Non-linear Dynamic Modelling and Optimal Control of Aerial Tethers for Remote Delivery and Capture of Payloads

A thesis submitted in fulfilment of the requirements for the degree of Doctor of Philosophy

Daniel E. Sgarioto
B. Eng.

School of Aerospace, Mechanical and Manufacturing Engineering
Science, Engineering and Technology Portfolio
RMIT University

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Declaration

I certify that except where due acknowledgement has been made, the work is that of the author alone; the work has not been submitted previously, in whole or in part, to qualify for any other academic award; the content of the thesis is the result of work which has been carried out since the official commencement date of the approved research program; and, any editorial work, paid or unpaid, carried out by a third party is acknowledged.

Daniel Emmanuel Sgarioto
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Abstract

Many potentially useful applications have been recently proposed for aerial towed-cable systems, including precise capture and delivery of payloads, water bombing of wildfires, remote geophysical sensing and environmental monitoring. These applications broadly fall under the umbrella of payload transportation operations for the aerial towed-cable system. There remain critical outstanding issues concerning the dynamics and control of aerial towed-cable systems that are inhibiting the near-term demonstration of these proposed transportation operations. The development of simplified representations of aerial towed-cable systems, capable of capturing the important dynamics, yet simple enough to be suitably used for control system development purposes is limited, even in light of the substantial benefits such research would provide. Likewise, relatively little research has been undertaken into the development of control systems for aerial towed-cable systems, especially the development of towing strategies and cable-based control techniques for rendezvous and payload transportation. Thus, this thesis presents novel research into the development of control strategies and simulation facilities that redress these two major anomalies, thereby overcoming a number of hitherto unresolved issues.

The primary objective of this thesis is to develop innovative non-linear optimal control systems to manoeuvre a cable towed beneath an aircraft to transport payloads both to and from surface locations. To appropriately satisfy this objective, accurate and efficient modelling capabilities are proposed, yielding the equations of motion for numerous models of the aerial towed-cable system, each possessing varying degrees of sophistication. A series of techniques for improving the representativeness of the simpler dynamical models are introduced in this thesis. The benefits associated with using these procedures were shown to be significant and possible without undue complexity or considerable computational expense. Use of such techniques makes it possible for accurate simulations to be performed and representative control systems to be designed, using relatively simple dynamic models of the aerial towed-cable system.

A series of single and multi-phase non-linear optimal control problems for aerial towed-cable systems are formally proposed in this thesis, which in turn are converted into non-linear programming problems using direct transcription for expedient and rapid solution. The possibility of achieving accurate, numerous instantaneous rendezvous of the cable tip with desired surface locations on the ground, in two and three-dimensions, is successfully
demonstrated. This was achieved through the use of deployment and retrieval control of the cable and/or aircraft manoeuvring. More importantly, the capability of the system to safely and accurately transport payloads to and from the surface via control of the cable and/or aircraft manoeuvring is also established in this thesis. A series of parametric studies are conducted to establish the impact that various parameters have on the ability of the aerial towed-cable system to perform various rendezvous and payload transportation operations. These parametric studies allowed important insights into the nature of the system to be examined, appreciably expanding the knowledge of concepts related to the dynamics and control of aerial towed-cable systems.

In order for the aerial towed-cable system to perform rendezvous and payload transportation operations in the presence of time-varying wind gusts, a number of simple closed loop optimal feedback controllers are proposed in this work. These feedback controllers are based on the linear quadratic regulator control methodology. A preliminary indication of the robustness of the aerial towed-cable system to wind gusts is provided for through a succession of parametric investigations. The performance of the closed-loop system demonstrates that precision and robust control of the aerial towed-cable system can be achieved for a wide variety of operating conditions.

The research presented in this thesis constitutes a significant contribution to the current state of expertise and will provide a solid foundation for further advancing the development of aerial tether payload transportation technology. The dynamic modeling facilities and various control systems developed in this dissertation, together with existing technology and appropriately established infrastructure, should enable the experimental demonstration of aerial tether transportation systems in the near-term.
Nomenclature

Multi-Body System Modelling

\( \mathbf{F}_i \)  
External force vector for the \( i^{th} \) particle

\( m_i \)  
Mass of the \( i^{th} \) particle

\( \mathbf{a}_i \)  
Acceleration vector of the \( i^{th} \) particle

\( \mathcal{L} \)  
Lagrangian of the system

\( \mathcal{T} \)  
Kinetic energy of the system

\( \mathcal{V} \)  
Potential energy of the system

\( Q_i \)  
Generalized non-conservative forces acting on the system

\( q_i \)  
The \( i^{th} \) generalized coordinate of the system

\( \dot{q}_i \)  
First-order time derivative of the \( i^{th} \) generalized coordinate

\( \ddot{q}_i \)  
Second-order time derivative of the \( i^{th} \) generalized coordinate

\( d\mathbf{r}_i \)  
Virtual displacement of the \( i^{th} \) particle

\( dt \)  
Differential time

\( dW \)  
Virtual work performed by the system

\( \mathbf{v}_i \)  
Velocity vector of the \( i^{th} \) particle

\( u_i \)  
The \( j^{th} \) generalized speed of the \( i^{th} \) particle

\( \frac{\partial \mathbf{v}_i}{\partial u_j} \)  
The \( j^{th} \) partial velocity of the \( i^{th} \) particle

\( \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} \)  
The \( j^{th} \) generalized velocity of the \( i^{th} \) particle

\( F^*_k \)  
Generalized inertia forces of the system

\( F_k \)  
Generalized active forces of the system

\( \mathbf{\omega}_i \)  
Angular velocity vector of the \( i^{th} \) body

\( \mathbf{\alpha}_i \)  
Angular acceleration vector of the \( i^{th} \) body

\( \frac{\partial \mathbf{\omega}_i}{\partial u_k} \)  
\( k^{th} \) partial angular velocity of the \( i^{th} \) body

\( \mathbf{T}_i \)  
Active resultant moment on the \( i^{th} \) body

\( \mathbf{I}_i \)  
Principle inertia dyadic of the \( i^{th} \) body
Coordinate Systems and Reference Frames

\[ x-y-z \] Non-rotating Cartesian reference frame

\[ r-\theta-\phi \] Normal-tangential reference coordinate system

\[ b_1-b_2 \] Local body reference coordinate frame

e_i \quad \text{Unit vector in the } x\text{-direction}
e_j \quad \text{Unit vector in the } y\text{-direction}
e_k \quad \text{Unit vector in the } z\text{-direction}
e_r \quad \text{Unit vector in the } r\text{-direction}
e_\theta \quad \text{Unit vector in the } \theta\text{-direction}
e_\phi \quad \text{Unit vector in the } \phi\text{-direction}

e_D \quad \text{Unit vector defining the direction of the aerodynamic drag force}
e_L \quad \text{Unit vector defining the direction of the aerodynamic lift force}

\[ C_k^{I/B} \] Direction cosine matrix that relates the global inertial frame to the local body frame of the \( k \text{th} \) cable segment

Aerial Towed-Cable System Modelling

\( \theta \) In-plane angle the cable makes with the local vertical

\( \theta_k \) Orientation angle of the \( k \text{th} \) rigid/extensible cable segment

\( \alpha_k \) Angle of attack of the \( k \text{th} \) rigid cable segment

\( \phi \) Out-of-plane angle the cable makes with the local vertical

\( l \) Instantaneous cable length

\( l_k \) Length of the \( k \text{th} \) rigid cable segment

\( L_{ks} \) Unstretched length of the \( k \text{th} \) extensible cable segment

\( x \) Horizontal displacement of the aircraft

\( y \) Vertical displacement of the aircraft (planar model)/Lateral displacement of the aircraft (three-dimensional model)

\( z \) Vertical displacement of the aircraft (three-dimensional model)

\( \dot{\theta} \) In-plane angular velocity of the cable

\( \dot{\phi} \) Out-of-plane angular velocity of the cable

\( \dot{i} \) Cable reel rate

\( \dot{x} \) Horizontal velocity of the aircraft

\( \dot{y} \) Vertical velocity of the aircraft (planar model)/Lateral velocity of the aircraft (three-dimensional model)
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \dot{z} )</td>
<td>Vertical velocity of the aircraft (three-dimensional model)</td>
</tr>
<tr>
<td>( \ddot{\theta} )</td>
<td>In-plane angular acceleration of the cable</td>
</tr>
<tr>
<td>( \ddot{\phi} )</td>
<td>Out-of-plane angular acceleration of the cable</td>
</tr>
<tr>
<td>( \ddot{l} )</td>
<td>Cable reel acceleration</td>
</tr>
<tr>
<td>( \dddot{x} )</td>
<td>Horizontal acceleration of the aircraft</td>
</tr>
<tr>
<td>( \dddot{y} )</td>
<td>Vertical acceleration of the aircraft (planar model)/Lateral acceleration of the aircraft (three-dimensional model)</td>
</tr>
<tr>
<td>( \dddot{z} )</td>
<td>Vertical acceleration of the aircraft (three-dimensional model)</td>
</tr>
<tr>
<td>( \mathbf{p}_k )</td>
<td>Generalized coordinate vector for the ( k )th extensible cable segment within the flexible multi-link aerial towed-cable system model</td>
</tr>
<tr>
<td>( \mathbf{r}_p )</td>
<td>Position vector to the payload from the origin of the Cartesian reference frame</td>
</tr>
<tr>
<td>( \mathbf{v}_p )</td>
<td>Velocity vector for the payload</td>
</tr>
<tr>
<td>( \mathbf{v}_N )</td>
<td>Normal velocity vector for the payload</td>
</tr>
<tr>
<td>( \mathbf{v}<em>{N</em>\theta} )</td>
<td>Payload normal velocity vector parallel to the unit vector in the ( \theta )-direction</td>
</tr>
<tr>
<td>( \mathbf{v}<em>{N</em>\phi} )</td>
<td>Payload normal velocity vector parallel to the unit vector in the ( \phi )-direction</td>
</tr>
<tr>
<td>( \mathbf{v}_T )</td>
<td>Payload tangential velocity vector parallel to the unit vector in the ( r )-direction</td>
</tr>
<tr>
<td>( \mathbf{v}_k^c )</td>
<td>Velocity vector of the centre of the ( k )th rigid cable segment</td>
</tr>
<tr>
<td>( \mathbf{v}_k^b )</td>
<td>Velocity vector for the ( k )th extensible cable segment in the local body frame</td>
</tr>
<tr>
<td>( \Delta y )</td>
<td>Change in altitude of the payload (planar model)</td>
</tr>
<tr>
<td>( \Delta z )</td>
<td>Change in altitude of the payload (three-dimensional model)</td>
</tr>
<tr>
<td>( T_n )</td>
<td>Tension within the cable</td>
</tr>
<tr>
<td>( T_{k-1}, T_k )</td>
<td>Tension within the cable adjacent to the ( k )th extensible cable segment</td>
</tr>
<tr>
<td>( \mathbf{M} )</td>
<td>Mass matrix of the rigid multi-link aerial towed-cable system model</td>
</tr>
<tr>
<td>( \begin{bmatrix} M_k^I \end{bmatrix} )</td>
<td>Mass matrix of the flexible multi-link aerial towed-cable system model</td>
</tr>
<tr>
<td>( \mathbf{F}_k^g )</td>
<td>Resultant gravitational force acting on the ( k )th extensible cable segment</td>
</tr>
<tr>
<td>( \mathbf{F}_k^s )</td>
<td>Resultant spring force acting on the ( k )th extensible cable segment</td>
</tr>
<tr>
<td>( F_{2(i-1)+i}^c )</td>
<td>Generalized spring forces for the flexible multi-link cable model</td>
</tr>
<tr>
<td>( \mathcal{K}_{j-1}, \mathcal{K}_j )</td>
<td>Constants used to calculate the generalized spring forces for the flexible multi-link cable model</td>
</tr>
<tr>
<td>( \mathbf{F}_k^c )</td>
<td>Resultant damping force acting on the ( k )th extensible cable segment</td>
</tr>
<tr>
<td>( F_{2(i-1)+i}^c )</td>
<td>Generalized damping forces for the flexible multi-link cable model</td>
</tr>
</tbody>
</table>
\( \mathcal{D}_{i-1}, \mathcal{D}_i \) Constants used to calculate the generalized damping forces for the flexible multi-link cable model

\( \mathbf{F}^d_k \) Generalized aerodynamic drag force acting on the \( k \)th extensible cable segment

\( \mathbf{D}^h_k \) Generalized aerodynamic drag forces for the \( k \)th extensible cable segment in the local body frame

\( \mathbf{F}^{drag}_k \) Aerodynamic drag force acting on the \( k \)th rigid cable segment

\( \mathbf{F}^{lift}_k \) Aerodynamic lift force acting on the \( k \)th rigid cable segment

\( \mathbf{F}_{drag} \) Aerodynamic drag force acting on the payload

\( \mathbf{x}(t) \) State-space vector for the system

\( \dot{\mathbf{x}}(t) \) Time derivative of the state-space vector for the system

\( \mathbf{u}(t) \) Control vector for the system

**Aerial Towed-Cable System Parameters**

\( N \) Number of cable segments

\( m_p \) Mass of the payload

\( \rho_c \) Mass density of the cable

\( E \) Young’s Modulus for the cable material

\( C_{eq,c} \) Structural damping coefficient for the extensible cable

\( g \) Gravitational constant

\( \rho_{air} \) Atmospheric air density

\( C_D_r \) Aerodynamic drag coefficient of the payload

\( C_{n_k} \) Normal aerodynamic drag coefficient for the rigid cable

\( C_{t_k} \) Tangential aerodynamic drag coefficient for the rigid cable

\( C_{D_k} \) Aerodynamic drag coefficient of the \( k \)th rigid cable segment

\( C_{k_a} \) Aerodynamic lift coefficient of the \( k \)th rigid cable segment

\( C_{f_k} \) Skin friction drag coefficient of the \( k \)th rigid cable segment

\( C_{n_k} \) Cross-flow drag coefficient of the \( k \)th rigid cable segment

\( M_{n_k} \) Local normal Mach Number for the \( k \)th rigid cable segment

\( M_{t_k} \) Local tangential Mach Number for the \( k \)th rigid cable segment

\( a_k \) Speed of sound for the \( k \)th rigid cable segment

\( T_{air_k} \) Ambient air temperature in the vicinity of the \( k \)th rigid cable segment
\(d_p\)            Diameter of the payload
\(d_c\)            Diameter of the cable
\(A_p\)            Projected area of the payload
\(A\)              Cross sectional area of each extensible cable segment
\(x_0\)            Initial position of the aircraft in the \(x\)-direction
\(y_0\)            Initial position of the aircraft in the \(y\)-direction
\(z_0\)            Initial position of the aircraft in the \(z\)-direction
\(\theta_0\)        Initial cable in-plane configuration angle
\(l_0\)            Initial cable length
\(U_0\)            Initial horizontal velocity of the aircraft/towing speed
\(V_0\)            Initial lateral velocity of the aircraft/towing speed
\(W_0\)            Initial vertical velocity of the aircraft/towing speed
\(\dot{z}_1\)        Aircraft velocity in the \(x\)-direction for the flexible multi-link aerial towed-cable system model
\(\dot{z}_2\)        Aircraft velocity in the \(y\)-direction for the flexible multi-link aerial towed-cable system model

**Aerial Towed-Cable System Model Matching**

\(t_{\text{initial}}\) Initial time used to perform case study simulations
\(t_{\text{final}}\) Final time used to perform case study simulations
\(C_{d_{\text{new}}}\) Updated payload drag coefficient for the single-link aerial towed-cable system model
\(l_{\text{new}}\) Updated cable length for the single-link aerial towed-cable system model
\(\theta_E\)        Equivalent equilibrium configuration angle for the multi-link aerial towed-cable system models
\(\dot{\theta}_E\)    First order time derivative of the equivalent equilibrium configuration angle for the multi-link aerial towed-cable system models
\(\ddot{\theta}_E\)   Second order time derivative of the equivalent equilibrium configuration angle for the multi-link aerial towed-cable system models
\(l_E\)             Equivalent cable length for the multi-link aerial towed-cable system models
\(i_E\)             First order time derivative of the equivalent cable length for the multi-link aerial towed-cable system models
\(\dddot{i}_E\)      Second order time derivative of the equivalent cable length for the multi-link aerial towed-cable system models
\( L_{\text{REF}} \) Non-dimensional term used within the cost function for model “matching”

\( J \) Cost function used for optimization to achieve model “matching”

\( x_E \) \( x \)-component of the equivalent equilibrium cable tip position

\( y_E \) \( y \)-component of the equivalent equilibrium cable tip position

\( \theta \) Equilibrium angle of the \( k \)\(^{th} \) rigid cable segment

\( x_k \) \( x \)-coordinate of the \( k \)\(^{th} \) extensible cable segment at equilibrium

\( y_k \) \( y \)-coordinate of the \( k \)\(^{th} \) extensible cable segment at equilibrium

\( \theta_e \) Equilibrium cable angle for the single-link aerial towed-cable system model

\( x_e \) \( x \)-coordinate of the equilibrium cable tip position for the single-link aerial towed-cable system model

\( y_e \) \( y \)-coordinate of the equilibrium cable tip position for the single-link aerial towed-cable system model

\( \theta_{e\text{ax}} \) Updated equilibrium angle for the single-link aerial towed-cable system model

\( x_{e\text{ax}} \) Updated \( x \)-component of the equilibrium cable tip position for the single-link aerial towed-cable system model

\( y_{e\text{ax}} \) Updated \( y \)-component of the equilibrium cable tip position for the single-link aerial towed-cable system model

\( p_1 \) Adjustment to the payload drag coefficient required by the single-link aerial towed-cable system model

\( p_2 \) Adjustment to the cable length required by the single-link aerial towed-cable system model

\( V_{\text{predict}} \) Towing speed used to calculate parameter adjustments for the single-link aerial towed-cable system model

\( l_{\text{predict}} \) Cable length used to calculate parameter adjustments for the single-link aerial towed-cable system model

\( \dot{x}_{\text{initial}} \) \( x \)-component of the initial towing speed used to calculate parameter adjustment for the single-link aerial towed-cable system model

\( \dot{y}_{\text{initial}} \) \( y \)-component of the initial towing speed used to calculate parameter adjustment for the single-link aerial towed-cable system model

\( \dot{x}_{\text{final}} \) \( x \)-component of the final towing speed used to calculate parameter adjustment for the single-link aerial towed-cable system model

\( \dot{y}_{\text{final}} \) \( y \)-component of the final towing speed used to calculate parameter adjustment for the single-link aerial towed-cable system model

\( l_{\text{initial}} \) Initial cable length used to calculate parameter adjustment for the single-link aerial towed-cable system model
Final cable length used to calculate parameter adjustment for the single-link aerial towed-cable system model

$f(l_E)$ Empirical function used to update the adjustment to the payload drag coefficient required by the single-link aerial towed-cable system model

$K_1, K_2$ Tuning parameters for the empirical function $f(l_E)$

$\Delta \theta$ Percentage error in “match” of the equilibrium configuration angle

$\Delta x$ Percentage error in “match” of the $x$-component of equilibrium cable tip position

$\Delta y$ Percentage error in “match” of the $y$-component of equilibrium cable tip position

**Optimal Control Theory**

$J$ Performance criterion/cost function/performance index/optimality criterion

$M$ Terminal cost component of the optimality criterion

$L$ Chosen performance/weighting index component of the optimality criterion

$w_k$ Quadrature weights used to numerically integrate the performance/weighting index

$x(t_0)$ Initial conditions for the system

$\psi_f$ Final boundary conditions for the system

$\hat{u}(t)$ Optimal value of the control for the system

$\delta J$ First variation of the cost function for the system

$\lambda(t)$ Lagrange multipliers/co-states of the system

$\mathcal{H}$ Hamiltonian for the system

$x_0(t)$ Initial state space trajectory for the system

$u_0(t)$ Initial nominal control time history

$\alpha$ Step size for steepest descent algorithm

$\alpha^*$ Optimum step size for steepest descent algorithm

$s_i(t)$ Conjugate search direction for conjugate gradient algorithm

$\beta_i$ Conjugate search direction parameter used within the conjugate gradient algorithm

$N$ Discretization level/number of nodes for the non-linear optimization problem

$t_0$ Initial time for the non-linear optimization problem

$t_k$ Discretized time domain for the non-linear optimization problem

$t_{nf}, t_f$ Final time for the non-linear optimization problem
Discretized state space trajectory for the non-linear optimization problem

Discretized control trajectory for the non-linear optimization problem

Decision variables for the non-linear optimization problem

Initial conditions for the non-linear optimization problem

Event condition for the $i^{th}$ phase of the non-linear optimization problem

Final conditions for the non-linear optimization problem

Dynamical state equality constraints for the non-linear optimization problem

General non-linear path constraints for the non-linear optimization problem

Step size for local discretization schemes

Estimate of the state variables at node $k + 1$

Defects of the state space constraint function to be minimized during the non-linear optimization problem

Value of the states in the centre of each interval for the Hermite Simpson discretization method

Value of the controls in the centre of each interval for the Hermite Simpson discretization method

Computational time domain/nodal point spacing for the Pseudospectral discretization method

$N^{th}$ order Jacobi polynomial used during Pseudospectral discretization

Defining parameters for the $N^{th}$ order Jacobi polynomials

Approximation of the state variables using the $N^{th}$ order Jacobi polynomials

Approximation of the time derivative of state variables

Approximation of the control variables using the $N^{th}$ order Jacobi polynomials

Lagrange interpolating polynomials

Coefficients of the state interpolating polynomials

Coefficients of the control interpolating polynomials

Parameters used to define the Lagrange interpolating polynomials

Jacobi differentiation matrix for the Pseudospectral discretization method

Aerial Towed-Cable System Optimal Control

$x_r$ $x$-coordinate of the first phase cable tip rendezvous location

$y_r$ $y$-coordinate of the first phase cable tip rendezvous location
\( z_{T_1} \) \( z \)-coordinate of the first phase cable tip rendezvous location

\( x_{T_1} \) \( x \)-coordinate of the initial cable tip rendezvous location

\( y_{T_1} \) \( y \)-coordinate of the initial cable tip rendezvous location

\( z_{T_1} \) \( z \)-coordinate of the initial cable tip rendezvous location

\( t_{f_1} \) Final time of the first phase

\( x_{T_n} \) \( x \)-coordinate of the final aircraft/cable tip rendezvous location

\( y_{T_n} \) \( y \)-coordinate of the final aircraft/cable tip rendezvous location

\( z_{T_n} \) \( z \)-coordinate of the final aircraft/cable tip rendezvous location

\( t_{f_n} \) Final time of the final phase

\( x_{T_m} \) \( x \)-coordinate of the final aircraft rendezvous location

\( y_{T_m} \) \( y \)-coordinate of the final aircraft rendezvous location

\( z_{T_m} \) \( z \)-coordinate of the final aircraft rendezvous location

\( y_{AC} \) Altitude of the aircraft for planar optimal control problems

\( z_{AC} \) Altitude of the aircraft for three-dimensional optimal control problems

\( x_C \) \( x \)-coordinate of the cable tip position

\( y_C \) \( y \)-coordinate of the cable tip position

\( z_C \) \( z \)-coordinate of the cable tip position

\( y_{TOL} \) Cable tip collision tolerance margin used for planar optimal control problems

\( z_{TOL} \) Cable tip collision tolerance margin used for three-dimensional optimal control problems

\( h_T \) Elevation of the ground terrain

\( h \) Maximum height of the terrain separating rendezvous locations

\( \lambda \) Width of the elevated section of terrain

\( a_y \) Inertial acceleration of the payload in the \( y \)-direction

\( F_{drag,y} \) Aerodynamic drag acting on the payload in the \( y \)-direction

\( \dot{T}_n \) First order time derivative of the cable tension

\( m^* \) Correction to the inertia of the single-link aerial towed-cable system model

\( J_f \) Final value of the performance index

\( \mathbf{v}_{WIND} \) Velocity vector for the prevailing atmospheric winds
$W_x$  
$x$-component of the prevailing wind velocity

$W_y$  
$y$-component of the prevailing wind velocity

$W_{xy}$  
Magnitude of the velocity for the multi-component prevailing winds

$v_{WIND}$  
Acceleration vector for the prevailing atmospheric winds

$v_{REL}$  
Velocity of the payload relative to the prevailing wind

$v_{NREL}$  
Normal velocity of the payload relative to the wind

$m_{P_0}$  
Initial payload mass for payload transportation

$\Delta m_p$  
Percentage change in payload mass occurring immediately after rendezvous, signifying either payload capture or delivery

$\Delta u$  
Percentage change in control actuation occurring at the instant of payload capture or delivery

$\Delta T_n$  
Percentage change in cable tension occurring at the instant of payload capture or delivery

**Neighbouring Optimal Feedback Control**

$\delta J$  
Second order perturbation in the performance criterion

$\delta x(t)$  
First order perturbation in the state space variables

$\delta u(t)$  
First order perturbation in the control variables

$A(t)$  
Time-varying system state influence matrix

$B(t)$  
Time-varying system control influence matrix

$Q(t)$  
State design weighting matrix

$R(t)$  
Control design weighting matrix

$S(t_f)$  
Final state design weighting matrix

$K(t)$  
Optimal feedback gain for the closed loop controller

$P$  
Controllability grammian

$\Phi(t)$  
Sufficient matrix condition for controllability

$G_x$  
Magnitude of the horizontal wind gust component

$G_y$  
Magnitude of the vertical wind gust component

$t_g$  
Wind gust commencement time

$\lambda_g$  
Total length of time a wind gust acts for

$\Delta x_c$  
Percentage change in $x$-coordinate of the cable tip at rendezvous
\( \Delta y_C \) Percentage change in \( y \)-coordinate of the cable tip at rendezvous
\( \Delta x_{d_1} \) Percentage change in \( x \)-coordinate of the aircraft at initial rendezvous
\( \Delta y_{d_1} \) Percentage change in \( y \)-coordinate of the aircraft at initial rendezvous
\( \Delta x_{C_1} \) Percentage change in \( x \)-coordinate of the cable tip at initial rendezvous
\( \Delta y_{C_1} \) Percentage change in \( y \)-coordinate of the cable tip at initial rendezvous
\( \Delta x_{C_\Pi} \) Percentage change in \( x \)-coordinate of the final cable tip position
\( \Delta y_{C_\Pi} \) Percentage change in \( y \)-coordinate of the final cable tip position
\( \Delta x_{d_\Pi} \) Percentage change in \( x \)-coordinate of the final aircraft position
\( \Delta y_{d_\Pi} \) Percentage change in \( y \)-coordinate of the final aircraft position

**Abbreviations**

ATC Aerial Towed-Cable
CG Centre of Gravity
MIN Minimum
MAX Maximum
CBAS Cable Body Airborne Simulator
TACAMO Take Charge and Move Out
KST Kelly Space and Technology
ISA International Standard Atmosphere
SL Sea Level
ODE Ordinary Differential Equation
TPBVP Two Point Boundary Value Problem
NLP Non-linear Programming Problem
SQP Sequential Quadratic Programming
Quad Quadratic
Interp Interpolation
Const Constant Tow Speed and Altitude
Rect Rectilinear Manoeuvring
Plan Planar Manoeuvring
SVD Singular Value Decomposition
LQR Linear Quadratic Regulator
REF Reference Condition
OL Open Loop
CL Closed Loop
1 INTRODUCTION

1.1 Preface

An Aerial Towed-Cable (ATC) system generally consists of three primary elements; a towing vehicle that is usually a fixed wing aircraft, although it can be a rotorcraft or dirigible; a long thin cable or tether; and a towed body, which may be either passive or active. However henceforth, the ATC systems that are the subject of the research that concerns this dissertation consist of a passive payload connected to the tip of a cable being towed beneath an aircraft. Similarly, the terms “cable”, “wire” and “tether” are considered as equivalent and will be used interchangeably from hereon in.

The apparent simplicity of ATC systems is deceptive, as their behaviour is complex and can be difficult to accurately predict. While the continuous three-dimensional, lightly damped and highly flexible nature of aerial tethers is the primary reason behind their complicated behaviour, it is also responsible for their lightweight and high structural efficiency. Indeed, it is this ideal combination of properties that not only render tethers valuable to the myriad of applications involving ATC systems; it ensures that they are possible in the first instance. The dynamic response of ATC systems has interested scientists since antiquity [1, 2], however it is the potential physical applications that interest and motivate this research. Such applications include; the capture of payloads [3, 4], precision delivery of payloads [2, 5-11], towing arrays, radar or decoys from aircraft [12-20], towing gliders and kites into the upper atmosphere [21-25], airborne refuelling hoses and trailing static pressure hoses [26, 27], towed sonar devices/charged cables for mine sweeping activities [26, 28], water bombing of wildfires [29, 30], environmental management using towed remote sensing/monitoring instrumentation [26, 27, 31, 32] and tethered balloons (aerostats) as inexpensive station-keeping platforms for such instrumentation [33, 34]. The intention of this thesis is to investigate in detail the broad airborne tether application known as “payload transportation”, concomitantly developing non-linear optimal control strategies and simulation facilities to help surmount some of the systemic impediments that prevent the full implementation of this important aerial tether application.
The capability to rapidly and accurately transport payloads to remote areas is of critical importance to countries such as Australia, with its widely strewn population, extensive coastlines and vast areas of inhospitable, but generally well-frequented terrain. Examples of the significance of this capability lie in the rescue of distressed mariners or bush-walkers, aerial wildfire fighting, and the delivery of supplies to isolated and drought or flood-ravaged communities. The recent devastating drought and ever-omnipresent wildfire threat have very publicly exposed the deficiencies that exist in the present means Australia and other comparable nations use to attain this capability. These deficiencies include the absence of a single system capable of both retrieving and deploying payloads, the ability to transport payloads of any considerable size over large distances and the inherent inaccuracies, dangers and expenses associated with present systems. Surprisingly, there is little research within the current body of knowledge that specifically addresses the issue of positioning the tip of an aerial tether over a surface location and delivering or capturing a payload, even in light of the potential windfalls the development of such a system may yield. Therefore, this thesis attempts to address this void through the development of novel non-linear optimal control systems for the ATC system for payload transportation operations.

1.2 Literature Review

A comprehensive review of the current body of knowledge concerning the modelling, control and applications of ATC systems is the subject of this sub-section. A partial review of methods used to formulate the equations of motion for multi-body systems, of which ATC systems are a subset, is separately provided in Section 2.2-2.3. A brief review of the numerical methods available for formulating and solving problems arising in optimal control is also provided for in Section 4.3-4.5.

1.2.1 Review of Modelling Aerial Towed-Cable Systems

With respect to the accumulated literature regarding the modelling of ATC systems, the research can be demarcated into two unique areas that are characterized by the manner in which the ATC system is utilized and the general operating environment it exists within. Previous research into the modelling of ATC systems has considered the system to be either travelling within a uniform flow field or repeatedly circling/orbiting in the sky about a location on the ground.
The current body of knowledge within this area can be further subdivided into three distinct areas: stability, steady-state equilibrium and dynamic modelling, which will be discussed separately below.

1.2.1.1 Stability Analysis

Glauret [32] is credited as the first researcher to comprehensively study the stability of towed bodies moving through a uniform flow field. In this seminal work, a simple method using “cable derivatives” was used to investigate both the longitudinal and lateral stability of a massless, dragless, inextensible wire towing an aerodynamically stable body. By employing a series of simplifying assumptions regarding the nature of the force experienced by the cable, Glauret was able to show that in addition to the usual pitching and yawing oscillations of the towed body, there are in fact three types of oscillation the ATC system is capable of performing. The most important type of oscillation was found to be the one associated with an in-plane bowing of the cable, which may become unstable for sufficiently short cables or if the drag experienced by the body is unsatisfactorily low compared to the drag experienced by the cable. This instability was found to be possible even if the towing body has a wide static stability margin.

Following from Glauret’s pioneering work, further research into the stability of kites and towed gliders was carried out in England during World War Two, from which Bryant and Brown [24] surveyed and collated a collection of hitherto unpublished results and findings. In Part I of their report, the conditions for complete stability of a kite with a high lift-to-drag ratio were determined, assuming that the kite is aerodynamic and the cable is massive and experiences drag perpendicular to its orientation, with small perturbation theory employed during the theoretical analysis. It was found that lateral stability decreases as the altitude of the kite decreases, whilst the reverse was true in the case of longitudinal stability. Careful choice of the attachment point position was found to increase the lateral stability of high lift-to-drag ratio kites, given sufficient wind strength, whilst longitudinal stability is easily secured at low altitudes and improved at higher altitudes if the lift-to-drag ratio of the kite is reduced. Part II of their report summarizes wind tunnel work performed using kites and further theoretical analysis. This resulted in the finding that a single tail fin could be used to improve the stability of high efficiency kites. Parts III and IV introduced the inclusion of towed gliders in the stability analysis, where it was found that the lateral stability of kites and gliders deteriorates as the cable is shortened and is a minimum as the cable length approaches...
the span of the kite/glider. Similarly, Brown concluded that for high wing loading (typically 10 lb/ft²), complete stability of towed gliders is assured only when the towing speed is high and the attachment point is close to the glider centre of gravity (CG), with stability margins increasing as the wing loading is reduced. For towing at low speeds with large attachment off-set away from the glider CG, unstable lateral oscillations arise, yet their period is sufficiently long to enable the pilot to appropriately respond. Parts V, VI and VII deal with the longitudinal and lateral stability of gliders and aerofoils being towed on twin booms (cables) and are extensions to the calculations and inferences made on the longitudinal and lateral stability of gliders towed by single cables. Essentially the twin boom tendencies are the same as those of a single cable, except that with careful placement and design, definite longitudinal and lateral stability gains can be achieved through the use of twin booms.

During the post war period, O’Hara [23] extended the work of Bryant, Brown and other colleagues by considering the effects of cable elasticity and cable tangential drag on the configuration and longitudinal stability of a glider in towed flight. O’Hara found that cable elasticity has a marked effect on the Glauret “cable derivatives” and the period of the longitudinal oscillations of the system. In effect, the inclusion of cable elasticity lead to a doubling of the period of oscillations as compared to when an inextensible cable was used, which was found to be in favourable agreement with a series of then recent flight measurements.

The most comprehensive theoretical investigation into the dynamic stability of a towed cable during the post war period was undertaken by Phillips [35]. Philips proposed a comprehensive theory to account for the violent motion that towed bodies with stabilizing fins sometimes experience at certain towing speeds. His substantial theoretical analysis indicated that lateral oscillations travelling down the cable are amplified by the aerodynamic force experienced by the cable, when the towing speed is greater than the wave propagation speed along the cable. Light damping is provided by the aerodynamic drag on the cable for towing speeds less than the wave propagation speed, whilst upward (towards the aircraft) travelling waves were found to be always damped.

Reid [36] considered the scenario of a symmetrical object being towed horizontally in air at a constant speed by a straight cable of constant length. Linearized approximations to the resulting differential equations were developed and characteristic equations were obtained that served as tests for the longitudinal and lateral stability of the object.
The first in-depth wind tunnel investigation into the stability of a cable-towed body system available in the public domain appears to be that undertaken by Mettam [19]. A wind tunnel investigation into the instability of a body towed by a helicopter was performed, which reproduced a divergent lateral oscillation that had lead to cable failure in recent flights. Stability boundaries in terms of the towing speed and cable length were experimentally determined; from which it was found that the most effective means to reduce the instability was to increase the body drag, although reducing the body inertia was also effective. These measures, along with cable elasticity were found to only affect the high-speed stability boundary. Minimum towing speeds and retrieval rates were also proposed for the system, while a physical basis for the observed increase in stability margins for short, highly flexible cables was also provided.

The first modern comprehensive stability analysis of a cable-body system totally immersed in a fluid stream was presented by De Laurier [33]. De Laurier was the first researcher to realize that the ATC system can be mathematically represented as a first order non-homogeneous initial-boundary value problem within the partial differential wave equation. With the towed body dynamics providing the boundary conditions for the problem and assuming the cable is inextensible, the resulting uncoupled partial differential equations governing the longitudinal and lateral dynamics of the ATC system are linear with constant coefficients. The solution of these equations takes the form of a transcendental characteristic equation for the stability roots, which were numerically extracted using a computer. De Laurier compared his theoretical analysis with data resulting from a series of wind tunnel tests, which within the estimated error limits, resulted in the proposed theory comparing well with the experimental results. De Laurier also applied his first order stability analysis toward the development of a high performance tethered aerodynamically shaped balloon [34], where he concluded that good lateral stability was attainable for the system through the use of a large and aerodynamically efficient vertical fin.

Using a similar approach to that utilized by De Laurier, Cannon and Genin [37] developed a series of techniques to analyse the vibration characteristics of a three-dimensional, flexible, massive and structurally damped cable. Two proposed techniques were developed to obtain the basic equations governing the dynamics of the system, each an approximation to the exact closed form solution. One technique relied on linearization about equilibrium to obtain the necessary characteristic equation for the system, whilst the other employed dynamical principles associated with the angular momentum of the towed cable to arrive at a slightly different characteristic equation. The results for a candidate two-dimensional towed-cable
system using each of the two techniques were compared to those results obtained previously by Huffman and Genin [38] (to follow), where it was found that the two approximations represent lower and upper bounds to the exact solutions. Cannon and Genin [39] extended their stability analysis to incorporate both the longitudinal and lateral motion of the system.

Following on from the work undertaken during the pre and post war period, the stability of bodies (cargo containers) slung beneath rotorcraft began to attract interest from researchers. Poli and Cromack [9] undertook a stability analysis of non-streamlined bodies (a cargo container and circular cylinder) being towed by a helicopter at constant velocity. After undertaking wind tunnel tests to determine the aerodynamic characteristics of each towed body, a linearized small perturbation analysis was carried out on the system. Assuming the following assumptions were utilized; the towing craft flies steady and level, the lateral and longitudinal motion of the system can be decoupled, the dynamics of the towed body are uncoupled from the dynamical motion of the cable and that the mass of the cable can be ignored. With respect to the last assumption, an approximate method was outlined in the appendix of the reference to account for the cable dynamics in the stability analysis. Poli and Cromack were able to conclude that long cables, fast towing speeds and light payloads are required for overall stability, while the drag-to-weight ratio and cable length are the most important parameters affecting stability. Micale and Poli [8] extended the analysis performed by Poli and Cromack to include the effect that the use of a reaction wheel has on the stability of a slung load. Whilst the inclusion of a reaction wheel does not affect the longitudinal stability of the system, it was shown to have a significant positive effect on the lateral stability. As the radius, mass and speed of the wheel increases, the cable length required for lateral stability decreases, up until the point where longitudinal stability considerations dominate those corresponding to lateral stability.

Feaster, Poli and Kirchhoff [10] further consolidated the work of Poli, Micale and Cromack in the area of slung load stability by investigating the impact of attaching stabilizing fins and incorporating a reaction wheel for stability augmentation. This study also compared the stability of the slung load when either a single cable or dual cable tandem suspension system is used. Once again, it was found that the use of stabilizing fins is detrimental to the lateral stability of the system regardless of the suspension system employed, whilst the reaction wheel was found to greatly aid the lateral stability of the system. They concluded that the two-cable tandem suspension system offers a satisfactory means of transporting standard cargo containers throughout the speed range of most (then) present-day cargo helicopters, although they conceded that pilots generally prefer the simple single point suspension system.
More recently, the stability analysis of ATC systems became a main priority of research within the field, with significant contributions made by Nakagawa and Obata [40]. Their work involved an investigation into the longitudinal stability of a series of ATC systems consisting of a flexible, inextensible cable having a constant circular cross section, towing a symmetric rigid body. Considering small motions of the system about steady state conditions, the governing equations of motion were derived by the application of Lagrange’s Equations, under the assumption that the cable motion can be approximated by finite degrees of freedom (assumed modes method). Various modes and their stability were determined by solving an eigenvalue problem for the linearized equations of motion for a variety of different ATC system configurations. Stability analyses were performed on towed spheres of various sizes, various cable configurations, towed bodies with various aerodynamic properties and various attachment point locations. Analysis of the resulting parametric studies indentified two important conclusions, the first being that the motion of the ATC system about the steady state can be characterized as being of three types- a pendulum mode, pitching mode or various order vibration modes, which is consistent with the conclusions reached by Glauret. These modes were found to be strongly dependent on the properties and configuration of the towed body and cable. The second important deduction was that the size, aerodynamic properties and attachment point positions of the towed body significantly affect the longitudinal stability of towed system. Considerably fore attachment point locations and very small or large absolute values for certain towed body aerodynamic derivatives result in unstable pitching flutter to occur, whilst very small diameter towed bodies induce unstable flutter motion within the ATC system.

De Matteis [25] presented the results and findings of a more modern approach to the investigation of the longitudinal stability of a towed sailplane. Following a similar approach to that first proposed by De Laurier, de Matteis treated the cable as a continuous, elastic, one-dimensional body subjected to distributed mass and aerodynamic forces. A series of non-linear partial differential equations were then developed, whose boundary conditions were supplied by kinematic relations designating the full two-dimensional motion of the towing aircraft and sailplane. A differential quadrature method was used as the solution procedure, which was found to be very effective at solving the non-linear motion equations for the system.
A stability analysis was carried out after linearizing the motion equations about the equilibrium condition, from which it was shown that there exists strong interactions between the cable, the towing vehicle and the towed sailplane motions. It was found that the short period modes of the two vehicles are only slightly affected by the cable, whilst the corresponding phugoid modes are strongly coupled to the dynamics of the cable. The stability of the system was found to improve if the sailplane remained below the aircraft and was attached to the cable at a point above its CG.

Following the work undertaken by Nakagawa and Obata, Etkin [26] undertook comprehensive and wide-ranging research into the stability of towed bodies. Etkin approached the problem in much the same way as De Laurier had, by formulating a series of exact partial differential equations for the flexible ATC system. Etkin considered both longitudinal and lateral motion for the towed body and preserved the ability to account for the elasticity of the cable in the resulting analysis. The main innovation of this work however was the utilization of a finite difference scheme to discretize the spatially dependent derivatives within the governing motion equations, thereby transforming the original partial differential equation set into a series of ordinary differential equations in time. This allowed the resulting equations of motion to be expressed in compact matrix format, resulting in a far more convenient solution procedure. Etkin was able to conclude that inherent lateral instabilities may arise if the towed body is equipped with fins. Etkin acknowledged this result had been found by many previous researchers (See Glauret [32] for example), however Etkin proposed that careful choice of attachment point position (sufficiently forward of and above the payload CG) can eliminate this instability. It appears that Etkin’s conclusions regarding the effect cable elasticity has on the stability of towed bodies is in strict disagreement with those obtained by Mettam [19], who maintains his claims have been independently and experimentally verified by French researchers at Breguet Aviation. Etkin concludes that cable elasticity has a destabilizing effect on the system and is more important for long cables as opposed to shorter cables. This is in direct contrast with Mettam, who concluded that very short, low stiffness cables experience enhanced lateral stability. In the absence of additional credible evidence directly supporting either party, one can only speculate as to where the truth lies in regards to this matter.
1.2.1.1.2 Steady-State and Equilibrium Modelling

McLeod [41] is widely credited as one of the first researchers to comprehensively study the equilibrium configuration of a heavy body being towed by a cable beneath an aircraft travelling in a uniform flow-field. Based on experimental work undertaken by Relf and Powell [42], McLeod formulated the classic “sine squared” equation for the normal aerodynamic force experienced by the cable and assumed in the resulting analysis that the cable was inextensible. McLeod acknowledged that Relf and Powell were able to measure a small tangential component to the aerodynamic force experienced by the cable, which he along with ensuing researchers chose to ignore. Although McLeod’s work was pioneering, it was restrictive and did not seek to obtain solutions suited to general practical use.

Glauret [31] was the first to provide systematic numerical results, in the form of a family of curves dependent on the weight-to-drag ratio of the cable, to obtain the equilibrium characteristics (tip position and cable tension) of the cable for a wide range of practical problems involving the towing of heavy bodies. Glauret’s simplified analysis took into account both the mass of the cable and the drag acting normal to it.

Landweber and Protter [43] built upon the method proposed by Glauret to include the effect of the tangential (skin friction) component of the drag force acting on the cable, although they neglected the mass of the cable. Glauret’s method was found to be inadequate at accurately representing the cable configuration when it is towed through water at moderately high speeds, where skin friction drag is prevalent. A series of functions and resulting curves were presented to determine the equilibrium characteristics of cables used for towing or mooring operations (loop cables).

The next significant advancement in the field of steady state cable modelling in uniform flow fields was due to Genin and Cannon [44]. Their treatment was motivated somewhat by their belief that the important work undertaken by Landweber and Protter was not being cited as widely by other researchers as they believed it should. As a result they revised the work of Landweber and Protter and derived the basic equations governing the equilibrium and tension of a flexible, massive cable subject to both normal and tangential drag in a relatively straightforward manner. Genin and Cannon also had the significant advantage of being able to use a computer to obtain solutions and tailored their approach accordingly. Their subsequent analysis patently quashed the then widely accepted assumption that the cable’s tangential drag force has a negligible effect on the equilibrium characteristics of cables towed below aircraft.
Narkis [45] proposed a simple analytical method to approximate much of the analysis proposed by Genin and Cannon, although the mass of the cable and both the normal and tangential drag forces were appropriately accounted for. This straightforward technique was found to give satisfactory results for the cable equilibrium configuration and tension when compared against more exact results.

### 1.2.1.1.3 Dynamic Modelling

There have been two comprehensive reviews of the research carried out in the broad area of predicting the motion of cable systems under hydro/aerodynamic loading. Although these surveys are now somewhat dated and were carried out in close succession using references dealing with underwater towed-cable systems, they represent an important benchmark and are useful primary references to consult and access more detailed specialist publications.

The first survey of investigations on the configuration and motion of cable systems under hydrodynamic loading was undertaken by Casarella and Parsons [46]. By chronologically and systematically presenting the methods and main findings of past studies in this area, they were able to elucidate the state of the art on the formulation of analytical models applicable to a wide range of physical problems concerning towed-cable systems. The main focus of the review was to compare and contrast the manner in which hydro/aerodynamic forces acting on the cable are considered. A discussion of the physical nature of these forces and a survey of existing analytical models was included. The details of various steady-state and dynamic cable system models, in both two and three dimensions, were critically reviewed and in some instances directly contrasted. A comparison of the results obtained through the use of these analytical methods with limited experimental data was also carried out. The main conclusions drawn from the survey was that although the two-dimensional steady-state analysis of cable systems is well established and effective models for these scenarios exist, there was a clear need for effective strategies for performing representative three-dimensional steady-state and dynamic analyses on cable systems. The main reason offered for this dearth is the lack of understanding on the true nature of the hydro/aerodynamic forces that cable systems encounter and the limited availability of test data in the open literature.

The second survey of analytical methods for dynamic simulation of cable-body systems was undertaken by Choo and Casarella [47]. Instead of highlighting the differences between methods, they grouped them according to the manner by which the cable is simulated. The central idea, advantages and disadvantages of each class of methods were concisely explained. They reviewed a slightly larger collection of resources than the first survey and
identified four main approaches then currently employed to model the dynamics of a variety of towed-cable system both in air and water- method of characteristics, linearization methods, finite element methods and equivalent lumped mass methods. The conclusions they drew from this survey included that the finite element method was the most versatile of the group, whilst the method of characteristics required the most computational resources. Interestingly, the only disadvantage associated with the equivalent lumped mass method given was how best to lump the parameters of the cable with the parameters of the towed body, although the resulting equations of motion are greatly simplified and easy to solve. As will be demonstrated later in this thesis, such a dilemma is easily overcome. Finally, Choo and Casarella concluded the survey with a statement outlining the need for an easy, simple method that can solve any towed-cable dynamic modelling problem, whilst requiring a small amount of computational resources, which to this day, is the goal of any newly proposed towed-cable modelling methodology.

The beginning of a flood of research into the dynamic behaviour of a flexible cable in a uniform flow field was initiated by Huffman and Genin [38]. A non-linear mathematical model of a two-dimensional extensible cable was formulated to study the dynamics of the system when it undergoes large-scale displacements due to suddenly applied loads. The method of characteristics was utilized to obtain time history solutions for a range of towing speeds and cable lengths. It was found that the system undergoes a transition from overdamped to oscillatory motion as the towing speed and cable length increase and decrease respectively. The cable dynamics were deemed important in considering the overall stability of the system, as resonance between cable and towed-body motions may be encountered for certain values of various system parameters.

Genin and Citron [48] investigated the coupling between the longitudinal and transverse motions of a flexible, extensible cable in a uniform flow field. Essentially using the same approach as Huffman and Genin, but including transverse dynamics, they used the method of characteristics to study the nature of the coupling between the longitudinal and transverse modes of motion for the system. They found that the primary reason for the coupling is a feedback effect whereby the transverse motion affects the cable tension variation due to the centripetal acceleration of the cable, which in turn affects the transverse motion and so on.
Paul and Soler [49] employed a finite element-type strategy to study the dynamics of underwater towed-cables. By making a series of assumptions regarding towing conditions and the drag force acting on the towed body, discretized equations for the planar non-steady motion of the cable, subjected to gravity and fluid drag forces were developed in a format readily suitable for numerical solution. Although preceding it by some twenty years, the dynamic cable modelling formulation employed by Paul and Soler shares distinct parallels with the generic approach developed later by Fleiss et al. [50] to efficiently study the dynamics of a large and varied class of important non-linear systems.

Ketchman and Lou [51] developed a systematic finite element procedure to study the two-dimensional dynamical motion of a towed-cable system, accounting for effects due to cable bending and stretch, along with acceleration, gravitational and hydrodynamic forces acting on the cable. The main innovation of this formulation lay in the manner in which the hydrodynamic nodal forces were treated as well as the inclusion of cable bending stiffness. Similarly, Ketchman and Lou introduced an innovative measure to prevent unwanted longitudinal oscillations (due to the inherent elastic nature of the cable) from occurring within the cable. This greatly improved the efficiency of the solution procedure. They also compared the effect of using either lumped or distributed mass and hydrodynamic loading representations, for which little difference was found to occur in the particular problems under investigation. Bending stiffness effects were found to be most dominant at the trailing end of the cable where the tension is nominally low.

Winget and Huston [52] developed a new methodology, termed “finite segment approach” to model the non-linear, three-dimensional dynamic motion of various classes of cables. In their methodology, the cable is physically discretized into a series of links connected to each other by ball and socket joints. The number of links, their structural and inertial properties, as well as their applied force model, is arbitrary and may be distinct for each link. This approach is heavily indebted to the revolutionary innovations and significant improvements attributed to Kane [53, 54] concerning the determination of the equations of motion for multi-body systems (see Section 2.2 and subsequent sub-sections thereof for details). The algorithmic nature of this formulation allows the rapid determination of the dynamic motion of a large class of cable systems, which was demonstrated by two distinct examples in this paper.
Huston and Kamman [55] developed more comprehensive expressions for the fluid dynamic forces acting on the cable links constituting finite segment models of cable systems. Representing the state of the art in modelling the fluid forces cable systems experience, the elegance of this approach was the ease to which it could incorporate changes and modifications to the hydrodynamic properties of the links. Different models of the normal and tangential drag coefficients for the cable segments, dependent on important flow parameters such as Reynolds or Mach Numbers, could easily be employed without major modifications. Huston and Kamman also outlined a simple procedure that combines all the forces experienced by each cable segment into a single force and moment passing through each link centre.

The expressions for the fluid dynamic forces due to Huston and Kamman’s research were found to afford an accurate, yet efficient simulation capability when used within an algorithmic equation of motion formulation such as that advanced by Winget and Huston. Huston and Kamman [56] undertook a study to validate the finite segment modelling strategy and their fluid dynamic force model. The results concerning the finite segment model were found to be comparable with known theoretical results and in excellent agreement with experimental data. Kamman and Huston [57] also followed up this work with a study involving the modelling of submerged cable dynamics for advanced structural applications (anchor drop and buoy release). Once again the results provided by the finite segment model were found to be comparable with those obtained through finite element modelling and were in excellent agreement with experimental data.

Sanders [58] outlined a formulation for modelling the three-dimensional motion of a towed-cable-body system, developed to account for and easily implement non-uniform cable properties and arbitrary tow-point motion, which (for example) the formulation proposed by Cannon and Genin [39] could not handle. Ignoring cable stretching, bending, twisting, side forces, inertia and added mass, this formulation used a similar approach to that proposed by Paul and Soler. The set of coupled first-order, non-linear ordinary differential equations that govern the motion of the system are reduced to a set of coupled non-linear algebraic equations of the same dimension. An iterative procedure incorporating an integration routine was then used to obtain the desired motion time histories for the system at selected, discrete times. The application of this modelling strategy was demonstrated on a towed system undergoing a typical manoeuvre. The effects and consequences of including bending stiffness and inertial effects in the methodology were also briefly discussed.
Jones and Krausman [22] provided a detailed analysis of the dynamic response of tethered aerostats, refining the modelling work undertaken by De Laurier by maintaining full non-linearity in the analysis. The effects of various turbulence models and other environmental disturbances were also investigated. In verifying the performance and accuracy of the model against experimental data obtained from a fully instrumented aerostat, the numerical model was found to be representative of the real aerostat in most instances. When and where the model was not representative, it was found that this was caused by limitations and deficiencies in the turbulence models. The cable model was shown to be conservative when used as a design tool to estimate the tension distribution within the tether.

May and Connell [59] looked into modelling the non-linear dynamics of an ATC system when the cable is deployed from the aircraft. Their system model comprised of an aircraft flying steady and level, towing a thin, inextensible and variable length cable with a homogeneous sphere attached to one end. They formulated a mathematical model of the ATC system in terms of a series of coupled non-linear partial differential equations, whose solution was obtained through a finite differencing technique. The deployment profile was simulated by directly specifying the payout time history. It was found that the tension reduces as the cable is deployed, increases as the payout rate reduces, and remains steady when deployment ceases. Verifying the results given by the model for the tension within the cable against rather limited experimental data was unsuccessfully attempted, whilst a series of improvements to the modelling strategy were also proposed, namely the removal of a series of limiting assumptions.

May et al. [60] studied the dynamic coupling of a six degree of freedom quasi-linear mathematical model of an aircraft with an ATC system model that is a variant of the Cable Body Airborne Simulator (CBAS), known as TRCBAS. CBAS is a program developed at the University of Bath to simulate a body attached to a fixed length, inextensible cable being towed by an aircraft. This investigation mostly focussed on how the aircraft dynamics are affected by the changes in tension within the cable as the aircraft manoeuvres.

Following on from this work, May et al. [61-64] undertook a series of investigations into various aspects concerning the dynamics of the coupled aircraft-TRCBAS system, the main findings and results of which were later revised and published elsewhere (see May et al. [65]). In the first of these studies, the effect of various towing aircraft was examined, namely how the towing aircraft to towed body mass ratio affects the dynamics of the aircraft-towed flexible cable system. As total dynamic coupling between the aircraft and the cable-body
system was not employed and each aircraft essentially performed a different manoeuvre during the comparative analysis, the results obtained from this study were treated with caution. However for a given towed body mass, it appears that the smaller the mass of the aircraft, the greater the effect the cable-body system has on the aircraft. It was also found that the further along the span of the wing the cable attachment point is located at, for a given towed body mass, the effect that the cable-body system has on the aircraft is greatest, the smaller the mass of the aircraft.

The second study looked at incorporating variable drag coefficients, cable elasticity and bending stiffness into the ATC system model. The well-known Mach Number-dependent relationships for the normal and skin friction drag coefficients for cables, first proposed by Cochran et al. [17], were used in the modelling procedures, as was the bending stiffness of the cable, which was assumed to be proportional to the local orientation (curvature) of each cable segment. The effects of these modelling alterations were assessed for the scenario when the aircraft performs a symmetrical pull-up manoeuvre. Whilst the bending stiffness of the cable was found to have virtually no effect on the tension and position of the payload, the variable drag coefficient model was found to significantly affect the tension and altitude of the payload. This was mainly due to the fact that the previously used estimate for the cable drag coefficients were noticeably different from those given by the Mach Number-dependent relationships. High bending stiffness values were found to cause instability problems in the execution of the TRCBAS program. Cable elasticity was also found to have a significant effect on the tension and position of the payload. As a result, it was concluded that only the inclusion of cable elasticity and variable cable drag coefficients was justified.

The penultimate enquiry focussed on modifying the TRCBAS model to allow for cable deployment or retrieval. Deployment and retrieval was simulated by subdividing the maximum length of the cable into a maximum number of segments, which were added or removed at the aircraft end of the cable as the length of the cable is increased or decreased accordingly. A maximum, minimum and nominal cable segment length are specified prior to a simulation commencing. For deployment, if and when the link closest to the aircraft is longer than the maximum allowed, it is subdivided into two co-linear links, the first having a length equal to the difference between the maximum and nominal cable segment length. An analogous technique is employed during retrieval operations for the cable. This strategy was successfully implemented and was found to be robust for a wide range of scenarios and aircraft manoeuvres. Unwanted oscillations were encountered for very short cables, attributed to the fact that the original CBAS code is unsuitable for modelling such scenarios.
The final study investigated stability limitations associated with the TRCBAS code, as instabilities were often experienced and erroneous results obtained when the cable was short or counter-intuitively, when a sufficiently large number of cable segments was used. Stability boundaries for a number of simulation parameters were determined, but the overwhelmingly clear conclusion drawn from the study was that further investigations were required, as the root cause of most instabilities was not determined.

