama and Clements on Building Blocks and Confrey’s work on Maths-Mapper point to an exciting future for LT/P research and development, particularly in relation to technology.

The implications of developing a dynamic digital learning system built around LTs represents a new paradigm of research and opens new possibilities for networked improvement models (Confrey & Maloney, 2015). This is because the design rests on an explicit learning theory (the LT/Ps) while the tool scaffolds curriculum flexibly and adaptably. In the case of Maths Mapper, the research team continuously monitors the tool’s use in a variety of ways—how and when it is used, how long students need to complete the items and assessments, how the items perform, and which psychometric models provide the best data models to inform the tool’s use. The communities of practice (students, teachers, curriculum specialists and administrators) are also leveraging the tools to plan, to develop new forms of instructional practice, to form student groups (or reteach) and to try out and refine materials. The focus is on student growth and on how different subgroups and individuals are able to get assistance and opportunities to learn as needed.

Research on LT/Ps is becoming more ambitious in its scope and intent. While this has the potential to transform the teaching and learning of mathematics through the provision of evidence-based frameworks, validated tools and quality instructional materials, reconceptualise the curriculum, and deepen teacher knowledge of the rich connections between different but related aspects of mathematics, at the end of the day it is the decisions teachers and students make every day that have the greatest impact on learning. For this work to have a sustainable influence on practice, it needs the support of school leadership, administrators working in close collaboration with researchers as partners.

References