Roberts et al. [66] extended the mathematical model developed by May and Connell by incorporating cable elasticity, three-dimensional motion of the system, manoeuvring of the aircraft and cable deployment/retrieval. Results for various aircraft manoeuvres and cable length profiles were presented, including a direct comparison with the earlier results obtained by May and Connell. This direct comparison showed that the inclusion of cable elasticity in the cable model had little effect on the tension within the cable and altitude of the payload when the aircraft flies steady and level. However for cases when the aircraft undergoes considerable manoeuvring, the effect of cable elasticity is significant, having the favourable effect of quickly damping out any tension oscillations within the cable, induced by sudden changes to the aircraft’s velocity. The numerical solution scheme employed in this work, based on an implicit finite differencing method was found to be inadequate for modelling inelastic cables in some instances. This is offset by the fact that in reality, all cables have some degree of elasticity, however modelling highly rigid cables using this approach may lead to erroneous results being obtained.

Hover and Yeorger [67] examined the possibility of modelling the behaviour of an underwater towed vehicle with a simple low-order differential equation set. Stressing the point that while high-resolution non-linear partial differential equations can be used to model the two-dimensional dynamical motion of a towed body, they are of limited use in the design of controllers for such a system. This is especially pertinent when the order of the system is very high or infinite, as is the case with towed-cable systems. With the aim of formulating a more tractable control problem, the cable was modelled as spring-mass-dashpot system, having one, two or four nodes. The springs were assumed to be linear, the cable uniform, deflections and angles small, whilst the drag models were either quadratic or linear. The resulting equations of motion for the various system representations were obtained through bond-graph modelling of the linear system, whose important defining parameters were estimated using an identification/verification process based on a Learning Model Method. The parameters of the low-order systems were tuned to ensure that these simple system models were representative of the more sophisticated towed-cable system models known to
compare well against real towed-cable systems. The performance of the reduced-order models were verified via a comparison of the output from the models with actual sea trial data, where it was found that a massless dual-link representation, or a single massive link cable model is capable of accurately simulating the dynamics of the real physical system. It was found that the quadratic drag model was superior to the linear variant, whilst the dual-link model slightly out-performed the simpler single-link model. Although they emphasized that single or dual link models may not always be valid for all towed-cable-body systems, their results and conclusions suggest that examining the possibility of using low-order dynamic models for ATC system is unquestionably warranted, an endeavour pursued in earnest in Section 3 of this dissertation.

Williamson and Govardhan [68] undertook a series of experiments into the dynamics and forcing of a tethered sphere in a fluid flow using a buoyant sphere attached to a short wire placed inside water tunnel with variable flow speed. It was found that the sphere oscillated vigorously with an amplitude of around twice the diameter of the sphere (peak-to-peak), leading to an increase in drag and tether angle of over 100 % when compared to non-oscillating drag measurements. In-line oscillations were phased locked with the out-of-plane oscillations at twice the frequency of the out-of-plane oscillations; whereas theory predicts that the two frequencies should be equal.

Kamman and Huston [69] revisited the area of towed cable dynamics and modelling by considering a variable length, multiple branch towed-cable system with multiple end-bodies and variable hydrodynamic effects. Their modelling strategy utilized the same finite segment approach previously used to study the dynamics of cable systems, yet the range of application and the nature of the cable systems investigated was greatly expanded in this work. Deployment and retrieval were modelled by having the links closest to the towing vehicle change length, all by the same amount, a scenario not truly reflected in reality. The effects of fluid drag, lift and buoyancy were included for both the cable and the end-body/s, whilst added mass forces and moments were considered only for the towed bodies. A number of simulations were carried out on a typical branch underwater towed-cable system, where the effects of various deployment rates were investigated. It was found that the tension within the cable drops dramatically during deployment and that the faster the deployment speed, the larger the offshoot in the position of the towed body/s (as compared to the equilibrium configuration), and the longer the settling time. As expected, suddenly arresting the deployment of the cable creates tension impulses within the cable, which lengthen the time required by the system to reach equilibrium.
Chin et al. [70] revisited the research performed by Roberts et al. and incorporated a six degree of freedom model of the towed body, which was found to improve the accuracy of the towed cable-body system when undergoing a series of varied manoeuvres. Although full dynamic coupling between the towed body and the cable was maintained, the motions of the aircraft and the cable-body system were not dynamically coalesced, hence discontinuities in the acceleration of the aircraft induced oscillations in the tension within the cable. To conclude, it was proposed that appropriate interaction between the motion of the aircraft and the cable-body system through the incorporation of an aircraft model dynamically coupled to the cable-body system, would eliminate these unwanted oscillations from occurring.

Similarly, the dynamically-induced recoil forces the cable experiences during sudden termination in the deployment of the cable was investigated by Narkis [71]. Assuming only two-dimensional motion, an approximate analytical formula for the upper limit in the tension peak for the cable during deployment was developed.

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The existing corpus of expertise within this sub-area can be partitioned into areas similar to those outlined previously for cables in uniform flow. Each will again be subject to detailed consideration.

\subsection*{1.2.1.1.4 Stability Analysis and Equilibrium Modelling}

The stability of “whirling” cables has been the subject of inquiry since the eighteenth century, drawing the attention of prominent applied mathematicians of the likes of Leonard Euler and Daniel Bernoulli. Neglecting aerodynamic drag, they performed an eigenvalue analysis on a cable spinning about its longitudinal axis (zero tow-point radius) after an appropriate linearization technique was applied to the governing non-linear dynamical equations.

Some two hundred years later, Kolodner [72] resumed work in this area by closely investigating the nature of the non-linear eigenvalue problem for heavy, inextensible and drag-free rotating strings. Kolodner showed that for a given mode, above the critical linear speed for that particular mode (critical frequency), the mode shape (deflection) is a continuous function of the speed of rotation. Wu [73] continued the investigations performed by Kolodner by studying the asymptotic nature of the solutions obtained as the angular velocity of rotation was increased to large values.
Caughey [74] obtained approximate non-linear solutions to the problem of a heavy chain being forced to whirl at a constant rotational speed, again neglecting the effect of aerodynamic drag. It was shown that for a given mode, multi-valued solutions are obtained for speeds of rotation above the linear critical speed for that mode. Caughey was able to further show that two of these multi-valued solutions are stable, whilst the third is unstable and leads to small amplitude solutions becoming unstable in the vicinity of the critical linear speeds, giving rise to mode conversion.

Skop and Choo [1] investigated the equilibrium configuration of a flexible and inextensible cable with a spherical drogue attached to it, being towed below an aircraft performing repeated circling manoeuvres. The equations of equilibrium and the boundary conditions that govern the cable configuration were derived by assuming that only the component of aerodynamic drag normal to the cable is present. These governing equations were then non-dimensionalized so that geometrically similar configurations could be appropriately compared and important parameters that affect the equilibrium configuration of the cable could be isolated. For certain values of the system parameters, a region of multi-valued equilibrium solutions was found for the system, the size of which was shown to decrease as the mass of the drogue increased. Further numerical results showed that a relationship exists between academic multi-valued solution regions and the more practical solution regions, which manifest as simultaneous large altitude changes and small radii of motion for the drogue.

Cannon and Genin [75] developed a series of simple analytical techniques to study the equilibrium configurations assumed by a cable being towed by an aircraft undergoing a coordinated turn. By assuming a rigid massive cable that experiences normal and tangential drag proportional to the sine square and cosine square of the local angle of attack respectively, as well as other simplifying assumptions, closed form solutions for the governing non-linear equations were obtained. An exact solution to the two-point boundary value problem was also found numerically without invoking some of the simplifying assumptions, which when compared against the results given by the simpler analytical method, both results were found to be in agreement with each other. Cannon and Genin commented on the need to check the stability of their solutions, in light of the work undertaken by Skop and Choo, although they assigned this task to a future study.
Russell and Anderson [76] employed a finite element method to investigate the equilibrium and stability of whirling inelastic cables. They utilized linear elements to study the circularly towed cable subjected to viscous drag, assuming that the end-body could be adequately represented as a point mass. Multi-valued solutions for certain towing speeds were again obtained, showing that dynamic instabilities can occur within the system when drag forces are not considered in the model.

Russell and Anderson [2] then extended their work by employing a non-linear finite element method to investigate the equilibrium and stability of an elastic cable being towed by an aircraft flying in a horizontal circular path at a constant angular velocity. Their study accounted for both the normal and tangential drag the cable encounters, which they took as being proportional to the square of the normal and tangential velocities respectively. Phenomena they encountered during this study included static and dynamic instabilities, jumps between various equilibrium conditions and detached solution branches. They likened the static instabilities and jumps to the behaviour exhibited by typical damped, hardening, non-linear spring-mass systems, whilst they believed the dynamic instabilities and detached solution branches warranted further investigation.

Cohen and Manor [27] investigated the dynamical equilibrium configurations of a cable-drogue system when towed in a helical motion (repeated circling with simultaneous vertical descent). Using a continuous model for the cable, assuming it was inelastic, its viscous drag force could be ignored and it was towing a spherical drogue, they too arrived at a series of results that were multi-valued in nature. The influence of angular (tow-point) velocity, tow-point radius and vertical rate of descent on various cable configurations in dynamic equilibrium was investigated. It was found this ATC system could reach three different dynamic equilibrium configurations, all of them for the same boundary conditions. It was assumed that one solution is unstable whilst the other two are stable and that jumps from one kind of configuration to the other is possible, a phenomenon previously postulated by Russell and Anderson.

Zhu et al. [77] performed a theoretical and experimental study into the non-linear dynamic response of a linearly elastic string, fixed at one end and undergoing constant speed circular motion at the other, ignoring aerodynamic drag. The theoretical procedure began with a non-linear steady-state analysis, for which the strings were linearized about, in order to perform an eigenvalue analysis on the system. They found that single loop balloons have high tension and vibrating frequencies that decrease as the length of the string increases. Highly extensible
strings were found to whirl in the first and higher modes, whilst inextensible strings whirled only in higher modes. Long and highly elastic strings form low-tension double balloons that are stable, whilst inextensible double balloons are unstable and experience flutter. Similarly, one-and-a-half loop balloons are divergently unstable. Steady state balloons, string tension and balloon stability were all successfully verified in a series of subsequent experiments.

Zhu and Rahn [78] turned their attention to the stability of a circularly towed cable system, where they employed a Galerkin approximation to the perturbed modes of vibration for the cable and linearized the result about the steady-state cable shape. Small cable strains were assumed and tangential drag was ignored in the model. The steady state equations were solved numerically using a shooting method. Zhu and Rahn were also able to find that multiple equilibrium solutions existed when the drag is small, the end-body mass is large, or the tow-point rotation speed is high. Divergently unstable solutions, jumps and flutter were all observed for certain combinations of system parameters. Sufficiently slow tow-point rotation always ensures stable single-valued steady state solutions, whilst increasing the drag generally increases the size of the stable solution regions.

More recently, Coomer et al. [79] accounted for the dynamical whirling motion of an inextensible open string, neglecting the effect of aerodynamic drag. Modelling the cable as a simple one-dimensional continua under variable tension, the resulting non-linear eigenvalue problem corresponding to the whirling modes of vibration for the system were solved numerically using a “shooting” method over a range of whirling speeds. These results were found to favorably compare to those observed from a series of experimental tests, despite that fact that drag, friction and structural damping were all ignored in the theoretical analysis.

Lemon and Fraser [80] extended the work of Coomer et al. by considering the steady-state bifurcations and dynamic stability of an inextensible cable being towed in a circular fashion, subjected to general aerodynamic drag but not accounting for a payload/body attached to the extremity of the cable. By constructing bifurcation diagrams for the quasi-steady configurations of the cable, significant and varied bifurcation behaviour was found to occur as the angular rotation speed, air drag and radius of the circular towing path was altered. A linearized stability analysis of these quasi-steady solutions was also carried out, finding that a transition from stable to unstable behaviour occurs if the rotational angular velocity of the tow-point is sufficiently large. Regions of instability were also found to occur for certain large circular towing radii, the size of which reduces if the aerodynamic drag acting on the cable is large enough.
Most recently, Williams and Trivailo [81-83] investigated the equilibrium and stability of the quasi-stationary solutions for an orbiting ATC system, with the particular innovation of using a high-drag device to enhance the overall performance of the system. Williams and Trivailo noted that for applications involving the capture and delivery of payloads during repeated aircraft circling, tight turns (large bank angles) at high rotational speeds lead to near stationary motion of the cable tip about a desired location on the ground. Such a scenario was found to be limited by the aircraft maximum lift coefficient, bank angle, and thrust/power available. After carrying out an equilibrium and stability analysis of the original orbiting ATC system using a variety of aircraft, high drag forces acting on the towed body were found to produce small orbit radii for the towed body. This indicated that the use of a high drag device placed somewhere along the cable may be warranted. A subsequent detailed numerical analysis addressing the location along the cable in which the wind-sock should be placed, as well as its geometric and aerodynamic properties, confirmed that the orbit radii for the towed body could be reduced. However, it was concluded that the reduction in cable tip orbit radii provided by the wind-sock would not prevail over the shortcomings of using the device, since the wind-sock system produces lightly damped, low frequency modes that may adversely influence the position of the cable tip. Similarly, it was found that the relatively small gains made possible by using the wind-sock device would not be worth the effort associated with overcoming the additional complexity of deploying the cable with the wind-sock attached.

1.2.1.1.5 Dynamic Modelling

Crist [84] appears to be the first researcher to provide detailed time-history results in the open literature for the dynamics of cables towed beneath orbiting aircraft. Crist overcame the difficulties of obtaining a solution when modelling the cable as a continuous system and used a lumped mass model for the cable. The application of Lagrange’s Equations was then employed to obtain the set of ordinary differential equations that govern the motion of the towed-cable orbiting system, which were solved using a fourth-order Runge-Kutta technique. Crist focused his analysis on the case where the aircraft undergoes large vertical oscillations, whilst orbiting at a constant radius and altitude. The steady-state shape of the cable was determined by assuming the angular rate of orbit, altitude and radius of the end-body is initially known, which allows all the forces for the payload to be determined. By applying equilibrium and Hooke’s Law, the forces and displacements for the next adjacent segment can be found; this process being repeated sequentially up the cable until the aircraft’s location is found.
Very low tension at the payload end of the cable (slack cable) was found to occur for vertical oscillations of a certain magnitude, the effects being especially pronounced on the motion of the rest of the system. By considering the important practical case when the aircraft undergoes the transition from orbit to steady-level flight, it was found that the tension within the cable changes significantly and rapidly during this transition.

Matuk [85] investigated the dynamic stresses that can occur in towed-cables when subjected to sudden accelerations. Limiting the analysis to one-dimension, the dynamic state of stress in the cable was investigated when the aircraft was accelerated to a higher constant velocity than it initially had. The maximum static, dynamic and total stress was found to occur in the cable at the aircraft end, whilst the curved profile the cable takes up due to the presence of drag was found to reduce the value of the maximum dynamic stress. It was also found that below a certain magnitude of acceleration, failure of the cable is not possible, regardless of the duration of the acceleration.

Matuk [86] followed up this study with an experimental investigation into the distribution and magnitude of the forces occurring within a towed-cable when the aircraft undergoes a 180 ° turn manoeuvre (race track pattern). In total, twelve flights with different aircraft speeds and turn radii were performed, with the force at the aircraft end of the cable and the path of the aircraft and payload tracked. The maximum force in the cable was found to occur immediately after the aircraft finishes the turn, due to the so-called “whip” effect. A theoretical treatment of the problem was developed and these results compared favourably against the experimental results, indicating that the theory could be successfully used to avoid cable failure in the field.

Clifton et al. [5] developed a modelling procedure for simulating the dynamics of a wire trailing an orbiting aircraft, treating the cable as a continuous, inelastic system capable of undergoing only small motion about its equilibrium configuration. Excitation due to non-uniform winds causing a once-per-orbit cycle harmonic aerodynamic input was applied to the system. The accuracy of the model was examined against flight test data, which suggested that the model could perform well. Although the magnitude of the separation between the aircraft and the end-body (verticality) predicted by the model was found to be significantly different from that measured during the test (8 % error), the general oscillatory nature of the verticality given by the model (frequency and phase) was synonymous with that recorded in the tests.
Murray [87] utilized the concept of differential flatness developed by Fleiss et al. to rapidly determine trajectories for a non-linear system consisting of a cable being towed below an orbiting aircraft. Murray successfully recognized that under certain assumptions, the orbiting ATC system is differentially flat, with the motion of the end-body acting as the flat output for the system. This enabled the entire motion of system to be characterized by the motion of the payload, a phenomenon Crist, Paul and Soler got close to realizing. Murray discretized the cable into a series of rigid links and demonstrated the necessary trajectories the aircraft must fly in order to achieve desired drogue trajectories. The effects of steady winds was also considered, which were found to cause a fixed offset in the horizontal plane and periodic oscillations in the vertical plane. Murray attempted to determine the required aircraft trajectory to eliminate the effect of winds, but the resulting trajectories were unrealistic. Transition between manoeuvres was also attempted, however the proposed algorithm became unstable and suffered numerous deficiencies.

1.2.2 Review of Control For Aerial Towed-Cable Systems

Most research previously undertaken into the development of control systems for ATC systems has focussed on either passive or direct active control of the payload itself. In the atmospheric environment, very few studies have focussed on the development of control laws for the cable and/or the cable-payload combination, although tether-based control systems for space tethered systems have been widely studied (see the survey undertaken by Misra and Modi [88] for early examples, selected recent approaches and techniques can be found in [89-96]). The distinct lack of research into the development of cable-based control laws for ATC systems is one of the primary motivating factors behind this dissertation.

One of the first control studies to appear in the public domain was carried out by Szustak and Jenney [11]. This work investigated many practically motivated issues concerning the control of large crane helicopters, namely load stability and controllability. During this period, it was expected that larger and more expensive helicopters were being proposed, for which the study looked into measures to ensure that these newer craft could maintain the best in controllability. The nature of this analysis was qualitative as opposed to quantitative, although some results concerning stability were presented.
Asseo and Whitbeck [7] followed up the earlier work performed by Szustak and Jenney by discussing in more detail the control requirements needed by heavy lift helicopters in order to stabilize externally slung loads. The results presented in this work were obtained elsewhere by Asseo and others, with the main innovation being the consideration of winches to be used as active controllers for load stabilization purposes. Linearized equations of motion for the helicopter-winch-cable-load system, incorporating various suspension geometries, were developed for the helicopter flying steady and level at a constant speed. These models were then used to design several linear feedback control systems, each of which was able to successfully stabilize the load for a given type of suspension. Load stability, winch power requirements and the dynamic performance of the load were then compared for each suspension geometry. It was determined that it was feasible to use active winch controllers in the design of an automatic load stabilization system, and when utilized, the magnitude of the feedback gains for other controllers (longitudinal and lateral cyclic, rudder) were greatly reduced. Power requirements for the winches in all configurations were found to be modest for the cases considered in this study.

Guidance problems for ATC systems first appeared in the literature after Jun et al. [3] undertook a computer simulation study examining the feasibility of improving the utility and capabilities of cable pick-up systems. Simulating a “snatch” procedure, the towed-body was equipped with active control surfaces that were used to guide the cable tip to hook on to a stationary target on the ground. A two-dimensional, extensible lumped mass model was used for the cable, while deployment and retrieval was carried out through a tension control mechanism that allowed the length of the cable segment closest to the aircraft to vary. A digital linear quadratic regulator was then designed to provide the guidance capability for the ATC system. Dynamic coupling between the cable dynamics and the towed vehicle control was not included in the control formulation; a priori knowledge of the cable forces was required. This was achieved by estimating the cable loads from steady state equilibrium, and then allowing them to vary sinusoidally at the natural frequency of the cable. Although not an ideal formulation, the study showed that a guidance system could be developed for the ATC system capable of achieving sound performance and excellent accuracy, whilst requiring relatively little control power.
Cochran et al. [17] investigated the development of an autopilot for stability augmentation and manoeuvring of a small vehicle whilst being towed below a much larger aircraft. A lumped mass model of the cable was synthesized using Newton’s Law, assuming an inextensible, massless cable. This required the imposition of numerous unwieldy constraint force equations requiring elimination, a significant disadvantage often encountered when equations of motion for multi-body systems are obtained by direct application of Newton’s second law (see Section 2.2.1 for more details). An alternative method based on substitution of variables was proposed and used to overcome the deficiencies associated with the original modelling approach. The dynamics of the cable-towed body was assumed not to affect the dynamics of the aircraft in any way. A classical control system was then developed using a linear short-period dynamic model of the towed vehicle. This control system consisted of a stability and control augmentation sub-system, a manoeuvre autopilot and a station-keeping autopilot. Good stability characteristics at trim in the presence of environmental disturbances were maintained by the towed vehicle during an illustrative example, as was good manoeuvrability for changing between trim conditions. The lateral stability boundary found from the model was compared to one obtained experimentally, resulting in a very good agreement between the two sets of results. This in turn led to the simulation capability being utilized as part of the successful development of a real manoeuvrable tow target.

Hover [97] demonstrated how dynamic inversion could be used to control a class single-input/single-output, non-linear boundary-controlled distributed systems, such as towed-cable systems. Using this open-loop inverse input-shaping technique, Hover was able to determine the necessary trajectory of the towing ship that would enable the tow-fish to precisely follow desired pre-determined trajectories. The resulting system trajectories found using this theoretical technique for a typical towing operation were then compared to those obtained during full-scale tests. Excellent agreement between the two data sets was observed.

Banerjee and Do [98] developed a simple, but effective cable control law for an underwater cable dynamics model used to deploy, retrieve and regulate the in-plane position of an underwater vehicle. An order-N algorithm, based on a lumped parameter model for the cable was proposed, assuming it to be massive, highly flexible, yet inextensible. The cable was subjected to general hydrodynamic loading and motion constraints, whose deployment and retrieval was simulated through a thrust-type force acting on the link closest to the ship. The cable dynamics non-linearly affected the dynamics of the towed body through the tension in the cable at the towed vehicle, creating a non-linear constraint force that is iteratively determined using a constraint stabilization procedure.
A simplified cable winch dynamics model, with both acceleration and velocity limits placed on it, was incorporated into the overall dynamics of the system. A simple linear feedback control law was then used to appropriately manage the cable length, which when used in conjunction with a tension compensator, appropriately managed deployment, station keeping and retrieval operations for the towed system.

Bourmistrov et al. [18] created a control law for a manoeuvrable towed target based on the inversion of the non-linear dynamic and kinematic equations of motion for the towed body, whilst taking into consideration cable tension forces. Utilizing a full three dimensional, non-linear dynamic model of the towed vehicle, along with a linear aerodynamics model, they reduced the total system into a series of low order sub-systems, each with different time scales for their dynamic variables. This allowed for the faster and slower portions of the system to be inverted sequentially to control the flight path of the towed vehicle. Little is mentioned of the cable dynamics model that was utilized, except that it was a point mass-type model that assumed the cable was flexible and inextensible.

The model was verified against more sophisticated and computationally expensive models (such as the one developed by May et al. [59]) and showed satisfactory performance matching. The excellent performance of the towed vehicle control law was demonstrated by commanding the vehicle to perform a three-dimensional spiral manoeuvre about the trim position, followed by a station keeping operation when the vehicle reaches its target location. These results were then compared to those obtained by Cochran et al. [17]. It was found that although the towed body required slightly more control using the non-linear inversion technique, compared to the guidance algorithm proposed by Cochran et al., when the non-linear inversion technique was employed, the resulting motion of the towed vehicle was significantly less pronounced and damped out much faster.

Kamman et al. [99] outlined the application of linear quadratic Gaussian/loop transfer recovery control design to the development of depth, roll and yaw rate controllers for a vehicle being towed underwater by a cable. A lumped parameter model for the cable was developed, discretizing the cable into a series of rigid links connected by frictionless spherical joints. Inertial, gravitational, normal and tangential fluid dynamic forces were assumed to be uniformly distributed over each link, then halved and lumped at connecting joints. With arbitrary three-dimensional motion possible for the towing vehicle, the non-linear dynamic equations of motion for the cable were formulated using Newton’s Law, with accompanying constraint forces defining the rigidity of the cable links and the towed vehicle. The resulting
non-linear dynamic equations of motion for the cable were then linearized about the steady state equilibrium condition and used in the design of the controllers for the towed vehicle, based on linear quadratic Gaussian/loop transfer recovery control. The resulting controllers could be systematically tested against model uncertainties, sensor noise, towing vehicle disturbances, unsteady or non-linear towed vehicle hydrodynamics and actuator dynamics. Through a practically motivated example, it was shown that with appropriate design parameters, the linear quadratic Gaussian/loop transfer recovery control strategy can produce robust controllers, which Kamman et al. report has been reinforced in real life sea trials.

Henderson et al. [12] examined the effect that various passive and active control measures have on the dynamics of an airborne towed target. They studied a particular ATC system, whose towing aircraft (Jindivik) is highly manoeuvrable. As a result, rather violent oscillations are often experienced by the towed vehicle at the beginning and end of manoeuvres, characterized as a conical swinging motion; a combination of lightly damped yaw and pitch pendulum modes of comparable frequency. The cable model used in this study was CBAS, the forerunner and basis of the widely used TRCBAS model thoroughly investigated by May et al. [61-64]. A series of passive actions were proposed in an attempt to reduce the unwanted oscillations the towed body experiences, from moving the tow point forward, increasing the drag and rear fin area, to reducing the mass of the towed body. Of all these options, moving the tow point forward was the only modification found to produce any worthwhile improvement, although this was modest at best.

Consequently, attention was focussed on the development of active control measures for the towed vehicle. From a wide variety of candidates, Henderson et al. chose to augment the relative velocities between the towed body and the towing vehicle, as this represented the most effective, cheapest and simplest method to physically implement. Using simple linear feedback control laws, it was demonstrated that significant improvements to the behaviour of the towed body were possible (conical swinging motion drastically reduced), through simulations representing a wide range of operating conditions for the ATC system.

Williams and Trivailo [6] developed a fuzzy logic controller designed to control missions for the ATC system involving deployment, station keeping and retrieval. A three-dimensional dynamical model of the system was developed, incorporating cable elasticity, normal and tangential aerodynamic drag, gravity, structural damping, arbitrary tow point motion and ground impact forces. The cable was modelled using a lumped parameter approach that employed Kane’s Equations to formulate the dynamical equations of motion for the system.
Deployment and retrieval of the cable was accounted for by carefully and systematically adding or removing cable elements as required. The equilibrium configuration of the system was determined by setting all the time dependent derivatives of the generalized coordinates in the motion equations to zero, then using a non-linear solver to determine a solution to the resulting large-scale root finding problem. For the ground impact forces, the ground was essentially treated as a highly stiff spring, which transmits a force to the payload if it drops below the ground level.

The fuzzy logic controller developed for the system was demonstrated through the example of delivering a payload to a ground location, whilst the aircraft repeatedly circles around and above the drop-off point. With this application in mind, Williams and Trivailo proposed a new empirical, non-dimensional relationship for the radius of the payload’s motion about the drop-off point. This was used to deduce that very long cables being towed at high rotation rates will produce very small orbit radii for the payload. The fuzzy logic controller was shown to work well in the individual deployment, station keeping and retrieval phases of the demonstrative case study, but when a portion of the payload was released from the cable tip, the resulting transient motion caused the payload/cable tip to violently strike the ground, indicating that further refinements and developments are required for the proposed control method.

Quisenberry and Arena [20] carried out a three-dimensional simulation of an ATC system performing ocean wave tracking operations. Using a finite element method and discretizing the cable into an arbitrary number of rigid segments, the equations of motion for the system were derived using Lagrange’s Equations. Interestingly, they found that a large number of segments were needed to accurately model the system. Detailed mathematical models of the towing vehicle were assumed to be independent of the cable/towed vehicle system. The towed body was treated as a cable segment with rotational inertia and an appropriate aerodynamics model. The resulting high degree of freedom, nonlinear equations of motion were solved efficiently and accurately using the method of Lagrange.

After providing appropriate models for the towed body autopilot, based on the ideas of Proportional-Integral-Derivative Control and the waveforms, a simulation suite was proposed to verify the effects that various system and environmental features have on the ability of the ATC system to track ocean waves. Such effects studied included varying oceanic waveforms, towing vehicle cruise altitude, side gusts, and towing vehicle oscillations. The results of the simulation reproduced the well-known longitudinal dynamic modes of the system, which
were coupled with the in-plane motion of the towing vehicle, the degree of which was found to be dependent on the aft position of the towed vehicle (relative to the aircraft) and length of the cable. Overcoming this dynamic coupling between the cable and the payload was found to require significant effort from the autopilot of the towed vehicle, the least being when the cable is long and the towed vehicle is sufficiently well aft of the aircraft. In such a configuration, it was found that close tracking of ocean waves could be maintained in the event of a side gust or an extensive disturbance to the towing vehicle.

Lawhon and Arena [100] performed several design studies specifically addressing issues concerning the inherent stability of an aerially-towed vehicle and its ability to hold altitude when a single wing-type control surface is attached. The cable model used in this investigation was that proposed and utilized by Quisenberry and Arena. The manner in which the towing cable interacts with the flight performance and dynamics of the aircraft were investigated through the use of two-dimensional static and dynamic simulations of the cable, coupled with well-known aircraft performance and stability techniques. A series of cases were then carried out to assess the influence of cable geometry, wing and tail sizing, cable and wing attachment location, center of gravity location, and towed-vehicle weight. The possible effects on the performance and stability of the towed vehicle when a ram air turbine is used to power onboard equipment housed within it, was also looked into through a series of wind tunnel tests.

Lawhon and Arena showed that in certain instances, the cable can sag well below the towed-vehicle when the towed vehicle is not producing lift. The sizing and aspect ratio of the towed vehicle wing was found to involve compromises that significantly differ from those encountered in the design of wings for conventional aircraft. This was mainly due to the action of the tension forces acting on the cable at the towed vehicle. Their study did not observe the presence of longitudinal modes when the cable attachment point is moved forward of the towed vehicle CG, although they found that larger tail areas are required for the towed vehicle as the cable attachment point is moved ahead of the towed vehicle CG.

Williams et al. [101] significantly extended the work initially undertaken by Murray through the development of an alternative path planning algorithm for the ATC system. This study investigated the scenario of attempting to precisely rendezvous the towed-body/cable tip with multiple surface-based locations. By utilizing the motion of the towed-body as the flat output, the motion of the cable and the aircraft that provides the desired trajectory for the towed body can be rapidly determined. This process is simplified by using pseudospectral methods based
on multiple time segments to differentiate the trajectories of the ATC system. The motion of the towed-body is approximated using Chebyshev polynomials between the desired rendezvous points, which are formulated so that the resulting curvature of the towed-body trajectory is minimized. This corresponds to minimum overall acceleration for the towed body. Both fixed length and variable length cables can be incorporated into the path planning technique; either by allowing the aircraft to vary in altitude (fixed length), or by fixing the aircraft altitude and varying the number of discrete cable elements (variable length). The aircraft controls and/or the cable reel rate were computed via a series of inverse techniques. The ability of the cable tip/towed body to travel to multiple locations in three-dimensional space, even in the presence of significant variable wind fields, was clearly demonstrated in this study. The numerical results presented in this work confirm the importance of including the effect that the cable tension has on the motion of the aircraft. Since the trajectory of the ATC system can be found rapidly (less than one second of computing time) using the proposed algorithm, Williams et al. suggest that real-time generation of trajectories for the system should be possible.

Following on, Williams et al. [102, 103] continued from their previous research into path planning algorithms for ATC systems by addressing issues associated with the robustness of the differential flatness-based algorithm. More specifically, this new algorithm parameterized the trajectories of the system using spatial coordinates as the independent variables, instead of time (as used in previous work). As a result, the inversion process was not sensitive to the time at which the towed-body passed through the prescribed waypoints, significantly improving the robustness of the path-planning formulation. This new work also saw the development of a suitable means to incorporate aircraft performance constraints into the path-planning methodology. The main conclusion drawn from this study was that for the majority of practical cases conceived, the aircraft does not operate sufficiently close to the constraints, for aircraft control requirements to be of much concern during guidance or rendezvous-type operations.

Williams et al. [104] consolidated their previous research into path planning algorithms for ATC systems by combining the robust open loop control algorithm with linear receding horizon feedback control in closed loop simulations. The application of the path planning algorithm for the real-time regeneration of trajectories for the ATC system, due to payload pickup and drop-off, as well as changes in the steady wind strength, was demonstrated in this work.
Feedback control was applied to the aircraft only, assuming that if its motion can be adequately controlled, then the towed-body should follow the desired path. The excellent performance of the closed loop system demonstrated that precision control of the ATC system is possible under a wide variety of operating conditions, involving fixed length and variable length payload transportation operations.

Most recently, Williams and Trivailo [105, 106] extended the work initially carried by Matuk and Murray by investigating the transitional dynamics of an ATC system as the aircraft changes from straight and level flight to repeated circular flight. Using a lumped parameter approach to model the cable as a series of spring-dashpot sub-systems that experience general aerodynamic drag, their results showed that the cable could become slack (zero tension) during the transition from straight to circular flight, if the aircraft turns too rapidly. A series of parametric investigations were then carried out to study how the path of the towed-body varies during both towed-in (straight to circular) and towed-out (circular to straight) manoeuvres. Measures were proposed to reduce the large amplitude tension waves the cable experiences due to the “whip” effect. One such technique is based solely on controlling the motion of the tow-point (aircraft). It was found that the most important parameters affecting the dynamics of the cable during transitional flight are the initial and final radii of the aircraft, along with the time required to undertake the transitions into or out of circular flight.

Alternatively, Williams and Trivailo looked into deploying the cable as the aircraft circles, developing two deployment control systems based on the reel acceleration of the cable; one based on heuristics, the other formulated using fuzzy logic. For the case of heuristic-based deployment control, it was found that an additional instability could arise in the lateral motion of the cable if the deployment rate is sufficiently large and the cable length is beyond a critical value. Williams and Trivailo subsequently showed how this instability can be significantly reduced by adjusting the deployment rate of the cable as a function of the deployed length. On the other hand, the fuzzy logic deployment scheme reduces the deployment rate of the cable as the towed body approaches the target, providing good closed loop performance and suppressing the unwanted lateral vibrations of the cable. The fuzzy logic deployment control scheme, along with appropriate changes to the tow point motion, were found to significantly moderate the tension within the cable during both types of transitional flight, as well as accelerate the rate at which the towed-body reaches and settles to the target position.
Finally, Williams and Trivailo [107] investigated the concept of sliding a payload down a cable being towed by an aircraft in constant circular motion. Employing the lumped parameter framework, the cable was discretized into a series of sequentially connected viscoelastic springs, subject to aerodynamic drag, gravitational loads, and frictional effects due to the sliding payload. The sliding mass was modeled as a free mass, subject to frictional forces from the cable, aerodynamic drag and gravity. The forces acting on the cable due to the geometric deformation caused by the sliding mass were calculated using a constitutive law. Noting that the equilibrium configuration of the cable whilst orbiting is strongly dependent on the weight of the cable tip/towed body, Williams and Trivailo hypothesize that large payloads traveling along the cable are likely to induce instabilities and result in large orbit radii for the cable tip/towed body.

Such phenomena would adversely affect the accuracy and operational performance of their proposed payload delivery system; hence they proposed that the cable tip should be anchored in either water or to the ground during operation. Investigations of the cable dynamics during the operation of the concept, revealed that it would be necessary to employ a braking device to slow the descent of the payload down the cable, since it can accrue large velocities (in the order of 25 m/s). Similarly, it was found that if the payload velocity is too large, the cable dynamics can become unstable and tension forces within the cable can be large. The study concluded that further work is necessary to improve the performance of the concept, in particular possible design changes, optimized aircraft maneuvers and measured braking profiles for the payload.

1.2.3 Review of Selected Applications for Aerial Towed-Cable Systems

As demonstrated by the literature review currently underway, a vast number of applications for ATC systems have been proposed and given considerable theoretical treatment by a large number of researchers. Attention will now be given to a select number of physical applications involving ATC systems that have been, and continue to be, successfully utilized in both civilian and defence domains.
**Defence Applications**

One of the most widely utilized defence applications involving ATC systems is the TACAMO (Take Charge and Move Out) system. This system has been operated by the United States Navy for over fifty years in the area of fleet ballistic missile communication operations [108]. The project was born out of the Cold War in 1961 to serve as the link between the United States Commander in Chief (president) and force command authorities with nuclear ballistic missile capabilities. The TACAMO system has flown over 28 years and 400,000 flight hours of safe missions ever since [109].

Over this time, many researchers in the field of ATC systems have either worked on the development of the TACAMO system, or used experimental flight test data concerning TACAMO to develop and verify similar ATC system designs (see [5, 84] for examples). The TACAMO system involves one of several classes of aircraft flying in a repeated circular manner (orbiting), towing an antenna via a very long (approximately five mile) dual trailing wire. The system is used for very low frequency/long distance communications, principally the one-way relaying of signals to submerged strategic missile submarines, providing an emergency communication capability [110]. As it currently stands, TACAMO provides the United States Navy with continuous airborne alert coverage for both the Atlantic and Pacific oceans; it can communicate with a submarine that is submerged in the Pacific Ocean while the aircraft is in flight over the Atlantic Ocean [108]. Initially, the TACAMO system was deployed using EC-130Q aircraft, although as these aircraft have aged, they have been replaced by the faster and longer range modified Boeing 707 aircraft, designated as E-6 A. These aircraft have maintained the same basic equipment that was installed in the EC-130Q variants, although significant performance improvements have been made to the E-6 A TACAMO system, in order to meet current and projected future requirements of the system (see [111, 112] for examples of enhancements to the E-6 A TACAMO system).

In the early nineties, Kelly Space and Technology (KST) proposed and subsequently patented the use of a towed airborne cable as part of a low cost method of launching satellites. The concept involves the use of an aircraft to tow a piloted space launch vehicle to a desired altitude. The tow rope is then severed at the launch vehicle end allowing the towing aircraft to return back to base, whilst the launch vehicle accelerates to high altitude using its rocket engines. At a sufficiently high altitude (low earth orbit), the nose of the launch vehicle is opened and a satellite payload is released into orbit. The space launch vehicle then re-enters and flies through the atmosphere using jet engines, before landing back at base.
Further support for KST was provided by the United States Air Force, which led to a feasibility study and experimental demonstration of the aerotow concept. This founded a flight test program for the aerotow concept that was initiated and successfully completed throughout the middle to late nineties by engineers at NASA [113]. The aerotow configuration consists of a C-141A Starlifter towing aircraft that tows a QF-106A Delta Dart aircraft by means of a 1000 ft long liquid crystal polymer tow rope. The flight test program consisted of a series of structural modifications made to the towing and towed aircraft, in order for the towing-induced loads to be appropriately accommodated. A series of ground-based tests were carried out on the tow rope, individual structural components and assemblies for the towing and towed aircraft, the towed aircraft release and towing aircraft guillotine mechanisms. Similarly, loose formation flying demonstrations were performed to confirm the minimal influence the towing aircraft flow field has on the towed aircraft.

A series of pre-flight simulation-based predictions were undertaken and compared with the flight test results, which were found to be promising in some respects and disappointing in others. The required take off distance and peak tow tension in the cable were well predicted, as was the fleeting and minimal effect the towed aircraft had on the towing aircraft. On the other hand, the results of dynamical simulations on the aerotow system using a single, straight and elastic segment to model the cable, was found to give erroneous results. The analytical simulation methods were found to under-predict and over-predict the sizes of the longitudinal and lateral-directional stability regions respectively, as compared to those observed in-flight. However, the handling qualities of the aerotow system in the vicinity of its optimal stability region were rated favourably by the test pilot.

**Civilian Applications**

Apart from the various defence related ATC system technologies, one of the first reported civilian applications of ATC systems was the “Air Pick Up” system developed by Lytle Adams and Boeing engineers in the late 1920’s. A system for picking up payloads, “Air Pick Up” consisted of a Stinson aircraft with a deployed boom that captured a cargo container attached to a cord running between two poles. A grappling hook at the end of the boom was used to capture the cord, thereby securing the container, with additional reserve rope deployed from a winch to nullify the shock associated with the pick-up operation. The cable and the payload were then retrieved back into the aircraft to complete the manoeuvre. All American Aviation used this technology until the 1950’s to delivery mail throughout the United States of America [4]. This system suffered from the disadvantage of having to fly the aircraft at very low altitude, and the pilot was required to provide the final guidance capability to
successfully secure the payload. Similarly, the presence of atmospheric turbulence and/or uneven terrain severely degraded the performance of the system, by reducing the probability of the hook mechanism successfully engaging the cord.

Another one of the first reported civilian applications of ATC systems was the payload delivery system demonstrated by Nate Saint in the early 1950’s [2]. A pilot working with the Missionary Aviation Fellowship deep in the jungles of Ecuador, Saint was able to successfully demonstrate that a light, fixed-wing aircraft could be used to deliver food and medical supplies to remote villages. Repeatedly orbiting over a delivery location, Saint used a cable with a basket attached to it to trade goods with Indians on the ground; the basket hovering almost stationary over the desired location on the ground.

Further developments and demonstrations were carried out in the United States of America confirming the feasibility, performance and suitability of the application as a means to safely deliver payloads both to and from surface locations without landing.

1.2.4 Deficiencies and Limitations Associated With Existing Research

With the comprehensive review of the dynamics, control and applications of ATC systems now complete, a number of conclusions can be immediately drawn from the review concerning the deficiencies and limitations associated with existing research. Firstly, it is clear that the steady state modelling and dynamic stability of ATC systems, in both uniform flow and when in orbital motion, has been thoroughly investigated by many researchers and is reasonably well understood by all. However, there is a distinct lack of time-history information in published works resulting from investigations into the dynamical behaviour of ATC systems in a variety of operating environments. In particular, research into the development of simplified representations of ATC systems, capable of accurately capturing the important dynamics of the physical system, is scant, even in light of the substantial benefits such research would provide. The reason for this one suspects, is partially because there has been little research undertaken into the development of control systems for ATC systems, with almost no work specifically addressing the development of towing strategies and cable-based control techniques during rendezvous and payload transportation-type operations. As a result, the primary objective of, and motivating force behind this research thesis, is the fitting redress of these two major anomalies in the existing body of knowledge concerning the dynamics and control of ATC systems.
1.3 Thesis Formulation

The extensive discourse concerning the dynamics, control and applications of ATC systems has elucidated some of the shortcomings associated with past and present technologies used to transport payloads to and from surface locations via a cable towed beneath an aircraft without landing. Accounting for this, the contents of this sub-section will delineate the central strategy employed to carry out this research, along with the obligatory constitution that serves to shape the ethos of this research thesis.

1.3.1 Research Questions

The following research questions govern the nature of the work proposed in this dissertation:

1. What are the feasible scenarios for transporting payloads to and from the Earth’s surface without landing, and if possible, how can the performance of these scenarios benefit through exploitation of the tether transient dynamics?

2. What are the most accurate and efficient mathematical models available to adequately simulate and control the complex dynamical behaviour of aerial tethers during payload transportation operations?

3. Is it possible to develop a systematic framework that can be used to improve the representativeness and accuracy of simplified mathematical representations of aerial tethers, and what form should this framework take?

4. Constrained by the available actuators, is it possible to create robust optimal control systems that adequately manipulate a tether towed from an aircraft to capture and deliver a payload, both to and from the earth’s surface?

1.3.2 Objectives

The primary objective of this thesis is to develop innovative non-linear optimal control systems to manoeuvre a cable towed beneath an aircraft, in order to transport payloads both to and from surface locations. To this end, it is necessary to develop accurate and efficient modelling capabilities that appropriately appraise and implement these control strategies for the ATC system.
Explicating further, the following have been identified as constituting the essence of the main objectives of the proposed research-

- To develop a variety of novel scenarios to be used for the purposes of transporting payloads to and from surface locations. These scenarios will account for the significant dynamic motion expected of the cable as it is methodically manoeuvred, so that its tip is in the near vicinity of the Earth’s surface for as long as practically possible. While a variety of scenarios are required to account for different operating situations, for simplicity and practicality, the aircraft flight regime will be restricted to straight and level flight or steady dynamic manoeuvres. The development of the scenarios will reveal the capabilities of, and set the reference environment for, the dynamics and control of the ATC system.

- To develop, validate and compare various mathematical models of the airborne tethered system, suitable for capturing the significant dynamics of the system and aid the development and evaluation of any candidate ATC control system. Advanced non-linear dynamic models allowing large curvature, deployment/retrieval, elasticity and distributed mass of the cable will be developed. These will be complemented by less advanced, yet highly representative dynamic models used for control system development. The sophistication of these simpler models will be increased inline with the complexity of the control objectives and proposed scenarios. These models will also be used to appraise the feasibility of the novel scenarios and assess control system performance. Systematic techniques that ensure the simpler dynamic models remain representative of the more sophisticated system models will also be pursued.

- To develop autonomous, non-linear optimal control systems for the ATC system to carry out each of the developed scenarios nominated for payload capture and release operations. Control actuation will be achieved through careful management of the cable deployment rate, aircraft speed and directional heading. These control systems must be robust to environmental disturbances encountered by the ATC system, and where possible, attempt to prolong the rendezvous time of each scenario in order to maximize the chances of successful capture/delivery. Finally, these highly autonomous control systems should be accurate, reliable and safe, whilst maintaining as wide an operational envelope as physically possible.
1.3.3 Scope

The research scope of this thesis will primarily focus on the role the cable plays in the ATC system during payload transportation operations, making recommendations where appropriate as to how cable dynamics and control implementation interacts with the aircraft and the payload. The scope of this work is necessarily restricted to the development of non-linear mathematical models and optimal control strategies required by the ATC system in order to achieve the aforementioned research targets. The formal design of any anticipated procedures required to exploit this technology is not part of this research. Finally, highly intricate simulations and detailed control of the aircraft-tether-payload system post capture and release will not be considered in this research.

1.4 Thesis Organization

The purpose of this thesis is to tender novel research conducted by the author in the areas of non-linear dynamics and optimal control of ATC systems, which together with current state of savoir-faire, will serve to provide an intellectual foundation for advancing the development of aerial tether payload transportation technology.

In Section 1, the background to the problems investigated in thesis is introduced, first by way of a comprehensive review of the literature concerning the dynamics, control and applications of ATC systems, thereby identifying and discussing the void that exists within the current body of knowledge. A characterization of the research questions that govern the nature of the work presented in this thesis then follows, from which the objectives and scope of this thesis are outlined.

Section 2 systematically develops and presents the various models of the ATC system that are extensively utilized throughout this thesis. The section begins with a review of the various methods employed to formulate the equations of motion for ATC systems, then progresses to develop the equations of motion for numerous models of the ATC system, each possessing varying degrees of sophistication. These models are then used to design innovative, non-linear optimal controllers for the system.
The development of a series of systematic techniques that can be used to improve the representativeness and fidelity of simplified ATC system models is the subject of material presented in Section 3. The objective of this section is to provide an efficient framework for matching the dynamical motion of simple ATC system models to that of the more complex multi-link models. Two separate approaches to this “model matching” problem were successfully developed, the implementation and performance of which are explored in this section.

Section 4 is committed to the development of single phase, non-linear optimal control strategies for the ATC system. The section begins with a review of the various numerical methods available to formulate and solve problems arising in non-linear optimal control, and then progresses into a development of the methods employed in this thesis to develop non-linear, single-phase optimal controllers for ATC systems. More specifically, instantaneous rendezvous problems for the ATC system in two and three-dimensions are the subject of a detailed investigation in this section. The results presented in this section show that it is possible to achieve accurate, instantaneous rendezvous of the cable tip with a desired surface location, through deployment and retrieval control of the cable and/or aircraft manoeuvring, implemented via single phase, non-linear optimal control.

Section 5 contains the results of various case studies that were performed to assess the impact that a wide range of system and environmental parameters have on the ability of the ATC system to carry out rendezvous manoeuvres. From this investigation, a range of important issues were identified that required careful consideration when more complex operations for the ATC system were formally considered.

Section 6 and Section 7 constitute a two-part series devoted to the development of multi-phase, non-linear optimal control problems for the ATC system. The twin areas of multiple instantaneous rendezvous and payload delivery/capture problems for the ATC system, in both two and three-dimensions, are extensively explored separately within these sections. Numerous case study investigations are also performed in these sections to assess the impact that a wide range of system and environmental parameters have on the ability of the ATC system to perform multiple rendezvous and payload transportation operations. The results obtained show that via multi-phase, non-linear optimal control, not only is it possible to achieve accurate, rendezvous of the cable tip with multiple desired locations on the ground, it is also possible to both capture and deliver payloads at/to these locations through specific aircraft manoeuvres and length control of the cable.
Whereas the various non-linear optimal controllers developed for the ATC system in Sections 4 through to 7 are inherently open loop, Section 8 is dedicated to the development of a series of simple, closed loop optimal feedback controllers for the ATC system. The influence that variable wind gusts have on the ability of the ATC system to follow the optimal open loop trajectories developed in Sections 4 through 7 is examined in this section. The resulting performance of the closed loop ATC system demonstrated in this section confirms that precise and robust control of the ATC system can be achieved under a wide range of operating conditions during rendezvous and payload transportation operations.

Section 9 summarizes the main conclusions drawn from the research undertaken for this thesis, along with presenting a series of key recommendations for further work related to the development and implementation of non-linear optimal controllers for ATC systems. This section also identifies the manner in which the results obtained from this thesis can be used in the development of a variety of novel applications currently proposed for ATC systems in the near-term.

For the most part, the details, results and outcomes of the research presented within this dissertation have also been published in various peer-reviewed international journals and conference proceedings. The bibliographic details of these publications are listed below in chronological order, beginning with the most recent; most references also appear in Section 10 of this dissertation, as indicated:


- [115] [Submitted, October 2005] Sgarioto, D., Williams, P. and Trivailo, P. “Remote Payload Capture and Delivery Using an Aircraft-Towed Flexible Cable System”, *Australian and New Zealand Institute of Applied Mathematics Journal*.


- [103] [Accepted, October 2005] Williams, P., Sgarioto, D. and Trivailo, P. “Constrained Path Planning For an Aerial-Towed Cable System”, *Aerospace Science and Technology Journal*.


• Sgarioto, D. and Trivailo, P. “Cable Assisted Rendezvous for Aircraft With Surface Locations”, *Proceedings of the 16th International Federation of Automatic Control World Congress*, Prague, Czech Republic, July 4-8, 2005.


2 AERIAL TOWED-CABLE SYSTEM MODELLING

2.1 Preface

The principal objective of this section is to formally present the various models of the ATC system that will be extensively utilized throughout this thesis. The section begins with a review of the various methods employed to formulate the equations of motion for ATC systems. The equations of motion for models of the ATC system possessing varying degrees of sophistication will then be developed. These models will be used to study the fundamental dynamics of the physical ATC system and design innovative optimal controllers for the system, in order to demonstrate the wide range of potentially useful physical applications long expected of ATC systems.

2.2 Obtaining the Equations of Motion for Aerial Towed-Cable Systems

To derive the equations of motion for ATC systems, two distinct approaches are available to the analyst, those being scalar or vectorial approaches. Each approach employs various mathematical laws and principles that yield unique advantages, yet concede inevitable disadvantages. These issues and others specifically pertaining to ATC systems are explored in this sub-section.

2.2.1 Newton’s Second Law

The application of Newton’s Second Law is the essence of engineering mechanics and serves as the most fundamental and widely used of all modelling tools to formulate equations of motion for dynamical systems. For a system of particles, Newton’s Second Law is represented mathematically as:

\[ \sum_{i=1}^{n} F_i = \sum_{i=1}^{n} m_i a_i \]  

(2.2.1)

where \( F_i \) is the external force acting on particle \( i \), having mass \( m_i \) and acceleration \( a_i \). Essentially a vectorial approach utilizing an inertial reference frame, this method uses concepts such as force and momentum to analyse the motion of a system. All motion
variables (displacements, velocities and accelerations) are expressed in this reference frame and with the aid of a free-body diagram, all external forces acting on each body must be identified, including non-working constraint forces. Whilst this is a relatively straightforward exercise for simple systems with low degrees-of-freedom, for higher degree-of-freedom systems such as ATC systems, the consideration of all constraint forces is cumbersome and the resulting vectorial algebra is arduous. This is especially so as Newton’s method requires the full calculation of the acceleration of each particle.

### 2.2.2 Lagrange’s Equations of Motion

While the derivation of the equations of motion for a dynamical system using Newton’s Second Law is based solely on vectorial mechanics, the Lagrangian formulation of the equations of motion is based entirely on a scalar procedure derived from Hamilton’s Principle. This formulation departs from its Newtonian counterpart in that the use of generalized coordinates is permitted, as opposed to physical coordinates, enabling versatility not possible with the Newtonian approach. Following on, the scalar quantities of kinetic energy, potential energy and work are formulated in terms of these generalized coordinates. Hence, Lagrange’s formulation has the advantage that only the calculation of first order time derivatives (velocities) for each particle/body is required, which are inherently much simpler and appreciably easier to determine than the accelerations that are required when Newton’s method is used. Since the Lagrangian formulation is based on Hamilton’s principle, the method remains invariant to the coordinate system employed to formulate the equations of motion. Similarly, the non-working constraint forces that plague Newton’s formulation are not present in the equations of motion derived using Lagrange’s method. As a result, the method of Lagrange provides an efficient and expedient means of deriving the equations of motion for high degree-of-freedom systems.

Using either the generalized principle of D’Alembert or conserving the total energy of the system, the Lagrangian equations of motion for an \(n\)-degree-of-freedom, non-conservative system are:

\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i , \quad (i = 1, 2, \ldots, n) \quad (2.2.2)
\]

where: \(\mathcal{L}\) is the Lagrangian of the system given by equation (2.2.3), \(T\) is the kinetic energy of the system,
\( V \) is the potential energy of the system,

\( Q_i \) are the generalized non-conservative forces acting on the system,

\( q_i \) are the generalized coordinates of the system,

\( \dot{q}_i \) are the first-order time derivatives of the generalized coordinates.

The Lagrangian of the system, defined as the difference between the kinetic and potential energy of the system, is given by:

\[
\mathcal{L} = T - V
\]  

While the derivation of the equations of motion for a system using Lagrange's formulation enjoys a range of advantages that endorse its widespread use, it does suffer from several distinct drawbacks. Firstly, the number of partial derivatives required to produce the equations of motion in the form of equation (2.2.2) increases significantly as the degree of freedom of a system increases. This is particularly apparent for ATC systems, which tend to have a disproportionately high degree-of-freedom. While advances in the development of symbolic manipulation software have offset this by enabling symbolic differentiation to be readily performed on complex algebraic functions, the resulting partial derivatives can be large in number and complex for high degree-of-freedom systems such as ATC systems.

Additionally, the Lagrangian formulation does not allow for changes to the mathematical model of a system to be routinely incorporated. Such an endeavour almost always requires a complete reformulation of the equations of motion. This is because the scalar and complicated algebraic nature of the formulation hinders the analyst from performing intuitive changes to the equations of motion. This re-formulating process is labour-intensive and inefficient, which impedes the fine-tuning of dynamical system models.

Moreover, the equations of motion derived using the method of Lagrange tend to have significant coupling existing between the acceleration terms (second-order time derivatives of the generalized coordinates) in the equations of motion. In order to proceed to a solution, integration is required and the equations of motion need to be decoupled. In most instances, this requires matrix inversion, which is of negligible significance if such a matrix can be inverted analytically. However, for complex and high degree-of-freedom systems such as ATC systems, this is not always possible and computationally expensive numerical inversion is required at each time step during the integration. This renders solution times long and simulations unwieldy to perform.
2.2.3 Kane’s Dynamical Equations of Motion

The onset of the era in which the use of digital computing was rapidly increasing, resulted in the need for a more algorithmic approach to the modelling of dynamical systems, particularly those of high degrees-of-freedom and complex configuration such as multi-body systems. Kane [54] elegantly provided a systematic framework for such a task, in which quasi-velocity coordinates were used to describe the motion of dynamical systems. This provided inherent flexibility in the modelling procedure and allowed systems with non-holonomic constraints to be straightforwardly accommodated. The procedure for modelling dynamical systems outlined in [53, 54, 118, 119] is now popularly known as Kane’s Method or Kane’s Equations, which can be derived from Newton’s Second Law by invoking the principle of virtual work as follows:

Consider an \( n \)-particle system possessing \( r \) degrees-of-freedom, whose motion is governed by Newton’s Second Law. Reverting back to the vectorial approach previously adopted for Newton’s method, the virtual work performed by the vector force \( \mathbf{F}_i \) through the virtual displacement \( d\mathbf{r}_i \) is:

\[
dW = \sum_{i=1}^{n} \mathbf{F}_i \cdot d\mathbf{r}_i = \sum_{i=1}^{n} m_i \mathbf{a}_i \cdot d\mathbf{r}_i
\]  \hspace{1cm} (2.2.4)

Equation (2.2.4) is an extended statement of D’Alembert’s Principle. The first variation in the position of the \( i^{th} \) particle \( d\mathbf{r}_i \) may be rewritten in terms of differential time \( dt \) :

\[
dW = \sum_{i=1}^{n} \mathbf{F}_i \cdot \mathbf{v}_i \cdot dt = \sum_{i=1}^{n} m_i \mathbf{a}_i \cdot \mathbf{v}_i \cdot dt
\]  \hspace{1cm} (2.2.5)

where \( \mathbf{v}_i \) is the velocity of the \( i^{th} \) particle, which can be written in terms of the generalized coordinates \( q \), as follows [120]:

\[
\mathbf{v}_i = \sum_{j=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_j} \frac{dq_j}{dt} + \frac{\partial \mathbf{r}_i}{\partial t}
\]  \hspace{1cm} (2.2.6)

Kane’s equations essentially differ from equations of motion derived using D’Alambert’s principle in that an additional velocity coordinate is introduced: generalized speeds. Generally, generalized speeds are non-holonomic and are not a direct derivative of the generalized coordinates. They are a distinct quantity whose form is usually selected as a matter of convenience so as to simplify the formulation of the equations of motion [118, 120].
Generalized speeds are defined as linear combinations of the generalized velocities:

\[ u_j = \sum_{i=1}^{n} Y_{ji} q_i + Z_j , \quad (j = 1, 2, \ldots, n) \]  

(2.2.7)

where \( Y_{ji} \) and \( Z_j \) are known functions of the generalized coordinates \( q_i \) and time \( t \).

Equation (2.2.7) may be inverted to allow for the unique determination of the generalized velocities \( \dot{q}_i \), resulting in the so-called kinematic equations:

\[ \dot{q}_i = \sum_{i=1}^{n} Y'_{ji} u_j + Z'_j , \quad (j = 1, 2, \ldots, n) \]  

(2.2.8)

where \( Y'_{ji} \) and \( Z'_j \) are again known functions of the generalized coordinates \( q_i \) and time \( t \) and are dependent only on the geometry of the system. Substituting equation (2.2.8) into equation (2.2.6), the velocity of the \( i \)th particle can be written in terms of the generalized speeds:

\[ v_i = \sum_{j=1}^{n} \frac{\partial v_i}{\partial u_j} u_j + \frac{\partial r_i}{\partial t} \]  

(2.2.9)

where \( \frac{\partial v_i}{\partial u_j} \) is the \( j \)th partial velocity of the particle \( i \), normally found by inspection once equation (2.2.9) has been formulated. The vector \( \frac{\partial r_i}{\partial t} \) is usually due to prescribed time-varying inputs to the system [119]. The partial velocity \( \frac{\partial v_i}{\partial u_j} \) of the \( i \)th particle can be related to the generalized velocity \( \frac{\partial v_i}{\partial \dot{q}_j} \) via the following:

\[ \frac{\partial v_i}{\partial u_j} = \sum_{j=1}^{n} \frac{\partial v_i}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial u_j} = \sum_{j=1}^{n} \frac{\partial v_i}{\partial \dot{q}_j} Y'_{ji} \]  

(2.2.10)

By utilizing the modified form of D’Alembert’s Principle in equation (2.2.5) and performing various substitutions, Kane’s Equations can be written in their standard form as follows [119]:

\[ F_k^* + F_k = 0 , \quad (k = 1, 2, \ldots, r) \]  

(2.2.11)

where \( F_k^* \) are the generalized inertia forces defined by equation (2.2.12) and \( F_k \) are the generalized active forces defined by equation (2.2.13).

\[ F_k^* = -\sum_{j=1}^{n} m_i a_i \frac{\partial v_i}{\partial u_k} , \quad (k = 1, 2, \ldots, r) \]  

(2.2.12)
\[ F_k = \sum_{i=1}^{n} \mathbf{F}_i \frac{\partial \mathbf{v}_i}{\partial u_k}, \quad (k = 1,2,...,r) \]  

(2.2.13)

While the dynamical equations of motion given by equations (2.2.12) and (2.2.13) govern the dynamics of a system of particles, direct extension to a system of rigid bodies, capable of both rotation and translation is straightforward. The generalized inertia forces and generalized active forces now become:

\[ F_k^* = \sum_{i=1}^{n} \left[ m_i \mathbf{a}_i \cdot \frac{\partial \mathbf{v}_i}{\partial u_k} - \left( \mathbf{I}_i \cdot \mathbf{a}_i + \mathbf{\omega}_i \times \left( \mathbf{I}_i \cdot \mathbf{\omega}_i \right) \right) \cdot \frac{\partial \mathbf{\omega}_i}{\partial u_k} \right], \quad (k = 1,2,...,r) \]  

(2.2.14)

\[ F_k = \sum_{i=1}^{n} \left( \mathbf{F}_i \cdot \frac{\partial \mathbf{v}_i}{\partial u_k} + \mathbf{T}_i \cdot \frac{\partial \mathbf{\omega}_i}{\partial u_k} \right), \quad (k = 1,2,...,r) \]  

(2.2.15)

where: \( \mathbf{\omega}_i \) is the angular velocity vector of body \( i \),

\( \mathbf{a}_i \) is the angular acceleration vector of body \( i \),

\( \frac{\partial \mathbf{\omega}_i}{\partial u_k} \) is the \( k^{th} \) partial angular velocity of body \( i \),

\( \mathbf{T}_i \) is the active resultant moment on body \( i \),

\( \mathbf{I}_i \) is the principle inertia dyadic of body \( i \).

Another way to interpret the equations of motion given by equations (2.2.14) and (2.2.15) is that they represent the sum of the resultant force and moment for each body projected along the direction of each of the partial velocities [121].

The algorithmic nature of Kane’s formulation allows complex dynamical systems with high degrees-of-freedom to be treated with relative ease in a systematic manner, overcoming the main disadvantage associated with employing Lagrange’s formulation on such systems [118]. The large numbers of partial derivatives required in Lagrange’s formulation are not present in Kane’s equations. In most cases through the careful selection of the generalized speeds, no coupling exists between the acceleration terms in the equations of motion. Kane’s formulation also enjoys a unique advantage over other methods in that changes to the mathematical model of the system can be intuitively and relatively easily accommodated. This is especially salient in the case of ATC systems where a large number of different forces are present and various models of these forces can be applied, tested and compared almost interchangeably with relative ease.
2.3 Modelling Approaches

As the literature review presented in Section 1.2 can attest, the modelling of towed-cable systems in a wide variety of operating environments has, and continues to be, a popular field of inquiry. Each operative environment, be it underwater, atmospheric or space, has unique features that must be appropriately considered when a candidate modelling approach is contemplated. Two very distinct approaches to the dynamic modelling of ATC systems can be identified, these being continuous or discrete approaches. The factors one must consider when selecting an approach include issues such as solution accuracy, computation and development time, cost or personal experience. It is difficult to make generalizations as to which approach is “best” in all circumstances, since the reality is that various factors often conflict with each other and different evaluations may apply in numerous instances. As a result, this section will briefly outline these approaches as they apply to ATC system modelling and discuss the relative merits and limitations associated with each approach.

2.3.1 Implicit Continuous Models

Continuous mathematical models of ATC systems usually begin with the consideration of a differential element of the cable in space. The equations of motion are found by applying Newton’s Second Law to the element, resulting in a set of highly coupled, non-linear partial differential equations, with boundary conditions often represented as ordinary differential equations. The solution to this set of equations usually involves the application of an implicit finite differencing technique, where the tension and cable dynamics are solved simultaneously using a Newton-based numerical algorithm. Continuous ATC models have the advantage of being capable of accurately simulating the dynamics of the cable, however this comes at the cost of considerable computational expense associated with the numerical solution procedure. Similarly, singularities and numerical instabilities arise in the solution process as the tension within the cable approaches zero, although the inclusion of bending stiffness within the model eliminates this predicament [122]. Modifications to an existing model are frequently cumbersome to implement since it is often impossible to incorporate them without re-deriving the governing partial differential equations of motion. Many examples of continuous ATC models do exist [1, 5, 25-27, 33-35, 37-39, 44, 48, 59, 65, 66, 70, 78, 84, 122, 123] as this approach was once popular. Its use and appeal has diminished in recent times, in line with the development of discrete towed-cable models.
2.3.2 Explicit Discrete Models

Whilst the cable is mathematically discretized when an implicit continuous modelling strategy is used to model the dynamics of an ATC system, the discrete school of modelling adopts a different approach by physically discretizing the cable; breaking it up into a series of elements of finite length. Each element is then considered individually by writing force/moment equations to represent its dynamical motion, with appropriate measures adopted to deal with the boundary conditions of each element. Where as continuous models are characterized by highly coupled, non-linear partial differential equations with boundary conditions represented as ordinary differential equations, the equations of motion for discrete models are categorized by a collection of coupled, ordinary differential equations with boundary conditions represented as algebraic functions. By their very nature, the governing dynamical equations of motion for discrete cable models lend themselves more easily to solution via a digital computer, the degree of ease depending largely on the method of discretization and formulation. The most significant reason as to why the popularity and appeal of discrete cable models has surpassed that associated with continuous cable models is due to the recent advances made possible through the development of algorithmic modelling techniques for multi-body systems [52-54, 118, 124-126]. Using the explicit discrete modelling approach, the cable can be physically discretized in a number of ways, the two most popular of which are the finite element method and the lumped parameter method.

**Finite Element Method**

As the name suggests, the finite element method of physical discretization involves the cable being partitioned into a series of finite elements, which may be rigid or elastic, straight or curved. The distributed effects of cable inertia and externally applied forces are accounted for by applying equivalent representations at the ends (nodes) of each segment. Once appropriate element and force models have been developed, the finite element method can prove to be very flexible, although several disadvantages associated with this discretization scheme do exist. Firstly, the nodal accelerations are generally coupled, resulting in the entire mass matrix for the system needing construction and inversion. Similarly, if the mass matrix for the system is not constant (time varying), as it can be when certain formulations are used to determine the equations of motion for the system, numerical matrix inversion is required at each time step during the solution procedure, a process demanding considerable computational resources. Developmental times can be long if one does not have access to a numerical package capable of solving the motion equations for the system. Examples of towed-cable systems modelled using this method can be found in [2, 20, 46, 47, 51].
Lumped Parameter Method

Modelling towed-cables using the lumped parameter approach involves dividing the cable into a series of discrete segments, whose mass and forces are lumped at the end of each individual segment (usually half the contribution of two adjoining segments). As the inertial properties of each segment are lumped to a single point, the lumped parameter approach is able to surmount the elemental coupling associated with the finite element approach. If each segment is assumed to be extensible, the constitutive relation for the material yields the longitudinal constraint force present within each segment, allowing the motion of each node to be uniquely determined at each time step. For inextensible cables, an appropriate choice of method to formulate the equations of motion for the system can result in constraint forces being eliminated altogether form the analysis (see Section 2.2.3), although the geometry of the links (length of each link) can be used to properly account for the connectivity constraints of each link. Examples of towed-cable systems modelled using the lumped parameter approach can be found in [3, 6, 17, 46, 47, 52, 55-58, 69, 81, 84, 98, 101, 105, 114, 127, 128].

Comparison of Each Method

Although several comparative surveys have been published that contrast the various approaches available for modelling towed-cable systems [46, 47], no clear conclusions have been reached as to which method can be considered the “best” possible in all circumstances. Solution accuracy depends on the level of discretization and the representativeness of the force models employed, whilst the solution time depends on the choice and degree of discretization, elemental coupling, and the numerical integration technique. The lumped parameter method has numerous advantages over the continuous and finite element methods. In particular, the implicit nature of the solution procedure associated with continuous models is a considerable drawback, as is the elemental coupling and repeated numerical matrix inversion that is characteristic of finite element methods. In contrast, the lumped parameter method requires an explicit solution procedure, it is both easy to formulate and flexible, whilst being well equipped to deal with any motion constraints acting on the system.
2.4 Simplified Aerial Towed-Cable System Modelling

2.4.1 Model Development

The basic physical arrangement of the ATC system is given in Figure 2-1a, along with an idealized mathematical representation of the physical system, depicted in Figure 2-1b.

![Figure 2-1: Description of the Aerial Towed-Cable System- (a) Physical System (b) Idealized Model](image)

The simplified representation of the ATC system in Figure 2-1b was deduced only after careful consideration, whereby a decision regarding the complexity and representativeness of the model was weighed up against its feasibility and ease with which it could be used for control system design and development purposes. This required a series of assumptions to be made as to the nature of the system and the environment within it operates. The choice of assumptions was strategic and judicious; any arduous complexity of the system considered superfluous was removed, yet enough detail was preserved in the model to ensure that all the significant dynamics of the system are adequately captured.
2.4.2 Governing Assumptions

The development of a simple and representative ATC system model begins with the assumption that the aircraft has an infinite mass and its dynamical motion remains unaffected by the dynamics of the cable-payload combination. This assumption is reasonable provided the mass of the payload being towed is well below that of the towing aircraft, which is predominately true for all but the most extreme cases. For simplicity, the aircraft centre of gravity (CG) is assumed to be co-incident with the location of the attachment point at the aircraft end of the cable; this attachment point is assumed to act as a frictionless hinge. As a result, the motion of the aircraft is essentially a kinematic input for the cable-payload sub-system; more specifically a kinematic constraint ensuring that the motion of the attachment point at the aircraft end (tow-point) can be prescribed for all time. Standard sea level conditions as specified by the International Standard Atmosphere (ISA) are assumed to prevail at all times.

Following on, it is assumed that the cable is rigid and has a constant circular diameter. It is also assumed that the cable always takes up a planar two-dimensional profile and that it remains straight at all times. Hence, the instantaneous length of the cable can be characterized by the shortest possible distance between each of the attachment points. Similarly, the planar orientation of the cable is represented by the angle the cable makes with the local vertical (see Figure 2-2). The length of the cable is permitted to vary by prescribing the time rate of change at which the deployment/retrieval speed of the cable varies (reel acceleration). Furthermore, the cable is rigidly attached to a perfectly spherical payload, which is taken to be homogeneous and attached to the cable at its CG.

With respect to the external forces acting on the ATC system, only forces of an aerodynamic and gravitational nature are considered. Moreover, it is assumed that external forces acting on the payload are the most dominant; hence the aerodynamic and gravitational forces acting on the cable will be ignored here. The aerodynamic and gravitational forces acting on the payload are assumed to have a point of action coincident with the CG of the payload. Since the payload is spherical and not rotating, it is further assumed that the payload is incapable of producing lift and only the drag component of the aerodynamic force will be considered. Subcritical flow is assumed to always be prevalent and only simple velocity-dependent drag will be considered for the payload; unsteady, compressible and high-order aerodynamics are not treated in this study. Finally, the aerodynamic drag considered for the payload is assumed to be limited to the component acting normal to the instantaneous orientation of the cable.
2.4.3 Derivation of the Equations of Motion Using Lagrange’s Equations

A more complete analytical treatment of the simplified model of the ATC system introduced in Figure 2-1b is provided in Figure 2-2.

![Diagram of aerial towed-cable model](image)

**Figure 2-2: Complete Mathematical Description of the Simple Aerial Towed-Cable Model**

To assist in the derivation of the dynamical equations of motion for the ATC system depicted in Figure 2-2, two coordinate systems are utilized. One such coordinate system is the non-rotating Cartesian frame \( x-y \), which translates as the aircraft CG location translates. This reference frame is defined by the unit vectors \( e_i \) in the \( x \)-direction and \( e_j \) in the \( y \)-direction respectively. The other coordinate system employed is the normal-tangential coordinate system \( r-\theta \), which is both translating and rotating in accordance with the motion of the payload. This coordinate system is characterised by the unit vectors \( e_r \) in the \( r \)-direction and \( e_\theta \) in the \( \theta \)-direction respectively. The generalized coordinates for the ATC system are selected as the in-plane angle the cable makes with the local vertical \( \theta \) and the instantaneous cable length \( l \). Furthermore, the kinematic constraints representing the displacement of the aircraft \( \{x, y\} \) are also present. The total degree-of-freedom \( n \) for the system is four.

Briefly adopting a vectorial approach and employing the Cartesian reference frame, the position of the payload \( \mathbf{r}_p \) can be written as:

\[
\mathbf{r}_p = (x - l \sin \theta) \mathbf{e}_i + (y - l \cos \theta) \mathbf{e}_j
\]  

(2.4.1)
Differentiating with respect to time, the velocity of the payload $v_p$ can be expressed in the Cartesian frame as:

$$v_p = (\dot{x} - i \sin \theta - l \dot{\theta} \cos \theta)e_i + (\dot{y} - i \cos \theta + l \dot{\theta} \sin \theta)e_j \quad (2.4.2)$$

Alternatively, the velocity of the payload $v_p$ can be more conveniently expressed in the normal-tangential frame as:

$$v_p = v_r e_r + v_N e_N$$

$$= (i \dot{x} \sin \theta - \dot{y} \cos \theta)e_r + (\dot{y} \sin \theta + i \dot{x} \cos \theta)e_N \quad (2.4.3)$$

Now the total kinetic energy of the system, $T$ can be found:

$$T = \frac{1}{2} m_P v_p^2$$

$$= \frac{m_P}{2} \left[ (i \dot{x} \sin \theta - \dot{y} \cos \theta)^2 + (l \dot{\theta} - \dot{x} \cos \theta + \dot{y} \sin \theta)^2 \right] \quad (2.4.4)$$

$$= \frac{m_P}{2} \left[ x^2 + y^2 + l^2 - 2ix \sin \theta - 2iy \cos \theta - 2il \dot{\theta} \cos \theta + 2iy \dot{\theta} \sin \theta + i^2 \dot{\theta}^2 \right]$$

The potential energy of the system $V$ has the following form:

$$V = m_P g \Delta y \quad (2.4.5)$$

where $\Delta y$ is the change in altitude the payload experiences. Figure 2-3 shows how the altitude of the payload varies during a typical towing application, whereby the payload begins at point $P$ and moves through to point $P'$. 

![Figure 2-3: Depiction of Payload Altitude Variation](image-url)
With the assistance of Figure 2-3, the altitude change the payload experiences is:

\[
\Delta y = y_p' - y_p = l - \Delta x \tan \theta = l - \left( l \sin \theta \cdot \frac{\cos \theta}{\sin \theta} \right) = l(1 - \cos \theta)
\]  

(2.4.6)

Upon substitution of equation (2.4.6) into equation (2.4.5), the total potential energy of the system \( V \) can be found:

\[
V = m_pl \left( 1 - \cos \theta \right)
\]

(2.4.7)

With the help of equations (2.4.4) and (2.4.7), the Lagrangian of the system \( L \) can now be formulated from the total kinetic and potential energies of the system as follows:

\[
L = T - V = \frac{m_p}{2} \left[ \dot{x}^2 + \dot{y}^2 + l^2 \dot{\theta} - 2 \dot{x} \dot{x} \sin \theta - 2 \dot{y} \dot{\theta} \cos \theta + 2 \dot{y} \dot{\theta} \sin \theta + l^2 \dot{\theta}^2 \right] - \left[ m_pl \left( 1 - \cos \theta \right) \right]
\]

(2.4.8)

With the Lagrangian of the system given by equation (2.4.8), it is now possible to determine the partial derivatives that make up the left-hand-side of Lagrange’s Equations of motion given by equation (2.2.2):

Letting \( q_1 = x \), \( q_2 = y \), \( q_3 = \theta \), \( q_4 = l \), the required partial derivatives are:

\[
\frac{\partial L}{\partial q_1} = 0
\]

(2.4.9)

\[
\frac{\partial L}{\partial q_2} = 0
\]

(2.4.10)

\[
\frac{\partial L}{\partial q_3} = m_p \left[ \dot{x} \dot{\theta} \sin \theta - \dot{x} \dot{\theta} \cos \theta + \dot{y} \dot{\theta} \sin \theta + \dot{y} \dot{\theta} \cos \theta - gl \sin \theta \right]
\]

(2.4.11)

\[
\frac{\partial L}{\partial q_4} = m_p \left[ l \dot{\theta}^2 + \dot{y} \dot{\theta} \sin \theta - \dot{x} \dot{\theta} \cos \theta - g \left( 1 - \cos \theta \right) \right]
\]

(2.4.12)

\[
\frac{\partial L}{\partial \dot{q}_1} = m_p \left( \dot{x} - \dot{\theta} \sin \theta - l \dot{\theta} \cos \theta \right)
\]

(2.4.13)
\[
\frac{\partial L}{\partial \dot{q}_2} = m_p \left( \dot{y} - \dot{l} \cos \theta + l \dot{\theta} \sin \theta \right) \\
\frac{\partial L}{\partial \dot{q}_3} = m_p \left( \dot{l} \dot{\theta} - \dot{x} \cos \theta + j \dot{y} \sin \theta \right) \\
\frac{\partial L}{\partial \dot{q}_4} = m_p \left( \dot{x} \sin \theta - \dot{y} \cos \theta \right)
\]

(2.4.14)  
(2.4.15)  
(2.4.16)

The required time rate of change of the partial derivatives that make up Lagrange’s Equations can now be found:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_1} \right) = m_p \left( \ddot{x} - \ddot{l} \sin \theta - 2 \dot{l} \dot{\theta} \cos \theta - \dot{l} \dot{\theta} \sin \theta + l \ddot{\theta}^2 \sin \theta \right) \\
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_2} \right) = m_p \left( \ddot{y} - \ddot{l} \cos \theta + 2 \dot{l} \dot{\theta} \sin \theta + \dot{l} \dot{\theta} \sin \theta + l \dot{\theta}^2 \cos \theta \right) \\
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_3} \right) = m_p \left( -\ddot{x} \cos \theta - \ddot{l} \sin \theta - \dot{x} \dot{\theta} \cos \theta \sin \theta + j \dot{l} \dot{\theta} \sin \theta + j \dot{y} \dot{\theta} \cos \theta + 2 \dot{l} \dot{\theta} + l \ddot{\theta} \right) \\
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_4} \right) = m_p \left( \ddot{x} \sin \theta - \ddot{y} \cos \theta + \dot{y} \dot{\theta} \sin \theta - \dot{x} \dot{\theta} \cos \theta \right)
\]

(2.4.17)  
(2.4.18)  
(2.4.19)  
(2.4.20)

Completing the formulation of Lagrange’s Equations of motion requires the calculation of the generalized non-conservative forces \( Q_i \) acting on the system using the principle of virtual work. In lieu of the assumptions previously made in section (2.4.2), the only non-conservative force acting on the system is the aerodynamic drag acting on the payload. A depiction of the external non-conservative forces acting on the system, along with the tension in the cable \( T_n \), is given in Figure 2-4.
To assist in the determination of the aerodynamic drag force acting on the payload, the normal-tangential frame is employed. The simplest and most general form of aerodynamic drag acting on the payload can be written in vector form as:

\[ \mathbf{F}_{\text{drag}} = -\frac{1}{2} \rho_{\text{air}} C_{D_p} A_p \mathbf{v}_p \mathbf{v}_N \]  

(2.4.21)

where:  
\( \rho_{\text{air}} \) is the atmospheric air density,  
\( C_{D_p} \) is the drag coefficient of the payload,  
\( A_p \) is the projected area of the payload given by equation (2.4.22),  
\( \mathbf{v}_N \) is the normal velocity of the payload.

Since the payload is assumed to be perfectly spherical with a diameter \( d_p \), the projected area of the payload is given by:

\[ A_p = \frac{\pi}{4} d_p^2 \]  

(2.4.22)

Now substituting equation (2.4.3) and (2.4.22) into equation (2.4.21), the aerodynamic drag acting on the payload becomes:

\[ \mathbf{F}_{\text{drag}} = -\frac{1}{8} \rho_{\text{air}} C_{D_p} \pi d_p^2 \begin{bmatrix} l\dot{\theta} + \dot{y}\sin \theta - \dot{x}\cos \theta \end{bmatrix} \mathbf{e}_\theta \]  

(2.4.23)

The total non-conservative force vector can now be formulated:

\[ \mathbf{F} = \mathbf{F}_{\text{drag}} \]  

(2.4.24)

By definition, the generalized forces acting on the system can be found via the principle of virtual work from:

\[ Q = \sum_{i=1}^{n} \mathbf{F} \cdot \frac{\partial \mathbf{r}_p}{\partial q_i} \]  

(2.4.25)
By substitution of equation (2.4.1) and (2.4.24) into equation (2.4.25), the generalized forces for the system can be found:

\[ Q_1 = \frac{1}{8} \rho_{air} C_{D_p} \pi d_p^2 \left[ l\dot{\theta} + \dot{y} \sin \theta - \dot{x} \cos \theta \right] \left[ \sqrt{x^2 + y^2 + \dot{l}^2 + 2y\dot{l}\theta \sin \theta - 2\dot{y}\dot{l} \cos \theta} \right] \cos \theta \]  

(2.4.26)

\[ Q_2 = -\frac{1}{8} \rho_{air} C_{D_p} \pi d_p^2 \left[ l\dot{\theta} + \dot{y} \sin \theta - \dot{x} \cos \theta \right] \left[ \sqrt{x^2 + y^2 + \dot{l}^2 + 2y\dot{l}\theta \sin \theta - 2\dot{y}\dot{l} \cos \theta} \right] \sin \theta \]  

(2.4.27)

\[ Q_3 = -\frac{1}{8} \rho_{air} C_{D_p} \pi d_p^2 \left[ l\dot{\theta} + \dot{y} \sin \theta - \dot{x} \cos \theta \right] l \left[ \sqrt{x^2 + y^2 + \dot{l}^2 + 2y\dot{l}\theta \sin \theta - 2\dot{y}\dot{l} \cos \theta} \right] \]  

(2.4.28)

\[ Q_4 = 0 \]  

(2.4.29)

The derivation of the equations of motion for the ATC system using Lagrange’s Equations can now be finalized. Using equation (2.2.2) and the individual components given by equations (2.4.9) through to (2.4.12), equations (2.4.17) through to (2.4.20) and equations (2.4.26) through to (2.4.29), the dynamical equations of motion for the ATC system are:

\[ \ddot{x} = \frac{1}{8m_p} \rho_{air} C_{D_p} \pi d_p^2 \left[ l\dot{\theta} + \dot{y} \sin \theta - \dot{x} \cos \theta \right] \left[ \sqrt{x^2 + y^2 + \dot{l}^2 + 2y\dot{l}\theta \sin \theta - 2\dot{y}\dot{l} \cos \theta} \right] \cos \theta \]  

(2.4.30)

\[ \ddot{y} = -\frac{1}{8m_p} \rho_{air} C_{D_p} \pi d_p^2 \left[ l\dot{\theta} + \dot{y} \sin \theta - \dot{x} \cos \theta \right] \left[ \sqrt{x^2 + y^2 + \dot{l}^2 + 2y\dot{l}\theta \sin \theta - 2\dot{y}\dot{l} \cos \theta} \right] \sin \theta \]  

(2.4.31)
The equations of motion for the ATC system given by equations (2.4.30) through to (2.4.33) are highly non-linear and coupled second order ordinary differential equations (ODEs). These equations of motion are not in the necessary format to facilitate a solution via numerical integration, as equations (2.4.30) through to (2.4.33) are required to be recast in so-called state space form. In order for this to proceed, the following substitutions are employed:

\[
\begin{align*}
\ddot{\mathbf{x}}(t) &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} \\
\mathbf{u}(t) &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \\
\mathbf{x}(t) &= \begin{bmatrix} x \\ y \\ \theta \\ \dot{l} \end{bmatrix}, \quad \mathbf{u}(t) = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}
\end{align*}
\]

where: \(\mathbf{x}(t)\) is the so-called state vector for the system, \(\mathbf{u}(t)\) is the so-called control vector for the system.

The actuators available to control the dynamics of the ATC system are the horizontal and vertical acceleration of the aircraft \(\{\dot{x}, \dot{y}\}\) and the rate at which the deployment speed of the cable changes with time \(\dot{l}\). The \(n\) second order ODEs given by equations (2.4.30) through to (2.4.33) can now be transformed into \(2n\) first order ODEs, where \(n\) is the degree-of-freedom for the system, in this case equal to four.
The eight state space equations of motion for the ATC system can be represented by:

\[
\begin{bmatrix}
    x_3 \\
    x_6 \\
    x_7 \\
    x_8 \\
    u_1 \\
    u_2
\end{bmatrix}
= \begin{bmatrix}
    \frac{-x_4 C_{D_p} \pi d_p^2}{8 \rho} \left(x_4 x_7 + x_6 \sin x_3 - x_5 \cos x_3\right) \\
    \frac{1}{m_p x_4^2} \left(x_5^2 + x_6^2 + x_8^2 + 2x_4 x_6 x_7 \sin x_3 - 2x_4 x_8 \cos x_3\right) \\
    \frac{1}{m_p x_4^2} \left(-2x_5 x_8 \sin x_3 - 2x_4 x_5 \cos x_3 + x_4^2 x_7^2 + u_1 x_4 \cos x_3 - u_2 x_4 \sin x_3 - 2x_4 x_7 x_8 - g x_4 \sin x_3\right)
\end{bmatrix}
\tag{2.4.35}
\]

The equations given by (2.4.35) characterize a simple representation of the dynamical motion of the ATC system, subject to the assumptions justified in section (2.4.2). These equations can now be readily integrated using a numerical ODE solver to study the dynamics of the system or employed within a control system design methodology to synthesize a controller for the system. However, it is now pertinent to introduce a series of additional assumptions regarding the ATC system, in relation to the impact the motion of the aircraft has on the system. Assuming that the aircraft has an autopilot that maintains constant altitude for the aircraft, the state space equations of motion for the ATC system now reduce to:

\[
\begin{bmatrix}
    x_4 \\
    x_5 \\
    x_6 \\
    u_1 \\
    u_2
\end{bmatrix}
= \begin{bmatrix}
    \frac{-x_4 C_{D_p} \pi d_p^2}{8 \rho} \left(x_4 x_5 - x_4 \cos x_2\right) \\
    \frac{1}{m_p x_4^2} \left(x_5^2 + x_6^2 - 2x_4 x_6 \sin x_2 - 2x_4 x_5 \cos x_2 + x_4^2 x_5^2\right) \\
    \frac{1}{m_p x_4^2} \left(u_1 x_3 \cos x_2 - 2x_4 x_5 x_6 + g x_3 \sin x_2\right)
\end{bmatrix}
\tag{2.4.36}
\]

where the state and control vectors for this ATC system are given by:

\[
\begin{bmatrix}
    \mathbf{x}(t) \\
    \mathbf{u}(t)
\end{bmatrix} = \begin{bmatrix}
    [x_1 \ x_2 \ x_3 \ x_4 \ x_5]^T \\
    [u_1 \ u_2]^T
\end{bmatrix} = \begin{bmatrix}
    [x \ \theta \ \dot{l} \ \dot{\theta} \ \dot{\gamma}]^T \\
    [\dot{x} \ \dot{\gamma}]^T
\end{bmatrix}
\tag{2.4.37}
\]
As the aircraft is now restricted to fly at constant altitude, the total degree-of-freedom $n$ for the system is now reduced to three. Control actuation is limited to that imparted by the forward acceleration of the aircraft $\ddot{x}$ and the cable reel acceleration $\ddot{l}$.

Finally, consider the case when the aircraft is restricted to fly steady and level at all times with a constant velocity $U_0$, the horizontal displacement of the aircraft $x$ is not considered and the total degree-of-freedom $n$ for the system reduces to two. Control actuation for this scenario is wholly supplied by the reel acceleration of the cable $\ddot{l}$. The state space equations of motion for this ATC system are:

$$\begin{bmatrix}
    \dot{x}_1 \\
    \dot{x}_2 \\
    \dot{x}_3 \\
    \dot{x}_4 \\
    \dot{u}_1
\end{bmatrix} = \begin{bmatrix}
    x_3 \\
    x_4 \\
    \frac{1}{m_p x_2^2} \left(2x_2 x_3 + gx_2 \sin x_1 + \frac{x_2}{8} \rho_{air} C_{b_p} \pi d_{p}^2 \left(x_2 x_3 - U_0 \cos x_1 \right)\right) \\
    \frac{1}{m_p x_2^2} \sqrt{U_0^2 + x_4^2 - 2U_0 x_4 \sin x_1 - 2U_0 x_2 x_3 \cos x_1 + x_2^2 x_3^2} \\
    \end{bmatrix} \dot{\theta} + \begin{bmatrix}
    \theta \\
    l \\
    \dot{l}
\end{bmatrix}$$

(2.4.38)

where the state and control vectors for this ATC system are given by:

$$\begin{bmatrix}
    x(t) \\
    u(t)
\end{bmatrix} = \begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4 \\
    u_1
\end{bmatrix} = \begin{bmatrix}
    \theta \\
    l \\
    \dot{l}
\end{bmatrix}$$

(2.4.39)

Equations (2.4.35), (2.4.36) and (2.4.38) represent a suite of simplified two-dimensional models of the ATC system, which when deployed within optimal control system architectures, will elucidate valuable insights into how the aircraft and cable-payload combination interact with each other during a wide variety of rendezvous and payload transportation operations.

### 2.4.4 Three Dimensional Equations of Motion for the Simple Aerial Towed-Cable System

In reality, the motion of the ATC system is three dimensional in nature, yet the simplified models of the system developed in this section do not reflect this. In the interests of capturing the full three-dimensional motion of the ATC system, an additional simplified model will now be developed. By invoking essentially the same assumptions as previously outlined in Section 2.4.2, the derivation of the three dimensional equations of motion for the ATC system will proceed in the same manner as that followed for the two-dimensional case. Figure 2-5 details the complete analytical treatment of the simplified three-dimensional model of the ATC system.
To progress the derivation of the dynamical equations of motion, two coordinate systems are utilized, one centred at the aircraft CG and the other at the cable tip/payload CG location \( P \). The first coordinate system is the non-rotating Cartesian frame \( x-y-z \) that translates as the aircraft CG translates and is defined by the unit vectors \( e_x \) in the \( x \)-direction, \( e_y \) in the \( y \)-direction and \( e_z \) in the \( z \)-direction respectively.

The second coordinate system is the spherical coordinate system \( r-\theta-\phi \), which is both translating and rotating according to the motion of the payload/cable tip. This coordinate system is characterized by the unit vectors \( e_r \) in the \( r \)-direction, \( e_\theta \) in the \( \theta \)-direction and \( e_\phi \) in the \( \phi \)-direction respectively. The generalized coordinates for this system are selected as the in-plane and out-of-plane angles the cable makes with the local vertical \( \{ \theta, \phi \} \), along with the instantaneous cable length \( l \). Kinematic constraints representing the horizontal, lateral and vertical displacement of the aircraft \( \{ x, y, z \} \) also operate; hence the total degree-of-freedom for the ATC system increases to six.

Figure 2-5: Complete Description of the Three Dimensional Aerial Towed-Cable Model
The three-dimensional position of the payload $\mathbf{r}_p$ can be written in vector format as:

$$\mathbf{r}_p = (x-l \sin \theta) \mathbf{e}_i + (y-l \cos \theta \sin \phi) \mathbf{e}_j + (z-l \cos \theta \cos \phi) \mathbf{e}_k$$  \hspace{1cm} (2.4.40)

Differentiating with respect to time, the velocity of the payload $\mathbf{v}_p$ can be expressed in the Cartesian frame as:

$$\mathbf{v}_p = \left( \frac{\dot{x} - \dot{l} \sin \theta - l \dot{\theta} \cos \theta}{\mathbf{e}_i} \right)
+ \left( \frac{\dot{y} - \dot{l} \cos \theta \sin \phi + l \dot{\theta} \sin \theta - l \dot{\phi} \cos \theta \cos \phi}{\mathbf{e}_j} \right)
+ \left( \frac{\dot{z} - \dot{l} \cos \theta \cos \phi + l \dot{\phi} \sin \phi \cos \theta + l \dot{\phi} \cos \phi \sin \theta}{\mathbf{e}_k} \right)$$  \hspace{1cm} (2.4.41)

Alternatively, the velocity of the payload can be expressed in the spherical frame by:

$$\mathbf{v}_p = \mathbf{v}_r \mathbf{e}_r + \mathbf{v}_\theta \mathbf{e}_\theta + \mathbf{v}_\phi \mathbf{e}_\phi
= \left( \frac{\dot{l} - \dot{x} \sin \theta - \dot{y} \cos \theta \sin \phi - z \cos \theta \cos \phi}{\mathbf{e}_r} \right)
+ \left( \frac{\dot{l} \dot{\theta} - \dot{x} \cos \theta + \dot{y} \sin \theta \sin \phi + \dot{z} \sin \theta \cos \phi}{\mathbf{e}_\theta} \right)
+ \left( \frac{\dot{l} \dot{\phi} \cos \theta - \dot{y} \dot{\phi} \sin \phi \cos \theta + \dot{z} \dot{\phi} \cos \phi \sin \theta - 2 \dot{z} \dot{l} \cos \theta \cos \phi}{\mathbf{e}_\phi} \right)$$  \hspace{1cm} (2.4.42)

Now the total kinetic energy of the system $T$ can be found:

$$T = \frac{1}{2} m_p \mathbf{v}_p^2
= \frac{m_p}{2} \left( \dot{x}^2 + \dot{y}^2 + \dot{z}^2 + \dot{l}^2 + \dot{l}^2 \dot{\theta}^2 + \dot{l}^2 \dot{\phi}^2 \cos^2 \theta - 2 \dot{x} \dot{l} \sin \theta - 2 \dot{x} \dot{l} \dot{\theta} \cos \theta \right)$$  \hspace{1cm} (2.4.43)

The potential energy of the system $V$ has the following form:

$$V = m_p g \Delta z
= m_p g l (1 - \cos \theta \cos \phi)$$  \hspace{1cm} (2.4.44)

where $\Delta z$ is the change in altitude the payload experiences. Using equations (2.4.43) and (2.4.44), the Lagrangian of the system $L$ can now be formulated as follows:

$$L = T - V
= \frac{m_p}{2} \left( \dot{x}^2 + \dot{y}^2 + \dot{z}^2 + \dot{l}^2 + \dot{l}^2 \dot{\theta}^2 + \dot{l}^2 \dot{\phi}^2 \cos^2 \theta - 2 \dot{x} \dot{l} \sin \theta - 2 \dot{x} \dot{l} \dot{\theta} \cos \theta \right)
- m_p g l (1 - \cos \theta \cos \phi)$$  \hspace{1cm} (2.4.45)
It is now possible to determine the partial derivatives that make up the left-hand-side of Lagrange’s Equations of motion for the system. To simplify matters, the partial derivatives relating to the cable length and aircraft variables will not be calculated. Since it is known a priori that the components of the equations of motion attributed to the cable length and the aircraft velocity are written solely in terms of the actuators for the system [see equation (2.4.35)], it is superfluous to proceed with a formalized derivation in a manner similar to that employed previously for the two-dimensional model. Instead, only the derivation of the components of the equations of motion concerning the in-plane and out-of-plane cable angles \( \{\theta, \phi\} \) will be presented.

Letting \( q_1 = \theta, \ q_2 = \phi \), the required partial derivatives are given below:

\[
\frac{\partial L}{\partial q_1} = m_p \left( \begin{array}{c}
\dot{x}l \sin \theta + \dot{y}l \cos \theta + \dot{z}l \cos \theta - \ddot{x}l \cos \theta \\
-\dddot{l} \cos \phi + \dot{y}l \sin \phi \cos \theta + \dot{z}l \phi \sin \theta \cos \phi \\
-\dot{z}l \phi \sin \theta \cos \phi - \dot{z}l \cos \phi \cos \theta - \ddot{g}l \sin \theta \cos \phi
\end{array} \right) \quad (2.4.46)
\]

\[
\frac{\partial L}{\partial q_2} = m_p \left( \begin{array}{c}
\dot{y}l \cos \theta \cos \phi + \dot{y}l \phi \sin \phi \cos \theta - \dot{y}l \cos \phi \cos \theta - \ddot{g}l \cos \theta \cos \phi \\
-\dot{z}l \phi \cos \theta \cos \phi + \ddot{z}l \cos \theta \sin \phi
\end{array} \right) \quad (2.4.47)
\]

\[
\frac{\partial L}{\partial \dot{q}_1} = m_p \left( \begin{array}{c}
\dot{y}l \sin \phi \sin \theta + \dot{z}l \cos \phi \sin \theta - \ddot{x}l \cos \theta + \dddot{l} \theta \\
-\dddot{z}l \theta \sin \phi + \dddot{z}l \theta \cos \phi + \dddot{z}l \cos \theta \sin \phi
\end{array} \right) \quad (2.4.48)
\]

\[
\frac{\partial L}{\partial \dot{q}_2} = m_p \left( \begin{array}{c}
\dddot{l} \phi \cos^2 \theta - \dddot{y}l \cos \theta \cos \phi + \dddot{z}l \phi \sin \phi \cos \theta
\end{array} \right) \quad (2.4.49)
\]

Now the required time rate of change of the partial derivatives that make up Lagrange’s Equations can now be found:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial q_1} \right) = m_p \left( \begin{array}{c}
\dddot{l} \theta - \dddot{x}l \cos \theta - \dddot{y}l \cos \theta + \dot{z}l \phi \sin \theta + \dddot{y}l \sin \theta \sin \phi \\
+ \dddot{y}l \sin \theta \sin \phi + \dddot{y}l \phi \sin \theta \cos \phi + \dddot{y}l \phi \cos \theta \sin \phi \\
+ \dddot{z}l \sin \theta \cos \phi + \dddot{z}l \phi \sin \theta \cos \phi - \dddot{z}l \phi \sin \theta \cos \phi
\end{array} \right) \quad (2.4.50)
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial q_2} \right) = m_p \left( \begin{array}{c}
2 \dddot{l} \phi \cos^2 \theta + l^2 \dddot{\theta} \cos^2 \theta - 2 \dddot{l} \dddot{\theta} \phi \cos \theta \sin \theta \\
- \dddot{y}l \cos \theta \cos \phi - \dddot{y}l \phi \cos \theta \sin \phi + \dddot{y}l \phi \cos \theta \sin \phi + \dddot{z}l \phi \cos \theta \sin \phi + \dddot{z}l \phi \cos \theta \sin \phi
\end{array} \right) \quad (2.4.51)
\]
All that remains now in order to complete the derivation of the three-dimensional equations of motion for the simplified ATC system, is the determination of the non-conservative generalized forces acting on the system $Q_i$. As was the case with the two-dimensional simplified ATC model, the only non-conservative force acting on the system is the aerodynamic drag acting on the payload. The three-dimensional aerodynamic drag model for the payload is essentially the same as the corresponding two-dimensional model, except that in this case there are two directional cosines normal to the payload/cable tip, hence two normal velocity components exist; one acting normal to the payload in an in-plane sense and the other in an out-of-plane sense. Utilizing the spherical reference frame, the general form of aerodynamic drag acting on the payload can be written in vector form as:

$$ F_{\text{drag}} = -\frac{1}{2} \rho_{\text{air}} C_{D_p} A_P \left| v_P \right| \left( v_{N_{\theta}} + v_{N_{\phi}} \right) \quad (2.4.52) $$

where: $\rho_{\text{air}}$, $C_{D_p}$, $A_P$ are the same as that given previously for the two-dimensional model,

$v_{N_{\theta}}$ is the in-plane normal velocity of the payload,

$v_{N_{\phi}}$ is the out-of-plane normal velocity of the payload.

Since the payload is perfectly spherical, the drag coefficient is the same in both an in-plane and out-of-plane sense. Substituting equation (2.4.22) and the appropriate components of equation (2.4.42) into equation (2.4.52), the drag acting on the payload now becomes:

$$ F_{\text{drag}} = -\frac{1}{8} \rho_{\text{air}} C_{D_p} \pi d_P^2 \left( \hat{\theta} \cos \phi + \hat{\phi} \sin \phi \right) \cdot \mathbf{e}_{\theta} $$

$$ -\frac{1}{8} \rho_{\text{air}} C_{D_p} \pi d_P^2 \left( \hat{\phi} \cos \phi + \hat{\phi} \sin \phi \right) \cdot \mathbf{e}_{\phi} $$

$$ = -\frac{1}{8} \rho_{\text{air}} C_{D_p} \pi d_P^2 \left( \hat{x} \cos \theta + \hat{y} \sin \theta \right) \cdot \mathbf{e}_{\theta} $$

$$ -\frac{1}{8} \rho_{\text{air}} C_{D_p} \pi d_P^2 \left( \hat{x} \cos \theta + \hat{y} \sin \theta \right) \cdot \mathbf{e}_{\phi} $$

$$ = -\frac{1}{8} \rho_{\text{air}} C_{D_p} \pi d_P^2 \left( \hat{x} \cos \theta + \hat{y} \sin \theta \right) \cdot \mathbf{e}_{\theta} $$

$$ -\frac{1}{8} \rho_{\text{air}} C_{D_p} \pi d_P^2 \left( \hat{x} \cos \theta + \hat{y} \sin \theta \right) \cdot \mathbf{e}_{\phi} $$

By inspection, the total non-conservative force vector $F$ for the system is equal to the aerodynamic drag force vector $F_{\text{drag}}$ given by equation (2.4.53). The generalized forces acting on the system can be found via the principle of virtual work as follows:
\[ Q_1 = F \frac{\partial r_p}{\partial q_1} = \begin{bmatrix} \frac{-l}{8} \rho_{air} C_{D_p} \pi d_p^2 \left( l \dot{\theta} - x \cos \theta + y \sin \theta \sin \phi + z \sin \theta \cos \phi \right) \\
\sqrt{x^2 + y^2 + z^2 + l^2 \dot{\theta}^2 + l^2 \dot{\phi}^2 \cos^2 \theta + 2yl \dot{\theta} \sin \theta \sin \phi} \\
-2yl \dot{\phi} \cos \theta \cos \phi + 2zl \dot{\phi} \cos \theta \sin \phi + 2zl \dot{\theta} \sin \theta \cos \phi \\
-2zl \theta \cos \theta - 2zl \sin \theta - 2yl \cos \theta \sin \phi - 2zl \cos \theta \cos \phi \end{bmatrix} \] (2.4.54)

\[ Q_2 = F \frac{\partial r_p}{\partial q_2} = \begin{bmatrix} \frac{-l \cos \theta}{8} \rho_{air} C_{D_p} \pi d_p^2 \left( l \dot{\phi} \cos \theta - y \cos \phi + z \sin \phi \right) \\
\sqrt{x^2 + y^2 + z^2 + l^2 \dot{\phi}^2 \cos^2 \theta + 2yl \dot{\phi} \sin \theta \sin \phi} \\
-2yl \dot{\phi} \cos \theta \cos \phi + 2zl \dot{\phi} \cos \theta \sin \phi + 2zl \dot{\theta} \sin \theta \cos \phi \\
-2zl \dot{\phi} \cos \theta - 2zl \sin \theta + 2yl \cos \theta \sin \phi + 2zl \cos \theta \cos \phi \end{bmatrix} \] (2.4.55)

The derivation of the equations of motion for the ATC system using Lagrange’s Equations can now be finalized. Using equation (2.2.2) and the individual components given by equations (2.4.46) through to (2.4.51), equations (2.4.54) and (2.4.55), the dynamical equations of motion for the three-dimensional ATC system are:

\[ \ddot{q}_1 = \ddot{\theta} = \frac{1}{l^2} \begin{bmatrix} \frac{-l}{8m_p} \rho_{air} C_{D_p} \pi d_p^2 \left( l \dot{\theta} - x \cos \theta + y \sin \theta \sin \phi + z \sin \theta \cos \phi \right) \\
\sqrt{x^2 + y^2 + z^2 + l^2 \dot{\theta}^2 + l^2 \dot{\phi}^2 \cos^2 \theta + 2yl \dot{\theta} \sin \theta \sin \phi} \\
-2yl \dot{\phi} \cos \theta \cos \phi + 2zl \dot{\phi} \cos \theta \sin \phi + 2zl \dot{\theta} \sin \theta \cos \phi \\
-2zl \theta \cos \theta - 2zl \sin \theta + 2yl \cos \theta \sin \phi + 2zl \cos \theta \cos \phi \end{bmatrix} \] (2.4.56)

\[ \ddot{q}_2 = \ddot{\phi} = \frac{1}{l^2 \cos^2 \theta} \begin{bmatrix} \frac{-l \cos \theta}{8m_p} \rho_{air} C_{D_p} \pi d_p^2 \left( l \dot{\phi} \cos \theta - y \cos \phi + z \sin \phi \right) \\
\sqrt{x^2 + y^2 + z^2 + l^2 \dot{\phi}^2 \cos^2 \theta + 2yl \dot{\phi} \sin \theta \sin \phi} \\
-2yl \dot{\phi} \cos \theta \cos \phi + 2zl \dot{\phi} \cos \theta \sin \phi + 2zl \dot{\theta} \sin \theta \cos \phi \\
-2zl \dot{\phi} \cos \theta - 2zl \sin \theta + 2yl \dot{\theta} \sin \phi + 2zl \dot{\theta} \cos \phi \end{bmatrix} \] (2.4.57)
To cast the equations of motion for the system into state space form, the following substitutions are employed:

\[
\begin{align*}
\{x(t)\} &= \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} \end{bmatrix}^T \\
\{u(t)\} &= \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix}^T \\
\{x\} &= \begin{bmatrix} x & y & z & \theta & \phi & l & \dot{x} & \dot{y} & \dot{z} & \dot{\theta} & \dot{\phi} & \dot{l} \end{bmatrix}^T \\
\{\dot{x}\} &= \begin{bmatrix} \ddot{x} & \ddot{y} & \ddot{z} & \dddot{l} \end{bmatrix}^T
\end{align*}
\]

(2.4.58)

where: \(x(t)\) is again the state vector for the system, 
\(u(t)\) is again the control vector for the system.

The actuators that control the dynamics of the ATC system are the acceleration components of the aircraft \(\{\dot{x}, \dot{y}, \dot{z}\}\) and the cable reel acceleration \(\ddot{l}\). Finally, the twelve state space equations of motion for the three-dimensional ATC system can be represented by:

\[
\begin{align*}
\{\ddot{x}(t)\} &= \begin{bmatrix} \ddot{x}_1 & \ddot{x}_2 & \ddot{x}_3 & \ddot{x}_4 & \ddot{x}_5 & \ddot{x}_6 & u_1 & u_2 & u_3 & \dddot{q}_1 & \dddot{q}_2 & u_4 \end{bmatrix}^T
\end{align*}
\]

(2.4.59)

after the relations given by equation (2.4.58) are substituted into equations (2.4.56) and (2.4.57).

2.5 Complex Aerial Towed-Cable System Modelling

This sub-section concerns the development of the dynamical equations of motion governing complex multi-link ATC system models. In general, complex ATC system models are often characterized by the mechanical nature of the cable, in particular whether or not the cable is capable of longitudinal extension. For each class of complex multi-link system model, very different approaches and methodologies are used to determine the governing equations of motion, each of which will be explored in detail in the following sub-sections.

2.5.1 Equations of Motion for the n-Rigid Link Aerial Towed-Cable System Model

The process and methodology used to derive the two-dimensional equations of motion for the single rigid link ATC system can be extended to the more general case of n-rigid links. Figure 2-6 depicts the ATC system utilizing n-rigid links to physically discretize the cable.
As compared to the modelling procedure associated with the simplified models of the ATC system, the use of prevailing assumptions is restricted and the requirements for inherent simplicity are relaxed, hence significant sophistication to the modelling procedure is employed here. This sophistication relates mostly to the modelling regime employed to capture the external forces acting on the ATC system, in particular the aerodynamic forces acting on the cable. The cable is modelled using the lumped parameter approach, which concentrates the distributed inertial and external forces acting along the entire length of the cable to point mass elements $m_k$, which are articulated by frictionless hinges. In reality, as the cable would be expected to be constructed with a high stiffness material, its longitudinal stretching is neglected, which allows the cable segments to be modelled as rigid link elements. Although this assumption is neglected in Section 2.5.2 to follow, its utilization does not jeopardise the accuracy of the modelling regime, since sufficient precision is maintained in order to capture the significant dynamical motion of the ATC system.
The equations of motion are derived using Kane’s equations, which as demonstrated previously, automatically eliminates non-working constraint forces and does not require the arduous process of differentiating the lengthy energy expressions involved when using Lagrange’s Equations. The generalized coordinates for this ATC system are selected as the orientation angles of each cable segment $\theta_k$ and the instantaneous length of the first cable segment $l_1$. The cable is capable of deploying or retracting by prescribing the reel acceleration of the first cable segment $\ddot{l}_1$, while all other cable links are assumed to have a constant length $l$. Prescribing the horizontal and vertical acceleration of the aircraft $\{\dot{x}, \dot{y}\}$ allows the two-dimensional motion of the aircraft to be captured, which in turn is modelled as a pair of kinematic constraints. The non-rotating Cartesian reference frame $x-y$ shown in Figure 2-6 is used to assist in the derivation of the equations of motion for the rigid multi-link ATC system. This coordinate system translates as the aircraft CG translates and is defined by the unit vectors $e_i$ in the $x$-direction and $e_j$ in the $y$-direction. The derivation of the dynamical equations of motion for the rigid multi-link ATC system owes much of its existence to the collaborative research lead by Dr. Paul Williams at RMIT University [114]. A more thorough and expanded formal derivation will be presented here, as follows:

The inertial positions of the point masses, relative to the reference axis, may be written in terms of the generalized coordinates as:

$$\mathbf{r}_1 = (x-l_1 \sin \theta_1) \mathbf{e}_i + (y-l_1 \cos \theta_1) \mathbf{e}_j$$

$$\mathbf{r}_k = \mathbf{r}_{k-1} + (-l \sin \theta_k) \mathbf{e}_i + (-l \cos \theta_k) \mathbf{e}_j, \quad (k = 2, \ldots, n) \tag{2.5.1}$$

The corresponding velocities of each inertial mass are found by differentiating equation (2.5.1) with respect to time:

$$\mathbf{v}_1 = (\dot{x} - \dot{l}_1 \sin \theta_1 - l_1 \dot{\theta}_1 \cos \theta_1) \mathbf{e}_i + (\dot{y} - \dot{l}_1 \cos \theta_1 + l_1 \dot{\theta}_1 \sin \theta_1) \mathbf{e}_j$$

$$\mathbf{v}_k = \mathbf{v}_{k-1} + (-l \dot{\theta}_k \cos \theta_k) \mathbf{e}_i + (l \dot{\theta}_k \sin \theta_k) \mathbf{e}_j, \quad (k = 2, \ldots, n) \tag{2.5.2}$$

The partial velocities for the ATC system, as required by Kane’s Equations, can be found as follows:

$$\frac{\partial \mathbf{v}_k}{\partial \theta_r} = \begin{cases} \frac{\partial \mathbf{v}_k}{\partial \theta_r}, & k \geq r \\ 0, & k < r \end{cases}$$

$$\frac{\partial \mathbf{v}_k}{\partial \theta_k} = (-l \cos \theta_k) \mathbf{e}_i + (l \sin \theta_k) \mathbf{e}_j, \quad (k = 2, \ldots, n) \tag{2.5.3}$$
The inertial accelerations of the ATC system are determined via differentiation of equation (2.5.2) with respect to time:

\[ a_i = \left( \ddot{x} - \ddot{l} \sin \theta_i - 2\dot{l} \dot{\theta}_i \cos \theta_i + l \dddot{\theta}_i \sin \theta_i - l \dddot{\theta}_i \cos \theta_i \right) e_i + \left( \ddot{y} - \ddot{l} \cos \theta_i + 2\dot{l} \dot{\theta}_i \sin \theta_i + l \dddot{\theta}_i \sin \theta_i + l \dddot{\theta}_i \cos \theta_i \right) e_j \quad (k = 2, \ldots, n) \]  

The generic form of Kane’s Equations for the rigid multi-link ATC system can be written as:

\[ F^{\ast}_k + F^{\ast}_k = 0 \quad (k = 1, \ldots, n) \tag{2.5.5} \]

where \( F^{\ast}_k \) are the generalized inertia forces for the system:

\[ F^{\ast}_k = \sum_{s=1}^{a} -m_s a_s \cdot \frac{\partial \mathbf{v}_s}{\partial \theta_k} \quad (k = 1, \ldots, n) \tag{2.5.6} \]

\( F_a \) are the generalized active forces for the system:

\[ F_a = \sum_{s=1}^{a} F_s \cdot \frac{\partial \mathbf{v}_s}{\partial \theta_k} \quad (k = 1, \ldots, n) \tag{2.5.7} \]

\( \mathbf{F}_s \) is a vector of external forces acting on the system given by equation (2.5.8)

\[ \mathbf{F}_s = F_{2s-1} e_i + F_{2s-2} e_j \quad (s = 1, \ldots, n) \tag{2.5.8} \]

**Generalized Inertia Forces**

The generalized inertia forces for the rigid multi-link ATC model presented in equation (2.5.6) can be expanded into the following format:

\[
F^{\ast}_k = \sum_{j=1}^{a} m_j \left[ l \left( \dot{x} \cos \theta_l - \dot{y} \sin \theta_l - 2\dot{l} \dot{\theta}_l - l \ddot{\theta}_l \right) \right] \\
+ \sum_{i=2}^{a} \left[ \sum_{j=1}^{a} m_j \left( \dot{l} \right) \left( \dot{\theta}_i \cos \theta_l - \dot{\theta}_i \sin \theta_l - \ddot{\theta}_i \sin \theta_l - \ddot{\theta}_i \cos \theta_l \right) \right] \tag{2.5.9}
\]

\[
F^{\ast}_k = \sum_{j=1}^{a} m_j \left[ l \left( \dot{x} \cos \theta_k - \dot{y} \sin \theta_k + \dot{l} \sin \theta_k \right) + l \dot{\theta}_k - l \ddot{\theta}_k \cos \theta_k \right] \\
+ \sum_{i=2}^{a} \left[ \sum_{j=1}^{a} m_j \left( \dot{l} \right) \left( \dot{\theta}_i \cos \theta_k - \dot{\theta}_i \sin \theta_k - \ddot{\theta}_i \sin \theta_k - \ddot{\theta}_i \cos \theta_k \right) \right] \tag{2.5.10}
\]
Generalized Active Forces

With the assistance of equation (2.5.8) and the partial velocities given by equation (2.5.3), the generalized active forces for the rigid multi-link ATC model can be expanded to yield:

\[ F_{\theta_j} = \sum_{j=1}^{n} \left( F_{2(j-1)+1} l_1 \sin \theta_j - F_{2(j-1)+1} l_1 \cos \theta_j \right) \]  
\[ F_{\theta_k} = \sum_{j=k}^{n} \left( F_{2(k-1)+1} l_1 \sin \theta_k - F_{2(k-1)+1} l_1 \cos \theta_k \right) , \quad (k = 2, \ldots, n) \]  

(2.5.11)  
(2.5.12)

Exterrnally Applied Forces

It is now well established that there exists two classes of external forces acting on the rigid multi-link ATC system, namely aerodynamic and gravitational, each of which will be treated separately.

2.5.1.1 Gravitational Forces

By inspection, the gravitational forces acting on each of the inertial masses that constitute the rigid multi-link ATC model are:

\[ F_k^g = -\left( m_k g \right) e_j \]  

(2.5.13)

2.5.1.2 Aerodynamic Forces

The process of calculating the aerodynamic forces acting on the cable begins by treating each cable segment as a cylinder inclined at some angle to the prevailing flow field. This idealization is aptly illustrated in Figure 2-7. Suitably representative mathematical models of the aerodynamic forces acting on cables/inclined cylinders are well developed [46, 129] and have been known to produce good results [46], although research within this area is dated and suggestions have been made recently as to whether a revision of this work is required [26]. The local Reynolds Number of the flow encountered by the cable, for the anticipated operational envelope available to the ATC system, is such that subcritical flow will always prevail. Therefore it is reasonable to invoke the cross-flow principle for the calculation of the aerodynamic coefficients of each cable segment. If the local flow regime in the vicinity of the cable is supercritical or the surface texture of the cable is coarse, a more representative model of such a scenario based on turbulent flow should be employed (see [46, 130] for details).
Treating each cable segment as an inclined cylinder, the aerodynamic coefficients for the cable can be written as functions of the angle of attack $\alpha_k$ of each segment [129]:

\[
C_{D_k} = C_{f_k} + \frac{1}{2} \rho c_s^3 \sin^3(\alpha_k) \\
C_{L_k} = C_n \sin^2(\alpha_k) \cos(\alpha_k)
\]  

(2.5.14)

(2.5.15)

where $C_{f_k}$ and $C_{n_k}$ are the skin friction and cross-flow drag coefficients for the cable, which are nonlinearly dependent on the local normal $M_{n_k}$ and tangential $M_{t_k}$ Mach Numbers of each segment according to the following formulae [17]:

\[
C_{n_k} = 1.17 + \frac{M_{n_k}}{40} - \frac{M_{n_k}^2}{4} + \frac{5M_{n_k}^3}{8} \\
C_{f_k} = \begin{cases} 
0.038 - 0.0425M_{t_k}, & M_{t_k} < 0.4 \\
0.013 + 0.0395(M_{t_k} - 0.85)^2, & M_{t_k} \geq 0.4 
\end{cases}
\]  

(2.5.16)

(2.5.17)

In order to calculate the Mach Numbers for each cable segment, firstly the normal and tangential velocity of each cable segment is required:

\[
v_{n_k} = (v_{j_k} \sin \theta_k - v_{j_k} \cos \theta_k) e_{\theta_k} \\
v_{t_k} = (-v_{j_k} \sin \theta_k - v_{j_k} \cos \theta_k) e_{\theta_k}
\]  

(2.5.18)

(2.5.19)

By definition, the Mach Numbers for each cable segment are inversely proportional to the speed of sound $a_k$, which in turn is dependent on air temperature $T_{air}$ accordingly:

\[
a_k = 340.3 \frac{T_{air}}{T_{air_0}}
\]  

(2.5.20)
The ISA model for the temperature of the prevailing flow at the altitude of each cable segment is given by the following linear relationship:

\[ T_{\text{air}_k} = T_{\text{air}_{\text{SL}}} - \frac{6.5}{1000} y_k \]  

(2.5.21)

where the subscript “SL” refers the appropriate value at sea level. The local normal and tangential Mach Numbers of each segment can now be computed:

\[ M_{n_k} = \frac{|v_{n_k}|}{a_k} \]  

(2.5.22)

\[ M_{t_k} = \frac{|v_{t_k}|}{a_k} \]

Following on, the angle of attack of each cable segment \( \alpha_k \) may be calculated from:

\[ \cos(\alpha_k) = \frac{-I_k \cdot v^c_k}{|I_k||v^c_k|} = \frac{v^c_{x_k} \sin \theta_k + v^c_{y_k} \cos \theta_k}{\sqrt{(v^c_{x_k})^2 + (v^c_{y_k})^2}} \]  

(2.5.23)

where \( v^c_k \) is the velocity vector of the centre of each cable segment, approximated from the motion of adjacent masses by:

\[ v^c_k = \frac{(v_k + v_{k-1})}{2} = v^c_{x_k} e_j + v^c_{y_k} e_j, \quad (k = 1, ..., n) \]  

(2.5.24)

and \( I_k \) is given by the following:

\[ I_k = -l_k \sin \theta_k e_j - l_k \cos \theta_k e_j \]  

(2.5.25)

In equation (2.5.24), \( v_0 \) represents the velocity vector for the aircraft. The unit vectors \( e_L \) and \( e_D \) define the direction of the lift force and drag force vectors respectively, whose magnitude and direction can be found from:

\[ e_D = -\frac{v^c_k}{|v^c_k|} \]  

(2.5.26)

\[ e_L = -\frac{v^c_k \times I_k \times v^c_k}{|v^c_k \times I_k \times v^c_k|} \]  

(2.5.27)
Now the general form of the lift and drag force vectors for each cable segment is:

\[
F_{k}^{\text{drag}} = \frac{1}{2} \rho_{\text{air}} C_{D_k} l_k d |v_{k}^e|^2 e_D = -\frac{1}{2} \rho_{\text{air}} C_{D_k} l_k d \left|v_{k}^e\right|^2 \left((v_{x_k}^e e_j + v_{y_k}^e e_j)\right)
\]

(2.5.28)

\[
F_{k}^{\text{lift}} = \frac{1}{2} \rho_{\text{air}} C_{L_k} l_k d |v_{k}^e|^2 e_L = \frac{1}{2} \rho_{\text{air}} C_{L_k} l_k d \left|v_{k}^e\right|^2 \left((-v_{y_k}^e e_i + v_{x_k}^e e_j)\right)
\]

(2.5.29)

where \(\rho_{\text{air}}\) is the atmospheric air density of the prevailing flow at the altitude of each cable segment given by the following ISA model:

\[
\rho_{\text{air}} = \rho_{\text{air}} \left(1 - \frac{6.5 y_{k}}{1000 T_{\text{air}}^k}\right)^{4.256}
\]

(2.5.30)

With respect to the towed body, as it is assumed to be perfectly spherical, it will only experience aerodynamic drag. The resulting drag force vector for the towed body can be expressed as:

\[
F_{n+1}^{\text{drag}} = 0
\]

(2.5.31)

\[
F_{n+1}^{\text{lift}} = \frac{1}{2} \rho_{\text{air}} C_{D_{T}} A_P |\mathbf{v}_{n}^e|^2 e_D = -\frac{1}{2} \rho_{\text{air}} C_{D_{T}} A_P \left|\mathbf{v}_{n}^e\right|^2 \left((v_{x_n}^e e_i + v_{y_n}^e e_j)\right)
\]

(2.5.32)

Assuming the lift and drag force vectors are constant across each cable segment, the distributed aerodynamic force the cable experiences is lumped to each point mass by using the now well-known procedure of appropriating one half of the aerodynamic force acting on adjacent segments. The equivalent aerodynamic force each cable segment experiences is:

\[
F_{k}^{\text{aero}} = \frac{F_{k}^{\text{drag}} + F_{k+1}^{\text{drag}}}{2} + \frac{F_{k}^{\text{lift}} + F_{k+1}^{\text{lift}}}{2}, \quad (k = 1, \ldots, n)
\]

(2.5.33)

Finally, the external forces \(F_k\) acting on the system can be assembled:

\[
F_k = F_k^{\mathbf{g}} + F_k^{\text{aero}}, \quad (j = 1, \ldots, n)
\]

(2.5.34)

Now Kane’s Equations of motion for the rigid multi-link ATC system can be compiled and expressed conveniently in matrix form as follows:

\[
\mathbf{M}\ddot{\mathbf{q}} = \mathbf{B}
\]

(2.5.35)

where \(\ddot{\mathbf{q}} = [\ddot{q}_1, \ldots, \ddot{q}_n]^T\).
\( \mathbf{M} \) is the symmetrical “mass” matrix having upper triangular entries governed by:

\[
\begin{align*}
\mathbf{M}_{ij} = & \sum_{j=1}^{n} m_i l_j^2 , & k = 1, i = 1 \\
& \sum_{j=i}^{n} m_i l_j \cos (\theta_j - \theta_i) , & k = 1, i \neq 1 \\
& \sum_{j=1}^{n} m_i l_i^2 , & k = i, k \neq 1 \\
& \sum_{j=1}^{n} m_i l_i \cos (\theta_j - \theta_i) , & k \neq i, k \neq 1
\end{align*}
\]

The vector \( \mathbf{B} \) has entries that are obtained using the following relations:

\[
\begin{align*}
\mathbf{B}_i = & \left[ \sum_{j=1}^{n} m_j \right] l_i \left( \ddot{x} \cos \theta_i - \ddot{y} \sin \theta_i - 2 \dot{\theta}_i \right) + \sum_{j=1}^{n} \left( \sum_{j=1}^{n} m_j \right) l_i l_j \ddot{\theta}_j^2 \sin (\theta_j - \theta_i) \\
& + \sum_{j=1}^{n} \left( F_{2(j-1)+1} \sin \theta_i - F_{2(j-1)+1} \cos \theta_i \right)
\end{align*}
\]

\[
\begin{align*}
\mathbf{B}_k = & \left[ \sum_{j=1}^{n} m_j \right] l \left[ \ddot{x} \cos \theta_k - \ddot{y} \sin \theta_k + \ddot{l}_j \sin (\theta_j - \theta_k) \right] \\
& + \sum_{i=2}^{n} \left( \sum_{j=1}^{n} m_j \right) l_i \ddot{\theta}_i^2 \sin (\theta_i - \theta_k) \\
& + \sum_{j=1}^{n} \left( F_{2(j-1)+1} \sin \theta_k - F_{2(j-1)+1} \cos \theta_k \right), \quad (k = 2, \ldots, n)
\end{align*}
\]

The equations of motion for the rigid multi-link ATC system follow once the terms making up equations (2.5.36) and (2.5.37) are determined and equation (2.5.35) is inverted. The resulting equations of motion can be rewritten in state space format, once the following substitutions are employed:

\[
\begin{align*}
\mathbf{x}(t) = & \begin{bmatrix} x_1 & \cdots & x_{2(n+3)} \end{bmatrix}^T \\
\mathbf{u}(t) = & \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T
\end{align*}
\]

\[
\begin{bmatrix} x \ y \ \theta_1 \ \cdots \ \theta_n \ l_1 \ \ddot{x} \ \ddot{y} \ \dot{\theta}_1 \ \cdots \ \dot{\theta}_n \ \ddot{l}_1 \end{bmatrix}^T
\]

where \( \mathbf{x}(t) \) is the state vector and \( \mathbf{u}(t) \) is the control vector formed using the horizontal and vertical acceleration of the aircraft \( \{\ddot{x}, \ddot{y}\} \) and the reel acceleration of the cable \( \ddot{l}_1 \).
Finally, the \(2(n+3)\) state space equations of motion for the rigid multi-link ATC system can be concisely written as:

\[
\mathbf{x}(t) = \begin{bmatrix} \dot{x}_1 & \dot{x}_2 & \dot{\mathbf{q}}^T(t) & \dot{x}_{n+3} & u_1 & u_2 & \dot{\mathbf{q}}^T(t) & u_3 \end{bmatrix}^T
\]  

(2.5.39)

### 2.5.2 Equations of Motion for the n-Flexible Link Aerial Towed-Cable System Model

As alluded to in the previous section, to ensure the safe and effective operation of the ATC system, the cable should be constructed using a strong and highly stiff material. This would ensure that the risks associated with excessive stretching of the cable or large dynamic forces induced by longitudinal oscillations in the cable are suitably minimized. In reality however, the cable will undergo some degree of stretching and in particular instances, it may be critical to account for the longitudinal flexibility of the cable. To appropriately simulate such a situation, the continuous cable can be physically discretized into a series of segments consisting of interconnected spring-dashpot elements, resulting in a model of the ATC system that is illustrated in Figure 2-8.

![Mathematical Model of the n-Flexible Link Aerial Towed-Cable System](image)

Figure 2-8: Mathematical Model of the n-Flexible Link Aerial Towed-Cable System
Once again, a lumped parameter approach was used to model the flexible multi-link ATC system. The springs allow the elastic longitudinal extension of the cable to be modelled, the dashpots are used to simulate the internal structural damping of the cable, while the point masses model the inertia of the cable and serve as nodes to which the distributed external forces acting on the cable are applied. The Cartesian reference frame $x-y$ shown in Figure 2-8 is located at the aircraft CG and is allowed to translate, but not rotate, at a prescribed velocity and acceleration. This Cartesian reference frame is defined by the unit vectors $e_i$ in the $x$-direction and $e_j$ in the $y$-direction. Hence, for the purposes of deriving the equations of motion for the flexible multi-link ATC system, this coordinate system can be considered as the inertial reference frame, the movement of which allows arbitrary motion constraints to be placed on the system. These kinematic constraints represent the horizontal and vertical acceleration of the aircraft $\{\ddot{x}, \ddot{y}\}$. Contrary to the numbering system employed for the rigid multi-link ATC system, the numbering system for the cable segments constituting the flexible multi-link system begin at the payload end of the cable (1) and incrementally increase up the cable towards the aircraft ($n$). A schematic of this numbering system is shown in Figure 2-9. Furthermore, the development of the equations of motion for each multi-link ATC system is markedly different, even though both are formulated using Kane’s Equations. The manner in which the equations of motion are postulated is the primary distinction between the two: normal-tangential coordinates $(\theta_k, l_k)$ are used when cable segments are rigid; Cartesian coordinates $(x_k, y_k)$ are used when elastic extension is considered.

Figure 2-9: Numbering System Used for Visco-Elastic Cable Elements
The following derivation of the dynamical equations of motion for the flexible multi-link ATC system borrows heavily from the derivation carried out for autonomous underwater towed-cable systems attributed to Trivailo et al. [127]. A less sophisticated and inherently simpler two-dimensional derivation will be presented here.

**Kinematic Equations**

Consider the motion of the $k^{th}$ mass in Figure 2-8 and Figure 2-9. The generalized coordinates of the flexible multi-link ATC system are chosen as the relative displacement of mass $k$ from mass $k+1$. This vector can be mathematically represented as:

$$p_k = (q_{2k-1})^i + (q_{2k})^j$$

A series of local coordinate systems, aligned parallel to the Cartesian inertial reference frame, is attached to each inertial mass. The vectors from the origin of the inertial reference frame to the origin of each of these local frames is given by:

$$r_k = \left(\sum_{s=k}^n q_{2s-1}\right)^i + \left(\sum_{s=k}^n q_{2s}\right)^j$$

As the inertial reference frame is translating and not rotating, the velocity of each lumped mass can be found by differentiating equation (2.5.41) with respect to time and appropriately accounting for the velocity of the aircraft:

$$v_k = \dot{r}_k = \left(\dot{z}_1 + \sum_{s=k}^n \dot{q}_{2s-1}\right)^i + \left(\dot{z}_2 + \sum_{s=k}^n \dot{q}_{2s}\right)^j$$

where $\dot{z}_1$ and $\dot{z}_2$ are the aircraft velocity components in the $x$-direction and $y$-direction respectively. The generalized speeds $u$, as required in the formulation of Kane’s Equations of motion, are selected as the inertial velocity of each lumped mass:

$$v_k = (u_{2k-1})^i + (u_{2k})^j$$

Finally, the time derivatives of the generalized coordinates are found by equating equations (2.5.42) and (2.5.43), upon which the difference between consecutive terms is obtained to yield:

$$\dot{q}_{2(s-1)+r} = \begin{cases} u_{2(s-1)+r} - u_{2s+rr} , & (s = 1, \ldots, n-1) , (r = 1, 2) \\ u_{2(s-1)+rr} - \dot{z}_r , & (s = n) , (r = 1, 2) \end{cases}$$

Force Equations

In order to accurately simulate the relevant dynamics of the flexible multi-link ATC system, appropriate internal and external force models are required. Departing somewhat from the limited breadth of forces previously considered to have a pronounced effect on the dynamics of the ATC system, the relevant forces considered for the flexible multi-link system are inertial, elastic, gravitational, structural damping and aerodynamic. The relevant equations that constitute these force models are individually presented below.

2.5.2.1.1 Elastic Forces

Assuming that the material the cable is manufactured from is a material that has mechanical properties synonymous with Hooke’s Law, the tension within the cable segments adjacent to mass $k$ is:

$$T_{k-1} = \frac{EA}{l_{k-1}} \left( \sqrt{q_{2k-3}^2 + q_{2k-2}^2} - l_{k-1} \right)$$

$$T_k = \frac{EA}{l_k} \left( \sqrt{q_{2k-1}^2 + q_{2k}^2} - l_k \right)$$

(2.5.45)

where $E$ is the Young’s Modulus for the cable material and $A$ is the cross sectional area of each cable segment, both of which are assumed to be constant across the entire length of the cable. The resultant spring force on mass $k$ can be expressed as:

$$F_{k}^{s} = T_{k-1} \frac{p_{k-1}}{p_{k-1}} - T_{k} \frac{p_{k}}{p_{k}}$$

(2.5.46)

The generalized spring forces for the flexible multi-link ATC system now become:

$$F_{2(j-1)+i}^{s} = \mathcal{K}_{j-1} q_{2(j-2)+i} - \mathcal{K}_{j} q_{2(j-1)+i}, \ (i = 1, 2), \ (j = 1, ..., n)$$

(2.5.47)

where the constants $\mathcal{K}_{j-1}$ and $\mathcal{K}_{j}$ are given by:

$$\mathcal{K}_{j-1} = \frac{EA}{l_{j-1}} \left( 1 - \frac{l_{j-1}}{\sqrt{q_{2j-3}^2 + q_{2j-2}^2}} \right)$$

$$\mathcal{K}_{j} = \frac{EA}{l_{j}} \left( 1 - \frac{l_{j}}{\sqrt{q_{2j-1}^2 + q_{2j}^2}} \right)$$

(2.5.48)
2.5.2.1.2 Gravity Forces

Assuming that the gravitational force is invariant with altitude and acts towards the centre of the earth, the generalized gravitational forces are:

\[ F_{2k-1}^g = 0 \]
\[ F_{2k}^g = -m_k g \]  \hspace{1cm} (2.5.49)

2.5.2.1.3 Inertia Forces

The generalized inertia force for the \(k\)th lumped mass is calculated assuming that it is independent of the generalized inertia forces acting on any segments adjacent to the \(k\)th segment. Although this is a simplifying assumption, it allows the accelerations for \(k\)th lumped mass to be decoupled from the accelerations of any other lumped mass. The importance of this assumption is paramount as it allows the equations of motion for the flexible multi-link ATC system to remain uncoupled and analytically invertible. The resulting equations of motion for each lumped mass can be written entirely in terms of the system states and their time derivatives, the controls and any physical parameters. This eliminates the need for computationally expensive and time consuming matrix inversion to be performed during the integration of the equations of motion for the system.

The generalized inertia forces for the flexible multi-link ATC system are:

\[
\begin{bmatrix}
F_{2k-1}^r \\
F_{2k}^r
\end{bmatrix} = \frac{1}{2} \sum_{k=1}^{n} \left[ C_{k}^{i/b} \right] \begin{bmatrix} m_k & 0 \\ 0 & m_k \end{bmatrix} \left[ C_{k}^{l/b} \right]^T \begin{bmatrix} \ddot{u}_{2k-1} \\ \ddot{u}_{2k} \end{bmatrix}
\]  \hspace{1cm} (2.5.50)

where \( C_{k}^{i/b} \) is the direction cosine matrix given by equation (2.5.51) which relates the motion of each cable segment in the global inertial frame to that in the local body frame of each cable segment. The direction cosine matrix \( C_{k}^{i/b} \) is given by [119]:

\[
\left[ C_{k}^{i/b} \right] = \begin{bmatrix} -\sin(\theta_{k}) & -\cos(\theta_{k}) \\ -\cos(\theta_{k}) & \sin(\theta_{k}) \end{bmatrix}
\]  \hspace{1cm} (2.5.51)

where \( \theta_{k} \) is the orientation of the \(k\)th cable segment, relative to the local vertical, given by the following trigonometric relation:

\[
\theta_{k} = \arctan 2(q_{2k-1}, q_{2k})
\]  \hspace{1cm} (2.5.52)

Figure 2-10 illustrates the relative orientations of the global inertial reference frame and the local body coordinate frame of the \(k\)th cable segment.
2.5.2.1.4 Structural Damping Forces

The internal dissipation of energy within the cable can be modelled using an appropriate structural damping force model. In certain circumstances, to maintain numerical stability during the integration of the flexible multi-link ATC system equations, it may also be necessary to introduce some artificial structural damping into the equations of motion.

The damping force for each flexible cable segment is assumed to be proportional to the first time derivative of the elongation of each segment. Given this damping constant \( c \), the damping force acting on each cable segment is:

\[
F_k^c = \frac{c}{L_{s_k}} \left[ \dot{p}_{k-1} \frac{p_{k-1}}{|p_{k-1}|} - \frac{c}{L_{s_k}} \left[ \dot{p}_k \frac{p_k}{|p_k|} - \frac{c}{L_{s_k}} \left[ \dot{p}_k \frac{p_k}{|p_k|} \right] \right] \right]
\]

(2.5.53)

The generalized damping forces are given by:

\[
F_{2(j-1)+i}^c = D_{j-1} q_{2(j-1)+i} - D_j q_{2(j-1)+i}, \quad (i = 1, 2), \quad (j = 1, ..., n)
\]

(2.5.54)

where the constants \( D_{j-1} \) and \( D_j \) are given by:

\[
D_{j-1} = \frac{c}{L_{s_{j-1}}} \left[ \frac{q_{2j-3} \dot{q}_{2j-3} + q_{2j-2} \dot{q}_{2j-2}}{q_{2j-3}^2 + q_{2j-2}^2} \right]
\]

(2.5.55)

\[
D_j = \frac{c}{L_{s_j}} \left[ \frac{q_{2j-1} \dot{q}_{2j-1} + q_{2j} \dot{q}_{2j}}{q_{2j-1}^2 + q_{2j}^2} \right]
\]
2.5.2.1.5 Aerodynamic Forces

As was the case with each of the previous ATC system models, the calculation of the aerodynamic drag forces acting on each cable segment can only proceed once the orientation of each cable segment is known. As the orientation of each cable segment is initially known in their respective local body reference frames, it is necessary to determine the orientation of each cable segment in the global inertial reference system. Such a goal can be easily achieved through the use of the appropriate direction cosine matrix given by equation (2.5.51) and the subsequent cable orientation relation given by equation (2.5.52). The approach taken to determine the aerodynamic drag forces for the flexible multi-link ATC system is essentially the same as that employed by Buckham et al. [131] in their study of the three-dimensional dynamics of underwater towed bodies, except that an extra dimension of motion is removed, as only the two-dimensional motion of the ATC system is of interest here.

The process of determining the aerodynamic drag force acting on each cable segment begins with the specification of the velocity of each lumped mass in its relevant local body frame:

\[
\begin{bmatrix}
    v^h_k \\
    v^b_k
\end{bmatrix} = \left[ C^{I/B}_k \right]^T \begin{bmatrix}
    u_{2k-1} \\
    u_{2k}
\end{bmatrix} \tag{2.5.56}
\]

Using Morrison’s Equation, which separates the total drag force acting on each cable segment into two completely independent components: normal and tangential, the general form of the aerodynamic drag force acting on each cable segment is given by:

\[
F^d_k = \frac{1}{2} \rho_{air} C_n d l \left| v^h_k \right| v^h_k + \frac{1}{2} \rho_{air} C_f \pi d l \left| v^h_k \right| v^h_k \tag{2.5.57}
\]

The components of the generic aerodynamic drag force acting on each cable segment, as well as the payload, in their appropriate body reference frames, can be written as:

\[
\begin{align*}
D^h_k &= -\frac{1}{2} \rho_{air} C_f \pi d l \left| v^h_k \right| v^h_k, \quad (k = 2, \ldots, n) \\
D^b_k &= -\frac{1}{2} \rho_{air} C_d d l \left| v^h_k \right| v^h_k, \quad (k = 2, \ldots, n) \\
D^b_1 &= -\frac{1}{8} \rho_{air} C_d d p \left| v^h_1 \right| v^h_1, \quad (k = 1)
\end{align*} \tag{2.5.58}
\]

Once again it is assumed in equation (2.5.58) that the payload is non-rotating and capable only of supporting a non-lifting aerodynamic drag force.
Now the generalized aerodynamic drag forces acting on the flexible multi-link ATC system can be calculated using:

\[
\begin{align*}
\left\{ F_{2k-1}^d \right\} &= \frac{1}{2} \left[ C_{k-1}^{I/B} \right]^T \left\{ D_{k-1}^h \right\} + \left[ C_{k}^{I/B} \right]^T \left\{ D_{k}^h \right\} \\
\left\{ F_{2k}^d \right\} &= \frac{1}{2} \left[ C_{k}^{I/B} \right]^T \left\{ D_{k}^h \right\} + \left[ C_{k+1}^{I/B} \right]^T \left\{ D_{k+1}^h \right\}
\end{align*}
\]  
(2.5.59)

2.5.2.1.6 Dynamical Equations

The direct application of Kane’s Equations to the flexible multi-link ATC system allows the derivatives of the generalized speeds to be found for each lumped mass, with the state space equations of motion for the system to follow. With a means to this end, the total sum of the generalized forces for the flexible multi-link ATC system can be found using:

\[
F_k = F_k^e + F_k^x + F_k^g + F_k^d
\]  
(2.5.60)

Now the derivatives of the generalized speeds are given by:

\[
\begin{align*}
\left\{ \dot{u}_{2k-1} \right\} &= \left[ M_k^I \right]^{-1} \left\{ F_{2k-1} \right\} \\
\left\{ \dot{u}_{2k} \right\} &= \left[ M_k^I \right]^{-1} \left\{ F_{2k} \right\}
\end{align*}
\]  
(2.5.61)

where \( \left[ M_k^I \right] \) is the “mass matrix” for the flexible multi-link ATC system found using:

\[
\left[ M_k^I \right] = \frac{1}{2} \sum_{k=j=1}^{n} \left[ C_k^{I/B} \right] m_k \left[ C_k^{I/B} \right]^T
\]  
(2.5.62)

Since the “mass matrix” for the flexible multi-link ATC system can be determined and inverted analytically, it is possible to further simplify the equations of motion for the system given by equation (2.5.61). By virtue of the lumped parameter approach adopted to model the flexible multi-link ATC system, there is another useful and advantageous feature on offer; the \( k \)th element is the only cable segment that contributes to the mass of the \( k \)th element. Hence, the dynamical equations of motion for the flexible multi-link ATC system can be written analytically as:

\[
\begin{align*}
\dot{u}_{2k-1} &= \frac{F_{2k-1}}{m_k} \\
\dot{u}_{2k} &= \frac{F_{2k}}{m_k}
\end{align*}
\]  
(2.5.63)

Additionally, the derivatives of the generalized speeds corresponding to the aircraft velocity are:

\[
\dot{\tilde{u}}_{2n+k} = \tilde{z}_k, \quad (k = 1, 2)
\]  
(2.5.64)
Now all the necessary constituents of the dynamical equations of motion for the flexible multi-link ATC system have been determined; the remaining task is to rewrite these equations into state space format. The time derivative of the state space vector for the flexible multi-link ATC system comprises of two separate components: the first is due to the difference equations given by equation (2.5.44); the second component is due to the derivatives of the generalized speeds for the system given by equations (2.5.63) and (2.5.64). Mathematically, the time derivative of state space vector for the flexible multi-link ATC system is:

\[
\begin{aligned}
\dot{x}_i(t) &= \{\dot{q}_i\} = \\
&= \begin{cases}
  u_{2k-1} - u_{2k+1}, & (k = 1, \ldots, n-1), (i = 1,3,\ldots,2n-3) \\
  u_{2k} - u_{2k+2}, & (k = 1, \ldots, n-1), (i = 2,4,\ldots,2n-2) \\
  u_{2i+n+1-k} - \ddot{z}_k, & (k = 1,2), (i = 2n-1,2n) \\
  \ddot{u}_{2k-1}, & (k = 1, \ldots, n), (i = 2n+1,2n+3,\ldots,4n-1) \\
  \ddot{u}_{2k}, & (k = 1, \ldots, n), (i = 2n+2,2n+4,\ldots,4n) \\
  \ddot{z}_k, & (k = 1,2), (i = 4n+1,4n+2)
\end{cases}
\end{aligned}
\]

(2.5.65)

where \(x(t)\) is the state vector for the system. The control vector \(u(t)\), formed using the horizontal and vertical acceleration of the aircraft \(\{\ddot{z}_1, \ddot{z}_2\}\), can be concisely written as:

\[
u(t) = [\ddot{z}_1, \ddot{z}_2]^T
\]

(2.5.66)

It is interesting to note the difference between the control vector for the flexible multi-link ATC system and the control vectors for the rigid multi-link and single link ATC systems. The former comprise of the direct velocities of the aircraft, whilst the latter consists of the inertial accelerations of the aircraft, along with the reel acceleration of the first cable segment. Currently, the flexible multi-link ATC system is incapable of handling any prescribed length changes to the cable (deployment or retrieval). This is in fact a major drawback of using the lumped parameter approach to model flexible ATC systems. Whilst such an approach has the advantages of relatively simple force calculations and significant flexibility to allow for a wide-range of scenarios to be modelled, it presents significant difficulties in simulating cables of varying length. These difficulties can be overcome and a variety of methods do exist that attempt to account for variable cable lengths in lumped parameter cable models (see Williams and Trivailo [6, 101, 114], Trivailo et al. [127], Kamman and Huston [69], Banerjee and Do [98], Jun et al. [3] and May and Connell [132] for details concerning a wide variety of approaches to this modelling aspect).
However, the use of flexible multi-link ATC system models in this thesis is somewhat restricted, due to their unsuitability for use within the development of controllers for the system using optimal control (see Section 4.6 for further details). As a result, it is not justified to devote the significant effort associated with overcoming the inherent complexity that arises when variable cable lengths are appropriately accounted for in flexible multi-link cable models. Utilizing the resulting modelling capability would be limited and would appreciably detract from the scope of this work. Consequently, only constant length flexible multi-link ATC systems will be considered in this dissertation.

2.6 Concluding Remarks

In this section, the various models of the ATC system that will be extensively utilized throughout other sections of this thesis were formally presented, with their governing equations of motion outlined in detail. These models have been proposed to study the fundamental dynamics of the physical ATC system and assist the design of innovative optimal controllers for the system. It is well known that simplified low order models, such as those offered in Sections 2.4.3 and 0, are ideal for control system design and development purposes, provided they remain representative of the real system, in light of the many simplifying assumptions invoked during their derivation. Contrastingly, the sophisticated models outlined in Sections 2.5.1 and 2.5.2 are more suited to modelling or simulation tasks, as their accuracy and fidelity ensure they can appropriately capture the significant dynamics of the real physical system. Yet their inherent complexity renders them onerous to use within modern control system design architectures. As a result, it would be highly valuable if it were possible to develop a means to certify that the simple single-link ATC system models are representative of the complex multi-link models. Measures to achieve this very aspiration will be the theme and focus of the following section.
3 IMPROVING THE ACCURACY OF SIMPLE AERIAL TOWED-CABLE SYSTEM MODELS

3.1 Preface

The contents of this chapter deal with the development of a series of systematic techniques that can be used to improve the representativeness of simple ATC system models. The primary objective of this chapter is to provide a systematic framework for matching the dynamical motion of a simple single-link ATC system model to that of the more complex multi-link models. Two separate approaches to this model “matching” problem were successfully developed, each based on the premise that for any given set of system parameters, if the equilibrium configurations of the simple and complex models were forced to be equal, then the representativeness of the simple model in a dynamic sense should approach that of the complex models. The essential difference between the two approaches is that one is based solely on static equilibrium conditions, whilst the other approach is able to incorporate dynamic inertial effects into the prediction.

3.2 Preliminaries

Highly complex mathematical models of ATC systems, such as those presented in Section 2.5, are suitably capable of capturing the significant dynamical motion of real physical ATC systems. Such models have been extensively used by researchers in the past with encouraging results (see [1, 5, 6, 25-27, 33, 35-40, 44, 48, 51, 52, 55-66, 69, 70, 78, 84, 101, 114, 127, 132, 133] for examples), although the absence of readily available experimental data has rendered the direct experimental validation of such models somewhat missing from the literature. More recently however, the control of ATC systems has begun to attract interest from researchers [3, 6, 7, 11, 12, 17, 18, 30, 60, 97-99, 101, 114, 132], which in the case of this dissertation, it is the non-linear optimal control of such systems that is of primary interest. Although the sophisticated models can accurately simulate and predict the motion of the physical system, their inherent complexity render them cumbersome to use for the purposes of control system design and development. Simple models tend to be more widely used in control system design and development, but unless verified or tested on complex models, their use can result in non-representative simulations and inept, poor performing controllers.
In light of these considerations, one can pose the following question:

*Is it possible to reconcile the competing demands of model simplicity and representativeness, in a systematic and relatively straightforward manner?*

As will be categorically demonstrated later in this chapter, the abovementioned question can be answered overwhelmingy in the affirmative. With several means to this end, it is proposed to “match” the dynamical motion of a simple ATC system model to that of correspondingly more complex models. The dynamic variables of primary interest here are the position and relative orientation of the cable tip, since knowledge of these variables is central to the development and successful performance of controllers for ATC systems. Only two-dimensional (in-plane) dynamical motion of the ATC system will be considered, although this by no means is a limitation of the proposed methodology, since extension into three-dimensions is straight-forward and does not require significant effort to be outlaid.

The various models of the ATC system that were comprehensively explored in Section 2, each possessing varying degrees of sophistication, are reproduced in Figure 3-1 (simple single-link model) and Figure 3-2 (complex multi-link models). The simple model attempts to represent an adequate compromise between accuracy and control system design feasibility. The dynamic equations of motion for the single-link model were derived in Section 2.4.3 by utilizing Lagrange’s Equations with the in-plane cable angle $\theta$ and the cable length $l$ used as generalized coordinates, whilst kinematic constraints $\{x, y\}$ were employed to represent the horizontal displacement and altitude of the aircraft. The actuators used for control are the horizontal and vertical acceleration of the aircraft $\{\ddot{x}, \ddot{y}\}$ and the cable reel acceleration $\ddot{l}$.

![Figure 3-1: Simple Single-Link Aerial Towed-Cable Model](image)
The state space equations of motion for the simple single-link ATC system are represented by equation (2.4.35) and were presented in Section 2.4.3. In order to assist in the development of the two proposed model “matching” procedures, these dynamical equations of motion will now be revisited and are reproduced by way of equation (3.2.1) through to equation (3.2.8):

\[ \dot{q}_1 = \dot{x} \]  
(3.2.1)

\[ \dot{q}_2 = \ddot{x} = u_1 \]  
(3.2.2)

\[ \dot{q}_3 = \dot{y} \]  
(3.2.3)

\[ \dot{q}_4 = \ddot{y} = u_2 \]  
(3.2.4)

\[ \dot{q}_5 = \dot{\theta} \]  
(3.2.5)

\[ \dot{q}_6 = \frac{1}{l} \left( \dot{x} \cos \theta - \dot{y} \sin \theta - 2l \dot{\theta} - g \sin \theta \right) \]

\[ - \frac{\pi \rho \omega_n d_r^2 C_{d_r}}{8m_p + \pi \rho \omega_n l d_c^2} \left( l \dot{\theta} - \dot{x} \cos \theta + \dot{y} \sin \theta \right) \]  
(3.2.6)

\[ \sqrt{\left( \dot{x} - l \sin \theta - l \dot{\theta} \cos \theta \right)^2 + \left( \dot{y} - l \cos \theta + l \dot{\theta} \sin \theta \right)^2} \]

\[ q_7 = l \]  
(3.2.7)

\[ \dot{q}_8 = \ddot{l} = u_3 \]  
(3.2.8)
where: \( \rho_c \) is the mass density of the cable,

\( d_c \) is the diameter of the cable.

In the interests of clarity and duplicity, the state space equations of motion for the complex flexible and rigid multi-link ATC system models, first presented in the concluding stages of Section 2.5.1 and Section 2.5.2, will not be reproduced here.

In order for the equilibrium configurations of the complex multi-link models to be suitably compared to that of the single-link model, an “equivalent” equilibrium configuration angle \( \theta_E \) and an “equivalent” cable length \( l_E \) need to be determined. For the simple single-link model, the equilibrium configuration of the system is designated simply by the cable orientation angle \( \theta \) and final cable length \( l \) at equilibrium. With respect to the complex multi-link models, the action of the aerodynamic drag forces acting on the cable cause it to take up a curved profile as shown in Figure 3-2. As a result, the cable tip appears to be in a position closer to the aircraft than the actual length of cable suggests. Thus at equilibrium, the general orientation of each cable segment will be unique and the position of the cable tip cannot be directly inferred from the instantaneous length of the cable, hence the need for an “equivalent” equilibrium configuration for the multi-link models. This very “equivalent” equilibrium configuration for the complex multi-link ATC system models is depicted in Figure 3-3.

![Figure 3-3: Equivalent Equilibrium Configuration of the Multi-Link System Models](image)
To determine the “equivalent” equilibrium configuration of each multi-link ATC model, optimization-based root finding procedures are used to determine the equilibrium configuration of each model. For the rigid multi-link model, the equilibrium configuration is specified by the equilibrium angle of each individual cable segment, as opposed to the flexible multi-link model, whose equilibrium configuration is denoted by the Cartesian coordinates of each cable segment at equilibrium. The root finding procedures determine the equilibrium configuration of each model by forcing the time-varying states of the system to their equilibrium values. For example, the equilibrium configuration of the rigid multi-link model occurs when the angular velocity and acceleration of each cable segment is zero and the velocity of the aircraft is the appropriate steady-state value. Using the state space equations of motion for the rigid multi-link model, the root finding procedure calculates the orientation angle of each cable segment at the steady state towing speed for zero values of the angular velocity and acceleration of each cable segment. A similar process is used in the case of the flexible multi-link model, where the root finding procedure calculates the relative displacement of each cable segment at the chosen towing speed for zero values of the relative velocity and acceleration of each segment.

Once the equilibrium configuration of each model is known, the “equivalent” position of the cable tip at equilibrium \( \{x_E, y_E\} \) can be inferred from equations (3.2.9) and (3.2.10) for the rigid multi-link model, or from equations (3.2.11) and (3.2.12) for the flexible model:

\[
x_E = -\sum_{k=1}^{n} l_k \sin(\theta_{\mathit{ek}}) \tag{3.2.9}
\]

\[
y_E = -\sum_{k=1}^{n} l_k \cos(\theta_{\mathit{ek}}) \tag{3.2.10}
\]

\[
x_E = -\sum_{k=1}^{n} x_k \tag{3.2.11}
\]

\[
y_E = -\sum_{k=1}^{n} y_k \tag{3.2.12}
\]

where: 
- \( l_k \) is the length of the \( k^{\text{th}} \) rigid cable segment,
- \( \theta_{\mathit{ek}} \) is the equilibrium angle of the \( k^{\text{th}} \) rigid cable segment,
- \( x_k \) is the \( x \)-coordinate of the \( k^{\text{th}} \) flexible cable segment at equilibrium,
- \( y_k \) is the \( y \)-coordinate of the \( k^{\text{th}} \) flexible cable segment at equilibrium.
The equivalent equilibrium configuration of each of the multi-link models can then be found using equations (3.2.13) and (3.2.14):

\[ \theta_E = \tan^{-1}\left( \frac{x_E}{y_E} \right) \]  
\[ l_E = \sqrt{x_E^2 + y_E^2} \]

With each of the various mathematical models of the ATC system that help constitute the proposed “model” matching procedures, along with the process employed to calculate and appropriately compare the equilibrium configuration of each class of models, the two proposed model “matching” procedures can now be formally presented.

### 3.3 Method 1

The first approach to model “matching” for the ATC system begins with a parametric study carried out on the various system models to see how the main physical parameters affect the equilibrium of the physical system. The physical parameters that were identified to have the strongest effect on equilibrium are the cable length, towing speed, payload drag coefficient, and payload mass. Plots were generated to see how such parameters affect the equilibrium configuration of the physical system. Then for each of these parameters, the change in the cable length and payload drag coefficient of the simple model in order to match its equilibrium configuration to that of the complex model/s was determined via an optimization-based technique. Next, interpolating polynomials were constructed to map the changes required of the simple model to the various physical parameters of the system. Hence, for any given physical parameter set, the changes to the cable length and payload drag coefficient of the simple model can be rapidly determined offline and applied prior to integration, in order to render the simple model representative of the correspondingly more complex model/s.

The underlining objective of this proposed model “matching” technique is to compel the equilibrium characteristics of the simple single-link model to be equal to the equilibrium characteristics of the complex multi-link model/s. The central premise behind this approach is that greater agreement between the equilibrium configurations of each class of model should lead to a more representative simple single-link model in a dynamic sense. The characteristics of the simple single-link model of interest are the equilibrium orientation angle \( \theta_e \) and position of the cable tip \( \{x_e, y_e\} \), which will be artificially forced to “match” the “equivalent”
equilibrium orientation angle $\theta_E$ and position of the cable tip $\{x_E, y_E\}$ corresponding to the complex multi-link model/s. This “match” will be attempted by prescribing changes to the length of the cable $l$ and the payload drag coefficient $C_{dp}$ for the simple single-link model. The physical parameters identified that have the strongest effect on equilibrium of the simple ATC system are the cable length $l$, towing speed $U_0$, payload drag coefficient $C_{dp}$, and payload mass $m_p$. Parametric studies have identified that the air density, cable aerodynamic drag coefficient, cable mass density, cable diameter and payload diameter all have a negligible effect on the equilibrium configuration of the single-link ATC system, compared to the manner in which the cable length, towing speed, payload mass and drag coefficient affect the equilibrium of the system. Of these system parameters, from a physical viewpoint, common sense suggests that the most suitable parameters to be manipulated are the cable length and payload drag coefficient. Changing the cable length is easily achieved through reel acceleration augmentation, in general the drag coefficient of the payload is likely to be uncertain or unsteady anyway, whilst the mass of the payload and aircraft towing speed in the field are likely to be well known.

Due to its inherently simple nature, the equilibrium configuration of the simple ATC system model can be found analytically. For the simple single-link ATC model, the equilibrium cable angle $\theta_e$ and the position of the cable tip $\{x_e, y_e\}$ can be mathematically represented by equation (3.3.1) through to equation (3.3.3):

$$\theta_e = \tan^{-1}\left(\frac{\pi \rho_{at} U_0^2 C_{dp} d_p^2}{g \left(8 m_p + \pi \rho \pi d_c^2\right)}\right)$$ (3.3.1)

$$x_e = -l \sin \theta_e$$ (3.3.2)

$$y_e = -l \cos \theta_e$$ (3.3.3)

Prior to conducting model “matching”, a parametric study was carried out on the various system models to see how the main physical parameters affect the equilibrium of the ATC system. Trace plots were generated to elucidate how these parameters affect the equilibrium configuration of the system, the results of which are presented in Figure 3-4 through to Figure 3-7. In each of the aforementioned figures, the red line denotes the results for the simple one-link ATC model, while the green and blue lines depict the results for the rigid and flexible multi-link models respectively. As the absolute magnitudes and nature of the trajectories presented in Figure 3-4 through to Figure 3-7 are comparable, it can be deduced that each of
the physical system parameters affect the equilibrium configuration of each of the models in exceedingly analogous ways. This outcome is especially pertinent, since it will greatly facilitate the subsequent “matching” of equilibrium configurations between each class of models.

Figure 3-4: Effect of Cable Length on the Equilibrium Configuration of the System

Figure 3-5: Effect of Payload Mass on the Equilibrium Configuration of the System
Figure 3-6: Effect of Payload Drag Coefficient on the Equilibrium Configuration of the System

The central strategy behind this proposed model “matching” method is as follows: for a given set of physical system parameters, determine the change in cable length and payload drag coefficient of the simple model to match its equilibrium configuration to that of either complex multi-link model. This is carried out using an optimization-based technique, which performs the “match” mathematically by calculating the required changes to the simple model that enforce the equality relations given by equation (3.3.4) through to equation (3.3.6):

\[
\theta_{ref} = \tan^{-1}\left(\frac{\pi\rho \mu U_0^2 C_{d_p} \left(1 + p_1\right) d_p^2}{g \left(8m_p + \pi\rho c l d_c^2\right)}\right) = \theta_E
\]

(3.3.4)

\[
x_{ref} = -(1 + p_2) \sin(\theta_{ref}) = x_E
\]

(3.3.5)

\[
y_{ref} = -(1 + p_2) \cos(\theta_{ref}) = y_E
\]

(3.3.6)

where: \(\theta_{ref}\) is the updated equilibrium angle for the simple (refined) model,

\(x_{ref}\) is the updated x-component of the equilibrium position for the simple (refined) model,

\(y_{ref}\) is the updated y-component of the equilibrium position for the simple (refined) model,

\(p_1\) is the payload drag coefficient change required by the simple model,

\(p_2\) is the length change required by the simple model.
The decision variables during optimization are $p_1$ and $p_2$, whilst the cost function $J$ used to achieve the model “match” is:

$$J = \left(\theta_e - \theta_{vue}\right)^2 + \left(\frac{x_e - x_{e_{REF}}}{L_{REF}}\right)^2 + \left(\frac{y_e - y_{e_{REF}}}{L_{REF}}\right)^2$$  \quad (3.3.7)$$

where the term $L_{REF}$ is used to non-dimensionalize the cost function. The main outcome of the above optimization process is the determination of how the changes to the simple model are affected by the physical parameters of the system. The sparse nonlinear solver SNOPT [134, 135], implemented in MATLAB 7® via mex files, was utilized for the optimization task. To reduce solver workload and improve convergence, zero values for $p_1$ and $p_2$ were used as an initial estimate for the solution. The Jacobians for the problem constraints were estimated using sparse finite differences. The results obtained after the optimization-based “matching” procedure was performed on both the rigid and flexible multi-link models appear in Figure 3-8 through to Figure 3-15. Figure 3-8 through to Figure 3-11 refer to the results obtained for the rigid multi-link cable model, whilst Figure 3-12 through to Figure 3-15 concern the flexible multi-link model. In each of these figures, the results for the simple model are given by the red line, the blue line denotes the results for either the rigid or flexible multi-link model where appropriate, whilst the green line signifies the results for the “refined” model; that is the results for a single-link model with adjustments made to the payload drag coefficient and cable length. The success of the “matching” procedure is aptly demonstrated in Figure 3-8 through to Figure 3-15. In each case the results for the “refined” model overlap the results of either complex model, which demonstrates that an exact match occurred in all circumstances.
Figure 3-8: Results for Rigid Multi-Link Cable Matching Procedure - Cable Length

Figure 3-9: Results for Rigid Multi-Link Cable Matching Procedure - Payload Mass
Figure 3-10: Results for Rigid Multi-Link Cable Matching Procedure- Payload Drag

Figure 3-11: Results for Rigid Multi-Link Cable Matching Procedure- Towing Speed
Figure 3-12: Results for Flexible Multi-Link Cable Matching Procedure- Cable Length

Figure 3-13: Results for Flexible Multi-Link Cable Matching Procedure- Payload Mass
To allow for the rapid determination of the required parameter changes for the simple model, numerical interpolating polynomials were constructed. The category of polynomial chosen for the interpolation were Chebyshev polynomials. These are globally orthogonal polynomials belonging to the Jacobi class of polynomials. To closely map the relationship between the parameter changes $p_1$ and $p_2$ and the physical parameters of the ATC system, sixteenth-order Chebyshev polynomials were utilized. Chebyshev polynomials within the range of five to twenty were trialed, but sixteenth-order polynomials were found to be the lowest capable of delivering a sufficiently close approximation to the relevant mapping.

![Figure 3-14: Results for Flexible Multi-Link Cable Matching Procedure- Payload Drag](image1)

![Figure 3-15: Results for Flexible Multi-Link Cable Matching Procedure- Towing Speed](image2)
The required changes to the simple model, as functions of the physical parameters of the ATC system are shown in Figure 3-16 through to Figure 3-19, with the results for the rigid multi-link model given by the red line, whereas the blue line denotes the results for the flexible multi-link model.

**Figure 3-16: Required Parameter Changes as a Function of Cable Length**

**Figure 3-17: Required Parameter Changes as a Function of Payload Mass**
It can be seen from Figure 3-16 through to Figure 3-19 that increasing cable length, payload mass, drag coefficient and towing speed all “lengthen” (stretch) the cable when the longitudinal elasticity of the cable is included in the complex multi-link model. To capture this with the simple single-link model, a net increase in length \(+p_2\) is applied. Contrastingly, in Figure 3-16, the action of aerodynamic drag on the rigid multi-link cable model tends to “shorten” the cable since the cable experiences increased curvature, reducing the separation between the payload and the aircraft. Thus a net reduction in cable length \(-p_2\) is applied to the simple single-link model to simulate this phenomenon. In summarizing, the simple model requires an otherwise longer cable and more aerodynamic
drag applied to the payload in order for it to “match” the rigid multi-link cable model, as compared to the case when a “match” with the flexible multi-link cable model is sought after.

In general, for each class of multi-link ATC system model, there are four different functions yielding a value for the parameters $p_1$ and $p_2$. Hence for any given set of physical parameters, there will be four unique predictions of which $p_1$ and $p_2$ value to use. As a result, an additional course of action is required to determine which proposed $p_1$ and $p_2$ estimate should be used within the model “matching” procedure. Supported by extensive background simulations, in the overwhelming majority of cases, all four predictions for $p_1$ and $p_2$ are similar, regardless of what combination of physical parameters were used (limited to parameter sets considered representative of real ATC systems) and an average value of each could be confidently used. This applies only if the control inputs applied to the ATC system are not excessively large, which in practice is almost always the case. Further supporting evidence of this can be inferred from Figure 3-16 through to Figure 3-19. In each of these figures, the functions returning a value for $p_1$ and $p_2$ do not have significantly large second-order derivatives and each behaves in a relatively quasi-linear manner. Small changes to the physical parameter set used to produce an estimate of $p_1$ and $p_2$, will correspondingly result in small changes to the actual $p_1$ and $p_2$ values returned by each of the functions in Figure 3-16 through to Figure 3-19. Hence, the seemingly arbitrary decision to use a mean $p_1$ and $p_2$ value within the model “matching” procedure is justified and can be appropriately qualified.

Furthermore, since the action of the control inputs for the ATC system varies the towing speed and cable length over the duration of a simulation, an additional procedure is needed to select the values of the towing speed and cable length for which the predictions $p_1$ and $p_2$ are carried out at. Again supported by extensive background simulations, a close match between the simple single-link model and the desired complex multi-link model is achieved when the predictions of $p_1$ and $p_2$ are carried out at a towing speed and cable length given by equations (3.3.8) and (3.3.9):

\[
V_{\text{predict}} = \frac{\dot{x}_{\text{final}}^2 + \dot{y}_{\text{final}}^2}{\sqrt{\dot{x}_{\text{initial}}^2 + \dot{y}_{\text{initial}}^2}} \tag{3.3.8}
\]

\[
I_{\text{predict}} = \frac{l_{\text{final}}^2}{l_{\text{initial}}} \tag{3.3.9}
\]
The offline estimation of the parameters $p_1$ and $p_2$ is carried out only once, prior to the commencement of the simulation, with no update required as the simulation progresses from the initial to final time.

### 3.4 Application of Method 1 for Model “Matching”

This sub-section concerns the wide-ranging application of the proposed model “matching” technique to the various complex multi-link ATC system models. The performance and robustness of this model “matching” procedure will be demonstrated through a series of dynamic simulations. This will elucidate the validity of the central premise that underlies the model “matching” technique: how well the matching of equilibrium conditions between the simple and complex models translates into representativeness of the simple single-link model in the dynamic environment.

#### 3.4.1 Model “Matching” for the Rigid Multi-Link Aerial Towed-Cable System

The following simulation case studies will demonstrate how the representativeness of the simple single-link ATC system model compares to the rigid multi-link system model for a range of physical parameters under the action of a variety of control inputs. With respect to the figures provided to display the forthcoming results, the results concerning the simple single-link model are given by the red line, the blue line denotes the results for the complex rigid multi-link model, whilst the green line signifies the results for the “refined” single-link model.

**Case 1**

This simulation begins with the cable hanging vertically below the aircraft and progresses under the action of applied control. The control applied to each of the ATC system models for this case is given by equations (3.4.1) through to (3.4.3) and shown in Figure 3-20:

\[
\ddot{x} = u_1 = \frac{1}{2} e^{-4t} \cos \left( \frac{t}{16t_{\text{final}}} \right) \tag{3.4.1}
\]

\[
\dot{y} = u_2 = 0 \tag{3.4.2}
\]

\[
\dot{I} = u_3 = \frac{1}{4} e^{-4t} \sin \left( \frac{t}{16t_{\text{final}}} \right) \tag{3.4.3}
\]

The simulation/physical parameters for this case study are given in Table 3-1.
Table 3-1: Rigid Cable Case Study 1- Simulation and Physical Parameters

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_0$</td>
<td>Initial Towing Speed (x-direction)</td>
<td>50 m/s</td>
</tr>
<tr>
<td>$W_0$</td>
<td>Initial Towing Speed (y-direction)</td>
<td>0 m/s</td>
</tr>
<tr>
<td>$t_{\text{initial}}$</td>
<td>Starting Simulation Time</td>
<td>0 s</td>
</tr>
<tr>
<td>$t_{\text{final}}$</td>
<td>Final Simulation Time</td>
<td>150 s</td>
</tr>
<tr>
<td>$m_P$</td>
<td>Payload Mass</td>
<td>100 kg</td>
</tr>
<tr>
<td>$C_{d_P}$</td>
<td>Nominal Payload Drag Constant</td>
<td>0.47</td>
</tr>
<tr>
<td>$d_P$</td>
<td>Payload Diameter</td>
<td>0.75 m</td>
</tr>
<tr>
<td>$l_0$</td>
<td>Initial Cable Length</td>
<td>500 m</td>
</tr>
<tr>
<td>$\rho_c$</td>
<td>Cable Mass Density</td>
<td>3000 kg/m$^3$</td>
</tr>
<tr>
<td>$d_c$</td>
<td>Cable Diameter</td>
<td>2.5 mm</td>
</tr>
<tr>
<td>$C_{n_c}$</td>
<td>Cable Normal Drag Constant</td>
<td>1.1</td>
</tr>
<tr>
<td>$C_{t_c}$</td>
<td>Cable Tangential Drag Constant</td>
<td>0.022</td>
</tr>
<tr>
<td>$N$</td>
<td>Number of Cable Elements</td>
<td>5</td>
</tr>
<tr>
<td>$\rho_{\text{air}}$</td>
<td>Air Density</td>
<td>1.225 kg/m$^3$</td>
</tr>
<tr>
<td>$g$</td>
<td>Gravity Constant</td>
<td>9.81 m/s$^2$</td>
</tr>
</tbody>
</table>

Figure 3-20: Rigid Cable Case Study 1- Applied Control

The predictions of $p_1$ and $p_2$ for this case are carried out at a towing speed and cable length given by equation (3.4.4):

$$V_{\text{predict}} = \frac{\sqrt{49.0773^2 + 0^2}}{\sqrt{50^2 + 0^2}} = 48.1717 \text{ m/s}$$

$$l_{\text{predict}} = \frac{855.1159^2}{500} = 1462.4 \text{ m}$$

(3.4.4)
For the towing speed and cable length given by equation (3.4.4), along with the rest of the physical parameters governing the simulation, the four estimates of $p_1$ and $p_2$ and the average value used within the model “matching” procedure are:

$$
\begin{align*}
\{ p_1 \} = \{ 1.2652, 2.9939, 1.2866, 1.265 \} \\
\{ p_2 \} = \{ -13.2515, -43.0699, -13.2293, -13.2502 \} \\
\Rightarrow p_1^{av} = 1.7027 \\
p_2^{av} = -20.7003 
\end{align*}
$$

(3.4.5)

The results of the model “matching” procedure for this case are depicted in Figure 3-21 through to Figure 3-25. Figure 3-21 through to Figure 3-23 detail a comparison of how the “equivalent” cable kinematics vary between each class of model; Figure 3-24 shows the trajectory the cable tip follows for each model, whilst Figure 3-25 outlines the percentage error in the “equivalent” cable kinematics for the “refined” single-link model as compared to the rigid multi-link model. With respect to results presented in Figure 3-25, the error in “equivalent” cable angle $\Delta \theta$ has been non-dimensionalized with respect to the maximum allowable cable orientation angle (90°), whilst the errors in the equivalent position of the cable tip $\{ \Delta_x, \Delta_y \}$ have been non-dimensionalized with respect to the initial cable length.

![Figure 3-21: Rigid Cable Case Study 1 Results- “Equivalent” Cable Angle](image)

The first conclusion that is evident from Figure 3-21 through to Figure 3-25 is that the simple single-link model is totally unrepresentative of the complex rigid multi-link model of the ATC system. What is equally apparent from the same figures is that with the appropriate modifications to the simple model’s payload drag coefficient and cable length given by equation (3.4.5), the significant dynamics of the simple model can be made to closely track those of the more sophisticated model.
Although the trends in the “equivalent” dynamics between the simple and complex models are generally similar, in order for representativeness to be achieved, adjustments to the simple model are necessary. What is especially encouraging about the results contained within Figure 3-21 through to Figure 3-25 is that the representativeness of the “refined” single-link model is maintained over the entire simulation time span, even though the ATC system undergoes considerable control augmentation and dynamical motion. Similarly, the ATC system does not reach a final equilibrium configuration, although it slowly approaches such a configuration.

Figure 3-22: Rigid Cable Case Study 1 Results- “Equivalent” $x$-coordinate of the Cable Tip

Figure 3-23: Rigid Cable Case Study 1 Results- “Equivalent” $y$-coordinate of the Cable Tip
After close analysis of Figure 3-25, it can be concluded that the results obtained for the “refined” single-link model do not significantly deviate from those pertaining to the complex multi-link model. In fact, the maximum error in the position of the cable tip for the “refined” single-link model throughout the above simulation is approximately 47.86 m or 8.91 %, whilst the absolute mean average error is approximately 19.74 m or 3.12 %. Compare this to the case when the necessary adjustments to the simple single-link model are not performed, where the maximum and mean errors in the position of the cable tip are a profuse 552.99 m (70.09 %) and 383.69 m (57.18 %) respectively. This further confirms the effectiveness and sound performance of the proposed model “matching” procedure.

Figure 3-24: Rigid Cable Case Study 1 Results- “Equivalent” Cable Tip Trajectory

Figure 3-25: Rigid Cable Case Study 1 Results- Percentage Error of Model “Match” (a) Angle (b) x-coordinate (c) y-coordinate
Finally, the differences between the configuration of the “refined” single-link and complex multi-link models do not diverge as the simulation progresses. Instead the configuration of the “refined” model approaches that of the complex model, although total convergence is not achieved; a small residual error of 26.73 m (3.23 %) exists in the position of the cable tip at the end of the simulation.

**Case 2**

This simulation begins with the ATC system in equilibrium and progresses under the action of applied control until a new equilibrium configuration is reached. The control applied to each of the ATC system models for this case is given by equations (3.4.6) through to (3.4.8) and shown in Figure 3-26:

\[
\begin{align*}
\ddot{x} &= u_1 = \frac{1}{2} e^{-q t} \cos \left( \frac{t}{4t_{final}} \right) \\
\ddot{y} &= u_2 = \frac{1}{4} e^{-q t} \sin \left( \frac{t}{2t_{final}} \right) \\
\ddot{\theta} &= u_3 = \frac{1}{20} \sin \left( \frac{2\pi t}{t_{final}} \right)
\end{align*}
\]  

The simulation parameters for this case study are given in Table 3-2. The predictions of \( p_1 \) and \( p_2 \) for this case are carried out at a towing speed and cable length given by equation (3.4.9):

\[
V_{predict} = \left( \frac{47.3776^2 + 2.1541^2}{\sqrt{45^2 + 0^2}} \right) = 49.9839 \text{ m/s}
\]

\[
l_{predict} = \frac{518.3033^2}{200} = 1343.2 \text{ m}
\]

For the towing speed and cable length given above, along with the rest of the physical parameters governing the simulation, the four estimates of \( p_1 \) and \( p_2 \) and the average value used within the model “matching” procedure are:

\[
\left\{ \begin{array}{c}
p_1 = \{1.2588, 2.8057, 1.2654, 1.0296\} \\
p_2 = \{-12.7047, -40.4123, -13.2315, -8.6905\}
\end{array} \right\} \rightarrow p_1^{av} = 1.5899, \quad p_2^{av} = -18.7598
\]
Table 3-2: Rigid Cable Case Study 2- Simulation and Physical Parameters

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<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
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<tbody>
<tr>
<td>$U_0$</td>
<td>Initial Towing Speed (x-direction)</td>
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<tr>
<td>$W_0$</td>
<td>Initial Towing Speed (y-direction)</td>
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<tr>
<td>$t_{initial}$</td>
<td>Starting Simulation Time</td>
<td>0 s</td>
</tr>
<tr>
<td>$t_{final}$</td>
<td>Final Simulation Time</td>
<td>200 s</td>
</tr>
<tr>
<td>$m_p$</td>
<td>Payload Mass</td>
<td>50 kg</td>
</tr>
<tr>
<td>$C_{dp}$</td>
<td>Nominal Payload Drag Constant</td>
<td>0.5</td>
</tr>
<tr>
<td>$d_p$</td>
<td>Payload Diameter</td>
<td>0.5 m</td>
</tr>
<tr>
<td>$l_0$</td>
<td>Initial Cable Length</td>
<td>200 m</td>
</tr>
<tr>
<td>$\rho_c$</td>
<td>Cable Mass Density</td>
<td>7850 kg/m$^3$</td>
</tr>
<tr>
<td>$d_c$</td>
<td>Cable Diameter</td>
<td>2 mm</td>
</tr>
<tr>
<td>$C_{dn}$</td>
<td>Cable Normal Drag Constant</td>
<td>1.1</td>
</tr>
<tr>
<td>$C_{tn}$</td>
<td>Cable Tangential Drag Constant</td>
<td>0.022</td>
</tr>
<tr>
<td>$N$</td>
<td>Number of Cable Elements</td>
<td>6</td>
</tr>
<tr>
<td>$\rho_{air}$</td>
<td>Air Density</td>
<td>1.225 kg/m$^3$</td>
</tr>
<tr>
<td>$g$</td>
<td>Gravity Constant</td>
<td>9.81 m/s$^2$</td>
</tr>
</tbody>
</table>

The results of the model “matching” procedure for this case are depicted in Figure 3-27 through to Figure 3-31.

Figure 3-26: Rigid Cable Case Study 2- Applied Control
Once again, it can be clearly inferred from Figure 3-27 through to Figure 3-31 that the simple single-link model cannot be confidently used to accurately reproduce the “equivalent” dynamics of the complex model and of the physical system itself. In a similar manner to the previous case, it was possible to perform modifications to the payload drag coefficient and cable length within the simple model so that the important dynamics of the simple model are representative of the dynamics of complex multi-link model. In general, the trends in the “equivalent” dynamics between the simple single-link and complex multi-link models are again similar, but adjustments to the simple model are again required in order for total representativeness to occur. Even though the control augmentation and subsequent dynamical motion of the ATC system is less for this case than the previous one, the representativeness of
the “refined” model is maintained over most of the simulation time span, although the degree of closeness in the “match” is significantly less in this case than the previous one. Figure 3-27 through to Figure 3-29 show that the model “matching” procedure was best able to match the “equivalent” coordinates of the cable tip, as opposed to the “equivalent” cable orientation angle. However, as evidenced by Figure 3-31, the results obtained for the “refined” model deviate considerably from those given by the complex model in the initial stages of the simulation. The differences between the results given by each model tend to stabilize and reduce as the simulation progresses and final equilibrium is attained, although as in the previous case study, total convergence is not achieved.

Figure 3-29: Rigid Cable Case Study 2 Results- “Equivalent” y-coordinate of the Cable Tip

Figure 3-30: Rigid Cable Case Study 2 Results- “Equivalent” Cable Tip Trajectory
The maximum error in the position of the cable tip for the “refined” model is approximately 43.52 m or 20.69% throughout this case study simulation, whilst the absolute mean average error is approximately 24.23 m or 8.73%. Once again, when compared to the case when no adjustments to the simple model are performed, the maximum and mean errors in the position of the cable tip are 309.13 m (61.03%) and 184.64 m (49.59%) respectively, reinforcing the need and efficacy of the proposed model “matching” procedure. In a similar manner to the previous case, the differences between the configuration of the “refined” single-link and complex multi-link models do not diverge as the simulation progresses, as demonstrated by Figure 3-28, Figure 3-29 and Figure 3-31 in particular. Instead, the configuration of the “refined” model slowly approaches that of the complex model, with a small residual error of 26.83 m (5.3%) existing in the position of the cable tip at the conclusion of the simulation.

### 3.4.2 Model “Matching” for the Flexible Multi-Link Aerial Towed-Cable System

The following simulation case studies demonstrate how the representativeness of the simple single-link model compares to the flexible multi-link system model for a range of physical parameters under the action of a variety of control inputs. With respect to the figures provided to display the forthcoming results, the results concerning the simple single-link model are given by the red line, the blue line denotes the results for the complex flexible multi-link model, whilst the green line signifies the results for the “refined” single-link model.
Case 3

This simulation begins with the cable hanging vertically below the aircraft and progresses under the action of applied control until final equilibrium is achieved. The control applied to each of the ATC system models for this case is given by equations (3.4.11) and (3.4.12) and is shown in Figure 3-32:

$$\ddot{x} = u_t = \frac{3t}{4e^t_{\text{final}}} \left[ 1 - \cos \left( \frac{t}{20t_{\text{final}}} \right) \right]$$  \hspace{1cm} (3.4.11)

$$\ddot{y} = u_2 = \ddot{t} = u_3 = 0$$  \hspace{1cm} (3.4.12)

The simulation/physical parameters for this case study are given in Table 3-3.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_0$</td>
<td>Initial Towing Speed (x-direction)</td>
<td>50 m/s</td>
</tr>
<tr>
<td>$W_0$</td>
<td>Initial Towing Speed (y-direction)</td>
<td>0 m/s</td>
</tr>
<tr>
<td>$t_{\text{initial}}$</td>
<td>Starting Simulation Time</td>
<td>0 s</td>
</tr>
<tr>
<td>$t_{\text{final}}$</td>
<td>Final Simulation Time</td>
<td>200 s</td>
</tr>
<tr>
<td>$m_P$</td>
<td>Payload Mass</td>
<td>100 kg</td>
</tr>
<tr>
<td>$C_{d_P}$</td>
<td>Nominal Payload Drag Constant</td>
<td>0.47</td>
</tr>
<tr>
<td>$d_P$</td>
<td>Payload Diameter</td>
<td>0.75 m</td>
</tr>
<tr>
<td>$l_0$</td>
<td>Initial Cable Length</td>
<td>500 m</td>
</tr>
<tr>
<td>$\rho_c$</td>
<td>Cable Mass Density</td>
<td>3000 kg/m³</td>
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<tr>
<td>$d_c$</td>
<td>Cable Diameter</td>
<td>2.5 mm</td>
</tr>
<tr>
<td>$C_{n_c}$</td>
<td>Cable Normal Drag Constant</td>
<td>1.1</td>
</tr>
<tr>
<td>$C_{t_c}$</td>
<td>Cable Tangential Drag Constant</td>
<td>0.022</td>
</tr>
<tr>
<td>$E$</td>
<td>Young’s Modulus</td>
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<tr>
<td>$C_{eQ}$</td>
<td>Structural Damping Constant</td>
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<tr>
<td>$N$</td>
<td>Number of Cable Elements</td>
<td>20</td>
</tr>
<tr>
<td>$\rho_{\text{air}}$</td>
<td>Air Density</td>
<td>1.225 kg/m³</td>
</tr>
<tr>
<td>$g$</td>
<td>Gravity Constant</td>
<td>9.81 m/s²</td>
</tr>
</tbody>
</table>

Table 3-3: Flexible Cable Case Study 3- Simulation and Physical Parameters

The predictions of $p_1$ and $p_2$ for this case are carried out at a towing speed and cable length given by equation (3.4.13):

$$V_{\text{predict}} = \frac{(56.3553^2 + 0^2)}{\sqrt{50^2 + 0^2}} = 63.5183 \text{ m/s}$$  \hspace{1cm} (3.4.13)

$$l_{\text{predict}} = \frac{500^2}{500} = 500 \text{ m}$$
The four estimates of $p_1$ and $p_2$ and the average value used within the model “matching” procedure are:

$$
\begin{align*}
\{ p_1 &= 1.1802 \ 1.1801 \ 1.0210 \ 1.1808 \} \\
p_2 &= 33.6891 \ 33.6101 \ 35.593 \ 33.6738
\end{align*}
\rightarrow \begin{align*}
p_1^{av} &= 1.1405 \\
p_2^{av} &= 34.1415
\end{align*}
\tag{3.4.14}
$$

The results of the model “matching” procedure for this case are depicted in Figure 3-33 through to Figure 3-37. As was encouragingly experienced in the previous case study simulations, it can be fittingly concluded from Figure 3-33 through to Figure 3-37 that it was again possible to modify the payload drag coefficient and cable length of the simple single-link model in order for it to be more representative of the sophisticated multi-link model. Figure 3-33 through to Figure 3-36 again provide an eloquent example of why the simple single-link model cannot be reassuringly utilized to accurately capture the “equivalent” dynamics of the physical ATC system.
Figure 3-33: Flexible Cable Case Study 3 Results- “Equivalent” Cable Angle

Figure 3-34: Flexible Cable Case Study 3 Results- “Equivalent” x-coordinate of the Cable Tip

In a similar manner to the previous case study simulations, the “equivalent” dynamics of the simple and complex models follow similar trends, but the matching procedure is still required to ensure wholesale representativeness occurs. The representativeness of the “refined” single-link model is maintained over most of the simulation time span, except shortly after the simulation begins.
Figure 3-35: Flexible Cable Case Study 3 Results- “Equivalent” y-coordinate of the Cable Tip

Figure 3-36: Flexible Cable Case Study 3 Results- “Equivalent” Cable Tip Trajectory

As evidenced by Figure 3-37, the results obtained for the “refined” single-link model deviate significantly from those given by the complex multi-link model in the initial stages of the simulation. The large and rapidly changing control inputs and subsequent significant dynamical motion of the system are the reason why the model “matching” procedure performs poorly in the initial stages of the simulation. However, the differences between the results given by each model rapidly stabilize and reduce quickly as the simulation progresses and final equilibrium is attained, although as in previous case study simulations, total convergence is not achieved.
Throughout this case study simulation, the maximum error in the position of the cable tip for the “refined” single-link model is approximately 218.74 m or 39.02 %, whilst the absolute mean average error is approximately 26.43 m or 4.88 %. Once again, when compared to the case when no adjustments to the simple model are performed, the maximum and mean errors in the position of the cable tip are 310.86 m (56.69 %) and 265.28 m (49.58 %) respectively. Although the absolute maximum error in the position of the cable tip for each class of model is comparable (39.02 % versus 56.69 %), the absolute mean error in the tip positions is not; 4.88 % for the “refined” model versus 49.58 % for the unadjusted model. Since the absolute mean tip position error is considerably less for the “refined” single-link model compared to the unadulterated single-link model, where the maximum and mean errors are comparably large, the utilization of the proposed model “matching” procedure is worthwhile and justified.

As was found in previous case studies, the differences between the configuration of the “refined” single-link and complex multi-link models do not diverge as the simulation progresses. The configuration of the “refined” model rapidly approaches that of the multi-link model, with a small residual error of 15.16 m (2.84 %) existing in the position of the cable tip at the conclusion of the simulation.

Case 4

This simulation begins with the ATC system in equilibrium and progresses under the action of applied control until a new equilibrium configuration is reached. The control applied to each of the ATC system models for this case is given by equations (3.4.15) through to (3.4.17) and shown in Figure 3-38.
\[
\ddot{x} = u_1 = e^{\frac{-5t}{5t_{\text{final}}}} \sin \left( \frac{t}{25t_{\text{final}}} \right)
\]

(3.4.15)

\[
\ddot{y} = u_2 = e^{\frac{-5t}{5t_{\text{final}}}} \cos \left( \frac{t}{25t_{\text{final}}} \right)
\]

(3.4.16)

\[
\ddot{l} = u_3 = 0
\]

(3.4.17)

The simulation/physical parameters for this case study are given in Table 3-4.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(U_0)</td>
<td>Initial Towing Speed (x-direction)</td>
<td>50 m/s</td>
</tr>
<tr>
<td>(W_0)</td>
<td>Initial Towing Speed (y-direction)</td>
<td>0 m/s</td>
</tr>
<tr>
<td>(t_{\text{initial}})</td>
<td>Starting Simulation Time</td>
<td>0 s</td>
</tr>
<tr>
<td>(t_{\text{final}})</td>
<td>Final Simulation Time</td>
<td>100 s</td>
</tr>
<tr>
<td>(m_p)</td>
<td>Payload Mass</td>
<td>100 kg</td>
</tr>
<tr>
<td>(C_{d_r})</td>
<td>Nominal Payload Drag Constant</td>
<td>0.5</td>
</tr>
<tr>
<td>(d_p)</td>
<td>Payload Diameter</td>
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</tr>
<tr>
<td>(l_0)</td>
<td>Initial Cable Length</td>
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</tr>
<tr>
<td>(\rho_c)</td>
<td>Cable Mass Density</td>
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</tr>
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<td>(d_c)</td>
<td>Cable Diameter</td>
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<td>(C_{n_c})</td>
<td>Cable Normal Drag Constant</td>
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<td>(C_{c_t})</td>
<td>Cable Tangential Drag Constant</td>
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<td>(E)</td>
<td>Young’s Modulus</td>
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<tr>
<td>(C_{eq})</td>
<td>Structural Damping Constant</td>
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<td>(N)</td>
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</tr>
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<td>(\rho_{\text{air}})</td>
<td>Air Density</td>
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<tr>
<td>(g)</td>
<td>Gravity Constant</td>
<td>9.81 m/s²</td>
</tr>
</tbody>
</table>

Table 3-4: Flexible Cable Case Study 4- Simulation and Physical Parameters

The value of the towing speed and cable length for which the predictions \(p_1\) and \(p_2\) are carried out in this case are given by equation (3.4.18):

\[
V_{\text{predict}} = \frac{(50.7609^2 + 3.8199^2)}{\sqrt{50^2 + 0^2}} = 51.8253 \text{ m/s}
\]

(3.4.18)

\[
l_{\text{predict}} = \frac{500^2}{500} = 500 \text{ m}
\]
The four estimates of the parameter adjustments \( p_1 \) and \( p_2 \) and the average value used within the model “matching” procedure are:

\[
\begin{align*}
\{ p_1 \} &= \{ 1.1752, 1.1801, 1.1592, 1.1808 \} \\
\{ p_2 \} &= \{ 34.2762, 33.6101, 33.5798, 33.6738 \}
\end{align*}
\]

\[ p_{1\text{av}} = 1.1738 \quad p_{2\text{av}} = 33.785 \]  \quad (3.4.19)

The results of the model “matching” procedure for this case are depicted in Figure 3-39 through to Figure 3-43.
Figure 3-39 through to Figure 3-43 provide further conclusive proof that not only is it possible to ensure that the simple single-link model is better equipped to capture the significant dynamical motion of the physical system, but that the outcome of such an endeavour can also be successfully accomplished. Figure 3-39 through to Figure 3-42 provide a final example of the need to develop some means of rendering the simple single-link model to be more representative of the sophisticated multi-link models. Such a means is given by equation (3.4.19), which denotes the required changes to the payload drag coefficient and cable length of the simple single-link model to force it to be more representative of the complex flexible multi-link model.

Figure 3-40: Flexible Cable Case Study 4 Results- “Equivalent” $x$-coordinate of the Cable Tip

Figure 3-41: Flexible Cable Case Study 4 Results- “Equivalent” $y$-coordinate of the Cable Tip
The accuracy of the “refined” model is maintained over the duration of the simulation; the relevant results do not deviate appreciably from those given by the complex model. The close nature of the “match” is best evidenced by Figure 3-39 and Figure 3-41, as opposed to Figure 3-40, where relatively speaking, the “match” is not as close. The reason why the model “matching” procedure performs so well in this case is that although the control inputs for the ATC system are large and rapidly changing, the consequent dynamical motion of the ATC system is not. This allows the equilibrium-based “matching” procedure to accurately specify changes to the simple model that render it more representative of the complex model, and in doing so, cause it to be increasingly representative of the real physical system.

Figure 3-42: Flexible Cable Case Study 4 Results- “Equivalent” Cable Tip Trajectory

Figure 3-43: Flexible Cable Case Study 4 Results- Percentage Error of Model “Match” (a) Angle (b) x-coordinate (c) y-coordinate
Throughout this case study simulation, the maximum error in the position of the cable tip for the “refined” single-link model is approximately 34.81 m or 6.8 %, whilst the absolute mean average error is approximately 28.16 m or 5.5 %. When compared to the case when no adjustments to the simple single-link model are performed, the maximum and mean errors in the position of the cable tip are 256.17 m (50.01 %) and 240.5 m (46.95 %) respectively. The configuration of the “refined” model is always sufficiently close to that of the complex multi-link model, although a small residual error of 29.87 m (5.83 %) in the position of the cable tip occurs between each class of model at the finale of the simulation.

3.5 Method 2

The second approach to the model “matching” task for the ATC system is a refinement of the first method. It involves prescribing the changes to the payload drag coefficient and cable length of the simple model directly as analytical functions of the system parameters and equilibrium configuration of the complex model. This enables rapid online computation to be made within the integration loop of the simulation, doing away with the offline parameter study and function interpolation stages associated with the first approach. As a result, this new approach has more of a rigorous and unadulterated mathematical basis compared to the previous approach, although as will become evident from hereon in, this additional approach is not totally devoid of convenient empirical intervention.

Once again, the underlining objective of this newly proposed model “matching” technique is to force the equilibrium characteristics of the simple single-link model to approach the equilibrium characteristics of the complex multi-link model/s. As was the case for the previous method, this additional model “matching” technique relies on the assertion that if a greater agreement between the equilibrium configurations of each class of model can be achieved, then this should lead to a more representative simple single-link model in a dynamic sense. However, in order to embed the capability of capturing dynamic inertial effects within the model “matching” process, a significant innovation has been introduced into this new model “matching” procedure. In the previous model “matching” approach, the changes required by the simple single-link model, $p_1$ and $p_2$, were determined offline, prior to the commencement of the simulation and not updated as the simulation progressed. If the ATC system is subject to large control inputs, in general, the cable may undergo significant dynamic motion, to such an extent that this may lead to significant divergence between the simple and complex system models when model “matching” takes place.
Since the simple single-link model has low fidelity, with the estimates of $p_1$ and $p_2$ used, the model may not be able to converge to the correct equilibrium position corresponding to the complex multi-link model. A closer “match” between the models would be possible if the parameters $p_1$ and $p_2$ were updated to reflect the changes undergone by the system due to the action of the control inputs. As a result, this additional model “matching” procedure has the capability of directly updating the required values for $p_1$ and $p_2$ throughout the simulation, in order to properly account for the changes to the system (cable length and towing speed) that large control inputs induce. Thus, the dilemmas that plague the previous model “matching” process: which estimate of $p_1$ and $p_2$ to use and at which towing speed and cable length to estimate them at, are circumvented by this newly proposed technique. However, the basic modus operandi of the new approach is the same as the previous one: to artificially force the equilibrium orientation angle $\theta_e$ and position of the cable tip $\{x_e, y_e\}$ for the simple single-link model to “match” the “equivalent” equilibrium orientation angle $\theta_{eE}$ and position of the cable tip $\{x_{eE}, y_{eE}\}$ for the complex multi-link model/s. Once again this “match” will be attempted by prescribing changes to the length of the cable $l$ and the payload drag coefficient $C_{dE}$ for the simple single-link model.

As outlined during the previous model “matching” procedure, for the simple single-link model, the equilibrium cable angle $\theta_e$ and the position of the cable tip $\{x_e, y_e\}$ can be found analytically from equation (3.3.1) through to equation (3.3.3). Similarly, the “equivalent” equilibrium configuration of each complex multi-link model is given by equations (3.2.13) and (3.2.14) after the relevant optimization-based root finding procedures are employed to determine their equilibrium configuration [given by equations (3.2.9) through to (3.2.12)]. The following derivation that underpins the mathematical basis of the refined model “matching” technique is acknowledged here as being largely due Dr. Paul Williams, and is a product of collaborative research undertaken by Dr. Paul Williams and the author at RMIT University [116].
The changes to the parameters of the simple single-link model, $p_1$ and $p_2$, needed to render the simple model more representative of the complex multi-link model/s can be found analytically. Let the chosen parameters, $p_1$ and $p_2$, be defined as:

$$C_{d_{new}} = C_{d_1} (1 + p_1)$$

$$l_{new} = l (1 + p_2)$$

(3.5.1)

(3.5.2)

For a successful “match” to take place, the left hand side of equation (3.5.2) must be equal to the “equivalent” length $l_E$ of the chosen complex multi-link model. This will allow the explicit determination of the parameter $p_2$ from equation (3.5.2) as follows:

$$p_2 = \left( \frac{l_E}{l} \right) - 1$$

(3.5.3)

The determination of the parameter $p_1$ can be found from equation (3.2.6), assuming that as the ATC system approaches equilibrium, the angular velocity of the cable approaches zero ($\dot{\theta} \to 0$) and the angular acceleration of the cable is approximately zero ($\ddot{\theta} \approx 0$). In general, the parameter $p_1$ can be isolated from equation (3.2.6) in an “exact” mathematical sense, provided that the time-dependent derivative terms such as $\dot{\theta}_E$, $\ddot{\theta}_E$ and $l_E$ are known. Recalling that for each of the complex multi-link models:

$$\tan \theta_E = \frac{x_E}{y_E}$$

(3.5.4)

The “equivalent” angular velocity of the complex multi-link model $\dot{\theta}_E$ can be found by differentiating equation (3.5.4) with respect to time:

$$\dot{\theta}_E = \frac{\left( \frac{x_E}{y_E} - \frac{x_E y_E}{y_E^2} \right)}{1 + \tan^2 \theta_E}$$

(3.5.5)

Similarly, the “equivalent” angular acceleration $\ddot{\theta}_E$ is obtained via further time differentiation of the “equivalent” angular velocity as follows:

$$\ddot{\theta}_E = \frac{\left( \frac{x_E}{y_E} - 2 \frac{x_E y_E}{y_E^2} + 2 \frac{x_E y_E^2}{y_E^3} - \frac{x_E y_E}{y_E^2} \right)}{1 + \tan^2 \theta_E} - 2 \dot{\theta}_E^2 \tan \theta_E$$

(3.5.6)
The “equivalent” radial velocity $\dot{i}_E$ is given by differentiating equation (3.2.14):

$$
\dot{i}_E = \frac{\dot{x}_E x_E + \dot{y}_E y_E}{l_E}
$$

(3.5.7)

For the simple single-link model, the “equivalent” radial velocity $\dot{i}_E$ is equivalent to the deployment rate of the cable $\dot{\bar{d}}$. For the complex multi-link models, these two dynamic variables are not the same due to the action of the aerodynamic forces providing curvature to the cable. Hence for mathematical completeness, some means is required to map how the “equivalent” radial velocity of the cable relates to the cable deployment rate for each of the complex multi-link models. Such an exercise is not attempted in this work. It is anticipated that the difference between the two quantities is likely to be small, but nevertheless they are unique and a methodology to address this void could prove to be an important and worthy future endeavor.

Now the parameter $p_1$ can be found by inverting equation (3.2.6) as follows:

$$
p_1 = \frac{2\left(m_P + \frac{\pi}{8} \rho l_E d_c^2 \left(\frac{\dot{x}}{l_E} \cos \theta_E - \frac{\dot{y}}{l_E} \sin \theta_E - \frac{\dot{i}_E}{l_E} \dot{\theta}_E - \frac{\ddot{E}}{l_E} \sin \theta_E - \ddot{\theta}_E\right)\right)}{ho_a C_d \frac{\pi}{4} d_p^2 \left(l_E \dot{\theta}_E + \dot{y} \sin \theta_E - \dot{x} \cos \theta_E\right) \sqrt{\left(\dot{x} - \dot{i}_E \sin \theta_E - \dot{l}_E \dot{\theta}_E \cos \theta_E\right)^2 + \left(\dot{y} - \dot{i}_E \cos \theta_E + \dot{l}_E \dot{\theta}_E \sin \theta_E\right)^2}}
$$

(3.5.8)

The parameter $p_1$ can be fully determined after making the appropriate substitutions using equation (3.2.14) and equation (3.5.4) through to (3.5.7). Using a variety of physical parameters and considering a wide range of operating conditions, after performing a series of simulations and constructing interpolating functions for equations (3.5.4) through to (3.5.7), it is possible to obtain a more accurate estimate of the parameter $p_1$ from the actual instantaneous values of $\theta_E$, $\dot{\theta}_E$, $\ddot{\theta}_E$, $l_E$ and $\dot{i}_E$. Alternatively, a series of artificial neural networks could be trained to approximate the multi-dimensional relationship existing between the physical parameters of the ATC system and the “equivalent” kinematics $(\theta_E, \dot{\theta}_E, \ddot{\theta}_E, l_E, \dot{i}_E)$ of the complex multi-link cable models. Once again, such a venture would be worthy of further research and development within the near future. In the absence of such work however, it is possible to introduce an important simplification into the methodology, which by no means detracts considerably from the mathematical completeness of the proposed
model “matching” procedure. It is possible to obtain a simpler expression than equation (3.5.8) for the parameter \( p_i \) by solving for \( p_i \) in equation (3.2.6) as a series expansion of the angular velocity of the cable \( \dot{\theta} \). The result can be expressed as:

\[
p_i = \frac{2\left(m_p + \frac{\pi}{8} \rho \, l \, d_e^2 \right) \left(\frac{x}{l} \cos \theta - \frac{\dot{y}}{l} \sin \theta - \frac{g}{l} \sin \theta \right)}{\rho_{air} \, C_d \, \frac{\pi}{4} \, d_e^2 \left(\dot{y} \sin \theta - \dot{x} \cos \theta \right) \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{\theta}^2 - 2 \dot{x} \dot{y} \sin \theta - 2 \dot{\theta} \dot{\theta} \sin \theta}} - 1
\]

(3.5.9)

where the term \( f(\dot{\theta}, \dot{\theta}^2, \ldots) \) denotes the infinite series approximation to any left over terms containing cable angular velocity, once the inversion of equation (3.2.6) has taken place. When the ATC system is in equilibrium, the towing speed is constant and the acceleration of the aircraft, as well as the cable deployment speed are nominally zero. Similarly, for the purposes of this study, second order or higher terms within the infinite series \( f(\dot{\theta}, \dot{\theta}^2, \ldots) \) can be ignored. With these simplifications invoked, the parameter \( p_i \) can be updated “online” during simulation process using the following expression:

\[
p_i = f(l_E) \dot{\theta} + \left[ \frac{g \left(8m_p + \pi \rho l_E d_e^2 \right)}{\rho_{air} U_0^2 C_d l_p^2 d_e^2} \right] \tan \theta_E
\]

(3.5.10)

where the function \( f(l_E) \) is an empirical approximation given by:

\[
f(l_E) \approx \frac{l_E}{K_1 + \frac{K_2}{l_E}}
\]

(3.5.11)

The values of \( K_1 \) and \( K_2 \) are dependent on the physical parameters of the ATC system and are selected to ensure the “match” between the complex models and the simple model is sufficiently close. It is possible to use an optimization procedure to determine the values of \( K_1 \) and \( K_2 \) to ensure the closest possible match between the models for the governing physical parameters of the ATC system. However an iterative, “reverse-engineering” approach (trial and error) was found to be sufficient and was easily accomplished. The inclusion of the \( \dot{\theta} \) term in equation (3.5.10) should ensure a closer “match” between each model, since this enables the dynamic motion of the ATC system to be accounted for during the “matching” procedure.
The implementation of this additional model “matching” procedure begins with the determination of the “equivalent” equilibrium configuration of the chosen complex multi-link model using the previously outlined optimization procedure and equations (3.2.13) and (3.2.14). Next the calculation of the parameters \( p_1 \) and \( p_2 \) can proceed using equation (3.5.3) and equations (3.5.10) and (3.5.11), which is carried out online during the integration loop of the simulation. Then the parameters \( p_1 \) and \( p_2 \) are used to augment the payload drag coefficient and cable length of the simple model during integration with appropriate values of \( K_1 \) and \( K_2 \), thereby rendering the simple single-link model more representative of its complex multi-link counterpart.

### 3.6 Application of Method 2 for Model “Matching”

This sub-section deals with the wide-ranging application of the newly proposed model “matching” technique to the various complex multi-link ATC system models. Similar the demonstration of the previous model matching technique, the performance and robustness of this model “matching” procedure will be established through a series of dynamic simulations. This will address the issues relating to the two underlying questions that distinguish the revised model “matching” technique from its preceding cousin: whether a rigorous and unadulterated mathematical basis, along with the online parameter updates, improves the performance and ultimate accuracy of the revised model “matching” procedure.

#### 3.6.1 Model “Matching” for the Rigid Multi-Link Aerial Towed-Cable System

The following simulation case studies will demonstrate how the representativeness of the simple single-link model compares to the rigid multi-link system model when using the newly proposed model “matching” procedure. With respect to the figures provided to display the forthcoming results, the results concerning the simple single-link model are given by the red line, the blue line denotes the results for the complex rigid multi-link model, whilst the green line signifies the results for the “refined” single-link model.

**Case 5**

This simulation begins with the cable hanging vertically below the aircraft and there is no control applied to each of the models during this case study simulation. The simulation and physical parameters for this case study are given in Table 3-5.
The values of $K_1$ and $K_2$ used within the model “matching” procedure are:

$$\{K_1, K_2\} = \{20, 0.1\} \quad (3.6.1)$$

Since there is no reel acceleration applied to the system in this simulation, the value of the parameter $p_2$ is constant and has a value given by equation (3.6.2):

$$p_2 = \left(\frac{l_E}{l_0}\right) - 1 = \left(\frac{987.92}{1000}\right) - 1 = -0.0121 \quad (3.6.2)$$

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<td>$t_{final}$</td>
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<td>$m_P$</td>
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**Table 3-5: Rigid Cable Case Study 5- Simulation and Physical Parameters**

The results of the model “matching” procedure for this case are depicted in Figure 3-44 through to Figure 3-49. Figure 3-44 through to Figure 3-46 detail a comparison of how the “equivalent” cable kinematics vary between each class of model; Figure 3-47 shows the trajectory the cable tip follows for each model; Figure 3-48 outlines the percentage error in the “equivalent” cable kinematics for the “refined” single-link model as compared to the flexible multi-link model, whilst Figure 3-49 specifies the changes to the payload drag coefficient required by the simple model to improve its accuracy. It is evident from Figure 3-44 through to Figure 3-48 that with the appropriate modifications to the simple model’s payload drag coefficient and cable length, the significant dynamics of the simple model can be made to closely track those of the more sophisticated model.
Similarly, it can be seen from Figure 3-44 through to Figure 3-48 that the simple ATC system model is totally unrepresentative of the complex multi-link model, even though the trends in the “equivalent” dynamics between the simple and complex models are essentially identical. To warrant representativeness, adjustments to the simple model are required, which as seen in Figure 3-49, these changes are sizable and pronounced, yet tend to reduce in magnitude as the dynamics of the system become less pronounced when the system approaches equilibrium.
Encouragingly, it can be seen from Figure 3-44 through to Figure 3-48 that the representativeness of the “refined” single-link model is maintained over the entire simulation time span, even though the ATC system undergoes considerable dynamical motion during the simulation, particularly initially when the static vertical cable encounters the impulsive effects due to the motion of the aircraft. Throughout this case study simulation, the maximum error in the position of the cable tip for the “refined” single-link model is approximately 24.14 m or 2.44 %, whilst the absolute mean average error is approximately 11.58 m or 1.17 %. When compared to the case when no adjustments to the simple single-link model are performed, the maximum and mean errors in the position of the cable tip are a sizable 328.81 m (33.28 %) and 254.11 m (25.71 %) respectively, detailing astutely the unrepresentativeness of the simple
model and the requirement for a model “matching” procedure. The configuration of the “refined” model is always sufficiently close to that of the complex multi-link model, with the differences between the configuration of each class of model not diverging as the simulation progresses. Instead the configuration of the “refined” model rapidly approaches that of the complex model, although total convergence is not achieved and a small residual tip position error of 19.79 m (2 %) occurs for the “refined” model at the conclusion of the simulation.

Figure 3-48: Rigid Cable Case Study 5 Results- Percentage Error of Model “Match” (a) Angle (b) x-coordinate (c) y-coordinate

Figure 3-49: Rigid Cable Case Study 5 Results- $p_1$ Adjustment for Simple Model
Case 6

This simulation begins with the ATC system in equilibrium and progresses under the action of applied control until a new equilibrium configuration is reached. The control applied to each of the ATC system models for this case is given by equations (3.6.3) through to (3.6.5) and shown in Figure 3-50:

\[
\ddot{x} = u_1 = \frac{17}{2000} \cos \left( \frac{2\pi}{t_{\text{final}}} t \right)
\]  \hspace{1cm} (3.6.3)

\[
\ddot{y} = u_2 = \frac{17}{2000} \cos \left( \frac{2\pi}{t_{\text{final}}} t \right)
\]  \hspace{1cm} (3.6.4)

\[
\ddot{I} = u_3 = \frac{9}{100} \cos \left( \frac{2\pi}{t_{\text{final}}} t \right)
\]  \hspace{1cm} (3.6.5)

The simulation/physical parameters for this case study are given in Table 3-6, whilst the values of \( K_1 \) and \( K_2 \) used within the model “matching” procedure are:

\[ \{K_1, K_2\} = \{1000, 100\} \]  \hspace{1cm} (3.6.6)

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<td>Payload Mass</td>
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Table 3-6: Rigid Cable Case Study 6- Simulation and Physical Parameters
The results of the model “matching” procedure for this case are depicted in Figure 3-51 through to Figure 3-56. Figure 3-51 through to Figure 3-53 compare how the “equivalent” cable kinematics vary between each class of model; Figure 3-54 illustrate the trajectory of the cable tip for each model; Figure 3-55 provides the percentage error in the “equivalent” cable kinematics for the “refined” single-link model, whilst Figure 3-56 denotes the changes to the payload drag coefficient and cable length required by the simple model in order to improve its representativeness.

**Figure 3-50: Rigid Cable Case Study 6- Applied Control**

**Figure 3-51: Rigid Cable Case Study 6 Results- “Equivalent” Cable Angle**
Preserving the consistency of unassailable and satisfying performance, it can be seen from Figure 3-51 through to Figure 3-55 that the important dynamics of the simple single-link model closely track those of the more sophisticated multi-link model, even whilst the system is subject to significant control inputs and undergoes considerable dynamic motion. The required modifications to the simple model’s payload drag coefficient and cable length indicated by Figure 3-56 are by no means substantial, yet their application is absolutely necessary for representativeness to be achieved, since the simple single-link model is totally unrepresentative of the ATC system. Such an inference is sufficiently corroborated by the evidence presented by Figure 3-51 through to Figure 3-55. Interestingly, the “equivalent” configuration angle of the cable is not notably altered as the simulation progresses as shown by Figure 3-51. However, the simple single-link model erroneously predicts that such a variation should occur, even suggesting that the cable configuration angle should be negative in the early stages of the simulation, a scenario physically unrealisable given the nature of the ATC system. This further illustrates the perils of making inferences about the ATC system when using a simplified single-link model without sufficient validation or any accuracy-enhancing measures, such as the model “matching” procedures proposed in this chapter.
As depicted in Figure 3-51 through to Figure 3-55, the representativeness of the “refined” single-link model is maintained over the entire simulation time span, although the model slowly diverges from the complex multi-link model in the initial half of the simulation, the degree of which is relatively small (approximately 10%). The rate at which the refined model diverges away from the complex model terminates halfway into the simulation and is recovered in the latter half of the simulation as the control inputs cease and the ATC system approaches equilibrium.
The maximum error in the position of the cable tip for the “refined” single-link model during this case study simulation is approximately 41.03 m or 10.53 %, whilst the absolute mean average error is approximately 20.53 m or 5.66 %. When compared to the unrefined single-link model, the maximum and mean errors in the position of the cable tip are unacceptable 261.85 m (83.24 %) and 177.87 m (51.74 %) respectively. The configuration of the “refined” model is always sufficiently close to that of the multi-link model, although a very small residual cable tip position error of 1.71 m (0.59 %) occurs between each class of model at the end of the simulation.
3.6.2 Model “Matching” for the Flexible Multi-Link Aerial Towed-Cable System

The following simulation case studies will demonstrate how the representativeness of the simple single-link system model compares to the flexible multi-link system model when using the newly proposed model “matching” procedure. With respect to the figures provided to display the forthcoming results, the results concerning the simple single-link model are given by the red line, the blue line denotes the results for the complex flexible multi-link model, whilst the green line signifies the results for the “refined” single-link model.

Case 7

This simulation begins with the cable hanging vertically below the aircraft and progresses under the action of no applied control. The simulation and physical parameters for this case study are given in Table 3-7.

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<td>100 s</td>
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<td>Initial Cable Length</td>
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<td>Cable Mass Density</td>
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<tr>
<td>$d_c$</td>
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</tr>
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<td>Cable Normal Drag Constant</td>
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</tr>
<tr>
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<td>Young’s Modulus</td>
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</tr>
<tr>
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<td>Structural Damping Constant</td>
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</tr>
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<td>Number of Cable Elements</td>
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</tr>
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</tr>
<tr>
<td>$g$</td>
<td>Gravity Constant</td>
<td>9.81 m/s$^2$</td>
</tr>
</tbody>
</table>

Table 3-7: Flexible Cable Case Study 7- Simulation and Physical Parameters

The values of $K_1$ and $K_2$ used within the model “matching” procedure are:

$$\{K_1, K_2\} = \{1 \times 10^{-3}, 1 \times 10^4\}$$  \hspace{1cm} (3.6.7)
Since there is no reel acceleration applied to the system in this simulation, the value of the parameter $p_2$ is constant and has a value given by equation (3.6.8):

$$p_2 = \left( \frac{I_E}{I_0} \right) - 1 = \left( \frac{444.47}{450} \right) - 1 = -0.0123$$

(3.6.8)

The results of the model “matching” procedure for this case are depicted in Figure 3-57 through to Figure 3-62.

Figure 3-57: Flexible Cable Case Study 7 Results- “Equivalent” Cable Angle

Figure 3-58: Flexible Cable Case Study 7 Results- “Equivalent” $x$-coordinate of the Cable Tip
Figure 3-57 through to Figure 3-61 demonstrate the successful application of the newly proposed model “matching” technique to the case study simulation outlined above, although as evidenced by Figure 3-57 through to Figure 3-59, the results obtained for the “refined” single-link model deviate significantly from those given by the complex multi-link model in the initial stages of the simulation. This is explained by the large and rapidly changing control inputs and subsequent significant dynamical motion of the ATC system. However, the differences in the results given by each model rapidly stabilize and reduce quickly as the simulation progresses and final equilibrium is attained.

![Figure 3-59: Flexible Cable Case Study 7 Results- “Equivalent” y-coordinate of the Cable Tip](image1)

![Figure 3-60: Flexible Cable Case Study 7 Results- “Equivalent” Cable Tip Trajectory](image2)
Figure 3-62 indicates that significant changes are required for the payload drag coefficient of the simple model to ensure that the “refined” model is representative of the complex model and of the physical system itself. The consequences of not applying the suggested changes to the simple model are aptly demonstrated in Figure 3-57 through to Figure 3-61, which again illustrate the poor performance of the unadulterated simple single-link model.

![Graphs showing percentage errors in model match for angle, x-coordinate, and y-coordinate](image)

**Figure 3-61: Flexible Cable Case Study 7 Results- Percentage Error of Model “Match” (a) Angle (b) x-coordinate (c) y-coordinate**

![Graph showing adjustment of model parameter P1](image)

**Figure 3-62: Flexible Cable Case Study 7 Results- P1 Adjustment for Simple Model**

The maximum error in the position of the cable tip for the “refined” single-link model during this case study simulation is approximately 59.2 m or 4.15 %, whilst the absolute mean average error is approximately 26.69 m or 1.87 %. When compared to the case when no adjustments to the simple single-link model are made, the maximum and mean errors in the
position of the cable tip are a sizable 1079.7 m (75.63 %) and 862.04 m (60.28 %) respectively, supporting the motivation underlying the application of the proposed model “matching” procedures. The configuration of the “refined” model is always sufficiently close to that of the complex multi-link model, except in the very initial stages of the simulation. As was the case with the previous case study simulations, total convergence is not quite achieved as a small residual error of 3.52 m (0.25 %) in the cable tip position occurs between each class of model at the culmination of the simulation.

Case 8

This simulation begins with the ATC system in equilibrium and progresses under the action of applied control until a new equilibrium configuration is reached. The control applied to each of the ATC system models for this case is given by equations (3.6.9) and (3.6.10) and is shown in Figure 3-63:

$$\ddot{x} = u_1 = \begin{cases} 
\frac{3}{4}, & t_{\text{initial}} \leq t \leq \frac{3}{16} t_{\text{final}} \\
\frac{1}{4}, & \frac{3}{16} t_{\text{final}} < t \leq \frac{9}{16} t_{\text{final}} \\
0, & \frac{9}{16} t_{\text{final}} < t \leq t_{\text{final}}
\end{cases}$$

$$\ddot{y} = u_2 = \ddot{\ddot}{x} = u_3 = 0 \quad (3.6.9)$$

The simulation/physical parameters for this case study are given in Table 3-8. The values of $K_1$ and $K_2$ used within the model “matching” procedure are:

$$\{K_1, K_2\} = \{29, 3 \times 10^4\} \quad (3.6.11)$$

Since there is no reel acceleration applied to the system in this simulation, the value of the parameter $p_2$ is constant and has a value given by equation (3.6.12):

$$p_2 = \left( \frac{l_E}{l_0} \right)^{-1} = \left( \frac{1432.4}{1500} \right)^{-1} = -0.0451 \quad (3.6.12)$$

The results of the model “matching” procedure for this case are depicted in Figure 3-64 through to Figure 3-69.
Table 3-8: Flexible Cable Case Study 8- Simulation and Physical Parameters

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_0$</td>
<td>Initial Towing Speed (x-direction)</td>
<td>50 m/s</td>
</tr>
<tr>
<td>$W_0$</td>
<td>Initial Towing Speed (y-direction)</td>
<td>0 m/s</td>
</tr>
<tr>
<td>$t_{initial}$</td>
<td>Starting Simulation Time</td>
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</tr>
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<td>$t_{final}$</td>
<td>Final Simulation Time</td>
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</tr>
<tr>
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<td>Payload Mass</td>
<td>200 kg</td>
</tr>
<tr>
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<td>Nominal Payload Drag Constant</td>
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<tr>
<td>$d_P$</td>
<td>Payload Diameter</td>
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<td>Cable Mass Density</td>
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<td>Cable Diameter</td>
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<td>Cable Normal Drag Constant</td>
<td>1.1</td>
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<tr>
<td>$C_{t_c}$</td>
<td>Cable Tangential Drag Constant</td>
<td>0.022</td>
</tr>
<tr>
<td>$E$</td>
<td>Young’s Modulus</td>
<td>120 GPa</td>
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<td>$C_{eq}$</td>
<td>Structural Damping Constant</td>
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<td>$N$</td>
<td>Number of Cable Elements</td>
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<td>$\rho_{air}$</td>
<td>Air Density</td>
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<tr>
<td>$g$</td>
<td>Gravity Constant</td>
<td>9.81 m/s²</td>
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</tbody>
</table>

As in all previous case study simulations presented in this chapter, it can be concluded from Figure 3-64 through to Figure 3-67 that the important dynamics of the simple single-link model closely track those of the more sophisticated complex model, even when the ATC system is subjected to significant control inputs and undergoes considerable dynamic motion.
The required modifications to the simple model's payload drag coefficient (indicated by Figure 3-69) for a successful “match” to take place are substantial and necessary for representativeness, as the simple single-link model is largely unrepresentative of the physical system. Contrary to the scenario experienced during most previous case study simulations, it can be seen from Figure 3-64 through to Figure 3-66 that the trends in the “equivalent” cable kinematics are not the same for each class of model.
The trends in the “equivalent” cable kinematics are the same for the complex and “refined” models, whilst the results for the simple model are unique and totally unrepresentative of the complex model. This further reinforces the degree of scepticism one should have for results obtained using the unrefined simple model. As depicted in Figure 3-64 through to Figure 3-67, the representativeness of the “refined” model is maintained over the entire simulation, although it slightly diverges from the complex model in the middle stages of the simulation. The rate at which the refined model diverges is small (approximately 4 %) and ceases quickly after the mid-stages of the simulation, before converging in the latter half of the simulation as the control inputs end and the ATC system approaches equilibrium.

Figure 3-66: Flexible Cable Case Study 8 Results- “Equivalent” $y$-coordinate of the Cable Tip

Figure 3-67: Flexible Cable Case Study 8 Results- “Equivalent” Cable Tip Trajectory
The maximum error in the position of the cable tip for the “refined” model during this case study simulation is approximately 192.30 m or 42.66%, whilst the absolute mean average error is approximately 29.66 m or 6.62%. Compare this to the unrefined model, where the maximum and mean errors in the position of the cable tip are large: 266.88 m (59.32%) and 154.98 m (34.8%) respectively. A small residual error of 3.52 m (0.79%) in the cable tip position occurs at the end of the simulation. These statistics and the information encapsulated in Figure 3-68 illustrate the power of the model “matching” technique, not to mention the outstanding ability to which it can expertly and accurately render the “equivalent” dynamics of the simple system model representative of the more sophisticated multi-link model.
3.7 Direct Comparison of Each Model “Matching” Procedure

This sub-section deals with a succinct discussion and comparison of the performance of each of the proposed model “matching” methodologies when deployed within each of the case study simulations previously presented in Section 3.4 and Section 3.6. The primary purpose of this sub-section is to compare and contrast the performance of each model “matching” procedure when employed on a variety of ATC systems across a wide variety of operating scenarios, in order to ascertain which methodology should be used and when, given the physical parameters and the associated control inputs.

Table 3-9 is a summary of the percentage errors in the “equivalent” cable kinematics for the “refined” model during each model “matching” procedure, when applied to the case study simulations presented elsewhere in this chapter. The various parameters that govern the application of each of the model “matching” techniques are given in Table 3-10. The entries of the first column in Table 3-9 and Table 3-10 are interpreted as follows- the first number represents the case study simulation number, whilst the second number denotes which of the two model “matching” procedures is used. Hence, the term in the fifth row of the first column in Table 3-9 (5.1) signifies that the first model “matching” procedure was applied to the ATC system governed by the physical parameters and applied control outlined in the Case 5 sub-section previously presented.

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<th>Case Study</th>
<th>Method</th>
<th>$\Delta_\theta [%]$</th>
<th>Mean</th>
<th>Max</th>
<th>$\Delta_\chi [%]$</th>
<th>Mean</th>
<th>Max</th>
<th>$\Delta_\gamma [%]$</th>
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Table 3-9: Direct Comparison Between Model "Matching" Procedures
Table 3-10: Model “Matching” Parameters Used During Each Case Study Simulation

<table>
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<th>Parameters</th>
</tr>
</thead>
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The most important deduction that can be made from Table 3-9 is that, as a general rule, the second model “matching” procedure is the most accurate and better performing model “matching” technique of the two proposed methodologies. Strictly speaking, the above statement is not totally true in all circumstances, as can be seen when the results for the first, second and fourth case study simulations are compared, yet in the majority of cases, the second model “matching” procedure clearly out-performs the first proposed model “matching” technique. As a result, it can be concluded that in the majority of case study simulations, it appears that the rigorous and more complete mathematical basis associated with the revised model “matching” procedure, along with the online parameter updates, is justified since it improves the performance and ultimate accuracy of the model “matching” method.

It should not be forgotten that the results contained within the upper portion of Table 3-9 (rows one to eight) validate the central premise that underlies the first of the two model “matching” techniques, since it is not unreasonable to conclude that matching the equilibrium configuration of the simple and complex models, results in the simple single-link model being more representative of the sophisticated multi-link model in the dynamic environment. In actual fact, as the first model “matching” technique out-performs the revised version in almost half of the scenarios considered in this chapter, the appeal and eminence of the initial model “matching” procedure should not be overlooked. It is and should remain an important
additional approach that should be deployed appropriately when the physical parameters and applied control inputs for the ATC system are such that the technique ensures a suitably representative simulation is produced.

Focussing now on the results concerning the first model “matching” technique given in Table 3-10. For the array of case study simulations considered within this chapter, the adjustments to the payload drag coefficient required by the simple single-link model all lie within a narrowly defined interval [1.1405, 1.7027], whilst the adjustments to the cable length are all within one degree of magnitude of each other: [-15.6011, -20.7003] for case study simulations involving the rigid multi-link model, [33.785, 61.249] for case study simulations involving the flexible multi-link model. With respect to the second model “matching” procedure, the results displayed in the lower portion of Table 3-10 clearly show that no noticeable or defined trend exists for the parameters $K_1$ and $K_2$ that govern the revised methodology. The relevant values fluctuate significantly for any given combination of physical parameters and applied control inputs, although this is somewhat expected as the choice of $K_1$ and $K_2$ is due to reverse-engineering and is not supported by exact mathematical reasoning.

### 3.8 Concluding Remarks

The rewards and benefits of the proposed model “matching” procedures are inherently valuable and highly significant, yet most alluringly, these advantages are possible without considerable complexity and computational resources required by either model “matching” procedure. A model “matching” procedure such as those developed and demonstrated within this chapter are unquestionably necessary if accurate and representative simulations of the ATC system are to be performed using simple single-link models. Similarly, one cannot attempt to design and develop innovative non-linear optimal controllers for the ATC system if the accuracy and representativeness of the relevant mathematical models are not validated or verified. As a result, either model “matching” technique, exactly which one is dependent on the system parameters and physical scenario considered, will be deployed from hereon in during the synthesis and demonstration of non-linear optimal controllers for the ATC system. The details and outcomes of which will be explored in more detail in the following chapters.
4 SINGLE PHASE OPTIMAL CONTROL OF AERIAL TOWED-CABLE SYSTEMS

4.1 Preface

This section is devoted to the development of single phase, non-linear optimal control strategies for the ATC system. The primary objective of this work is to investigate the possibility of achieving accurate, instantaneous rendezvous of the cable tip/towed body with desired surface locations on the ground, both in two and three-dimensions, using deployment and retrieval control of the cable and/or aircraft manoeuvring. The desired framework employed to formulate these rendezvous problems is single phase, non-linear optimal control.

4.2 Introduction

In many applications involving the use of ATC systems, a proposed control system may be called upon to guide the system from a recognized initial state to a particular desired final state, thereby achieving a desired set of objectives. The manner in which to best design and implement the particulars of such a control system may not be readily apparent, and in some cases, a specific control law may not be available to achieve the required objectives. Usually the task of achieving the control system objectives depends strongly on entities that serve to constrain the capabilities of the control system. Additionally, a candidate control system may have other restrictions placed on it in the form conflicting performance criteria, whereby some means of achieving a compromise is needed.

As will be extensively demonstrated later in this section, non-linear optimal control is the most suitable form of control one can utilize to develop multi-objective controllers for the ATC system, which are capable of satisfying multiple system and performance constraints. Optimal control theory provides an ideal framework for appropriately ensuring that the ATC system moves through a path that allows the desired objectives to be met, whilst at all times accounting for the inevitable constraints and performance limitations the system is subjected to. The particular appeal of optimal control is that, subject to the constraints and limitations of the system, if there exists a path through which the system can travel through in order to meet the desired goals, then optimal control is capable of determining this path and the control needed to do so.
Several additional advantages associated with formulating rendezvous problems for ATC systems using the optimal control framework will become readily apparent in subsequent subsections. The manner in which the control and resulting system trajectories are found using optimal control methods is not as intuitive as those used by alternative control methodologies, although this turns out to be a small price to pay when one considers what is possible through optimal control. Traditionally, designing control systems using optimal control theory was arduous and required large amounts of computational resources, especially for complex dynamical systems and those having large degrees of freedom. In certain instances, obtaining solutions was not even possible for sufficiently complex systems and one had to resort to adopting alternative control strategies. The onset of the digital age and recent advances in numerical trajectory optimization methods has provided both the impetus and the necessary tools to allow optimal control methods to be developed for more sophisticated dynamical systems, including ATC systems.

4.3 Optimal Control Theory

Consider the following general non-linear dynamical system, which is assumed to be fully controllable and observable:

\[
\dot{x}(t) = f[x(t), u(t), t]
\]  

(4.3.1)

where:

- \( x(t) \) is the vector of states for the system,
- \( u(t) \) is the control vector for the system,
- \( t \) is the time.

The objective of the optimal control method is to determine the control \( u(t) \), subject to any constraints, that control’s the states \( x(t) \), which may also have restrictions, in such a manner as to achieve a desired final state, whilst simultaneously minimizing/maximizing certain performance attributes of the system. In general, a performance criterion for the system, often referred to as the cost function, performance index or optimality criterion, can be mathematically constructed to satisfy one of any number of system and/or control performance characteristics.
Assuming that the system begins at time $t_0$ with an initial state $x(t_0)$ and progresses to a final state $x(t_f)$ at the final time $t_f$, the performance criterion for the system $J$ can be mathematically posed as follows:

$$J = M[x(t_f), t_f] + \int_{t_0}^{t_f} L[x(t), u(t), t]$$  \hspace{1cm} (4.3.2)$$

where:

- $M[x(t_f), t_f]$ is the terminal cost that provides an appropriate weighting on the final states of the system,
- $\int_{t_0}^{t_f} L[x(t), u(t), t]$ is a chosen performance/weighting index that is either minimized or maximized along the resulting optimal system trajectory.

If a desired final state for the system is to be attained, then these boundary conditions may be imposed as a set of equality constraints:

$$\psi[x(t_f), t_f] = 0$$  \hspace{1cm} (4.3.3)$$

The purpose of the optimal control law is to determine the control $u(t)$, that minimizes the performance criterion $J$, while satisfying the dynamical state equations for the system given by equation (4.3.1). Here the system dynamical state equations are considered as constraints that the system trajectory must satisfy during the process of determining the optimal control law. If a particular control law for the system produces the minimum value of the performance criterion, then that control is said to be optimal and denoted by $\hat{u}(t)$. The determination of the optimal control $\hat{u}(t)$ using the optimal control framework results in the construction of a variational problem, the solution to which requires the application of the calculus of variations. A number of necessary conditions must be satisfied in order for the optimal control condition to be met, the derivation of which is protracted and its inclusion here would adversely affect the objectives, scope and findings of this thesis. A detailed derivation and discussion of the application of the calculus of variations to the formulation of optimal control problems can be found in Kirk’s introductory treatise on the subject [136].
4.3.1 Conditions Required to Achieve Optimality

For a candidate system trajectory and control to be designated as optimal, the path must satisfy the necessary condition that the first variation in the cost be zero. The first variation of the cost $\delta J$ refers to the first order expansion of the variations in each of the individual variables making up the candidate optimal path. To ensure that the optimal path is an extremum, the first order variation is found from any neighbouring path (about the candidate optimal path) that satisfies the same constraints. Furthermore, to satisfy the condition requiring the candidate optimal path to be a local minimum extremum, the second order variation of the cost must be greater than zero.

Since the dynamical state space equations are considered as dynamical constraints that the system trajectory must satisfy, these state space equations can be re-cast in terms of an equality constraint as follows:

$$f[x(t), u(t), t] - \dot{x}(t) = 0 \quad (4.3.4)$$

Using a vector of Lagrange multipliers $\lambda(t)$ to appropriately account for the newly formed state space equality constraints, the original performance criterion given by equation (4.3.2) can be augmented to incorporate the dynamical constraints in the following manner:

$$J = M[x(t_f), t_f] + \int_{t_0}^{t_f} \left\{ L[x(t), u(t), t] + \lambda^T(t)[f[x(t), u(t), t] - \dot{x}(t)] \right\} \quad (4.3.5)$$

The Lagrange multipliers $\lambda(t)$ are referred to as the co-state variables for the continuous differential equation constraints. It will now be convenient to define the Hamiltonian for the system $\mathcal{H}$ as:

$$\mathcal{H} = L[x(t), u(t), t] + \lambda^T(t)f[x(t), u(t), t] \quad (4.3.6)$$

The condition requiring the first variation in the performance criterion to be zero leads to the Euler-Lagrange equations, the so-called necessary conditions for optimality given by equations (4.3.7) through to (4.3.9). The first of these conditions provides the state equations for the system:

$$\frac{\partial \mathcal{H}}{\partial \lambda} = \dot{x}(t) \quad (4.3.7)$$
The second condition defines the co-state equations, known as the adjoint equations:

\[
\frac{\partial \mathcal{H}}{\partial \mathbf{x}} = \frac{\partial f^T}{\partial \mathbf{x}} \lambda + \frac{\partial \mathcal{L}}{\partial \mathbf{x}} = -\dot{\lambda}(t)
\]  

(4.3.8)

The third condition ensures that the control causes the performance criterion to be an extremum and is known as the control equations:

\[
\frac{\partial \mathcal{H}}{\partial \mathbf{u}} = \frac{\partial f^T}{\partial \mathbf{u}} \lambda + \frac{\partial \mathcal{L}}{\partial \mathbf{u}} = 0
\]

(4.3.9)

The boundary conditions for the optimal control problem are determined from the following expression [136], which is an alternate form of the transversality conditions that govern the final conditions of the optimal control problem:

\[
\left[ \frac{\partial M}{\partial \mathbf{x}} \mathbf{x}(t_f), t_f \right] + \lambda(t_f) \right]^T \delta \mathbf{x} + \left[ \mathcal{H}\big|_{t=t_f} + \frac{\partial M}{\partial t} \mathbf{x}(t), t \right] \big|_{t=t_f} \delta t_f = 0
\]

(4.3.10)

The expression given by equation (4.3.10) applies for a wide range of boundary conditions, in the various instances where the final time and states of the system are unconstrained or required to satisfy given conditions (and all combinations thereof). The entire range of boundary conditions possible for the problem is found by solving equation (4.3.10) with appropriate substitutions. The required substitutions and resulting boundary conditions can be found in Kirk’s text [136].

The control equations given by equation (4.3.9) can be generalized further to accommodate more general and practical control scenarios, where discontinuities and/or saturation limits may be experienced. The application of Pontryagin’s minimum (or maximum) principle [137] leads to a more generalized expression for the control equations given by:

\[
u = \arg \min_{\mathbf{u} \in \mathcal{U}} \mathcal{H} \quad \mathbf{u} \in \mathcal{U}
\]

(4.3.11)

where \( \mathcal{U} \) defines the domain of feasible controls. Essentially, the Hamiltonian of the system must be minimized with respect to the controls at every instant of time along the optimal path and the unconstrained optimal control will always have a Hamiltonian that is less than or equal to the Hamiltonian of the constrained optimal control case. This statement can be more intuitively expressed using the following equation:

\[
\mathcal{H}\big|_u \leq \mathcal{H}\big|_u
\]

(4.3.12)
The differential-algebraic equation system comprising of the coupled dynamical state and co-state equations given by equations (4.3.7) and (4.3.8), along with the control equations given by equation (4.3.9), together with the boundary conditions governing the problem given by equations (4.3.3) and (4.3.10), constitute a two-point boundary value problem (TPBVP). A closed form solution to this TPBVP is generally not possible for all but the simplest of dynamical systems. In fact even a numerical solution to this TPBVP may be difficult to obtain, especially for highly non-linear dynamical systems and those capable of large degrees-of-freedom. In general, the optimal control for the system can be found by solving equation (4.3.8) for \( \lambda(t) \) by integrating backwards in time and using the boundary condition given by equation (4.3.10), appropriate to the problem at hand. The desired optimal control \( \mathbf{u}(t) \) can then be found by solving equation (4.3.9). In almost all cases, this solution process needs to be carried out using suitable numerical means, for which many techniques are available, some of which will be discussed in the sub-section to follow.

Although the general theory governing optimal control problems presented here appears to demonstrate the simplicity of the field, the underlying theory is very complicated and solutions are notoriously difficult to obtain in the traditional sense. The major reason for this is the strong degree of coupling between the state variables in the state and co-state equations. When the system is sufficiently complex and integration needs to be performed numerically, the requirement that the co-state equation be integrated backwards in time often leads to an unstable integration procedure, since the values of the co-states can be very small and of comparable order of magnitude to numerical errors associated with the integration routine. Reliably estimating the final value of the co-states in order to initiate the backwards integration process is difficult, particularly if the final states are unrestrained. The fact that the co-states have no intrinsic physical meaning and serve merely as a means-to-an-end, only reinforces the tribulations associated with optimal control problems.
4.4 Available Solution Techniques for Non-Linear Optimal Control

The vast amount of research concerning the development, theory and application of techniques to solve optimal control problems renders it practically impossible to identify, let alone review, all the available materials that are relevant to this dissertation. In this subsection, a partial review of the various numerical methods available to formulate and solve problems arising in non-linear optimal control will be presented, with a particular focus on the more recent contributions to this field. Effort is made to demonstrate the general nature of these methods, as opposed to critically appraising the finer details of each of these techniques.

As can be attested by the exhaustive review undertaken by Betts [138], there exist many techniques for numerically solving non-linear optimal control problems. These techniques generally fall into two distinct groups, being either indirect or direct methods. The advantages and drawbacks of both methods will now be discussed, along with a brief overview of a selection of solution techniques that each fall within these two categories.

4.4.1 Indirect Methods

This class of techniques is characterized by their attempt to explicitly solve the equation set given by equations (4.3.7) through to (4.3.11) that constitute the necessary optimality conditions, the state and co-state differential equations, Pontryagin’s maximum principle and the associated boundary conditions. As shown in Section 4.3, the formulation and general solution procedure for the optimal control problem involves the calculus of variations, achieved by setting the first variation in the Hamiltonian for the system to zero. Generally, there are a number of significant disadvantages associated with indirect methods to solve optimal control problems, the first being the very requirement to obtain closed form expressions for the necessary optimality conditions. For highly non-linear and large degree-of-freedom systems such as ATC systems, such a task is laborious and demanding at best. Similarly, most indirect methods are extremely sensitive to the initial guess provided for the co-states to initiate the solution procedure, since extremal solutions are often very sensitive to small changes in the unspecified boundary conditions [139]. Finally, arguably the most arduous difficulty associated with indirect methods is the need to define both the constrained and unconstrained sub-arcs a priori when solving optimal control problems incorporating path constraints [138]. As will be demonstrated later in this chapter, this is an important consideration when solutions to optimal control problems for ATC systems are sought, since path constraints are readily employed for a variety of reasons.
Indirect Shooting

Also known as the Variation of Extremals method, the objective of indirect shooting techniques is to convert the original TPBVP governing the optimal control problem into a standard initial value problem and iterate on the unknown initial conditions [140]. In general, a solution found using indirect shooting will be optimal in the sense that it minimizes the Hamiltonian for the system, but it may not necessarily completely satisfy the terminal or unknown boundary conditions for the problem.

An indirect shooting technique for solving the optimal control problem usually begins by guessing values of the co-states at the initial time, although exactly how to do this is the biggest difficulty associated with all indirect methods, since the co-states are mathematical constructions with little (if any) physical meaning. To overcome this, usually a simple candidate control time history is constructed and used to integrate the state equations forward in time. The co-state equations can then be integrated backward in time, assuming that the final value of the co-states are zero, which will produce an estimate of the initial value for the co-states that can serve as an initial guess. Next the initial conditions for the states and co-states are used to integrate both the state and co-state equations forward in time, using an estimate of the optimal control provided by solving the control equation. The estimate of the optimal control, not necessarily complying with the boundary conditions, is calculated at each integration time step and usually involves the solution of a non-linear algebraic equation. The resulting trajectory for the system will be “optimal” but most likely violate the boundary conditions, particularly the final condition for the co-states. Hence the whole process is repeated by making changes to the initial values of the co-states until an ensuing trajectory and control is found that results in the desired final conditions for the co-states being met. Hence indirect shooting methods use the error in the known final conditions to perturb the unknown initial conditions until convergence is reached, which occurs if and only if, the initial co-state estimate is sufficiently close to the optimum value.

The issue of dealing with the sensitive nature of indirect shooting methods has achieved widespread attention from researchers, with particular attention being devoted to the development of continuation or homotopy methods [141, 142]. The central premise behind continuation/homotopy methods is to incrementally solve a series of problems, using the solution to the previous problem as a guess of the solution for the next problem. By placing a given optimal control problem into a family of such problems parameterized by a given parameter, the continuation/homotopy method tracks the solution by varying the parameter from the initial problem with a known solution to the original problem [142].
The continuation/homotopy method is usually initiated by obtaining a solution to the problem using a value of the parameter (usually the value zero) that ensures the initial solution can be readily obtained. The parameter is increased slightly and another solution is obtained, using the previous solution as the initial guess. This process is repeated until the final value of the parameter is reached (usually the value one) and a solution representing the actual solution to the original optimal control problem is found. The homotopy/continuation approach can be used with any solution technique, although its attractiveness and value is especially pertinent for optimal control problems solved using indirect shooting methods [138].

Following on, additional difficulties often arise when indirect shooting problems are applied to highly non-linear systems. Small changes to the initial value of the co-states introduced at the beginning of the solution procedure can propagate non-linearly towards the end of the trajectory of the system, producing erroneous system state and co-state trajectories. The concept of multiple shooting was developed to overcome this [140], whereby the trajectory of the system is subdivided into a series of segments and simple shooting is used on each segment. Although this improves the robustness of the shooting procedure, it comes at the cost of a sharp increase in the number of variables and constraints that need to be considered [138]. Recent work undertaken by Holsapple et al. [143] proposed a technique that simultaneously combines the best aspects of simple and multiple shooting and reduces the unfavorable characteristics, resulting in a fast converging modified multiple shooting method.

**Gradient Methods**

Comprising of a whole host of related approaches, gradient-based methods of solution for optimal control problems are based on the tactic of simulating the dynamical equations of motion for a system with perturbed control trajectories until the optimal condition is reached. Gradient-based methods differ from shooting techniques in that the control time history is perturbed, as opposed to the initial value of the co-states that are perturbed in shooting-based techniques. The trajectory for gradient methods always satisfies the boundary conditions, but may not be optimal in the Hamiltonian sense, until the appropriate final solution is obtained. Contrast this with shooting point techniques where the trajectory is always optimal, but does not necessarily satisfy the boundary conditions until the desired final solution is found.

Gradient-based methods operate by specifying the initial conditions $\mathbf{x}(t_0)$ and a candidate, nominal control time history $\mathbf{u}(t)$. The dynamical state equations are then integrated forward in time and an initial trajectory $\mathbf{x}(t)$ is found, upon which an initial estimate of the
cost function is obtained. The co-state equations are integrated backwards in time to provide the initial estimate of the co-states \( \lambda_0(t) \). Now the control equation given by equation (4.3.9) can be applied to the system, yielding the variation of \( \frac{\partial H}{\partial u} \) along the initial system trajectory.

The initial nominal control \( u_0(t) \) is unlikely to be the optimal, so \( \frac{\partial H}{\partial u} \) will not be zero along the entire trajectory, hence the control \( u(t) \) is perturbed as a function of \( \frac{\partial H}{\partial u} \), until \( \frac{\partial H}{\partial u} \) is zero or sufficiently close to a desirable small tolerance.

Several techniques for determining the solution to optimal control problems fall within the umbrella of gradient-based methods, each differing in the manner by which the control history is utilized to obtain the optimum solution. The simplest approach is known as the method of steepest descent, which uses equation (4.4.1) to iterate on the control time history:

\[
u_{i+1}(t) = u_i(t) - \alpha \left[ \frac{\partial H}{\partial u} \right]^T
\]  

(4.4.1)

where \( \alpha \) is the step size, which may be arbitrarily selected, although it is usually determined via an internal optimization routine that ensures that maximum reduction occurs in performance criterion \( J \) at all times. The steepest descent method is not efficient as it utilizes the previous value of \( \frac{\partial H}{\partial u} \) to determine the current value of \( u(t) \), yet the direction of \( \frac{\partial H}{\partial u} \) is known to vary in between iterations. Consequently, the steepest descent method tends to oscillate between different conditions on the path to the optimum condition, resulting in slow convergence. In order to improve convergence, the steepest descent algorithm was modified by Fletcher-Reeves into a conjugate-gradient method, which was then adapted to control problems by Leadon, Mitter and Waren [144]. In the conjugate gradient method, equation (4.4.2) is used to iterate on the control time history in order to achieve the optimum value:

\[
u_{i+1}(t) = u_i(t) - \alpha^*_i s_i(t)
\]  

(4.4.2)

where \( \alpha^*_i \) is the optimum step size. The conjugate search direction \( s_i(t) \) is given by:

\[
s_i(t) = \left( \frac{\partial H}{\partial u} \right)_i + \beta s_{i-1}(t)
\]  

(4.4.3)

with \( \beta_0 = 0 \). There exists many formulae for calculating \( \beta_i \) (see [144-146] for details) that yield different estimates for the search direction and provide varying degrees of convergence.
The idea behind the conjugate gradient approach is to determine the local minimum for each conjugate gradient direction before the direction of the search is modified. Convergence is strongly dependent on the optimum step size as reductions in the performance criterion are greatly dependent on the step size. Determining the optimum step size is essentially a one-dimension search, as the direction of the search is initially known. The control time history is iterated upon until the conjugate gradient algorithm converges to some chosen tolerance.

Whilst the implementation of gradient-based methods for solving optimal control problems is more complicated than that associated with shooting-based methods, gradient-based methods tend to be more robust. However, arguably the biggest limitation associated with gradient-based methods is the difficulty in incorporating constraints for the state and control time histories. Although methodologies have been proposed to overcome this, namely the clipping-off method due to Quitana and Davison [147] and the penalty function approach proposed by Sargent [148], accounting for constraints using these strategies is quite difficult and compromises are usually sought in the areas of constraint violation and prompt convergence.

### 4.4.2 Direct Methods

In contrast to indirect methods, direct solution methods for optimal control do not require the analytical determination of the necessary conditions for optimality and do not require initial estimates to be provided for the co-states. Alternatively, the dynamic state and control variables for the system are adjusted in a manner that directly reduces the desired performance criterion, which introduces a certain flexibility and versatility into the solution procedure. As a result, direct methods of solution have recently become appealing for use on complicated applications [138], such as those relating to ATC systems. Although there exists direct shooting methods to solve optimal control problems, only direct methods pertaining to transcription or collocation are considered in this thesis. Numerous approaches to direct transcription/collocation methods for solving optimal control problems for a wide range of applications have previously been investigated by many researchers [30, 95, 96, 101, 114, 138, 148-160]. A similar approach to the formulation and implementation of direct methods of solution for optimal control problems involving ATC systems will be pursued in this dissertation.

The central premise behind most direct methods of solution for optimal control problems is to transform the original continuous optimal problem into a more manageable non-linear discrete parameter optimization problem or non-linear programming problem (NLP).
All direct methods require the introduction of some form of parametric representation of either the control variables, the state variables, or both. Hull [152] has characterized the various methods that exist to convert optimal control problems into parameter optimization problems into four categories, each differing by the number of unknowns, the numerical integration technique and its order. Generally, the conversion process begins with the time domain of the optimal control problem separated into a prescribed number of intervals, with the times at the ends of each interval serving as nodes. The states and controls for the system at the nodes serve as the unknowns (along with the final time if it is free), from which time histories for the state and control variables are constructed via interpolation. The dynamic state equations are then integrated via a pre-determined numerical procedure and a Newton-based constrained optimization procedure is then used to determine the solution to the resulting NLP through iteration with a fixed set of unknowns.

The evolution of optimal control solution methods has closely mirrored the advancement of algorithms used to solve non-linear programming problems, which themselves have been strongly dependent on the progress of digital computers [158]. The modern approach to solve NLPs containing generalized constraints usually involves the solution of a series of simpler quadratic programming sub-problems, a technique known as Sequential Quadratic Programming (SQP) [161]. Most modern optimal control problems involve complex dynamical systems that have large degrees-of-freedom and numerous control variables, and ATC systems are no exception. However, when these complex optimal control problems are converted to NLPs, it often transpires that the resulting NLP involves a very large number of variables requiring solution, particularly if both the dynamic state and control variables are parameterized. Without appropriate modifications, the solution to these large-scale NLPs can be computationally expensive and time consuming. Fortunately, the important matrices that govern the NLP (Jacobian and Hessian matrices) are sparse and recent advances in numerical linear algebra have made it possible to reduce storage requirements and computational times associated with large-scale NLPs by exploiting the sparsity of the Jacobian and Hessian matrices. A sparse SQP algorithm such as that developed by Gill et al. [134] enables the relatively rapid solution to NLPs characterized by large numbers of decision variables and constraints. Consequentially, this very powerful and indispensable solution technique will be utilized to solve the optimal control problems for ATC systems proposed in this thesis.
4.5 Trajectory Optimization

Consider the general continuous optimal control problem involving the minimization of the following performance index:

\[
\mathcal{J} = \mathcal{M}\left[\dot{x}(t_f), x(t_f), t_f\right] + \int_{t_0}^{t_f} \mathcal{L}\left[\dot{x}(t), x(t), u(t), t\right] \, dt
\]  

(4.5.1)

where \( t \in \mathbb{R} \) and the states \( x \in \mathbb{R}^p \) and controls \( u \in \mathbb{R}^q \) for the system are subject to the following dynamical constraints:

\[
f[x(t), x(t), u(t), t] = 0, \quad t \in [t_0, t_f]
\]  

(4.5.2)

the subsequent boundary conditions given by:

\[
\psi_0\left[\dot{x}(t_0), x(t_0), t_0\right] = 0 \quad \psi_0 \in \mathbb{R}^r \quad (r \leq n)
\]  

(4.5.3)

\[
\psi_f\left[\dot{x}(t_f), x(t_f), t_f\right] = 0 \quad \psi_f \in \mathbb{R}^s \quad (s \leq n)
\]  

(4.5.4)

along with general non-linear path constraints represented as:

\[
g\left[\dot{x}(t), x(t), u(t)\right] \leq 0, \quad g \in \mathbb{R}^s
\]  

(4.5.5)

As stated previously, the idea behind direct methods is to transform the continuous optimal control problem given by equations (4.5.1) through to (4.5.5) into a non-linear parameter optimization problem. Mathematically, this transformation process can be stated as follows: Find the decision variables:

\[
X = [x_0, ..., x_N, u_0, ..., u_N, t_N]
\]  

(4.5.6)

that minimizes the performance criterion:

\[
\mathcal{J} = \mathcal{M}[\dot{x}_N, x_N, t_N] + \sum_{k=0}^{N} \left( \mathcal{L}[\dot{x}_k, x_k, u_k, t] w_k \right)
\]  

(4.5.7)

subject to the dynamical state equality constraints:

\[
f[\dot{x}_k, x_k, u_k, t] = 0, \quad k = 0, ..., N
\]  

(4.5.8)

the given initial conditions:

\[
\psi_0[\dot{x}_0, x_0, t_0] = 0
\]  

(4.5.9)

the final boundary conditions:

\[
\psi_N[\dot{x}_N, x_N, t_N] = 0
\]  

(4.5.10)
along with any general non-linear path constraints:

\[ g[\dot{x}_k, x_k, u_k] \leq 0, \quad k = 0, \ldots, N \]  

(4.5.11)

It may also be pertinent to provide the following box constraints for the NLP variables:

\[ x_k^{MIN} \leq x_k \leq x_k^{MAX} \]
\[ u_k^{MIN} \leq u_k \leq u_k^{MAX} \]  

(4.5.12)

A numerical quadrature procedure is required to discretize the integral component of the performance criterion shown in equation (4.5.7), for which several classes of well-known methods exist. Fahroo and Ross [157] distinguish between various direct collocation techniques by the manner in which each discretizes the time history and the way the state equations are satisfied at the various discrete points. Direct methods for solving optimal control problems, namely the choice of discretization scheme, can be essentially grouped into two general classes, local or global methods, which will now be explored. Similarly, these methods can be further classified as either differentiation-based methods or integration-based methods.

### 4.5.1 Local Discretization Schemes

As Williams [149] eloquently points out, local discretization methods employ a series of arbitrarily spaced nodes at which both the state and control variables are collocated. The state equations are enforced as equality constraints at internal collocation points located between the nodes via an implicit integration technique. Several popular local discretization procedures that receive widespread treatment in the literature will now be considered.

**Heun Discretization**

One such implicit integration technique is known as Heun’s method, which is a predictor-corrector variation of Euler’s well-known integration method [138]. For \( N + 1 \) arbitrarily chosen nodal points and \( N \) segments satisfying the following:

\[ t_0 < t_1 < t_k < t_{k+1} < t_N \]
\[ h_k = t_{k+1} - t_k \]  

(4.5.13)

an estimate of the state variables at the nodes is found firstly by Euler’s method:

\[ x_{k+1}^* = x_k + h_k f(x_k, u_k, t_k) \quad k = 0, \ldots, N - 1 \]  

(4.5.14)

which is then corrected for using a trapezoidal-based routine.
Adopting constraint-type terminology, the defects for Heun’s method are:

\[
\Delta_k = x_k - x_{k+1} + \frac{h}{2} \left[ f(x_k, u_k, t_k) + f(x_{k+1}, u_{k+1}, t_{k+1}) \right] \quad k = 0, \ldots, N-1
\]  

(4.5.15)

In order for the direct method to enforce the dynamical state equality constraints, these defects are driven to zero by the chosen NLP solution algorithm. Most recently, Williams and Trivailo [162] introduced a higher order form of Heun’s method, that they term the Legendre-Gauss-Lobatto integration discretization method.

**Hermite Simpson Discretization**

This hybrid local discretization scheme first uses Hermite cubic interpolation to approximate the state variables at the centre of each interval:

\[
x^c_k = \frac{x_{k+1} + x_k + \frac{h}{8} \left[ f(x_k, u_k, t_k) - f(x_{k+1}, u_{k+1}, t_{k+1}) \right]}{2} \quad k = 0, \ldots, N-1
\]  

(4.5.16)

The procedure then uses Simpson quadrature to satisfy the dynamical state equality constraints and drive the following defect equation to zero:

\[
\Delta_k = x_k - x_{k+1} + \frac{h}{6} \left[ f(x_k, u_k, t_k) + 4f(x^c_k, u^c_k, t^c_{k+1}) + f(x_{k+1}, u_{k+1}, t_{k+1}) \right]
\]  

(4.5.17)

The controls in the centre of each interval \( u^c_k \) may be determined via linear interpolation:

\[
u^c_k = \frac{(u_{k+1} + u_k)}{2}
\]  

(4.5.18)

Alternatively, the controls in the centre of each interval may be determined by a Hermite cubic interpolation scheme, an approach used by Enright and Conway [163]. The Hermite Simpson discretization procedure has been widely used to successfully discretize optimal control problems for a variety of applications (see [138, 152, 153, 158, 163] for appropriate details). Similarly, there exists additional local discretization procedures that have been employed to discretize practical optimal control problems, namely Runge-Kutta transcription [163, 164] and high-order Gauss-Lobatto Quadrature techniques [165].

### 4.5.2 Global Discretization Schemes

Global discretization schemes, otherwise known as pseudospectral methods, have begun to attract widespread attention from researchers in recent times. Pseudospectral methods use globally orthogonal interpolating polynomials based on the so-called Gauss-Lobatto points to approximate the state and dynamic variables. The state equality constraints are enforced by differentiating the interpolating polynomials at the Gauss-Lobatto points, as opposed to
employing a numerical quadrature technique. The Gauss-Lobatto points are chosen to correspond with the zeros of the derivatives of the interpolating polynomials. The use of pseudospectral methods in optimal control has traditionally involved the use of either Legendre polynomials [156, 160] or Chebyshev polynomials [154, 155]. Williams formalized the approach by developing a general pseudospectral method for solving optimal control problems based on the use of Jacobi polynomials of a general form [149]. Collocation is based on the roots of the derivatives of the general Jacobi polynomials, from which the Legendre and Chebyshev methods reside as particular subsets of this general procedure. An outline of pseudospectral discretization as formulated by Williams [149] will now follow.

**Pseudospectral Discretization**

Pseudospectral methods involve the expansion of the dynamic state and control variables using Lagrange interpolating polynomials. The nodal points $\tau$ have a defined spacing, lying in the interval $[-1,1]$ and chosen as the Gauss-Lobatto points, which are the extrema of the $N$th order Jacobi polynomials $P^\alpha_\beta_N$. Jacobi polynomials are parameterized by the parameters $\alpha$ and $\beta$ ($\alpha > -1$, $\beta > -1$); Legendre polynomials are obtained by setting $\alpha = \beta = 0$, whilst Chebyshev polynomials are given using $\alpha = \beta = -0.5$. A linear transformation is required to relate the computational domain $\tau$ to the physical time domain $t$ as follows:

$$t = \left( \frac{t_f - t_0}{2} \right) \tau + \left( \frac{t_f + t_0}{2} \right)$$  

(4.5.19)

The state and control variables are approximated using the following $N$th order polynomials:

$$x_N(\tau) = \sum_{j=0}^{N} \tilde{x}_j \phi_j(\tau) \quad j = 0, ..., N$$  

(4.5.20)

$$u_N(\tau) = \sum_{j=0}^{N} \tilde{u}_j \phi_j(\tau) \quad j = 0, ..., N$$  

(4.5.21)

where $\tilde{x}_j$ and $\tilde{u}_j$ are the coefficients of the interpolating polynomial. The term $\phi_j(\tau)$ in equations (4.5.20) and (4.5.21) are the Lagrange interpolating polynomials, as it is required that the coefficients $\tilde{x}_j$ and $\tilde{u}_j$ be equal to $x_N(\tau_j)$ and $u_N(\tau_j)$ respectively. It can be shown that the Lagrange interpolating polynomials $\phi_j(\tau)$ are given by [149]:

$$\phi_j(\tau) = \prod_{k=0, k\neq j}^{N} \left( \frac{\tau - \tau_k}{\tau_j - \tau_k} \right) = \left( \frac{\tau - 1}{\tau_j - \tau} \right)^{c_j} \frac{\partial P^\alpha_\beta_N(\tau)}{\partial \tau}(\tau)$$  

(4.5.22)
where the term \( c_j \) is given by:
\[
c_j = \begin{cases} 
\beta + 1 , & j = 0 \\
1 , & 1 \leq j \leq N - 1 \\
\alpha + 1 , & j = N 
\end{cases}
\] (4.5.23)

In order to approximate the derivatives of the state variables, it is necessary to find expressions for the derivatives of the approximating polynomials. It can be shown that this process can be achieved through matrix multiplication alone, through the use of the so-called Jacobi differentiation matrix \( D(\alpha, \beta, \tau) \). Williams provides detailed formulae for the calculation of the Jacobi differentiation matrix and shows how higher order derivatives of the approximating polynomials can be found [149]. As the Jacobi differentiation matrix is parameterized with respect to \( \tau \), substitution of the linear transformation given by equation (4.5.19) is required to determine the derivatives of the state variables as follows [149]:
\[
\frac{d\bar{x}_\gamma}{d\tau} = \frac{d\bar{x}_\gamma}{dt} \frac{d\tau}{dt} = \frac{2}{(t_f - t_0)} \frac{d\bar{x}_\gamma}{d\tau} \frac{2}{(t_f - t_0)} \frac{d\bar{x}_\gamma}{dt} \quad (4.5.24)
\]

Equation (4.5.24) shows that the state derivatives at any point depend only on the values of the states at every node point, along with the initial and final time. This is a significant advantage associated with the pseudospectral method of discretization, as most other local collocation methods require the implicit determination of the state derivatives (see Section 4.5.1). In order to satisfy the dynamical state equality constraints, the following defect equation is enforced by the NLP solution algorithm:
\[
\Delta_k = \sum_{j=0}^{N} D_{\gamma j} x_j - \left(\frac{t_N - t_0}{2}\right) \left[ f(x_k, u_k, t_k) \right] \quad k = 0, ..., N
\] (4.5.25)

In recent years, the pseudospectral discretization procedure has been used to successfully discretize optimal control problems for a variety of applications, including orbit transfer manoeuvres [149, 156], re-entry trajectory design [166] and space tether-satellite control [95, 167].

### 4.5.3 Final Remarks

It is clear from the preceding discussion into the methods available for solving optimal control problems that the direct approach is preferable and far superior to that offered by the various indirect approaches. In the following section, the application of direct trajectory optimization to ATC systems will be thoroughly explored, as a means to demonstrate and investigate the important ATC system application known as rendezvous.
4.6 Instantaneous Rendezvous Scenarios for Aerial Towed-Cable Systems

This sub-section deals with the application of single-phase optimal control theory to instantaneous rendezvous problems involving ATC systems. Broadly speaking, the principal objective of rendezvous is to precisely place the tip of the cable (or the payload) in the vicinity of a desired location on the ground. The most appropriate means to achieve this outcome is an issue the optimal control methodology will pursue. Rendezvous problems incorporating the various dynamic models of the ATC system will be studied, from which important conclusions will be drawn as to the expediency and practicality of using ATC systems to perform more complex operations such as payload transportation.

It is important to explore the nature of the decision taken as to which class of ATC system model should be used within the proposed optimal control framework. In Section 2, the equations of motion for a variety of ATC systems were derived, including those for the case when the cable is considered to be extensible. The choice of model, in particular the chosen generalized coordinates and governing reference frame, all have major implications during the direct numerical solution of optimal control problems for the ATC system. The applicability, implementation and subsequent convergence of the NLPs arising from rendezvous problems for the ATC system was found to be significantly enhanced by eliminating altogether the non-working tension forces from the model of the ATC system. To do this, the motion of the cable must be specified using angular orientations, as utilized when the cable is assumed to be inextensible, in contrast to Cartesian coordinates used to classify the motion of the extensible ATC system. When using a flexible cable within the optimal control framework, it was found that the resulting tension within the cable was extremely sensitive to changes in the generalized coordinates, more specifically the Cartesian coordinates of the cable segments. Small changes to the generalized coordinates of the cable segments resulted in disproportionately large changes in the tension within each segment, rendering the resulting state equations (dynamic constraints) very stiff. This led to numerical instabilities when a solution to the resulting NLP was sought, in spite of the use of numerous scaling and non-dimensionalization treatments. In fact, even a solution to the most trivial of optimal control problems could not be determined when an elastic cable was utilized within the ATC system. Consequently, the non-linear optimal control problems considered in this thesis will be those for ATC systems incorporating an inextensible cable. Similarly, simple single-link models will be used to formulate these optimal control problems, with an appropriate model “matching” procedure deployed to improve the representativeness of the ensuing scenarios.
4.6.1 Instantaneous Rendezvous With Constant Towing Speed

The simplest possible rendezvous operation that the ATC system can perform is pictorially depicted in Figure 4-1. In this scenario, the aircraft flies steady and level at a constant speed and altitude and the cable is deployed in such a manner as to rendezvous the tip of the cable with some desired surface location \( \{x_T, y_T\} \), whilst preventing any collisions with the ground. Initially, the ATC system is in the equilibrium configuration governed by the physical parameters of the system, with the aircraft positioned at the following coordinates \( \{-1000, y_{AC}\} \). The manoeuvre terminates once rendezvous has taken place.

![Figure 4-1: Pictorial Representation of the Constant Tow Speed Rendezvous Scenario](image)

The dynamic mathematical model of the ATC system used in this investigation is that given by equation (2.4.38). Control actuation is limited to that provided by the cable reel only and rendezvous is assumed to take place instantaneously.

4.6.2 Rendezvous Problem Formulation

The single-phase optimal control problem that characterizes the rendezvous scenario shown in Figure 4-1 will now be formally presented, with the necessary components that constitute the optimal control problem receiving individual treatment as follows.

**Performance Criterion**

In order to obtain the optimal control \( \hat{u}(t) \) and corresponding state space trajectory for the ATC system during rendezvous, a performance index \( J \) is required to be minimized. A wide range of suitable options is available, each having a profound effect on the manner in which the rendezvous problem is executed. The effect and impact that various optimality criteria have on rendezvous is the subject of extensive investigation in Section 4.6.4 to follow.
Dynamic Constraints

The dynamical state-space equality constraints to be satisfied as part of the optimal control problem governing rendezvous are formulated using equation (2.4.38). The model “matching” technique that incorporates dynamic inertial effects (Method 2) is utilized to improve the representativeness of the resulting rendezvous problems.

Initial Conditions

As mentioned previously, the ATC system is initially in equilibrium, the particular configuration of which is determined by the chosen physical parameters for the ATC system. Mathematically, the initial conditions for the rendezvous problem are given by:

\[
\psi_N[x_0, x_0, t_0] = \left( \theta \ 0 \ l \ 1 \right)^T_{t_0=0} - \left( \theta_0 \ 0 \ l_0 \ 0 \right)^T = 0
\] (4.6.1)

where \( \theta_0 \) and \( l_0 \) are the chosen initial cable orientation angle and length.

Final Conditions

The final boundary conditions governing rendezvous are specified in terms of the in-plane position of the cable tip \( \{x_C, y_C\} \). The \( x \)-coordinate \( x_C \) and \( y \)-coordinate \( y_C \) of the cable tip are given by:

\[
x_C = U_0 t - l \sin \theta
\] (4.6.2)

\[
y_C = y_{AC} - l \cos \theta
\] (4.6.3)

where: \( y_{AC} \) is the altitude of the aircraft,

\( U_0 \) is the selected towing speed of the aircraft.

In addition to the final position of the cable tip specifying the conditions for rendezvous, the cable should cease deployment/retrieval at the instant rendezvous occurs, hence the reel rate is constrained to be zero. Mathematically, the final conditions for the rendezvous problem are given by:

\[
\psi_N[\dot{x}_N, \dot{y}_N, t_N] = \left( x_C \ 0 \ y_C \ y_C \right)_{t=t_0} - \left( x_T \ y_T \ 0 \ 0 \right) = 0
\] (4.6.4)

where \( x_T \) and \( y_T \) are the \( x \)-coordinate and \( y \)-coordinate of the desired location on the ground that rendezvous is to occur at. The physical significance of the final conditions given by equation (4.6.4) is that the cable tip trajectory is required to be tangential to the ground at the instant of rendezvous. Coupled with the non-linear path constraint given by equation (4.6.5), this will ensure that collision of the cable tip with the ground is prevented both before and after rendezvous has taken place.
Non-Linear Path Constraints

A non-linear path constraint is utilized to ensure that at all times the cable tip/towed body is prevented from colliding with the ground or localized elevated terrain. The elevation of the ground terrain \( h_g(x_c) \) is defined as a function of the \( x \)-coordinate of the cable tip position and forced to be sufficiently beyond the cable tip by specifying a collision tolerance \( y_{tol} \).

This non-linear path constraint can be mathematically written as follows:

\[
\mathbf{g}[\dot{\mathbf{x}}(t), \mathbf{x}(t), \mathbf{u}(t)] = [h_g(x_c) - y_c - y_{tol}] \leq 0
\]  

(4.6.5)

The value of the collision tolerance margin \( y_{tol} \) used in all rendezvous problems is 2 metres.

Box Constraints

The following dimensional box constraints are placed on the dynamical states and control variables for the ATC system to ensure that unrealistic/infeasible values are not attained during rendezvous manoeuvres:

\[
\begin{aligned}
\{\theta_{\text{MIN}}, & \dot{\theta}_{\text{MIN}}, i_{\text{MIN}}, i_{\text{MIN}}\}^T \leq \mathbf{x} \leq \{\theta_{\text{MAX}}, \dot{\theta}_{\text{MAX}}, i_{\text{MAX}}, i_{\text{MAX}}\}^T \\
\{-90^\circ \ \ -\infty \ 10 \ -5\}^T & \leq \mathbf{x} \leq \{90^\circ \ \ \infty \ 2000 \ 20\}^T \\
\{\dot{i}_{\text{MIN}}\} & \leq \mathbf{u} \leq \{\dot{i}_{\text{MAX}}\} \\
\{-2.5\} & \leq \mathbf{u} \leq \{2.5\}
\end{aligned}
\]  

(4.6.6)

Solution Methodology and Implementation Issues

There still exists a select number of issues that require attention in order to complete the entire procedure that transforms the continuous optimal control problem governing rendezvous into a resulting NLP for solution. Firstly, the choice of direct collocation technique to discretize the dynamic state and control time histories and satisfy the state equations at the various discrete points requires finalizing. The effect and impact that various discretization schemes have on rendezvous is officially pursued in Section 4.6.3 to follow.

All optimal control problems that are the subject of consideration and investigation within this dissertation are formulated and solved using the re-usable MATLAB®-based software package known as DIRECT [168]. Developed by Dr. Paul Williams at RMIT University, DIRECT is capable of solving non-linear optimal control, dynamic optimization, and parameter estimation problems based on direct transcription formulations incorporating a wide range of discretization methods. With respect to solving optimal control problems, DIRECT transcribes the original continuous optimal control problem into a finite dimensional NLP using a chosen discretization scheme.
Fortunately from a user perspective, this transcription process is fully automated and handled completely internally by DIRECT, with the user maintaining complete control over the physical formulation and description of the chosen optimal control problem. DIRECT employs the sparse non-linear quasi-Newton SQP algorithm SNOPT developed by Gill et al. [134, 135] to solve the resulting NLPs. Williams has provided DIRECT with an automated pattern generator to determine the locations of the non-zero entries for the Jacobian matrix, whilst the actual values themselves are estimated using a sparse finite differencing scheme.

All decision variables that constitute a particular NLP arising from optimal control problems for the ATC system, receive appropriate non-dimensionalization to ensure that their values either approximately fall within the interval \([-1,1]\) or are all at least of a similar order of magnitude. This is particularly important as, between them, the dynamic state and control variables for the ATC system vary over several orders of magnitude; cable lengths can be in the order of thousands of metres, whilst cable angular rates can be thousandths of a radian per second. Scaling issues have profound numerical implications during the NLP solution process and strongly affect the ability of SNOPT to seek and converge to optimal solutions [134].

The initial guess of the decision variables for large-scale NLPs is an important consideration. As it is not known a priori if a solution to the NLP actually exists for a given problem, to improve the efficiency and expediency of the underlying optimization process, a good quality estimate of the solution should be provided to SNOPT. Not only does this reduce the workload required of SNOPT, often a solution may not be returned by DIRECT if a poor initial guess is provided. To initiate the NLP solution process, the initial guess of the solution provided to SNOPT is the equilibrium configuration of the ATC system with no applied control inputs. The guess is generated by first numerically integrating the dynamic state equations for the ATC system with no control, then interpolating the appropriate values of the state variables at the node points corresponding to the chosen discretization scheme.

Similarly, the optimization process in DIRECT should commence with an initial level of discretization \(N\), set to a modest value (typically \(N = 10\) initially) to readily obtain an approximate initial solution to the optimal control problem. A first generation solution to the NLP problem is found at this modest discretization level using the aforementioned initial guess of the decision variables. An iterative process is then employed to obtain solutions at increasingly higher discretization levels. The level of discretization is increased and a solution is obtained using the NLP solution corresponding to the previous discretization level as an updated guess.
This iterative process continues until the solution converges; whereby the computational cost of further increasing the discretization level outweighs any improvements to the accuracy of the NLP solution. For rendezvous problems involving ATC systems, this threshold is usually reached when the discretization level approaches sixty nodes.

Once DIRECT has returned a solution to the NLP at the final discretization level that approximates the original optimal control problem, the feasibility of this discrete solution must be investigated further. To do this, DIRECT integrates the continuous dynamical state equations using the discrete controls returned by SNOPT. If the propagated state space trajectories pass through the discrete values of the states found using SNOPT, then the discrete solution can be considered to be feasible and representative of the real, closed-form solution one may obtain if it was possible to indirectly solve the original continuous optimal control problem. DIRECT goes a step further by calculating the maximum error that occurs between the “propagated” and “discrete” solutions for the dynamical state variables.

4.6.3 Effect of Various Discretization Schemes on Rendezvous

In this sub-section, various in-plane rendezvous problems for the constant tow speed ATC system will be presented as part of an investigation into which is the most appropriate discretization scheme to use for transcribing optimal control problems for the ATC system. Issues such as solution accuracy, solver convergence and associated computational expense will be explored in an attempt to deduce the most suitable discretization scheme for use on optimal control problems for the ATC system. The physical system parameters that govern the rendezvous problems studied in this sub-section, and those that subsequently follow are presented in Table 4-1.

The results obtained when the Hermite Simpson discretization scheme with linear control interpolation is used to transcribe the optimal control problem governing constant tow speed rendezvous are given in Figure 4-2 through to Figure 4-6. The performance index used in the determination of these optimal trajectories is one that ensures the control input (reel acceleration) for the system is minimized at all times throughout the rendezvous manoeuvre (see Section 4.6.4 for details).
In Figure 4-2 through to Figure 4-6, the solid circular markers (●) represent the discrete values of the dynamic states and control found using DIRECT, whilst the solid line (—) represents the propagated solution generated by integrating the governing state equations with the discrete controls found using DIRECT. The transparent circle (○) represents a final target condition where appropriate. It can be seen from the results offered in Figure 4-2 through to Figure 4-6 that the discrete optimal trajectories found using DIRECT are feasible and in close agreement with the propagated solution expected of the original continuous optimal control problem.
Similarly, it can also be deduced from Figure 4-4 and Figure 4-5 that all the final target states were attained by the ATC system, confirming that the desired objectives for the rendezvous problem were successfully met. The general nature and practical significance of the results given by Figure 4-4 through to Figure 4-6 are discussed in the Minimum Control sub-section to follow. The procedure used to calculate the tension within the cable is outlined in the Minimum Tension sub-section. With respect to the results presented in Figure 4-2, they confirm the desired requirement for rendezvous to occur at constant tow speed and flight altitude. From Figure 4-3, it can be seen that the in-plane orientation of the cable is not significantly altered throughout the rendezvous manoeuvre.

![Figure 4-3: Cable Angular Dynamics During Rendezvous (a) Angle (b) Velocity](image1)

Similarly, it can also be deduced from Figure 4-4 and Figure 4-5 that all the final target states were attained by the ATC system, confirming that the desired objectives for the rendezvous problem were successfully met. The general nature and practical significance of the results given by Figure 4-4 through to Figure 4-6 are discussed in the Minimum Control sub-section to follow. The procedure used to calculate the tension within the cable is outlined in the Minimum Tension sub-section. With respect to the results presented in Figure 4-2, they confirm the desired requirement for rendezvous to occur at constant tow speed and flight altitude. From Figure 4-3, it can be seen that the in-plane orientation of the cable is not significantly altered throughout the rendezvous manoeuvre.

![Figure 4-4: Cable Radial Dynamics During Rendezvous- (a) Cable Length (b) Length Rate (c) Control](image2)
A solution to the same constant tow speed rendezvous problem was obtained using the various local and global discretization schemes previously outlined in Section 4.5. In each case, the same problem formulation, system parameters and optimization procedure was employed. The optimization process began with ten grid points in all cases; this was increased to sixty for the final solution in subsequent intervals of ten. The same initial guess was provided in all instances. Numerical results concerning the accuracy, convergence and efficiency of the NLPs that constitute this enquiry are summarized in Table 4-2. Regardless of the discretization method employed, the nature of the physical results representing constant tow speed rendezvous, such as those shown in Figure 4-2 through to Figure 4-6, remain unaltered.
The percentage maximum state errors provided in Table 4-2 are determined as a percentage of the maximum value of the particular state the error is associated with. A review of Table 4-2 demonstrates that of the candidate discretization procedures, there is not one that clearly stands out as the “best”. With respect to Hermite-Simpson with linear control interpolation and the Heun class of discretization schemes, these procedures are the most accurate and cause SNOPT to produce relatively the same value for the final value of the performance index. However, the Hermite-Simpson method allows a solution to be found is significantly less time. The Euler scheme is the quickest to produce a solution and corresponds to the lowest final cost value, although these benefits are only possible if the accuracy of the NLP solution is sacrificed. With respect to the solution times, as MATLAB® 7 is a high-level interpreted language; the solution times themselves may not be totally accurate and should be treated with caution. The important point to consider is how the relative order of magnitude of the solution times compares for each discretization scheme. Surprisingly, the use of quadratic interpolation for the control time histories within the Hermite Simpson discretization procedure results in a higher final cost, longer execution time and greater errors between the NLP and propagated state variables. This renders the transcription procedure not particularly appealing, given the increased complexity associated with implementing the quadratic control interpolation. The two pseudospectral discretization methods produce similar results between them, with the Lagrange variant slightly more preferable to the Chebyshev-based technique due to the lower computational time. The results for both of these methods also compare favourably against those associated with the Heun or Hermite Simpson linear control interpolation methods.

In light of the findings deduced from Table 4-2, the Hermite Simpson discretization scheme with linear control interpolation is the preferred transcription method for use on optimal control problems involving ATC systems. Although lower final costs are returned from other candidate discretization methods, this is only a minor drawback in comparison to benefits of fast solution times and high discrete solution accuracy that is characteristic of the Hermite

<table>
<thead>
<tr>
<th>Discretization Scheme</th>
<th>Final Cost [m/s²]</th>
<th>Solution Time [secs]</th>
<th>Max State Error [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euler</td>
<td>10.5638</td>
<td>41.5097</td>
<td>9.7</td>
</tr>
<tr>
<td>Heun</td>
<td>13.7588</td>
<td>104.0196</td>
<td>0.16</td>
</tr>
<tr>
<td>Simpson (Linear Control Interp.)</td>
<td>13.6818</td>
<td>64.3425</td>
<td>0.12</td>
</tr>
<tr>
<td>Simpson (Quad Control Interp.)</td>
<td>17.2074</td>
<td>148.3032</td>
<td>1.31</td>
</tr>
<tr>
<td>Lagrange</td>
<td>12.3176</td>
<td>88.4672</td>
<td>0.27</td>
</tr>
<tr>
<td>Chebyshev</td>
<td>12.4726</td>
<td>104.6705</td>
<td>0.27</td>
</tr>
</tbody>
</table>

Table 4-2: Summary of Results Obtained During Discretization Study into Rendezvous
Simpson procedure. This is especially pertinent when one compares the numerical results (such as those given in Figure 4-2 through to Figure 4-6), determined when both the Hermite Simpson and pseudospectral discretization methods are used; it is almost impossible to identify an important distinguishing feature between each set of results. Small differences between the final costs returned by SNOPT, in general, do not translate into measurable differences in the resulting state and control trajectories found for the ATC system. This case study only considered the simple scenario of constant tow speed rendezvous, however these very same conclusions were drawn when the investigation was performed for a wide range of optimal control problems involving the ATC system, including multiple rendezvous and payload capture or delivery. Thus, the Hermite Simpson scheme with linear control interpolation is used to discretize all optimal control problems for the ATC system contained within this thesis.

4.6.4 Effect of Various Optimality Criteria on Rendezvous

The application of various optimality criteria in the determination of optimal trajectories for instantaneous, constant tow speed rendezvous is considered here. By studying the effect that various performance indices have on the dynamics and control of the ATC system during rendezvous, valuable insights into the dynamics of the system can be made, in particular the importance of careful and measured controller design. Ill-considered choice of the optimality criteria used during optimal control problems can lead to trajectories possessing undesirable characteristics such as unwanted oscillations, sharp changes in system states or unrealistic and infeasible controls.

There are a great number of optimality criteria that can be employed within a controller designed using open loop optimal control. Traditional choices of performance indices include those based directly on the system states and/or control and time. Consequently, a performance index may be an indirect function formulated in terms of the system states and control. The performance criteria used in this study are individually discussed in the following sub-sections. To ensure efficient computation and improved convergence for the resulting NLPs, these optimality criteria are quadratic and non-dimensionalized.
**Minimum Control**

Smooth or “well-behaved” dynamics for the ATC system is arguably one of the most important criteria to be satisfied by any controller designed for the system. One method to ensure that smooth system dynamics occur is to formulate a performance index that minimizes the accelerations of the system. Such an approach is desirable since it will prevent violent changes in the dynamics of the ATC system from occurring, which may induce or excite modes of vibration in the cable. For the two-dimensional ATC system, two accelerations are directly accessible; the cable reel acceleration $\ddot{l}$ and the angular acceleration of the cable $\dot{\theta}$. Focusing on radial acceleration, which happens to be the control input of the system, the performance index used to ensure minimum control is:

$$J_1 = \int_{t_0}^{t_f} u^2 \, dt$$

(4.6.8)

The results concerning the minimum control optimal control problem were given previously in Figure 4-4 through to Figure 4-6. In Figure 4-4, it can be seen that when the minimum control performance index is used, expectedly and encouragingly, smooth and conservative-valued radial dynamics for the cable result. The reel acceleration required to perform this manoeuvre is linear for approximately 30 seconds, which in turn leads to a quadratic-type reel rate and a cubic deployment profile for the cable over this time span. Non-linear reel acceleration is then required to manoeuvre the cable tip for rendezvous with zero reel velocity. Overall, it can be concluded that the amount of control required during this manoeuvre is modest, with associated energy needs required by the cable reel mechanism likely to be low. The profile the cable takes up during the rendezvous manoeuvre is given in Figure 4-5. The cable tip follows an unadulterated smooth trajectory, initially being retrieved slightly, before diving steadily towards the target and then pulling-up marginally as the target is approached. The final pull-up motion is consistent with the need for the cable to have zero deployment velocity at the instantaneous point of rendezvous.

The tension time history for the cable shown in Figure 4-6 indicates that the tension in the cable steadily increases during the initial quadratic deployment regime, before reducing slightly and then steadily increasing as the cable tip approaches the target position. Nevertheless, the actual tension levels in the cable are neither ominously high nor low.

**Minimum Time**

Optimal control problems formulated using minimum time performance indices form a potentially useful set of problems governing the instantaneous rendezvous of ATC systems.
Although solutions obtained using such a performance index often result in “bang-bang” type trajectories that are accompanied by undesirably large variations in the system controls, they set the lower bound for the time required to achieve rendezvous. The cost for this case is:

\[ J_2 = t_f \]  

(4.6.9)

The results for the minimum time optimal control problem are given in Figure 4-7 through to Figure 4-9. As previously anticipated, it can be seen from Figure 4-7 that use of the minimum time performance index results in a series of “bang-bang” type control inputs over the duration of the manoeuvre. Moreover, the control input is such that approximately equal amounts of cable deployment are required in the initial and final halves of the rendezvous attempt. Most cable reel acceleration is focussed towards the beginning and the end of each half of the manoeuvre, where the control input alternates rapidly from minimum to maximum, with a short period of rapidly oscillating control occurring in between.

![Graphs showing minimum time cost: Cable Length, Length Rate, and Control](image)

**Figure 4-7: Minimum Time Cost- (a) Cable Length (b) Length Rate (c) Control**

As can be inferred from Figure 4-8, the large “bang-bang” control results in dual periods where the cable alternates between being deployed and retrieved. As the periods of deployment are longer than the periods of retrieval, the cable tip is able to travel towards the target until rendezvous is achieved. Figure 4-9 demonstrates another problematic issue associated with large “bang-bang” type control inputs, namely significant and abrupt changes to the tension within the cable. The tension levels within the cable both sharply rise and fall in relatively short periods of time (approximately 2 seconds in certain instances); the magnitude of these changes are in the order of ± 600 Newtons. In practice, this may lead to the excitation of unwanted and self-exciting longitudinal modes of vibration in the cable.
Optimal control problems formulated using hybrid optimality criteria are used to deduce trajectories for the system that balance various and often competing requirements. A popular hybrid is the minimum time-minimum control performance index, which attempts to temper the often control-heavy trajectory rendered by a minimum time performance index and impart a sense of restraint to the system. Mathematically, the hybrid performance index is:

$$ J_3 = t_f + \int_{t_0}^{t_f} u^2 \, dt $$  \hspace{1cm} (4.6.10) 

Figure 4-10 to Figure 4-12 outline the results for the minimum time/control rendezvous case.
Contrasting the results obtained for the minimum control and minimum time rendezvous problems, with those acquired for the hybrid minimum time and control case, it is clear that the dominant constituent of the hybrid performance index is that associated with minimum control. It can be deduced from Figure 4-8 and Figure 4-11 that the cable is manoeuvred in a similar manner with comparable levels of control actuation, when either the minimum control or hybrid minimum time/minimum control performance indices are used. The inclusion of the minimum control term within hybrid index ensures that “bang-bang” type control inputs do not transpire and large, abrupt changes in the cable tension are avoided.
From a practical point of view, it may be necessary to keep the absolute deployment rate of the cable to a minimum. Although direct control is achieved via reel acceleration, the length rate may be indirectly minimized using a performance index of the form:

$$J_4 = \int_{t_0}^{t_f} \dot{L}^2 \, dt \quad (4.6.11)$$

The results concerning the minimum length rate optimal control problem are given in Figure 4-13 through to Figure 4-15.
Interestingly, the use of a minimum length rate performance index does not guarantee the prevention of large control inputs during the rendezvous manoeuvre, as evidenced by Figure 4-13. Large and rapidly changing reel accelerations occur at the very beginning and end of the manoeuvre, which consequently result in sharp tension spikes (± 300 Newtons) within the cable as depicted in Figure 4-15. However, for the overwhelming majority of the manoeuvre, no control input is forthcoming and the cable reel mechanism maintains a constant deployment rate of approximately 10 m/s, resulting in a quasi-linear cable length profile. As a result, the cable tip itself follows a quasi-linear trajectory throughout the duration of the manoeuvre as shown in Figure 4-14.
Minimum Cable Orientation Angle

The action of external aerodynamic drag on the cable tends to increase the cable orientation angle $\theta$, which could become a critical issue in certain instances, where the potential for contact with the aircraft superstructure is likely. Accordingly, it may be desirable to minimize the orientation angle using the following performance index:

$$J_5 = \int_{t_0}^{t_f} \theta^2 \, dt \quad (4.6.12)$$

The results pertaining to the minimum cable orientation angle optimal control problem are given in Figure 4-16 through to Figure 4-18.

![Figure 4-16: Minimum Cable Angle Cost- (a) Cable Length (b) Length Rate (c) Control](image1)

![Figure 4-17: Minimum Cable Angle Cost- Cable Profile](image2)
Comparing the cable profile obtained using the minimum length rate performance index displayed in Figure 4-14, with the profile obtained using the minimum cable orientation angle cost, shown in Figure 4-17, several similarities arise. As demonstrated in Figure 4-16, again large and rapidly changing reel accelerations occur at the very beginning and end of the manoeuvre, slightly more pronounced for the minimum cable angle cost as opposed to that for minimum length rate. The larger and faster changing reel accelerations consequently result in sharper and greater tension spikes (± 300-500 Newtons) within the cable as depicted in Figure 4-18. Although, once again almost no control input is required for most of the manoeuvre, resulting in constant deployment and a subsequent quasi-linear cable length profile and tip trajectory, as shown in Figure 4-16 and Figure 4-17.

**Minimum Cable Angular Rate**

Under some circumstances, it may be desirable to ensure that the “swing” motion of the cable is kept to a minimum. One way to do this is to employ the following performance index, formulated solely in terms of the cable angular rate:

$$J_\phi = \int_{t_0}^{t_f} \theta^2 \; dt \quad (4.6.13)$$

The results associated with the minimum cable angular rate optimal control problem are given in Figure 4-19 through to Figure 4-21.
Although minimizing the “swing” motion of the cable may seem laudable, the most obvious way to achieve this, through the use of a minimum cable angular rate performance index, does not necessarily produce favourable outcomes in practice. This is fittingly substantiated by Figure 4-19 through to Figure 4-21 for numerous reasons.

Overall, to ensure that the cable angular rate is minimized, the reel acceleration required is considerable, mimicking a pronounced version of “bang-bang” type control, where the control input rapidly cycles from minimum to maximum. Unpromisingly, this results in large amplitude-high frequency tension waves to be repeatedly induced within the cable, some in the order ±500 Newtons, which should be avoided at all times for obvious reasons. It appears
that the “bang-bang” nature of the controls is used as a strategy to counter the rate at which the cable swings in one radial direction by causing the cable to travel in another direction. This is demonstrated in Figure 4-19 where repeated short bursts of deployment and retrieval are observed to occur, resulting in the non-smooth tip trajectory illustrated in Figure 4-20.

![Figure 4-21: Minimum Cable Angular Rate Cost- Cable Tension](image)

**Minimum Cable Angular Acceleration**

As mentioned previously, an additional means to ensure smooth dynamics for the ATC system is to minimize the angular acceleration of the cable as follows:

\[
J_\gamma = \int_{t_i}^{t_f} \ddot{\theta}^2 \, dt \tag{4.6.14}
\]

The results for the minimum cable angular acceleration optimal control problem are given in Figure 4-22 through to Figure 4-24. It was expected that smooth dynamics for the ATC system would result when the minimum cable angular acceleration optimality criteria was used, and Figure 4-22 through to Figure 4-24 only partially confirm this hypothesis. The reel acceleration required for the minimum angular acceleration scenario is considerable and of a restrained “bang-bang” type as seen in Figure 4-22. The reel acceleration required in the very initial and final stages of the manoeuvre is large and rapidly changing. However as depicted Figure 4-23, the resulting trajectory the cable tip follows during the manoeuvre is smooth, approaching the profile given by Figure 4-5 for the case of minimum control. Similarly, for minimum angular acceleration, the cable deployment profile and length time history are relatively smooth, although to a lesser degree than the corresponding trajectories associated with the case for minimum control.
As substantiated by Figure 4-24 however, the point at which the results for minimum control and angular acceleration diverge, are those pertaining to the level and nature of the tension within the cable. The restrained “bang-bang” type control inputs associated with minimum angular acceleration produce large amplitude tension spikes within the cable, some in excess of ±400 Newtons. Overall, when the minimum angular acceleration performance index is employed, the tension levels within the cable fluctuate significantly, both in magnitude and frequency. As established from Figure 4-6 previously, these phenomena were not observed when the minimum reel acceleration performance criterion was used.
Therefore, it can be reasonably surmised that the minimization of the accelerations pertaining to the ATC system do not produce comparable and interchangeable results. Each option should be pursued and explored separately to ascertain which produces the most satisfactory scenario. Clearly, the minimum control/reel acceleration scenario is more preferable to that afforded by the minimum cable angular acceleration cost function.

**Minimum Tension**

For heavy or rapidly accelerating payloads, the tension in the cable can grow to alarmingly high levels, which obviously should be addressed. Although seldom attempted in practice, one way to do this is to directly minimize the tension within the cable. With the aid of Figure 2-4, the cable tension $T_n$ is found by applying Newton’s Second Law to the payload:

$$T_n = \frac{m_ya_y - (F_{\text{drag}_y} - F_g)}{\cos \theta}$$

(4.6.15)

where:

- $a_y$ is the inertial acceleration of the payload in the $y$-direction,
- $F_{\text{drag}_y}$ is the aerodynamic drag acting on the payload in the $y$-direction,
- $F_g$ is the gravity force acting on the payload.

The inertial acceleration of the payload in the $y$-direction is found by differentiating equation (2.4.2) with respect to time, and neglecting terms involving the aircraft, the result being:

$$a_y = -\ddot{\ell} \cos \theta + 2\dot{\theta} \sin \theta + \dot{\theta} \sin \theta + i\ddot{\theta}^2 \cos \theta$$

(4.6.16)
The aerodynamic drag acting on the payload in the $y$-direction can be found using:

$$
F_{\text{drag},y} = \frac{-\sin \theta}{8} \rho_{\text{air}} C_{Dy} \pi d_p^2 \left( l \dot{\theta} - U_0 \cos \theta \right) \left( U_0^2 + i^2 + l^2 \dot{\theta}^2 - 2U_0i \dot{\theta} \sin \theta - 2U_0 \dot{\theta} \cos \theta \right)^{1/2} \tag{4.6.17}
$$

By inspection, the gravitational force acting on the payload is simply:

$$
F_g = m_p g \tag{4.6.18}
$$

The minimum tension performance index, given by equation (4.6.19), can now be constructed after substitution of equation (4.6.15) through to equation (4.6.18), the result being:

$$
J_s = \int_{t_0}^{t_f} T_n^2 \, dt \tag{4.6.19}
$$

The results found using the minimum tension cost are given in Figure 4-25 to Figure 4-27.

![Figure 4-25: Minimum Tension Cost- (a) Cable Length (b) Length Rate (c) Control](image)

The most obvious characteristic that can be deduced from the results given by Figure 4-25 through to Figure 4-27 is that they are strikingly similar to those associated with the case of minimum control given previously in Figure 4-4 to Figure 4-6. The main noticeable difference between the two sets of results is the slight difference in the reel acceleration required for rendezvous, which in turn causes changes to the radial dynamics of the cable, the cable tip trajectory and tension within the cable. The use of a minimum tension cost results in a tip trajectory that is a heavily moderated version of that found when the minimum time performance index was used, yet similar to that corresponding to minimum reel acceleration.
Counter-intuitively, initially the tension levels within the cable are slightly higher for the minimum tension case as compared to the minimum control scenario, due to the slightly lower absolute magnitudes of the reel acceleration that are necessary in the initial stages of the manoeuvre. In spite of this, the overall variation of the tension within the cable is expectedly lower when the minimum tension performance criterion is employed.
Minimum Tension Rate

For safety and/or performance-related issues, it is often pertinent to avoid sudden changes in the tension within the cable and secure smooth and continuous cable dynamics. This can be achieved by minimizing the tension rate of the cable, which requires the introduction of an additional state variable $x_5$ representing the cable tension $T_n$, along with an additional control variable $u_2$ representing the cable tension rate $\dot{T}_n$. Similarly, an additional path constraint is required to force the additional state variable $x_5$ to follow the cable tension $T_n$ trajectory given by equation (4.6.15). The performance index required for the minimum tension rate problem is:

$$J_0 = \int_{t_0}^{t_f} u_2^2 \, dt$$  \hfil (4.6.20)

The results obtained for the minimum tension rate optimal control problem are given in Figure 4-28 through to Figure 4-30.

Together with the results found using the minimum control and tension performance indices, the results obtained for minimum tension rate rendezvous, as shown in Figure 4-28 through to Figure 4-30, are superlative. The well-behaved radial dynamics, coupled with the modest control input offered in Figure 4-28, and the continuous cable tip trajectory shown in Figure 4-29, all conspire to produce an attractive rendezvous scenario for the ATC system.
The inclusion of the additional state and control variables to allow for the direct minimization of cable tension rate deftly engineers a moderated tension time history within the cable, as illustrated in Figure 4-30. The ever-so slight increase in the sophistication of the ATC system model could be justified by the quality of the manoeuvre it produces.
Minimum Power

With an eye to the potential energy requirements needed by the electro-mechanical device employed to drive the reel mechanism for the cable, it may be advantageous to minimize the rate at which the cable does work. The cost function for this minimum power case is:

$$J_{10} = \int_{t_0}^{t_f} (T_n \dot{r})^2 \, dt$$  \hspace{1cm} (4.6.21)

The results pertaining to the minimum power optimal control problem are given in Figure 4-31 through to Figure 4-33.

![Figure 4-31: Minimum Power Cost- (a) Cable Length (b) Length Rate (c) Control](image1)

![Figure 4-32: Minimum Power Cost- Cable Profile](image2)
By contrasting the results offered in Figure 4-13 through to Figure 4-15 with those on hand in Figure 4-31 through to Figure 4-33, the use of a minimum length rate performance index produces a manoeuvre that is very similar to that produced through the use of a minimum power index. The major difference between the two sets of results is that the inclusion of the tension term within the minimum power performance criterion tends to smooth out the discontinuities in the cable reel acceleration. Increasingly continuous cable control, consequentially results in a more favourable deployment profile and less fluctuation in the tension levels within the cable. The overall nature of the two manoeuvres is essentially the same, which is not totally unforeseen since the instantaneous power of the cable is directly proportional to the length rate of the cable.

Minimum Altitude

In certain practical applications like remote sensing or subterranean mapping, it is desirable for the cable tip to closely track the surface of local terrain. To do this, the altitude of the payload should be minimized, achieved using the following cost function:

$$ J_{II} = \int_{t_i}^{t_f} \left( y_{TOL} - y_e \right)^2 \, dt $$ \hspace{1cm} (4.6.22)  

The results obtained for the minimum altitude optimal control problem are given in Figure 4-34 through to Figure 4-36.
To minimize the altitude of the cable tip/payload, it can be seen from Figure 4-34 that large control inputs are initially required to position the cable tip as close to the ground as possible. This is followed by significant “bang-bang” type control applied thereafter as the cable tip tracks the local flat surface terrain until the desired target point is reached. The opening control input consists of rapid deployment to the maximum deployment speed, then increasingly reduced deployment until no deployment is required, before alternating between modest deployment and retrieval when the tip is in the vicinity of the ground. This scenario is depicted in Figure 4-35, which illustrates how the cable tip initially loses altitude rapidly, before oscillating in altitude in the vicinity of the ground until rendezvous is achieved.
As can be seen in Figure 4-36, the control trajectory given in Figure 4-34 leads to an initial pronounced tension increase within the cable of approximately 450 Newtons, quickly followed by a short period of large amplitude oscillations (~300 Newtons). The tension levels then briefly stabilize as cable reel acceleration ceases, before rapidly oscillating with large amplitude (~550 Newtons) once again when the cable tip is close to the ground and the level of “bang-bang” type control is increased. The overall nature of the minimum altitude scenario is consistent with what one would intuitively expect: rapid initial control action to locate the cable tip close to the ground, although the significant control input required to keep the tip in the vicinity of the ground until rendezvous occurs is somewhat unexpected.

**Minimum Distance to Target**

Yet another way to achieve rendezvous between the cable tip and a desired location on the ground is to minimize the instantaneous distance separating the cable tip and the ground-based target. The performance index to achieve this is:

\[
J_{t2} = \int_{t_0}^{t_f} \left( (x_c - x_r)^2 + (y_c - y_r)^2 \right) \, dt \tag{4.6.23}
\]

The results depicting the minimum distance to the target scenario are given in Figure 4-37 through to Figure 4-39. Whilst the results obtained using the minimum altitude performance index were intuitively anticipated, the results acquired for this scenario are certainly not. Referring to the cable profile in Figure 4-38, one would expect that in order to reduce the separation between the cable tip and the target, the cable tip should follow a linear trajectory such as those outlined in Figure 4-14, Figure 4-17 and Figure 4-32, since this represents the shortest path between the initial position of the cable tip and the desired target on the ground.
Nevertheless, this situation is not reflected in Figure 4-38, where the cable tip follows a curved trajectory reminiscent of that observed for the minimum time manoeuvre. Figure 4-37 shows that significant reel acceleration is initially undertaken to rapidly retrieve the cable and re-position it closer to the aircraft, upon which the cable is swiftly deployed, causing the cable tip to follow the anticipated quasi-linear trajectory for approximately 25 seconds. Following on, the cable is then briefly retrieved for approximately 5 seconds, after which rapid deployment is repeated for a second time until rendezvous occurs. These rapidly changing, “bang-bang” type control inputs induce large amplitude tension peaks and troughs within the cable, some in the order of 600 Newtons, as revealed in Figure 4-39.
The effect that the chosen performance index has on the rendezvous problem for the ATC system is significant and warrants special and detailed attention, as has been demonstrated in this sub-section. To develop an additional layer of detail, various specifics concerning both the numerical solution and the output obtained from each performance index scenario are summarized in Table 4-3.

<table>
<thead>
<tr>
<th>Cost Function Name</th>
<th>Final Time [secs]</th>
<th>Solution Time [secs]</th>
<th>Min/Max Tension [N]</th>
<th>Min/Max Control [m/s²]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum Control</td>
<td>48.5677</td>
<td>64.3435</td>
<td>692.04</td>
<td>1167.8</td>
</tr>
<tr>
<td>Minimum Time</td>
<td>48.1394</td>
<td>114.6248</td>
<td>572.547</td>
<td>1186</td>
</tr>
<tr>
<td>Minimum Time &amp; Control</td>
<td>48.5675</td>
<td>91.4114</td>
<td>692.161</td>
<td>1167.8</td>
</tr>
<tr>
<td>Minimum Length Rate</td>
<td>48.3005</td>
<td>150.3462</td>
<td>572.547</td>
<td>1177.9</td>
</tr>
<tr>
<td>Minimum Cable Angle</td>
<td>48.366</td>
<td>142.5049</td>
<td>572.547</td>
<td>1175.9</td>
</tr>
<tr>
<td>Min. Cable Angular Rate</td>
<td>48.2899</td>
<td>192.9274</td>
<td>572.547</td>
<td>1183.8</td>
</tr>
<tr>
<td>Min. Angular Acceleration</td>
<td>48.4830</td>
<td>561.0367</td>
<td>708.189</td>
<td>1172.1</td>
</tr>
<tr>
<td>Minimum Tension</td>
<td>48.3337</td>
<td>448.1043</td>
<td>744.314</td>
<td>1176.8</td>
</tr>
<tr>
<td>Minimum Tension Rate</td>
<td>48.4445</td>
<td>482.3636</td>
<td>715.944</td>
<td>1171.2</td>
</tr>
<tr>
<td>Minimum Power</td>
<td>48.3076</td>
<td>616.9471</td>
<td>572.547</td>
<td>1177.6</td>
</tr>
<tr>
<td>Minimum Altitude</td>
<td>49.0314</td>
<td>80.0251</td>
<td>572.547</td>
<td>1189</td>
</tr>
<tr>
<td>Minimum Distance</td>
<td>48.2878</td>
<td>102.5675</td>
<td>573.71</td>
<td>1188.8</td>
</tr>
</tbody>
</table>

When rigorously and objectively compared to all other performance indices, it is apparent from Table 4-3 together with the preceding discussion, that the use of the minimum control performance index is preferred over all others. Not only does it enjoy the quickest solution time and require the smallest overall control input, only approximately 0.43 secs longer is needed to achieve rendezvous than the absolute minimum time associated with the minimum
time scenario. The scenario that takes the longest to perform is the minimum altitude manoeuvre, although this is only 0.89 secs longer than the minimum, hence one can conclude that the choice of performance index has little effect on the time required for rendezvous. In terms of the time required to obtain a solution, the minimum power performance index attracts the longest time period and there exists a large spread in the data pertaining to the rest of the scenarios. Once again the actual solution times themselves may not be totally accurate, but the important point to consider is relative order of the solution time and how the solutions for the various performance indices require different times to compute. With respect to the absolute magnitude of the tension within the cable, the minimum altitude and minimum distance to target performance indices produce the largest tension values, while the minimum time, angle, angular rate, length rate, power, and altitude optimality criteria experience the lowest. Similarly, the breadth of control required by all performance indices traverses from absolute minimum to absolute maximum except for the minimum control, minimum time and control, minimum angular acceleration, tension and tension rate cases. To consolidate further, it can be deduced from Table 4-3 that the choice of the performance index has a considerable effect on the solution time, magnitude of the tension within the cable and control required to successfully perform each rendezvous manoeuvre.

The main outcome uncovered from the results presented in this sub-section is that judicious choice of the performance index utilized in the formulation of the ATC system rendezvous problem is essential. Unless a unique and rather specialized objective such as minimum cable tip altitude is essential, all else being equal, the performance index that should be employed for rendezvous problems is that of minimum reel acceleration. The adoption of a minimum reel acceleration performance index will ensure well-behaved system radial dynamics result, leading to modest and smooth control inputs and low associated energy needs for the reel mechanism. The cable tip will follow an unadulterated smooth trajectory and the tension within the cable will be continuous and well behaved. Such a solution will be rapidly determined and require low computational overheads. Similar persuasive arguments can be formulated to support the case for use of the minimum tension and tension rate performance indices, however the minimum reel acceleration scenario is preferred due to its simpler implementation and significantly lower solution time.
4.6.5 Instantaneous Rendezvous Incorporating Aircraft Manoeuvres

This sub-section continues with the introduction, formulation and solution of optimal control problems governing rendezvous for different configurations of the ATC system. Principally, the restrictions governing the manner in which the aircraft is flown will be sequentially eased and the effect that this has on the nature of the rendezvous scenario will be investigated.

Rendezvous With Rectilinear Manoeuvring of the Aircraft

The simplest possible rendezvous operation that the ATC system can perform was pictorially depicted in Figure 4-1. In a similar vein to that scenario, the aircraft is still restricted to fly at a constant altitude, but is permitted to accelerate rectilinearly in the direction tangential to its longitudinal axis (in the \(x\)-direction). With this increased control actuation, the cable is once again deployed so as to rendezvous the tip with a desired location on the ground, whilst avoiding any collisions with the terrain en-route. This rendezvous problem begins with the ATC system in the equilibrium configuration governed by the physical parameters shown in Table 4-1, with the manoeuvre terminating once rendezvous takes place. The mathematical model of the ATC system used for this rendezvous problem is that given by equation (2.4.36). The single-phase optimal control problem that characterizes this rendezvous scenario will now be formally introduced.

4.6.5.1.1 Performance Criterion

The performance index \(J\) for this rendezvous problem is a variant of the minimum control optimality criteria given by equation (4.6.8). It is slightly modified here to incorporate the additional actuation made possible by the horizontal acceleration of the aircraft:

\[
J = \int_{t_0}^{t_f} \left( x^2 + \dot{x}^2 \right) dt = \int_{t_0}^{t_f} u_1^2 + u_2^2 dt \quad (4.6.24)
\]

4.6.5.1.2 Dynamic Constraints

The dynamical state-space equality constraints to be satisfied as part of the optimal control problem governing rendezvous are formulated using equation (2.4.36). Once again the model “matching” technique that incorporates dynamic inertial effects is utilized to improve the representativeness of the resulting rendezvous problem.

4.6.5.1.3 Initial Conditions

As mentioned previously, the ATC system is initially in equilibrium, the particular configuration of which is determined by the chosen physical parameters for the ATC system.
Mathematically, the initial conditions for the rendezvous problem are given by:

\[
\psi_0[\dot{x}_0, x_0, t_0] = \left( x \ \theta \ l \ \dot{x} \ \dot{\theta} \ i \right)^T \left( x_0 \ \theta_0 \ l_0 \ \dot{U}_0 \ 0 \ 0 \right)^T = 0 \tag{4.6.25}
\]

where \( x_0 \) is the initial position of the aircraft in the \( x \)-direction from a reference point, taken to be the origin of the global Cartesian reference frame, located at the coordinates \( \{0, 0\} \).

### 4.6.5.1.4 Final Conditions

Once again the final boundary conditions governing rendezvous are specified in terms of the in-plane position of the cable tip. The formula that calculates the \( y \)-coordinate of the cable tip remains unchanged [given by equation (4.6.3)], while the \( x \)-coordinate \( x_c \) is given by:

\[
x_c = x - l \sin \theta \tag{4.6.26}
\]

where \( x \) is the horizontal displacement of the aircraft. In addition to the final position of the cable tip specifying the conditions for rendezvous, the reel rate of the cable is constrained to zero at the instant rendezvous occurs. The final conditions for this rendezvous problem are the same as those formulated for the previous constant tow speed rendezvous problem [given by equation (4.6.4)], except that equations (4.6.3) and (4.6.26) are used to determine the appropriate position of the cable tip.

### 4.6.5.1.5 Non-Linear Path Constraints

The non-linear path constraint given by equation (4.6.5) is utilized for this rendezvous problem to ensure that the cable tip/towed body is prevented from colliding with the ground. Once again, equations (4.6.3) and (4.6.26) are used to determine the position of the cable tip in order to correctly specify the non-linear path constraint for this rendezvous problem. The value of the collision tolerance \( y_{TOL} \) remains unchanged from the value used previously.

### 4.6.5.1.6 Box Constraints

The following dimensional box constraints are placed on the state and control variables for the ATC system during this rendezvous procedure:

\[
\begin{align*}
\left\{ x_{\text{MIN}} \ \theta_{\text{MIN}} \ l_{\text{MIN}} \ \dot{x}_{\text{MIN}} \ \dot{\theta}_{\text{MIN}} \ i_{\text{MIN}} \right\}^T \leq x \leq \left\{ x_{\text{MAX}} \ \theta_{\text{MAX}} \ l_{\text{MAX}} \ \dot{x}_{\text{MAX}} \ \dot{\theta}_{\text{MAX}} \ i_{\text{MAX}} \right\}^T \\
\left\{ -\infty \ -90' \ 10 \ -\infty \ -5 \right\}^T \leq \left\{ \left\{ \left\{ x_{\text{MIN}} \ \theta_{\text{MIN}} \ l_{\text{MIN}} \ \dot{x}_{\text{MIN}} \ \dot{\theta}_{\text{MIN}} \ i_{\text{MIN}} \right\}^T \right\}^T
\end{align*}
\]

\[
\begin{align*}
\left\{ \ddot{x}_{\text{MIN}} \ \ddot{\theta}_{\text{MIN}} \ i_{\text{MIN}} \right\}^T \leq u \leq \left\{ \ddot{x}_{\text{MAX}} \ \ddot{\theta}_{\text{MAX}} \ i_{\text{MAX}} \right\}^T \\
\left\{ -2.5 \ -2.5 \right\}^T \leq u \leq \left\{ 2.5 \ 2.5 \right\}^T
\end{align*}
\tag{4.6.28}
\]
An actual solution to the optimal control problem governing rendezvous for the rectilinear ATC system is delayed temporarily. Since the nature of ATC system rendezvous with varying degrees of aircraft manoeuvring is to be explored; the results for each scenario will be compared simultaneously using a single set of results at the conclusion of the next subsection.

**Rendezvous With In-plane Manoeuvring of the Aircraft**

The rendezvous problem for the ATC system with in-plane aircraft manoeuvring enabled is shown in Figure 4-40. In this scenario, the aircraft is not restricted to fly at a constant altitude or speed, as it is free to perform any in-plane manoeuvre of its choosing (subject to appropriate performance and motion constraints).

![Figure 4-40: Depiction of the Rendezvous Scenario With In-plane Aircraft Manoeuvring](image)

With full in-plane control actuation available, the cable is deployed in a similar manner as before so that its tip can rendezvous with a desired location on the ground, without colliding with the prevailing terrain. The scenario begins with the ATC system in equilibrium with the same the physical parameters as given in Table 4-1. The mathematical model of the ATC system used for this rendezvous problem is that given by equation (2.4.35). The single-phase optimal control problem that characterizes this rendezvous scenario will now follow.

### 4.6.5.1.7 Performance Criterion

The performance index $J$ for this rendezvous problem is the minimum control optimality criterion given by equation (4.6.29):

$$J = \int_{t_0}^{t_f} \left( x^2 + y^2 + \dot{\theta}^2 \right) dt = \int_{t_0}^{t_f} u_1^2 + u_2^2 + u_3^2 dt \quad (4.6.29)$$
4.6.5.1.8 Dynamic Constraints
The dynamical state-space equality constraints to be satisfied for this optimal control problem are formulated using equation (2.4.35). The same model “matching” technique used for previous rendezvous problems is also used here.

4.6.5.1.9 Initial Conditions
Mathematically, the initial conditions for this rendezvous problem are given by:

\[ \psi_0(x_0, y_0, \theta_0, l_0, U_0, W_0, 0, 0)^T = 0 \]  \hspace{1cm} (4.6.30)

where \( y_0 \) and \( W_0 \) are the initial altitude and vertical velocity of the aircraft, equal to 600 metres and 0 m/s respectively. Since the initial equilibrium configuration of the ATC system is assumed to occur for the condition that the aircraft is in steady level flight, the initial vertical velocity of the aircraft is always zero.

4.6.5.1.10 Final Conditions
In addition to zero reel rate for the cable at rendezvous, the boundary conditions governing this rendezvous problem are given by the final position of the cable tip. Equation (4.6.26) is used to specify the \( x \)-coordinate of the cable tip, whilst the \( y \)-coordinate \( y_c \) is given by:

\[ y_c = y - l \sin \theta \]  \hspace{1cm} (4.6.31)

where \( y \) is the vertical displacement of the aircraft. The final conditions for this rendezvous problem are the same as those formulated for the previous rendezvous problems [given by equation (4.6.4)], except that equations (4.6.26) and (4.6.31) are used to determine the appropriate position of the cable tip. It is important to note that the final conditions for this rendezvous problem have a different physical meaning from those associated with the constant tow speed and rectilinear manoeuvring rendezvous problems. In this case, the motion of the cable tip is not required to be tangential to the ground at rendezvous, implying that the direction the cable tip approaches the target point is arbitrary due to the degree of freedom provided by aircraft manoeuvring. Even if coupled with an appropriate non-linear path constraint, the collision of the cable tip with the ground immediately after rendezvous is not strictly prevented. However, collisions before rendezvous takes place are always prevented through the use of non-linear path constraints. Measures that prevent collisions between the cable tip and the ground immediately after rendezvous are outlined in Chapter 6 of this thesis.
4.6.5.1.11 Non-Linear Path Constraints

In order to correctly specify the non-linear path constraint for this rendezvous problem, the condition outlined by equation (4.6.5) is employed, along with equations (4.6.26) and (4.6.31) to determine the position of the cable tip. The collision tolerance $y_{tol}$ remains unchanged.

4.6.5.1.12 Box Constraints

The same dimensional box constraints that were employed in previous rendezvous problems were placed on the state and control variables during this rendezvous procedure, except for the following inclusions and variations:

\[
\begin{align*}
\{y_{\text{MIN}}, \dot{y}_{\text{MIN}}\} & \leq x \leq \{y_{\text{MAX}}, \dot{y}_{\text{MAX}}\}^T \\
\{-\infty, -10\}^T & \leq x \leq \{\infty, 10\}^T
\end{align*}
\]

\[
\begin{align*}
\{\dot{x}_{\text{MIN}}, \ddot{x}_{\text{MIN}}, \dddot{x}_{\text{MIN}}\} & \leq \{\dddot{x}_{\text{MAX}}, \dddot{y}_{\text{MAX}}, \dddot{y}_{\text{MAX}}\}^T \\
\{-2.5, -2.5, -2.5\}^T & \leq \{2.5, 2.5, 2.5\}^T
\end{align*}
\]

A complete set of results pertaining to the various in-plane, instantaneous rendezvous problems the ATC system is capable of performing are presented in Figure 4-41 through to Figure 4-46. For a complete comparison to occur, the results for the constant tow speed rendezvous previously given in Figure 4-2 through to Figure 4-6 are included in this set. The results concerning the rendezvous problem with full in-plane aircraft manoeuvring are given by the blue line (denoted “Plan” in the legend of each figure), the results for the rendezvous problem with rectilinear aircraft manoeuvring are given by the green line (denoted “Rect” in the legend of each figure), whilst the results for the constant tow speed rendezvous problem are designated using the red line (denoted “Const” in the legend of each figure).

With respect to the angular dynamics of the cable during rendezvous, it can be seen from Figure 4-41 that the cable swings further and more freely when rectilinear aircraft manoeuvres are used as opposed to the cases for full planar or no aircraft manoeuvring. Regarding the time required for rendezvous, it can be seen from Figure 4-41 that when rectilinear manoeuvres for the aircraft are allowed, it takes 8.25 % longer to achieve rendezvous than would otherwise be the case if the tow speed were constant. Further time savings of 3.62 % are possible if full planar manoeuvres for the aircraft are employed.
Considering the radial dynamics of the cable, it can be deduced from Figure 4-42 that in order to reach the same rendezvous point, a significantly longer cable is required when the tow speed is restricted to be constant; with the length of the cable required for rendezvous reducing as the level of sophistication in aircraft manoeuvring is increased. Consequentially, the speed at which the cable should be deployed is significantly greater for the cases where the aircraft manoeuvring is restricted, with the largest deployment rates required when the aircraft speed is constant, followed by the case where the aircraft speed is allowed to vary rectilinearly. The overall nature of the deployment profile in each scenario is the same; each essentially differs by the absolute rates at which the cable should be deployed or retrieved.
With respect to the cable tip trajectory, Figure 4-43 shows that irrespective of the level of aircraft manoeuvring, the path the cable tip follows during rendezvous is smooth and well behaved due to the use of a minimum control cost function in each case. Although the cable tip trajectories are similar in each instance, the trajectory tends to flatten out as the level of manoeuvring for the aircraft is increased, indicating a tendency for the aircraft to dominate the execution of the rendezvous manoeuvre.

To successfully perform rendezvous, initially the cable should be deployed as quickly as possible to the maximum permitted value; the deployment rate is then slowed quickly as the cable tip approaches the target location to satisfy the zero reel rate final condition. If the aircraft is permitted to manoeuvre, then for rendezvous with minimum control, Figure 4-44 points out that the horizontal and vertical descent velocity of the aircraft should be reduced and increased respectively. If possible, the aircraft will tend to dive with an increasing rate of descent and reduced horizontal velocity in order to successfully carry out rendezvous for minimum overall control. Similarly, if the aircraft is permitted to dive, then the reduction in horizontal velocity is not as pronounced as is the case when only rectilinear motion is allowed. In reality, the minimum speed at which the aircraft can travel and the rate at which it is permitted to dive is limited by the stall, structural loading and other performance characteristics, hence the need for box constraints to be applied to the aircraft velocity components. If the steep dive at slow horizontal speed manoeuvre shown in Figure 4-44 cannot be physically performed by a candidate aircraft, a feasible way to overcome this would be to begin the rendezvous manoeuvre earlier, allowing more time available for rendezvous.
With respect to the control input required to perform in-plane rendezvous, Figure 4-45 confirms the previous claim that the aircraft will tend to dominate the execution of the rendezvous manoeuvre if allowed to do so. Referring to the portion of the figure denoting the cable reel acceleration time history $u_3$, it can be seen that significantly lower adjustments are made to the reel acceleration when the aircraft is permitted to manoeuvre. The overall nature of the reel acceleration time histories for each rendezvous case is similar, except that the absolute rate at which the reel acceleration is applied varies between each scenario. Certain additional corrections are required to the cable reel acceleration late in the constant tow speed scenario. Figure 4-45 suggests that in order to successfully rendezvous the tip of the cable with a desired location on the ground when the aircraft in-plane motion is unrestricted, the
cable reel acceleration should be linear, which in turn leads to the quadratic-type reel rate and cubic deployment profile for the cable, shown previously in Figure 4-42. The cable reel acceleration time history becomes increasing non-linear as the restrictions on the aircraft motion increase accordingly. Similarly, the use of the minimum control performance index ensures that the components of the aircraft acceleration vary linearly over the duration of the rendezvous manoeuvre.

During instantaneous rendezvous operations, the ATC system tends to allocate the control burden evenly across all its actuators. Thus, when full planar aircraft manoeuvres are allowed, the control required by the ATC system is relatively low, since the workload is dispersed uniformly across the aircraft and reel actuators. As the aircraft degree of freedom reduces, for the same rendezvous operation, progressively more actuation is required by a diminishing number of actuators. This results in incrementally more control effort to be performed by the cable reel mechanism. Overall, the level control actuation required by the ATC system is evenly distributed throughout rendezvous, in line with the degree of aircraft-related control.

As a result of the increased dynamic inertial loads acting on the cable, the value of the tension within the cable during rendezvous is generally much larger when the aircraft is restricted in manoeuvring, as evidenced by Figure 4-46. The dominant action of the control inputs conspires to lower the tension within the cable in the initial stages of rendezvous procedures involving constant speed and rectilinear aircraft manoeuvring. When the nature of the aircraft motion is restricted to being rectilinear, dynamic inertial loads then become significant as the scenarios progress, resulting in rapid increases in tension acting within the cable.
The large control inputs occurring late in the rendezvous manoeuvre that are produced when the tow speed is constant cause the tension within the cable to reach appreciably higher values than would otherwise be the case if the planar or rectilinear manoeuvring is permitted. The tension variation (increase) in the cable during rendezvous is quasi-linear when the aircraft performs planar manoeuvres and increasingly non-linear when the degree of freedom of the aircraft is incrementally reduced.

*Rendezvous With Three Dimensional Aircraft Manoeuvring*

Whilst the previous rendezvous problems for the ATC system were two dimensional in nature, the instantaneous rendezvous problem for the system can be generalized further by incorporating the full three-dimensional motion capabilities of the ATC system. Hence the aircraft is permitted to fly a three-dimensional trajectory and the cable is free to take up a resulting configuration that may have both in-plane and out-of-plane components. The instantaneous rendezvous problem with three-dimensional aircraft manoeuvring permitted is graphically depicted in Figure 4-47.

With full control actuation available, the cable is deployed in a similar manner as earlier so that its tip can rendezvous with a desired location on the ground, without colliding with the prevailing terrain, which in this instance is a surface. The scenario begins with the ATC system in equilibrium with the same physical parameters as those given in Table 4-1, hence no out-of-plane motion for the ATC system exists initially. Initially, the aircraft is located at
a position given by the coordinates \{-2000, -200, 600\}. The location of the desired rendezvous target has the coordinates \{-1000, 0, 1.5\}. The mathematical model of the ATC system used for this rendezvous problem is that given by equation (2.4.57). The optimal control problem that characterizes this rendezvous scenario now follows.

### 4.6.5.1.13 Performance Criterion

The performance index $\mathcal{J}$ for this rendezvous problem is the minimum control optimality criterion given by equation (4.6.34):

$$
\mathcal{J} = \int_{t_0}^{t_f} \left( \dot{x}^2 + \dot{y}^2 + \dot{z}^2 + \dot{l}^2 \right) \, dt = \int_{t_0}^{t_f} \left( u_x^2 + u_y^2 + u_z^2 + u_l^2 \right) \, dt
$$

(4.6.34)

### 4.6.5.1.14 Dynamic Constraints

The equality constraints to be satisfied as part of the optimal control problem governing rendezvous are formulated using equation (2.4.57). The model “matching” technique based entirely on equilibrium characteristics (Method 1) is used for this rendezvous problem.

### 4.6.5.1.15 Initial Conditions

The initial conditions for this rendezvous problem are given by:

$$
\psi_0 = \begin{bmatrix} x_0, y_0, z_0, \theta_0, \phi_0, l_0, \dot{x}_0, \dot{y}_0, \dot{z}_0, \dot{\theta}_0, \dot{\phi}_0, \dot{l}_0 \end{bmatrix}^T
$$

(4.6.35)

$$
-\begin{bmatrix} x_0, y_0, z_0, \theta_0, 0, l_0, U_0, V_0, W_0, 0, 0, 0 \end{bmatrix}^T = 0
$$

where $y_0$ and $V_0$ are the initial lateral position and velocity of the aircraft, whilst and $z_0$, $W_0$ are the initial altitude and vertical velocity of the aircraft.

### 4.6.5.1.16 Final Conditions

Like the previous cases, the boundary conditions governing this rendezvous problem are given by the final position of the cable tip and the zero reel rate constraint. When the ATC system has general three-dimensional motion, the coordinates of the cable tip are given by:

$$
x_c = x - l \sin \theta
$$

(4.6.36)

$$
y_c = y - l \cos \theta \sin \phi
$$

(4.6.37)

$$
z_c = z - l \cos \theta \cos \phi
$$

(4.6.38)
The final conditions for this rendezvous problem are the same as those formulated for the previous rendezvous problems [given by equation (4.6.4)], except that equations (4.6.36) through to (4.6.38) are used to determine the position of the cable tip.

**4.6.5.1.17 Non-Linear Path Constraints**

When the ATC system is capable of general three-dimensional motion, a slight modification is made to the non-linear path constraint represented by equation (4.6.5). In reality, the profile of the ground terrain is a surface; hence a multi-valued function or discrete point mapping approximation is required to mathematically represent such surfaces. In general, closed form multi-valued functions can be constructed to model only relatively simple terrain profiles, yet real terrain profiles are anything but simple. To appropriately account for generic terrain profiles, a routine is required to approximate and interpolate real surfaces. For this and all other rendezvous problems involving general three-dimensional motion for the ATC system, surface terrain profiles are approximated and interpolated using a matrix Singular Value Decomposition (SVD) technique developed by Long and Long [169].

In this method, SVD is used to extend planar function interpolation into three dimensions. This process is carried by first fitting a series of Hermite cubics to the left and right singular vectors of the terrain surface data matrix. These are then assembled and weighted according to the singular values of the terrain surface data matrix. The method is simple and easy to implement, whilst being relatively quick, provided the matrix of terrain data has low rank. If the rank of the terrain data matrix is high, then a low rank approximation may be used without compromising the fidelity of the technique. For a given terrain profile, in order to correctly specify the non-linear path constraint for rendezvous, the value of the local terrain height is interpolated from the reference terrain data set using the instantaneous position of the cable tip. Once the value of the local terrain height has been determined, the non-linear path constraint previously outlined in equation (4.6.5) is enforced to ensure that the cable tip is always sufficiently beyond the local terrain, with equations (4.6.36) through to (4.6.38) used to calculate the position of the cable tip. The collision tolerance $z_{TOL}$ again remains unchanged. The profile of the terrain used in this three-dimensional rendezvous problem, and those that follow, is graphically depicted in Figure 4-48. The surface appearing in Figure 4-48 is a more realistic representation, in terms of surface terrains encountered by ATC systems, of the three dimensional surface used by Long and Long to demonstrate their SVD matrix interpolation/approximation technique [169].
4.6.5.1.18 Box Constraints

The same dimensional box constraints that were employed in the previous rendezvous problems are utilized again, except for the following inclusions and variations:

\[
\begin{align*}
\{ y_{\text{MIN}}, z_{\text{MIN}}, \phi_{\text{MIN}}, \dot{y}_{\text{MIN}}, \dot{z}_{\text{MIN}}, \dot{\phi}_{\text{MIN}} \}^T \leq x \leq \{ y_{\text{MAX}}, z_{\text{MAX}}, \phi_{\text{MAX}}, \dot{y}_{\text{MAX}}, \dot{z}_{\text{MAX}}, \dot{\phi}_{\text{MAX}} \}^T \\
\{-\infty, -\infty, -90, -10, -10, -\infty \}^T \leq x \leq \{ \infty, \infty, 90, 10, 10, \infty \}^T
\end{align*}
\]

(4.6.39)

\[
\begin{align*}
\{ \dot{x}_{\text{MIN}}, \dot{y}_{\text{MIN}}, \dot{z}_{\text{MIN}}, \dot{\phi}_{\text{MIN}} \}^T \leq u \leq \{ \dot{x}_{\text{MAX}}, \dot{y}_{\text{MAX}}, \dot{z}_{\text{MAX}}, \dot{\phi}_{\text{MAX}} \}^T \\
\{-2.5, -2.5, -2.5, -2.5 \}^T \leq u \leq \{ 2.5, 2.5, 2.5, 2.5 \}^T
\end{align*}
\]

(4.6.40)

Figure 4-48: Terrain Profile Used for Three-dimensional Rendezvous

The results concerning the rendezvous problem with full three-dimensional aircraft manoeuvring are represented by Figure 4-49 through to Figure 4-56. Again the solid circular markers (●) represent the discrete values of the states and controls found using DIRECT, the solid line (−) represents the propagated solution, whilst the transparent circles (○) represent the final target conditions. It can be seen from the results offered by Figure 4-49 through to Figure 4-53 that the discrete trajectories found using DIRECT are in close agreement with the propagated solution as desired. Similarly, it can also be concluded from Figure 4-51, Figure 4-54 and Figure 4-55 that all the final target states were attained by the ATC system, confirming that the desired rendezvous objectives were successfully met.

With reference to Figure 4-49, initially the in-plane cable dynamics are excited by the action of the control inputs, although this excitation is small and results in the cable in-plane orientation changing briefly by no more than 1 degree per second.
Beginning from the equilibrium condition, the cable in-plane orientation angle increases slowly to a new quasi-equilibrium condition, differing from the initial configuration by approximately 5 degrees, suggesting that the in-plane aerodynamic drag acting on the payload increases as the ATC progresses towards rendezvous. Contrast Figure 4-49 with Figure 4-41 given previously, which shows that the cable in-plane orientation angle marginally decreases during in-plane rendezvous problems, although the absolute value of the angular rate of change in each case is comparable. With respect to the out-of-plane cable dynamics, the cable out-of-plane orientation angle increases steadily to a new quasi-equilibrium condition of approximately 14 degrees, in similar manner to the cable in-plane dynamics, although at a slightly faster rate.

Figure 4-49: In-Plane Cable Angular Dynamics During Rendezvous (a) Angle (b) Velocity

Figure 4-50: Out-of-Plane Cable Angular Dynamics During Rendezvous (a) Angle (b) Velocity
Figure 4-51 demonstrates that in order to successfully rendezvous the tip of the cable with a desired location on the ground, the cable reel acceleration should be linear, which in turn leads to a quadratic-type reel rate and a cubic deployment profile for the cable. This observation was first made when the rendezvous problem with in-plane aircraft manoeuvring was investigated. The use of minimum cable reel acceleration in the performance criterion ensures that smooth and conservative-valued radial dynamics are experienced by the cable.

![Figure 4-51: Cable Radial Dynamics During Rendezvous- (a) Length (b) Length Rate (c) Control](image)

Figure 4-52: Aircraft Velocity for Rendezvous- (a) x-component (b) y-component (c) z-component
Figure 4-52 proves that for successful rendezvous, the aircraft should dive steadily towards the target position in a similar vein as previously determined when the aircraft was restricted to in-plane manoeuvring only. Variations of up to 10 m/s in the three-dimensional velocity of the aircraft are initially required to ensure that the aircraft is heading in direction of the target location. This is required since the aircraft is initially flying steady and level with a directional heading that is offset laterally by approximately 11.31 degrees and inclined approximately 30.9 degrees above the target position. Once direct line of sight between the aircraft and the rendezvous point has been achieved, the aircraft maintains constant velocity until rendezvous takes place. A comparative flight path was determined for the case when the aircraft was restricted to in-plane manoeuvring.

![Figure 4-53: Aircraft Control for Rendezvous - (a) x-component (b) y-component (c) z-component](image)

Figure 4-53 further confirms the observation that the aircraft should dive steadily towards the target position in order to achieve rendezvous. In this instance, aircraft related control actuation is only required in the initial stages of the manoeuvre to ensure that the aircraft is heading in the direction of the rendezvous location, which is similar to what was observed when the aircraft was restricted to in-plane manoeuvring. After an initial bout of control activity, the aircraft merely coasts towards the rendezvous point for most of the manoeuvre.

Figure 4-54 outlines the in-plane trajectory both the aircraft (top line) and cable tip (bottom line) follow during the rendezvous manoeuvre, along with the in-plane configuration of the cable at various times. The cable tip follows a similar unadulterated smooth trajectory that was revealed previously during the in-plane rendezvous problem case study. Initially the cable tip dives steadily towards the target, before pulling-up marginally as the target
approaches, consistent with the requirement of zero deployment velocity at rendezvous. The lengthening of the cable during rendezvous can also be seen from Figure 4-54, as can the in-plane profile of the surface terrain located at the bottom of Figure 4-54. The in-plane trajectory the aircraft follows is in agreement with the trajectory required of the aircraft when rendezvous was restricted to occur in the plane the ATC system was initially in.

![In-Plane Aerial Towed-Cable System Configuration During Rendezvous](image)

**Figure 4-54: In-Plane Aerial Towed-Cable System Configuration During Rendezvous**

With respect to the out-of-plane configuration of the ATC system during the rendezvous procedure, Figure 4-55 clearly shows the three-dimensional turning manoeuvre the aircraft initially executes to ensure that it is heading in the line-of-sight direction to the target point. Once this manoeuvre has taken place, the aircraft continues to fly steadily towards the rendezvous location. The cable tip tracks a slightly different trajectory in the z-y plane compared to that given in Figure 4-54 for the z-x plane, although the out-of-plane trajectory is smooth, consistent and in good agreement with those observed throughout Section 4.6.5.

With respect to the tension within the cable, a modified form of equation (4.6.15) was used to appropriately incorporate the additional dynamic state and control variables in the cable tension calculation. It can be seen from Figure 4-56 that the initial control actuation provided to the ATC system causes the cable tension to drop moderately in the initial stages of the manoeuvre. As the aircraft begins to coast towards the rendezvous target, the tension within the cable begins to steadily increase, in line with the increases to the dynamic inertial loading of the cable in the mid to later stages of the rendezvous attempt.
4.7 Concluding Remarks

In this chapter, a variety of single phase, non-linear optimal control problems for the ATC system were introduced and subsequent solutions to these problems were obtained. As a result, the prospect of achieving accurate, instantaneous rendezvous of the cable tip with desired surface locations on the ground, in two and three-dimensions, using deployment and retrieval control of the cable and/or aircraft manoeuvring was unequivocally demonstrated. In the following section, for reasons that will become evident, a more detailed examination of various rendezvous problems for the ATC system will be presented.
5 PARAMETRIC STUDY OF RENDEZVOUS FOR AERIAL TOWED-CABLE SYSTEMS

5.1 Preface

This section is devoted to a systematic parametric study of the ATC system whilst undergoing the rendezvous-type operations previously introduced in Section 4. The primary objective of this work is to investigate the impact that a wide range of system and environmental parameters have on the ability of the ATC system to perform a variety of rendezvous manoeuvres. By addressing how these parameters affect the dynamics and control of the ATC system when performing various rendezvous operations, a greater appreciation of the dynamics and control of the system will be in the offing. Potential insights will be uncovered that enable the careful and more considered design of controllers for the ATC system, thereby allowing more practical-orientated tasks such as payload capture and delivery to be investigated. Due to the inherently similar nature of rendezvous performed when the aircraft is permitted fly in either two or three dimensions, for simplicity, the aforementioned parametric studies will be limited to the two-dimensional in-plane situation.

5.2 Effect of Physical System Parameters on Constant Tow Speed Rendezvous

The effect that the physical system parameters have on the instantaneous constant tow speed rendezvous problem is the subject of investigation in this sub-section. Again the model of the ATC system used for this study is that given by equation (2.4.38), except that one small adjustment to the inertia of the ATC system is made. The payload mass term \( m_p \) in the denominator of the factor appearing outside the bracketed term in equation (2.4.38) is replaced with the term \( m^* \) given by equation (5.2.1), to account for the mass of the cable.

\[
m^* = m_p + \frac{\pi}{4} \rho d_c^2 l
\]  

(5.2.1)

The physical parameters of the ATC system selected for thorough investigation are the mass of the payload \( m_p \), the selected towing speed of the aircraft \( U_0 \) and the initial nominal length of the cable \( l_0 \).
For the simplified version of the ATC system of the form given by equation (2.4.38), it can be deduced by inspection that these parameters (along with the payload drag coefficient $C_{D_p}$) will have the most significant effect on the system. For a given payload mass, changes to the payload diameter only affect the drag of the payload, which is artificially augmented via changes to the payload drag coefficient to ensure that the simple model is closely representative of the more sophisticated ATC system model. As a result, insights into the true effect that the payload diameter and drag coefficient have on the simplified ATC system will not be meaningful, hence pursuing such an objective will not be attempted here. Similarly, the mass density and diameter of the cable only affect the ATC system via the mass of the cable, which in turn has a similar effect on the ATC system as the payload mass, though to a much lesser extent. The physical system parameters that govern the rendezvous problems studied in this sub-section are the same as those given in Table 4.1, unless otherwise specified. Exactly the same “minimum control” optimal control formulation and solution methodology that was first outlined in Sections 4.6.1 and 4.6.2 is adopted here to solve the following constant tow-speed rendezvous problems. The domains considered for the various parameters are such that a solution to the rendezvous problem posed in Section 4.6.2 is still possible.

### 5.2.1 Payload Mass

The results concerning specifically how the payload mass affects the ATC system whilst performing instantaneous rendezvous are given in Figure 5-1 through to Figure 5-5. The payload mass values considered here lie in the interval $m_p \in [40, 250]$ [kg]. As can be seen from Figure 5-1, counter intuitively, it takes progressively longer to rendezvous the cable tip with the desired location on the ground for increasingly lighter payloads. One would expect that it would take longer to deliver heavy payloads as opposed to lighter payloads, yet this is not reflected in the results presented in Figure 5-1, which shows a non-linear exponential decay-type relationship existing between rendezvous time and payload mass. Pertinently, zero payload mass will cause a singularity to occur in the state equations for the ATC system given by equation (2.4.38), hence the use of the inertial correction granted by equation (5.2.1). The reason why it takes longer for rendezvous to occur with lighter payloads is due to the balance of external forces acting on the payload. For heavy payloads, the gravitational force on the payload dominates over the aerodynamic force, resulting in the payload/cable tip being positioned further beneath the aircraft and closer to the ground and the desired target itself. Hence, it takes less time for rendezvous to occur the heavier the payload is, within the payload mass range considered here.
Due to the use of the minimum control/reel acceleration performance index, the cable tip follows a smooth trajectory over the duration of the manoeuvre regardless of the mass of the payload. In Figure 5-2, the reel acceleration required to perform the rendezvous manoeuvre is entirely linear for a 250 kg payload mass. Similarly, for payload masses between 100 kg and 200 kg, the required reel acceleration is quasi-linear for most of the manoeuvre, with an additional piecewise portion of linear control needed to secure rendezvous as the cable tip approaches the desired target. However, it can be seen from Figure 5-2 that as the payload mass becomes sufficiently low, the nature of the applied reel acceleration undergoes significant transformation. For payload masses between 40 kg and 75 kg, as the mass of the payload decreases, the reel acceleration variation progressively switches from a quasi-linear nature to a “bang-bang” one. Then for a relatively long period of time, the longer the lighter
the payload is, no reel acceleration is required, upon which a sinusoidal variation in the reel acceleration is requisite for rendezvous to occur, the amplitude and frequency of which increases and decreases respectively, as the mass of the payload decreases.

![Figure 5-3: Effect of Payload Mass on Final Cost](image)

It can be observed from Figure 5-3 that an exponential-decay type relationship is shown to exist between the mass of the payload and the final value of the performance index. By virtue of Figure 5-3, although heavier payloads are less manoeuvrable due to the increased inertial forces acting on them, this does not necessarily translate into increased amounts of overall control for rendezvous, since the initial position of the cable tip will be nominally closer to the target for heavier payloads. This phenomenon is clearly shown in Figure 5-3, which illustrates that the overall rate of applied control rapidly diminishes as the mass of the payload continues to increase beyond 150 kg.

In contrast to Figure 5-1, lighter payloads disproportionately experience more aerodynamic drag force than gravitational force, which tends to result in the payload/cable tip lying at a higher altitude and closer to the aircraft, leading to longer rendezvous times. This phenomenon is clearly demonstrated in Figure 5-4, which shows the increasingly higher initial altitudes the cable tip begins the rendezvous manoeuvre at, the lighter the payload is. However, once the cable tip is in the vicinity of approximately 600 metres from the target, the action of the control input causes the lighter payloads to pass increasingly closer to the ground on their way to the target, as shown in the latter portion of Figure 5-4. Following on, lighter payloads then attain higher altitudes approximately 400 metres from the target position.
The effect that the payload mass has on the tension within the cable is displayed in Figure 5-5, which shows an increasingly non-linear relationship existing between the payload mass and cable tension as the mass of the payload decreases. In most cases, the tension within the cable slowly and steadily increases throughout most of the duration of the rendezvous manoeuvre, except in the final stages of some scenarios when the tension decreases, then rises sharply due to the large control inputs associated with reducing the reel rate to zero at rendezvous. However, when the payload mass is sufficiently heavy (250 kg), the tension variation in the cable is entirely quasi-linear. The overall magnitude of the tension within the cable rapidly and non-linearly decreases as the payload mass correspondingly decreases, since lighter payloads experience more drag than gravitational force, which conspires to lower the tension.
5.2.2 Aircraft Towing Speed

The results showing how the towing speed affects the ATC system whilst performing instantaneous rendezvous are given in Figure 5-6 through to Figure 5-10. The range of values considered for the towing speed lie in the domain $U_0 \in [40, 56]$ [m/s].

![Figure 5-6: Effect of Towing Speed on Rendezvous Time](image)

In Figure 5-6, the time required for rendezvous is shown to reduce substantially and in a quasi-linear manner, as the towing speed of the aircraft increases. Performing the rendezvous manoeuvre when flying at 56 m/s takes approximately 44.8 secs, as compared to approximately 57.4 secs when undertaking the same manoeuvre at a towing speed of 40 m/s, an approximate 28% time difference. However, upon contemplating Figure 5-7, it can be seen that this will come at a significant cost, being extensively increased reel acceleration. The reel acceleration required for rendezvous is minor for low towing speeds, the variation of which is quasi-linear for most of the manoeuvre. Once again, an additional accompanying piecewise control segment is required late in the rendezvous attempt, the magnitude of which increases as the towing speed increases. However, once a critical threshold is reached, a towing speed of 48 m/s, the control time history becomes increasingly non-linear in the initial stages of the manoeuvre. Further increases in the towing speed result in piecewise, highly non-linear and pronounced cable reel acceleration paths, reminiscent of those observed when the payload mass was less than 75 kg, although control is applied for progressively shorter periods of time as the towing speed increases. Nevertheless, a quadratic-type relationship exists between the towing speed and the final value of the performance index, as depicted by Figure 5-8. This indicates that overall, significant and progressively higher levels of reel
acceleration are needed to perform rendezvous for increased towing speeds. Again due to the use of the minimum control performance index, the cable tip follows a smooth trajectory over the duration of the manoeuvre regardless of the towing speed, as shown in Figure 5-9.

![Figure 5-7: Effect of Towing Speed on Control for Rendezvous](image)

For fast towing speeds, the aerodynamic force on the payload dominates over gravity, resulting in increasingly higher initial altitudes for the cable tip. Therefore, it is unsurprising as to why more control is required to manipulate the cable tip to rendezvous with the surface target as the towing speed increases. However, once the cable tip is approximately 450 m from the target, due to the action of the control, the cable tip passes increasingly closer to the ground en route to rendezvous as the towing speed increases, a scenario shown in Figure 5-9.

![Figure 5-8: Effect of Towing Speed on Final Cost](image)
The effect that the towing speed has on the tension within the cable is displayed in Figure 5-10, which in line with the control time histories shown in Figure 5-7, demonstrates an increasingly non-linear relationship existing between variations to the towing speed and the cable tension. For towing speeds less than 48 m/s, the tension within the cable increases quasi-linearly for most the rendezvous manoeuvre, with an additional piecewise high-gradient, quasi-linear segment occurring when the cable tip is in the vicinity of the target. Further increases in the towing speed result in highly non-linear and pronounced variations in the cable tension, both increases and decreases of up to 400 Newtons. These variations occur both early on and late in the rendezvous attempt, consistent with the sizable reel accelerations that are required during these periods of the manoeuvre. The tension within the cable
generally increases more rapidly throughout the duration of rendezvous as the towing speed increases, although the overall magnitude of the cable tension is lower as the towing speed increases. Once again, this is by virtue of the contribution the payload gravitational and aerodynamic forces have on the cable tension given by equation (4.6.15).

5.2.3 Initial Nominal Cable Length

The results edifying how the initial nominal cable length affects the ATC system when performing instantaneous rendezvous are given in Figure 5-11 through to Figure 5-15. The cable length range utilized to yield these results is $l_0 \in [150, 700]$ [m].

Contrary to what one would expect, as evident from Figure 5-11, the time required for rendezvous increases slightly, in a quasi-linear manner, as the nominal initial length of the cable increases. Although this effect is not significant, an approximate 11 % increase in rendezvous time for an approximate 360 % increase in cable length, it is an important finding that best illustrates the necessity for, and merit of performing studies such as the one concerning this sub-section. Conversely, the results concerning the control required to perform rendezvous as depicted in Figure 5-12 are inherently intuitive, showing progressively lower levels of reel acceleration are needed as the initial cable length increases. For cable lengths greater than 150 metres, the control for each scenario is relatively continuous throughout the duration of the manoeuvre. In the concluding stages of the manoeuvre, significant levels of reel acceleration are required to ensure that rendezvous occurs with zero reel rate, the amount of which decreases as the initial cable length increases. The rate at which
reel acceleration is applied decreases and the duration it is applied increases, as the initial cable length increases accordingly. Similar inferences can be made upon consideration of the final value of the performance index given in Figure 5-13, which depicts a strong quasi-linear type relationship existing between the initial cable length and final value of the performance index. This confirms that the total amount of control required decreases sharply for increasing initial cable lengths, even though this control is applied for longer periods of time.

![Figure 5-12: Effect of Initial Cable Length on Control for Rendezvous](image)

As observed previously, the cable tip follows a smooth trajectory during rendezvous regardless of the initial cable length. Referring to Figure 5-14, the longer the cable is initially, the closer it is to the target, yet there is no overlap in the various cable tip trajectories, as was
seen in the results for the payload mass and towing speed shown in Figure 5-4 and Figure 5-9. Instead, the effect of increasing the initial cable length manifests as a “flattening” of the trajectory, resulting in the cable tip following a shallower trajectory towards the target.

The effect that the initial cable length has on the tension within the cable is displayed in Figure 5-15. Except for the shortest cables (less than 200 m), a continuous relationship exists between the cable length and tension. As the initial length of the cable decreases, this relationship becomes progressively non-linear and increasing in magnitude. Overall, the magnitude of the tension within cable decreases as the initial cable length is increased, which is consistent with the cable tension terms given by equations (4.6.15) through to (4.6.18).
5.2.4 Effect of Atmospheric Winds on Rendezvous

The effect of assorted atmospheric winds on the ATC system when performing constant tow speed rendezvous will be the focus of comprehensive enquiry in this sub-section. Attending to the impact that various natured atmospheric winds have on the dynamics and control of the ATC system throughout rendezvous operations is a crucially important domain of research, as in reality the system is expected to encounter a wide range of environmental disturbances, the most prevalent of which are atmospheric winds. Furthermore, it will be possible to ascertain to some degree the damping and robustness the system possesses, which has obvious real world implications. Once again, the model of the ATC system used in this investigation is that given by equation (2.4.38), with the inclusion of atmospheric winds in the drag model for the payload. The aerodynamic drag force acting on the payload is calculated using the normal velocity of the payload relative to the velocity of the prevailing wind $v_{\text{WIND}}$. The velocity of the payload relative to the prevailing wind $v_{\text{rel}}$ is given by equation (5.2.2):

$$v_{\text{rel}} = v_p - v_{\text{WIND}}$$

$$= \left[ (\dot{x} - \dot{l} \sin \theta - l \dot{\theta} \cos \theta) - W_x \right] e_i + \left[ (\dot{y} - \dot{l} \cos \theta + l \dot{\theta} \sin \theta) - W_y \right] e_j$$  \hspace{1cm} (5.2.2)

where:

$W_x$ is the $x$-component of the prevailing wind velocity,

$W_y$ is the $y$-component of the prevailing wind velocity.

Following on, the normal velocity of the payload relative to the wind $v_{\text{Nrel}}$ is:

$$v_{\text{Nrel}} = \left[ (y \sin \theta + l \dot{\theta} - \dot{x} \cos \theta) - (W_x \cos \theta + W_y \sin \theta) \right] e_\theta$$  \hspace{1cm} (5.2.3)

Equation (5.2.3) is then substituted into equation (2.4.21) to determine the aerodynamic drag force acting on the payload. Following on, the time rate of change for the velocity of the prevailing wind $\dot{v}_{\text{WIND}}$ is required in the derivation of Lagrange’s Equations of motion for the ATC system. In general form, the time rate of change for the velocity of the prevailing wind $\dot{v}_{\text{WIND}}$ is mathematically expressed as:

$$\dot{v}_{\text{WIND}} = \frac{d}{dt} W_x e_i + \frac{d}{dt} W_y e_j$$

$$= \left( \frac{\partial W_x}{\partial x} \cdot \dot{W}_x + \frac{\partial W_x}{\partial y} \cdot \dot{W}_y \right) e_i + \left( \frac{\partial W_y}{\partial x} \cdot \dot{W}_x + \frac{\partial W_y}{\partial y} \cdot \dot{W}_y \right) e_j$$  \hspace{1cm} (5.2.4)
Finally, the physical system parameters that govern the rendezvous problems studied in this sub-section are the same as those given in Table 4.1. The same features and solution procedures used in the previous sub-sections are again adopted here.

*Constant Strength Horizontal Wind*

In this subdivision, the effect of constant strength horizontal winds on the ATC system during rendezvous is considered, the results of which appear in Figure 5-16 through to Figure 5-20. Mathematically, constant strength horizontal winds can be expressed as:

\[
\begin{align*}
v_{\text{WIND}} &= W_x e_i \\
\dot{v}_{\text{WIND}} &= 0
\end{align*}
\]

The magnitude domain considered for the constant strength horizontal winds is \(W_x \in [-12, 20]\) [m/s], with negative wind strengths representing head winds, whilst positive wind strengths manifest as tail winds.

![Figure 5-16: Effect of Constant Strength Horizontal Wind on Rendezvous Time](image)

With reference to Figure 5-16, a quasi-linear relationship exists between the strength of the horizontal wind and the time required for rendezvous. The presence of a tail wind tends to reduce the time required for the cable tip to reach the target, at a rate approximately invariant to the strength of the tail wind, while performing the rendezvous manoeuvre in the presence of a head wind requires a longer period time than otherwise would be the case for no winds. In Figure 5-17, the reel acceleration required to perform the rendezvous manoeuvre is entirely linear for the strongest of tail winds (greater than 15 m/s). Similarly, for tail winds between 10 m/s and 15 m/s, the required reel acceleration is quasi-linear for most of the manoeuvre, with an additional piecewise portion of linear control needed to secure rendezvous.
However, it can be seen from Figure 5-17 that as the strength of the tail wind reduces from 10 m/s to 0 m/s and the strength of the head wind increases from 0 m/s to 2 m/s, the nature of the applied reel acceleration undergoes a rapid transformation from being predominantly piecewise quasi-linear to being “bang-bang” in nature. For head winds in excess of 2 m/s, for a period of time, the length of which increases the stronger the head wind is, no reel acceleration is required, upon which a sinusoidal variation is needed for rendezvous to occur. The amplitude and frequency of this sinusoidal variation increases and decreases respectively as the strength of the head wind increases. Initially, higher levels of reel acceleration are required for longer periods of time as the strength of the head wind increases.

The longer rendezvous times associated with manoeuvres performed in the presence of head winds also require increased amounts of overall control, as demonstrated by Figure 5-18. A pronounced exponential-decay type dependency of the final value of the performance index on the magnitude of the constant strength wind is shown in Figure 5-18. This indicates that it becomes progressively easier to perform instantaneous rendezvous in the presence of increasingly stronger tail winds, as opposed to performing the same manoeuvre in the presence of a head wind. By virtue of equation (5.2.3), increasing the strength of a prevailing head wind will increase the aerodynamic drag acting on the payload, which by its very nature will drive the cable tip further away from the target. As a result, increased reel acceleration for longer time periods is required to overcome this, a scenario unmistakably reflected in both Figure 5-17 and Figure 5-18.

![Figure 5-17: Effect of Constant Strength Horizontal Wind on Control for Rendezvous](image)
Strong tail winds render the cable/payload system less manoeuvrable due to the increased inertial forces acting on them. However, this does not necessarily translate into increased amounts of overall control for rendezvous as outlined in Figure 5-18, since the initial position of the cable tip will be nominally closer to the target when rendezvous occurs in the presence of a tail wind. The physical basis for this lies in the action of the aerodynamic drag force, which shifts from tending to increase the altitude of the cable tip, to reducing the tip altitude and forcing the cable to be more “vertical”, as the prevailing wind switches from acting as a head wind to acting as a tail wind.
The cable tip follows a smooth trajectory throughout the duration of the rendezvous manoeuvre regardless of the strength and direction of the horizontal wind, as shown in Figure 5-19. Beginning from a common starting point, initially the cable tip attains higher altitudes as the prevailing wind switches from acting as a strong tail to a strong head wind. Once the tip is approximately 1750 metres from the target, the situation is reversed due to the large control forces, with the cable tip passing closer to the ground as the wind changes from being a strong tail to a strong head wind. This reign continues until the cable tip is approximately 400 metres from the target, when it reverts back to attaining increasingly higher altitudes as the wind changes from acting against, to acting in the direction of travel, until rendezvous is achieved.

Figure 5-20 shows that as the strength of the tail wind increases, so does the tension within the cable, since the lower domination of the aerodynamic drag over inertial forces when tail winds act, compels the cable tip to tend further beneath the aircraft, thereby increasing the tension within the cable. The tension within the cable always increases throughout the duration of the manoeuvre when a tail wind of 6 m/s or greater prevails. The large changes in reel acceleration that occur for weak tail winds and all head winds, produce a series of tension increases and decreases in the cable, the size of which are increasingly prominent as the wind tends from being a slight tail to a strong head wind.

![Figure 5-20: Effect of Constant Strength Horizontal Wind on Cable Tension](image)

Clearly, constant strength horizontal winds have a noticeable effect on all aspects of the ATC system during rendezvous. Sizable tail winds of were found to assist the execution of constant tow speed rendezvous, whilst the presence of pronounced head winds have an unfavourable influence on the performance of the ATC system during rendezvous.
Constant Strength Vertical Wind

The effect of constant strength vertical winds on the ATC system during rendezvous will now be considered, the results of which appear in Figure 5-21 through to Figure 5-25. These constant strength vertical winds have the following simple mathematical form:

\[ \mathbf{v}_{\text{WIND}} = W_y \mathbf{e}_j \]
\[ \mathbf{\dot{v}}_{\text{WIND}} = 0 \]  

(5.2.6)

The magnitude of the vertical winds considered in this investigation lie in the interval \( W_y \in [-12, 20] \) [m/s]. Negative winds prevail downward towards the ground, whilst positive winds act upwards towards the aircraft. Firstly, upon initial consideration of Figure 5-21 through to Figure 5-25, it can be concluded that vertical winds have a major effect on certain aspects of the ATC system when performing rendezvous, whilst having a minor effect on others. It is important to note that the initial nominal orientation of the cable is such that the normal component of the vertical wind velocity is much smaller than that of the tangential component. As a result, the change to the aerodynamic drag experienced by the payload due to a vertical wind is smaller compared to that corresponding to a constant strength horizontal wind, expectedly having less of an effect on the ATC system. Nevertheless, Figure 5-21 through to Figure 5-25 demonstrate that vertical winds do affect the ATC system in both obvious and subtle ways, which in itself is an important point to consider.

\[ \begin{align*}
\text{Figure 5-21: Effect of Constant Strength Vertical Wind on Rendezvous Time} \\
\text{Figure 5-21 illustrates that a quasi-quadratic relationship exists between rendezvous time and the nature of the vertical wind, hinting that downward vertical winds reduce the time required for rendezvous, while upward acting winds bring about longer rendezvous times.}
\end{align*} \]
In the presence of increasingly strong downward vertical winds, the action of aerodynamic
drag will ensure that the cable tip has a lower altitude, compared to the case when an upward
acting vertical wind prevails. Consequently, as the cable tip is closer to the target position
when downward winds are present, the time required for rendezvous should be lower, a
postulation aptly justified by the results outlined in Figure 5-21.

![Figure 5-22: Effect of Constant Strength Vertical Wind on Control](image)

Figure 5-22 illustrates similar-natured control time histories as those obtained in Figure 5-17
during the investigation into how constant strength horizontal winds affect constant tow speed
rendezvous. For approximately the first thirty seconds of the rendezvous manoeuvre, the cable
reel acceleration required is linear when downward acting winds and upward winds no larger
than 6 m/s are encountered. Following this period of linear control activity, quasi-sinusoidal
variations in the reel acceleration are needed for rendezvous to occur, the amplitude and
frequency of which increase and decrease respectively as the strength of the upward wind
increases. Once the strength of the upward vertical winds exceed 6 m/s, the initial period of
control activity transforms from being quasi-linear to “bang-bang” in nature. Initially, higher
levels of reel acceleration are required for longer as the strength of the upward wind is
increased, while no reel acceleration is then required for a period of time that increases in line
with increases to the strength of the upward vertical wind. Quasi-sinusoidal reel accelerations
are then employed to achieve rendezvous, increasing in amplitude and decreasing in
frequency as the upward wind increases. These claims are further substantiated by the
scenario depicted in Figure 5-23, which shows that significantly increased levels of overall
control are needed to perform rendezvous in the presence of strong upward winds, compared
to strong downward winds, which require lower total control to perform the same manoeuvre.
A strong, approximate quadratic relationship exists between the final cost of rendezvous and the nature of the vertical winds as indicated in Figure 5-23, confirming that prevailing vertical winds have a large effect on the overall level of reel acceleration required for rendezvous.

Figure 5-24 demonstrates that constant strength vertical winds have a small effect on the trajectory the cable tip follows during rendezvous. Initially the cable tip attains slightly higher altitudes when the winds act downward, but when the cable tip is approximately 600 to 1100 metres from the target, the cable tip travels marginally closer to the ground in the presence of upward acting winds. For approximately the last 400 metres, the cable tip maintains slightly higher altitudes in the presence of upward acting winds.
With respect to the tension within the cable, the magnitude and direction of the vertical wind has a noticeable effect, as can be seen in Figure 5-25. For all downward acting winds and upward wind strengths up to 6 m/s, the general variation in the tension within the cable is essentially the same; the only noticeable difference being the decrease in absolute tension level within the cable as the strength of the downward vertical wind decreases and corresponding strength of the upward wind increases. Once the strength of upward winds exceed 6 m/s, the tension variation in the cable is piecewise, remaining constant when no reel acceleration is applied, before rapidly increasing and decreasing in line with the sinusoidally applied cable reel acceleration. However, the absolute tension level within the cable continues to decrease as the strength of the upward acting wind increases.

**Combination Constant Strength Wind**

The combined effect of constant strength horizontal and vertical winds that are simultaneously encountered by the ATC system during rendezvous is the focus of this subsection. The combination constant strength wind considered here is represented mathematically by equation (5.2.7):

\[
\mathbf{v}_{\text{WIND}} = W_x \mathbf{e}_x + W_y \mathbf{e}_y \\
\dot{\mathbf{v}}_{\text{WIND}} = 0
\]  

(5.2.7)

The results of this investigation are presented in Figure 5-26 through to Figure 5-30. The strength of the horizontal and vertical winds considered in this study are such that the resulting magnitude of the combination wind lies in the interval \( W_y \in [-12, 20] \) [m/s]. Positive combination winds are interpreted as upward acting tail winds, whilst negative
combination winds represent downward acting head winds. To appropriately summarise the effect that the combination constant strength wind (blue line) has on the final cost and time required for rendezvous, the results concerning the individual contributions of the horizontal (red line) and vertical (green line) winds are included in Figure 5-26 and Figure 5-28. From Figure 5-26, it can be deduced that the time required for rendezvous increases as the strength of the downward head wind increases, whilst rendezvous occurs faster as the size of the upward acting tail winds increase. Figure 5-26 clearly shows how the individual contribution of the horizontal wind dominates the overall resultant effect that the combination wind has on the required rendezvous time. Although the magnitude of the horizontal and vertical wind components are equal, the horizontal component has a disproportionately stronger effect, since combined winds affect the time required for rendezvous in the same manner as horizontal winds do, although the extent to which is tempered slightly due to the presence of the vertical wind.

![Figure 5-26: Effect of Various Constant Strength Winds on Rendezvous Time](image)

Comparatively similar conclusions can also be drawn upon reflection of the results presented in Figure 5-28. Once again the individual contribution of the horizontal wind can be seen to dominate the overall resultant effect that the combination wind has on the final value of the performance index, despite the magnitude of both wind components being equal. The overall control required for rendezvous decreases as the strength of the downward head wind decreases, with further increased control input savings possible if rendezvous is performed in the presence of increasingly stronger upward tail winds. This conclusion is further supported by the evidence provided by the cable reel acceleration time histories shown in Figure 5-27.
For upward acting tail winds exceeding 14 m/s, the cable reel acceleration is entirely linear over the duration of the rendezvous manoeuvre, with additional piecewise linear control segments of increasing size and magnitude required late in manoeuvre as the strength of the tail wind reduces to 8 m/s. For upward acting tail winds and downward head winds less than 8 m/s and 6 m/s respectively, the cable reel acceleration is linear for the majority of the time, before quasi-sinusoidal variations are required late in the rendezvous attempt. The amplitude and frequency of these quasi-sinusoidal reel accelerations increase and decrease respectively, as the size of the downward acting head wind increases. For downward head winds in excess of 6 m/s, the cable reel acceleration is “bang-bang” in nature; first increased amounts of control, for longer periods of time are required as the strength of the head wind is increased. Then no reel acceleration is required for a period of time that increases in line with increases to the strength of the downward head wind. Finally, quasi-sinusoidal reel accelerations are then employed to achieve rendezvous, increasing in amplitude and decreasing in frequency as the head wind increases in strength.

![Figure 5-27: Effect of Combination Constant Strength Horizontal Wind on Control](image)

Figure 5-29 indicates that the combination winds have a sizable effect on the cable tip trajectory during rendezvous, consistent with the observations made during the investigations into the separate effect each individual wind has on rendezvous. Initially, the cable tip attains slightly higher altitudes when the wind acts downward, but when the tip is approximately 700 to 1600 metres from the target, it moves significantly closer to the ground in the presence of downward head winds. For approximately the last 300 metres, the cable tip maintains marginally higher altitude in the presence of stronger downward acting head winds.
With respect to the tension within the cable, it can be surmised from Figure 5-30 that the cable tension is strongly affected by both the strength and direction of the prevailing combination wind. For upward acting winds greater than 8 m/s, the tension within the cable continually increases over the duration of the rendezvous attempt, initially rather steadily before quickly tapering off as the target point is approached. As the size of the upward tail wind reduces further and the downward head wind increases to 4 m/s, the general variation in the tension within the cable is essentially the same; steadily rising for most of the manoeuvre, before decreasing and then rising again in the concluding stages of the rendezvous attempt. The only noticeable difference in the tension level within the cable when these prevailing winds are encountered is the decrease in absolute tension within the cable as the strength of
the tail and head winds decrease and increase respectively. Once the strength of upward wind exceeds 4 m/s, the tension variation in the cable is piecewise; initially increasing steadily, then briefly remaining constant when no reel acceleration is applied, before rapidly increasing and decreasing in line with the sinusoidally applied cable reel acceleration. The absolute tension within the cable continues to decrease as the size of the head wind further increases.

![Figure 5-30: Effect of Combination Constant Strength Wind on Cable Tension](image)

5.3 Effect of Physical Parameters on Rendezvous With In-Plane Aircraft Manoeuvring

The effect that the physical system parameters have on the instantaneous rendezvous problem involving two-dimensional aircraft manoeuvring is the subject of investigation in this subsection. The model of the ATC system used in this investigation is that given by equation (2.4.35), along with the adjustment to the inertia of the system given by equation (5.2.1) to appropriately account for the mass of the cable. Once again, the physical parameters of the ATC system selected for thorough investigation are the mass of the payload $m_p$, the initial horizontal towing speed of the aircraft $U_0$ and the initial nominal length of the cable $l_0$. The physical system parameters that govern the rendezvous problems studied in this sub-section are the same as those given in Table 4.1, unless otherwise specified. The optimal control formulation and solution methodology that was first outlined in Section 4.6.5 is again used to solve the following rendezvous problems. The domains considered for the various parameters are such that a solution to the rendezvous problem posed in Section 4.6.5 is still possible.
5.3.1 Payload Mass

The results showing how the payload mass affects the ATC system whilst performing rendezvous with in-plane aircraft manoeuvring are given in Figure 5-31 through to Figure 5-38. The values of the payload mass considered lie in the interval $m_p \in [25, 250]$ [kg].

![Figure 5-31: Effect of Payload Mass on Rendezvous Time](image)

Following a similar trend to the case when the tow speed was constant, it can be seen from Figure 5-31 that it takes progressively longer to rendezvous the cable tip with the desired location on the ground for increasingly lighter payloads. Once again, a non-linear exponential-decay type relationship exists between rendezvous time and payload mass. The fact that the aircraft is able to dynamically manoeuvre does not alter the very reason why it takes longer for rendezvous with lighter payloads; namely the dominance of aerodynamic drag over gravitational forces for lighter payloads, resulting in the payload/cable tip being positioned closer to the aircraft and further away from the target. Consequently, this results in more time required for rendezvous to occur, the lighter the payload is.

With respect to the planar aircraft acceleration, Figure 5-32 and Figure 5-33 demonstrate that the effect of increasing payload mass tends to reduce the overall amount of aircraft related control actuation during the initial stages of the rendezvous manoeuvre. However, the control actuation provided by the aircraft needs to be sustained for increasingly longer periods as the mass of the payload subsequently increases. Similar to constant tow speed rendezvous, Figure 5-34 shows that the heavier the payload is, the lower the amount of cable reel acceleration required in the initial and final stages of the rendezvous manoeuvre.
Similar to the case for aircraft related control, the control provided by the cable reel is required for longer periods of time, the heavier the payload becomes. Unlike the constant tow speed rendezvous cases, a quadratic-type relationship exists between the mass of the payload and the final value of the performance index, as shown in Figure 5-35. Towing a 250 kg payload at constant speed attracts the least amount of overall control, whilst the optimal mass of the payload, when in-plane aircraft manoeuvres are allowed is 175 kg. Similarly, when the results in Figure 5-3 and Figure 5-35 are contrasted, it can be seen that for an appropriate given payload mass, significantly less overall control (lower final cost) is required to perform rendezvous when aircraft manoeuvres are permitted, even though the number of actuators increases three-fold. This further demonstrates the preference of the ATC system to uniformly
spread the control burden required during rendezvous over all its available actuators if possible.

Figure 5-34: Effect of Payload Mass on Cable Reel Acceleration

![Graph showing the effect of payload mass on cable reel acceleration.](image)

Figure 5-35: Effect of Payload Mass on Final Cost

![Graph showing the effect of payload mass on final cost.](image)

Figure 5-36 demonstrates that the payload mass does not significantly alter the trajectory of the aircraft during rendezvous, although the aircraft tends to fly further and lower in the sky as the mass of the payload decreases. Contrastingly, it can be seen from Figure 5-37 that the altitude and overall distance the cable tip travels during rendezvous increases as the mass of the payload is reduced, although it can be postulated that the payload mass does not significantly alter the cable tip trajectory during rendezvous.
Figure 5-38 shows that initially the cable can become slack when payloads less than 40 kg are towed during rendezvous, a scenario which should be avoided since the safety of the ATC system could be compromised because instabilities are likely to occur when the cable is in this condition. Similarly, it is generally difficult to accurately predict, let alone actively control the behaviour of a slack cable. Low cable tensions were also observed when light payloads were used during rendezvous at constant tow speed, although the cable always remained taut regardless of the mass of the payload. Finally, Figure 5-38 confirms that with in-plane aircraft manoeuvres enabled, increasing payload mass non-linearly increases the value of the tension reached in the cable during rendezvous.
5.3.2 Initial Towing Speed

The results indicating how the initial horizontal towing speed of the aircraft affects the ATC system whilst performing instantaneous rendezvous with in-plane aircraft manoeuvring are given in Figure 5-39 through to Figure 5-46. The values of the initial tow speed considered in this investigation lie in the interval \( U_0 \in [40, 70] \) [m/s]. Figure 5-39 proposes that a quasi-linear relationship exists between the initial aircraft horizontal speed and the time required for rendezvous to occur when in-plane aircraft motion is allowed. This relationship is relatively strong in the sense that an almost doubling in the initial aircraft horizontal speed allows for an approximate 5 second time saving (20 %) to be available. A similar inference can be made upon reflection of the results contained within Figure 5-6.
Figure 5-39: Effect of Initial Tow Speed on Rendezvous Time

Figure 5-40 and Figure 5-41 suggest that for successful rendezvous to take place, the horizontal velocity of the aircraft should be reduced whilst the vertical descent velocity should be increased, the rates of which both increase as the initial horizontal velocity of the aircraft is increased. Global trends are difficult to infer from Figure 5-40, although for fast initial towing speeds (55-70 m/s), generally changes to the horizontal acceleration of the aircraft are required for longer periods of time as the initial horizontal velocity of the aircraft is increased. The magnitude of these changes decrease in line with reductions to the initial horizontal velocity of the aircraft. Figure 5-41 shows that changes to the vertical acceleration of the aircraft are required for longer periods of time as the initial horizontal velocity of the aircraft is reduced. The magnitude of these changes decrease as the tow speed is reduced.

Figure 5-40: Effect of Initial Tow Speed on Aircraft Horizontal Acceleration
Initially, the amount of the cable reel acceleration required for rendezvous decreases as the initial tow speed decreases, as shown in Figure 5-42. Smaller changes in the reel acceleration of the cable, in both the initial and concluding stages of the rendezvous procedure, are required for longer periods of time as the initial tow speed decreases.

As found during the parametric study into the constant tow speed rendezvous problem, a quadratic-type relationship exists between the initial towing speed and the final value of the performance index as depicted by Figure 5-43. This indicates that progressively higher levels of overall control actuation are required to perform rendezvous as the initial towing speed of the aircraft is increased.
With in-plane aircraft manoeuvres permitted, the trajectory of the aircraft is relatively unaffected by the initial towing speed, as demonstrated by Figure 5-44. However, as the initial towing speed is increased, the aircraft is required to fly a slightly shorter distance, at higher altitude, for successful rendezvous to occur.

Similarly, the initial towing speed does not significantly affect the cable tip trajectory during rendezvous as shown in Figure 5-45, although that the altitude and overall distance the cable tip travels during rendezvous increases, as the initial towing speed increases accordingly.
Figure 5-45: Effect of Initial Tow Speed on Cable Tip Trajectory

With respect to the tension within the cable during rendezvous, Figure 5-46 illustrates that the absolute value of the tension within the cable decreases for increasingly faster initial towing speeds. Although this may be counter-intuitive upon first inspection, this observation can be adequately explained by the large deployment accelerations and rapid horizontal deceleration the cable and aircraft experience, when the initial towing speed is sufficiently high. Following on, as the initial towing speed is reduced, not only does the absolute value of the cable tension increase, but the overall level of variation in the tension also reduces noticeably.

Figure 5-46: Effect of Initial Tow Speed on Cable Tension
5.3.3 Initial Nominal Cable Length

The results indicating how initial nominal cable length affects the ATC system whilst performing instantaneous rendezvous with in-plane aircraft manoeuvring are given in Figure 5-47 through to Figure 5-54. The values of the initial nominal cable length considered in this investigation lie in the interval \( l_0 \in [250, 700] \) [m].

Unexpectedly, the time required for rendezvous is relatively unaffected by the initial cable length when planar aircraft manoeuvres are permitted during rendezvous, as demonstrated by Figure 5-47. Although a quadratic-type relationship exists between the time required for rendezvous and the initial nominal cable length, this relationship is weak, since slightly more than 1.5 seconds separates the time required to rendezvous when the cable is initially 300 or 700 metres long. Nevertheless, for rendezvous to occur in the shortest possible time, the cable should be initially 300 metres long.

![Figure 5-47: Effect of Initial Cable Length on Rendezvous Time](image)

For successful rendezvous to occur, the absolute magnitude of the aircraft related control actuation should be reduced as the initial length of the cable is increased, as indicated by Figure 5-48 and Figure 5-49. Smaller changes to both the horizontal and vertical acceleration of the aircraft are required for longer periods of time as the initial length of the cable is increased. For initial cable lengths in excess of 450 metres, changes to both the horizontal and vertical acceleration of the aircraft are required over the entire duration of the rendezvous manoeuvre. Initially, the amount of cable reel acceleration required for rendezvous decreases as the initial cable length increases, as shown in Figure 5-50. Smaller changes in the reel
acceleration of the cable, in both the initial and concluding stages of the rendezvous procedure, are required for longer periods of time as the initial cable length increases.

![Figure 5-48: Effect of Initial Cable Length on Aircraft Horizontal Acceleration](image)

Figure 5-48: Effect of Initial Cable Length on Aircraft Horizontal Acceleration

![Figure 5-49: Effect of Initial Cable Length on Aircraft Vertical Acceleration](image)

Figure 5-49: Effect of Initial Cable Length on Aircraft Vertical Acceleration

Figure 5-51 confirms validity of these inferences since it shows that progressively lower levels of overall control actuation are needed as the initial cable length increases, with an exponential-decay type relationship existing between the initial cable length and final value of the performance index. This confirms that the overall amount of control required for rendezvous decreases markedly for increasing initial cable lengths. A similar inverse relationship between the initial cable length and the overall amount of control required for rendezvous was also found to occur when rendezvous was performed at constant tow speeds.
In contrast to the effects that the payload mass and initial towing speed have on rendezvous with in-plane aircraft manoeuvring, Figure 5-52 and Figure 5-53 show that the initial cable length significantly affects the trajectories the aircraft and cable tip follow during rendezvous. As the initial cable length increases, the aircraft is required to fly significantly further and at a much higher altitude in order for rendezvous to take place. The aircraft descent rate markedly decreases as the initial cable length is increased. Contrastingly, the cable tip travels noticeably further and visibly closer to the ground as the initial cable length is increased. Compared to the aircraft trajectory, the cable tip descent rate noticeably decreases as the initial cable length increases.
Figure 5-52: Effect of Initial Cable Length on Aircraft Trajectory

Figure 5-53: Effect of Initial Cable Length on Cable Tip Trajectory

Figure 5-54 demonstrates that although the absolute value of the tension within the cable increases for increasingly longer initial cable lengths, the overall variation in the tension reduces noticeably for initially longer cables. Again this may be counter-intuitive, but the trends in the cable tension shown in Figure 5-54 are in agreement with the findings reached when the effect of initial cable length was investigated for the constant tow speed rendezvous. Hence, this phenomenon can be satisfactorily explicated. Once again, the large deployment accelerations and rapid horizontal deceleration the cable and aircraft experience when the initial cable length is sufficiently high, both conspire to significantly reduce the tension within the cable.
5.3.4 Effect of Atmospheric Winds on Rendezvous

The effect of assorted atmospheric winds on the ATC system whilst performing instantaneous rendezvous with two-dimensional aircraft manoeuvring will be the focus of enquiry in this sub-section. The model of the ATC system used in this investigation is again that given by equation (2.4.35), with the inclusion of atmospheric winds in the aerodynamic drag force model for the payload, as given previously by equations (5.2.2) and (5.2.4). The mathematical form of the atmospheric winds considered in this sub-section is the same as those given by equations (5.2.5) through to (5.2.7). All other physical system parameters and solution procedures remain unchanged.

**Constant Strength Winds**

The effect of constant strength winds on the ATC system during rendezvous with planar aircraft manoeuvres is now considered, the results of which appear in Figure 5-55 through to Figure 5-62. In this sub-section, the effects of both constant strength horizontal (red line) and vertical (green line) winds are investigated separately, as well as the case when each of the winds are concurrently encountered by the ATC system during rendezvous (blue line). The magnitude domain considered for each form of the constant strength winds is \( W_x, W_y \in [-12, 12] \text{ [m/s]} \).
Figure 5-55: Effect of Constant Strength Wind on Rendezvous Time

With respect to Figure 5-55, it can be seen that the time required for rendezvous decreases as the strengths of the horizontal head wind and upward tail winds decrease. Quasi-linear relationships exist between the time required for rendezvous and the strength and direction of the winds. The stronger the horizontal tail wind, the faster rendezvous will occur, with the same being true as the strength of the combined upward tail wind increases. Alternatively, the stronger the horizontal head wind, combined downward head wind and upward vertical wind, the longer rendezvous takes. Once again, the horizontal winds tend to have a disproportionately stronger effect (than the vertical winds) on the time required for rendezvous when the system encounters concurrent horizontal and vertical prevailing winds.

Figure 5-56: Effect of Combination Constant Strength Wind on Aircraft Horizontal Acceleration
Since both the horizontal and vertical winds individually affect the control actuation, aircraft and cable tip trajectories as well as the cable tension in a similar manner, the results shown in Figure 5-56 through to Figure 5-58 and Figure 5-60 through to Figure 5-62 are for the case when both winds are concurrently encountered during rendezvous. Referring to Figure 5-56 through to Figure 5-58, the aircraft and cable related control actuators are relatively unaffected by the action of the constant wind. Rendezvous in the presence of strong head winds initially requires slightly more aircraft and cable reel acceleration compared to the case when a strong tail wind is encountered. With respect to aircraft control, slightly less control is required for slightly longer periods of time when strong tail winds are encountered, the opposite being true for the case of strong head winds, although aircraft-related control activity is focussed mostly in the initial portion of the rendezvous manoeuvre. At the finale of the rendezvous procedure, strong tail winds require the ATC system to apply greater cable reel acceleration for progressively shorter periods of time, compared to the case when strong head winds are experienced, where low levels of cable reel acceleration are applied for longer.

![Figure 5-57: Effect of Combination Constant Strength Wind on Aircraft Vertical Acceleration](image)

In Figure 5-59, the overall control required to perform rendezvous is a minimum when a 6 m/s horizontal tail wind occurs, as given by the quadratic relationship relating the strength and direction of the wind to the final cost. Disproportionably higher levels of control are required for rendezvous in the presence of strong horizontal head winds, compared to the case for large horizontal tail winds. With respect to the vertical wind, the control required for rendezvous increases quasi-quadratically as the magnitude of this wind increases and its direction shifts from acting downward to upward. The overall impact of the combination winds is the sum of the individual effects caused by the more dominant horizontal wind and
the weaker vertical wind, indicating that lower levels of control are required for rendezvous as
the prevailing wind swings from being a strong downward head wind to a strong upward tail
wind.

Figure 5-58: Effect of Combination Constant Strength Wind on Cable Reel Acceleration

Figure 5-59: Effect of Constant Strength Wind on Final Cost

Figure 5-60 and Figure 5-61 show the extent to which the aircraft and cable tip trajectories
remain unaffected by the action of the constant strength winds during rendezvous. The aircraft
flies marginally further and higher in the presence of strong tail winds, compared to the
trajectories flown when strong head winds are encountered. Alternatively, the cable tip travels
at a slightly lower altitude for most of the rendezvous manoeuvre in the presence of a tail
wind, as opposed to when strong head winds are encountered.
Figure 5-60: Effect of Combination Constant Strength Wind on Aircraft Trajectory

Figure 5-62 indicates that the tension in the cable is slightly higher when rendezvous is performed in the presence of strong head winds compared to tail winds. The overall variation in the cable tension is relatively unaffected by prevailing winds that occur during rendezvous.

Figure 5-61: Effect of Combination Constant Strength Wind on Cable Tip Trajectory
Figure 5-62: Effect of Combination Constant Strength Wind on Cable Tension

5.4 Concluding Remarks

The primary objective of this chapter was to investigate the impact that various parameters have on the ability of the ATC system to perform single, instantaneous rendezvous operations. Addressing how these factors affect rendezvous by undertaking a series of parametric studies, allowed for a better appreciation of the dynamics and control of the ATC system to be possible. In the following chapter, multi-phase non-linear optimal control problems for the ATC system are formally introduced, permitting the development of more general and potentially useful multiple rendezvous operations to be considered.
6 MULTI-PHASE OPTIMAL CONTROL OF AERIAL TOWED-CABLE SYSTEMS: PART A

6.1 Preface

This chapter is the first of a two-part series devoted to the development of multi-phase, non-linear optimal control problems for the ATC system. The area of multiple instantaneous rendezvous problems for the ATC system, in two and three-dimensions, are thoroughly explored within this section. In a similar manner to that employed in Section 5, numerous case study investigations will be performed to assess the impact that a wide range of system and environmental parameters have on the ability of the ATC system to perform multiple rendezvous operations.

6.2 Multi-Phase Non-Linear Optimal Control

In section 4.5, the general continuous single-phase optimal control problem was introduced by way of equation (4.5.1) through to equation (4.5.5). Using direct transcription, this continuous optimal control problem was discretized and transformed into the non-linear parameter optimization problem given by equations (4.5.6) through to equations (4.5.12). Single-phase optimal control problems can be straightforwardly extended to multi-phase optimal control problems consisting of \( m \) phases, by introducing a phase index \( i \) and using event conditions \( \psi_i \) that provide the appropriate connectivity between each phase. The use of phases within the non-linear optimal control framework allows for both non-smooth and hybrid optimal control problems to be considered without undue complexity and difficulties [151]. Non-smooth and hybrid optimal control problems are frequently formulated in engineering practice, since many technical problems involve jump discontinuities in the states, control switches, interior point constraints, and empirical models containing non-smooth “table-look up” data. Similarly, physical system parameters, the state equations, performance index and path constraints may be distinct during each phase of the entire optimal control problem, producing a general non-linear, hybrid optimal control problem. These issues can be investigated and ultimately addressed using multi-phase, non-linear optimal control theory.
The nodes that occur on the boundary between two adjoining segments are known as knots, which in general, allow both the state and control variables to be multi-valued at the junction of each phase. This allows discontinuities and non-smooth characteristics to be appropriately handled within the optimal control methodology. Information to nodes across each phase boundary is transferred via the event conditions localized at the knots [151]. Furthermore, the location of the event conditions may be arbitrary, allowing significantly increased flexibility in the formulation of the optimal control problem. This is because floating knots are able to accurately capture non-smooth behaviour in instances where corners occur in the dynamical states or control switches are used [151].

A formal mathematical formulation of multi-phase, non-linear optimal control problems such as that appearing in [170, 171] will not be presented in this work, as such a task would significantly detract from the objectives, scope and findings of this thesis. Instead a brief outline the general nature of multi-phase optimal control problems will be provided. With a means to this end, consider the general continuous multi-phase optimal control problem involving the minimization of the following performance index:

\[
\mathcal{J} = \mathcal{M}[\dot{x}(t_e), \dot{x}'(t_e), x'(t_e), t_e'] \\
+ \int_{t_0}^{t_f} \mathcal{L}[\dot{x}(t), \dot{x}'(t), x'(t), u'(t), t] \, dt, \quad (i \leq m)
\]

where the states and controls for the system are subject to the following dynamical constraints in each phase:

\[
f[\dot{x}(t), x'(t), u'(t), t] = 0, \quad t \in [t_i', t_f'] \quad (i \leq m)
\]

the subsequent initial and final conditions given by:

\[
\psi_0[\dot{x}(t_0), x'(t_0), t_0] = 0 \quad (6.2.3)
\]

\[
\psi_f[x^m(t_f), x'(t_f), t_f] = 0 \quad (6.2.4)
\]

appropriate event conditions having the form:

\[
\psi_e[\dot{x}'(t_i'), x'(t_i'), t_i'] = 0 \quad , \quad (i \leq m)
\]

along with any general non-linear path constraints represented as:

\[
g[\dot{x}(t), x'(t), u'(t)] \leq 0 \quad , \quad (i \leq m)
\]
The original, continuous multi-phase optimal control problem given by equations (6.2.1) through to (6.2.6) may be transformed by way of direct transcription into a non-linear parameter optimization problem ready for numerical solution. Using \( N \) nodes to discretize each phase of the optimal control problem, mathematically the transformation process can be stated as follows: Find the decision variables:

\[
X = \begin{bmatrix}
 x_0^i, \ldots, x_N^i, x_0^{i+1}, \ldots, x_N^{i+1}, \ldots, x_0^m, \ldots, x_N^m, \\
u_0^i, \ldots, u_N^i, v_0^{i+1}, \ldots, v_N^{i+1}, \ldots, v_0^m, \ldots, v_N^m, \\
t_N^i, t_N^{i+1}, \ldots, t_N^m
\end{bmatrix}, \quad (i = 1, \ldots, m)
\]

that minimizes the performance criterion:

\[
\mathcal{J} = \sum_{i=1}^m \left[ \mathcal{M}^i \left[ \dot{x}_N^i, \dot{x}_N^i, x_N^i, t_N^i \right] + \sum_{k=0}^N \left( \mathcal{L}^i \left[ \dot{x}_k^i, \dot{x}_k^i, x_k^i, u_k^i, t \right] w_k^i \right) \right]
\]

subject to the dynamical state equality constraints:

\[
f \left[ \dot{x}_k^i, x_k^i, u_k^i, t \right] = 0, \quad (k = 0, \ldots, N), \quad (i = 1, \ldots, m)
\]

the given initial conditions:

\[
\psi_0^i \left[ \dot{x}_0^i, x_0^i, t_0^i \right] = 0
\]

appropriate event conditions:

\[
\psi_e^i \left[ \dot{x}_k^i, x_k^i, t_k^i \right] = 0, \quad (k = 0, \ldots, N), \quad (i = 1, \ldots, m)
\]

the final boundary conditions:

\[
\psi_N^m \left[ \dot{x}_N^m, x_N^m, t_N^m \right] = 0
\]

along with any general non-linear path constraints:

\[
g^i \left[ \dot{x}_k^i, x_k^i, u_k^i \right] \leq 0, \quad (k = 0, \ldots, N), \quad (i = 1, \ldots, m)
\]

It may also be necessary to provide box constraints for the decision variables:

\[
x_k^i \mid_{MIN} \leq x_k^i \leq x_k^i \mid_{MAX}
\]

\[
u_k^i \mid_{MIN} \leq u_k^i \leq u_k^i \mid_{MAX}
\]

Multi-phase optimal control problems allow for much more varied and richer situations to be considered within the general optimal control framework, including those that arise in real-world optimal control problems. Such scenarios include discontinuous changes in the system.
dynamics, switching conditions and various generalized event conditions that are associated with hybrid optimal control problems [151]. In the context of ATC systems, the multi-phase optimal control framework allows for multiple rendezvous and payload transportation operations to be studied. The details and development of the former is the subject of inquiry in the following sub-section of this chapter.

6.3 Dual Instantaneous Rendezvous Problems for Aerial Towed-Cable Systems

This sub-section deals with the application of multi-phase, non-linear optimal control theory to the scenario when the ATC system is required to instantaneously rendezvous with multiple desired locations on the ground. In general, regions of elevated terrain may separate these rendezvous locations; hence the ATC system must be capable of appropriately ensuring that the cable tip can safely avoid any obstacles posed by regions of elevated terrain. Dual rendezvous problems incorporating the various dynamic models of the ATC system will be presented and contrasted. It will then be possible to uncover general characteristics that are associated with various multi-rendezvous problems for the ATC system.

6.3.1 Constant Tow Speed Dual Rendezvous

The simplest possible dual rendezvous operation that the ATC system can perform is that which is pictorially depicted in Figure 6-1.

Figure 6-1: The Constant Tow Speed Dual Rendezvous Scenario
In this scenario, the aircraft flies steady and level at constant speed and altitude and the cable is deployed so as to rendezvous the tip with a series of desired surface locations \([\{x_{r_1}, y_{r_1}\}, \{x_{r_n}, y_{r_n}\}]\), whilst at all times preventing the cable tip from colliding with the ground. Initially, the ATC system is in the equilibrium configuration governed by the physical parameters for the system and the manoeuvre terminates once the final rendezvous has taken place. As in previous constant tow speed rendezvous problems, the dynamic mathematical model of the ATC system used for this investigation is that given by equation (2.4.38), along with the adjustment to the inertia of the system given by equation (5.2.1). The cable reel provides all the control actuation and each rendezvous is assumed to take place instantaneously; hence no measures are utilized to prolong the window available for these rendezvous opportunities to occur. The physical system parameters that govern the dual rendezvous problems studied in this sub-section, and those that subsequently follow, are given in Table 6-1.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(U_0)</td>
<td>Aircraft Towing Speed</td>
<td>50 m/s</td>
</tr>
<tr>
<td>(y_{IC})</td>
<td>Aircraft Altitude</td>
<td>600 m</td>
</tr>
<tr>
<td>(m_p)</td>
<td>Payload Mass</td>
<td>100 kg</td>
</tr>
<tr>
<td>(C_{D,e})</td>
<td>Nominal Payload Drag Constant</td>
<td>0.5</td>
</tr>
<tr>
<td>(d_p)</td>
<td>Payload Diameter</td>
<td>0.8 m</td>
</tr>
<tr>
<td>(l_0)</td>
<td>Initial Cable Length</td>
<td>295.28 m</td>
</tr>
<tr>
<td>(\theta_0)</td>
<td>Initial Cable Orientation Angle</td>
<td>38.84°</td>
</tr>
<tr>
<td>(\rho_c)</td>
<td>Cable Mass Density</td>
<td>7850 kg/m³</td>
</tr>
<tr>
<td>(d_c)</td>
<td>Cable Diameter</td>
<td>2.5 mm</td>
</tr>
<tr>
<td>(\rho_{air})</td>
<td>Air Density</td>
<td>1.225 kg/m³</td>
</tr>
<tr>
<td>(g)</td>
<td>Gravity Constant</td>
<td>9.81 m/s²</td>
</tr>
<tr>
<td>(x_{r_1})</td>
<td>x-coordinate of First Rendezvous</td>
<td>1000</td>
</tr>
<tr>
<td>(y_{r_1})</td>
<td>y-coordinate of First Rendezvous</td>
<td>2</td>
</tr>
<tr>
<td>(x_{r_n})</td>
<td>x-coordinate of Final Rendezvous</td>
<td>1750</td>
</tr>
<tr>
<td>(y_{r_n})</td>
<td>y-coordinate of Final Rendezvous</td>
<td>2</td>
</tr>
<tr>
<td>(h)</td>
<td>Maximum Height of Terrain</td>
<td>0 m</td>
</tr>
<tr>
<td>(\lambda)</td>
<td>Width of Elevated Terrain</td>
<td>750 m</td>
</tr>
</tbody>
</table>

Table 6-1: System Parameters Governing Dual Rendezvous
6.3.2 Constant Tow Speed Dual Rendezvous Problem Formulation

The multi-phase optimal control problem that characterizes the dual rendezvous scenario shown in Figure 6-1 will now be formally presented. The same performance criterion, dynamic constraints, model “matching” procedure, initial conditions and box constraints that were used during the single constant tow speed rendezvous scenarios are all used again to formulate these dual rendezvous scenarios. Only the final conditions and non-linear path constraints differ, along with an appropriate event condition that is required in this case.

**Event Conditions**

The first rendezvous attempt shown in Figure 6-1 is governed by the so-called event conditions, which like the final boundary conditions, are specified in terms of the in-plane position of the cable tip and the reel rate of the cable. Using equations (4.6.2) and (4.6.3) for the coordinates of the cable tip, the event condition for dual rendezvous is:

\[
\psi_e \left[ \hat{x}_c, \hat{y}_c, t_e \right] = \begin{bmatrix} x_c \ y_c \ i \end{bmatrix}_{t_e} - \begin{bmatrix} x_{r_1} \ y_{r_1} \ 0 \end{bmatrix} = 0
\]  

(6.3.1)

All other states not affected by equation (6.3.1) are continuous across are the phase boundary.

**Non-Linear Path Constraints**

Once again, a non-linear path constraint is utilized to ensure that at all times the cable tip/towed body is prevented from colliding with any localized elevated terrain. For the constant tow speed dual rendezvous scenario, the elevation of the ground terrain is defined in terms of the cable tip and is given by the following piecewise sinusoidal function:

\[
h_T(x_c) = \begin{cases} 
0 & , \ x_c < x_{r_1} \\
\frac{h}{2} \left[1 - \cos \left( \frac{x_c - x_{r_1}}{\lambda/2} \right) \right] & , \ x_r < x_c < \left(x_{r_1} + \lambda \right) \\
0 & , \ x_c > \left(x_{r_1} + \lambda \right)
\end{cases}
\]  

(6.3.2)

where \( x_{r_1} \) is the \( x \)-coordinate of the first rendezvous location, \( h \) is the maximum height of the terrain, while \( \lambda \) is the width of the elevated section of the terrain. The 2 metre collision tolerance \( y_{TOL} \) is again employed to ensure the cable tip is always higher than the ground.

Using equation (4.6.2) to specify the \( x \)-coordinate of the cable tip, equation (6.3.2) to determine the local height of the terrain, the non-linear path constraint given by equation (4.6.5) is enforced during rendezvous to prevent the cable tip from colliding with the ground.

It is important to note that there exist situations where non-linear path constraints of the form
given by equation (4.6.5) are not totally sufficient in preventing the entire cable (not just the tip) from colliding with localized terrain i.e. for sufficiently high terrain and exceptionally large cable angles, and the non-linear path constraints need to be set in a different way. However, for all the results presented in this and subsequent chapters, such scenarios are not encountered, and the entire cable does not at any time collide with any elevated terrain.

**Final Conditions**

Like the event conditions, the final boundary conditions governing this dual rendezvous problem are designated by the in-plane position of the cable tip and the final cable deployment speed. The final conditions for dual rendezvous have the same form as those given previously by equation (4.6.4), although the cable tip is required to reach a different final target condition in this scenario. Again using equations (4.6.2) and (4.6.3) for the coordinates of the cable tip, the final conditions for dual rendezvous are:

\[
\psi_N \left[ \tilde{x}_N, x_N, t_N \right] \equiv \left( x_c, y_c, j \right)_{\text{final}} - \left( x_{\text{tf}}, y_{\text{tf}} \right) = 0
\]  

(6.3.3)

**Solution Methodology and Implementation**

In order to finalize the procedure used to transform the aforementioned continuous multi-phase optimal control problem into a resulting NLP to obtain a solution, a number of outstanding issues still require further attention. Firstly, for reasons previously outlined in Section 4.6.3, the discretization scheme employed to transform the original multi-phase optimal control problem into a NLP via direct transcription is the Hermite Simpson collocation technique with linear control interpolation. Thus, the dynamic state and control time histories are approximated using Hermite interpolation, with linear interpolation used to obtain a value of the controls at the center of each interval, while the state equality equations are satisfied at the various discrete points using Simpson quadrature.

All multi-phase optimal control problems proposed and investigated as part of this dissertation are formulated and solved using a modified version of the MATLAB®-based software package DIRECT, previously introduced in Section 4.6.2. Dr. Paul Williams has modified the original formulation of DIRECT so that the software suite is capable of solving non-linear multi-phase optimal control problems, again based on direct transcription formulations using numerous discretization techniques. The modified version of DIRECT transcribes continuous multi-phase optimal control problems in essentially the same manner as single-phase variants; the original optimal control problem is converted into a finite dimensional NLP using the chosen discretization scheme. Once again, the actual transcription process is fully automated and handled completely internally by the software package. The
user is still able to maintain complete control over the actual physical formulation and
description of the multi-phase optimal control problem. However, certain rights pertaining to
the discretization schemes and internal machinations are reserved and cannot be altered.

This modified version of DIRECT employs the same sparse non-linear SQP algorithm
(SNOPT) used previously to solve NLPs resulting from single-phase optimal control
problems for the ATC system. Similarly, to determine the locations of the non-zero entries for
the Jacobian matrix, an automated pattern generator, comparable to the one used for single-phase
optimal control problems, is also provided within the modified version of DIRECT.
The actual values of the constraints are again estimated using a sparse finite differencing
scheme. Scaling issues are just as critical during the direct solution of multi-phase optimal
control problems, as they are for single-phase problems. All decision variables constituting a
particular NLP arising from multi-phase optimal control problems for the ATC system receive
appropriate non-dimensionalization, to ensure that their values either lie within the interval
\([-1,1]\) or are at least of a similar order of magnitude.

To initiate the NLP solution process using the modified version of DIRECT, the initial guess
of the solution provided to SNOPT is again the equilibrium configuration of the ATC system
with no applied control inputs, assuming that the optimal control problem under consideration
is single-phase. Once more, this guess is generated by numerically integrating the state
equations with no control and interpolating the appropriate values of the state variables at the
corresponding node points. Similarly, the optimization process using the multi-phase version
of DIRECT commences with the initial discretization level \(N\) for each phase set to a low value
(typically \(N = 10\) initially) to readily obtain an approximate initial solution to the multi-phase
optimal control problem. The same iterative process employed during the single-phase
solution process is again utilized here to obtain a solution to multi-phase optimal control
problems at increasingly higher discretization levels for each phase. The iterative process
continues until the NLP solution converges, whereby the computational cost of further
increasing the discretization level of each phase outweighs any improvements to the accuracy
of the NLP solution. For multi-phase optimal control problems involving ATC systems, this
critical threshold is usually attained when the discretization level in each phase approaches
50-60 nodes. Finally, once a solution to the NLP has been returned by the modified version
of DIRECT at the final discretization level, the feasibility of this discrete solution is again
investigated further by integrating the continuous dynamical state equations using the discrete
values of the controls returned by SNOPT.
6.3.3 Dual Instantaneous Rendezvous With Aircraft Manoeuvring

Instantaneous dual rendezvous performed with the inclusion of aircraft manoeuvring is graphically illustrated in Figure 6-2. This scenario is exactly the same as that depicted in Figure 6-1, except that the aircraft is permitted to dynamically manoeuvre to assist in rendezvousing the cable tip with the desired surface locations. The mathematical model of the ATC system used to study this type of dual rendezvous problem is that provided by equation (2.4.35), as both the aircraft and the cable reel are capable of providing the control actuation needed for each rendezvous to take place. The physical system parameters that govern this dual rendezvous problem are the same as those appearing in Table 6-1.

![Figure 6-2: Dual Instantaneous Rendezvous With In-Plane Aircraft Manoeuvring](image)

6.3.4 Problem Formulation For Dual Rendezvous With In-plane Aircraft Manoeuvring

The multi-phase optimal control problem that characterizes the dual rendezvous scenario shown in Figure 6-2 will now be formulated. The performance criterion, dynamic constraints, model “matching” procedure, initial conditions and box constraints that were used during the single rendezvous scenarios with in-plane aircraft manoeuvring, are re-cycled in order to formulate these newly proposed dual rendezvous problems.
Only the final conditions, event conditions and path constraints are different, although they are very similar to those previously outlined for the dual constant tow speed rendezvous problem. Equations (6.3.1) and (6.3.2), along with equation (4.6.5) are used to specify the event conditions and path constraints for dual rendezvous with aircraft manoeuvring, except that equation (4.6.26) is used to determine the $x$-component of the cable tip position, instead of equation (4.6.2). Equation (6.3.3) is used to specify the final rendezvous conditions, with equations (4.6.26) and (4.6.31) used to appropriately determine final position of the cable tip.

The results outlining the nature of dual rendezvous scenarios for the ATC system with varying degrees of aircraft manoeuvring are shown in Figure 6-3 through to Figure 6-9. The results concerning the dual rendezvous problem with full in-plane aircraft manoeuvring are given by the blue line (denoted “Plan” in the legend of each figure), whilst the results for the constant tow speed dual rendezvous problem are designated using the red line (denoted “Const” in the legend of each figure). Although not formally presented in the results offered by Figure 6-3 through to Figure 6-9, the discrete optimal trajectories found using DIRECT were found to be feasible and in close agreement with the propagated solution expected of the original continuous optimal control problems. It can be seen from Figure 6-3 that the cable in-plane configuration angle does not vary considerably during the dual rendezvous manoeuvre, regardless of whether the aircraft velocity is permitted to vary or not. This phenomenon was also observed during the previous investigations into single-phase rendezvous problems for the ATC system. However, as Figure 6-3 attests, the cable in-plane configuration angle varies faster and by a slightly increased amount when the aircraft is permitted to manoeuvre.

![Figure 6-3: Cable Angular Dynamics During Dual Rendezvous- (a) Angle (b) Velocity](image)

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In order to perform dual rendezvous with aircraft manoeuvres, Figure 6-4 and Figure 6-5 indicate that initially the aircraft dives steadily towards the first target position with decreasing horizontal speed and increasing vertical dive speed. The aircraft then begins to pull up with deceasing dive speed and increasing horizontal speed as it approaches the first target location and successfully executes rendezvous. In the absence of any localized elevated terrain between the rendezvous locations, once the first rendezvous attempt has been completed, the aircraft flies at a relatively constant speed and altitude (slight shallow dive) until the final rendezvous is completed. The aircraft flies further and for a slightly longer time when dual rendezvous is attempted with aircraft manoeuvres enabled.
Expectedly, Figure 6-6 indicates that significantly larger amounts of cable are deployed when dual rendezvous is performed at a constant tow speed, compared to the case when the same operation is performed with variable aircraft manoeuvring. Consequentially, the cable deployment rates are appreciably higher during both phases of the dual rendezvous procedure when the towing speed is fixed, with the cable length remaining relatively constant during the final phase of the rendezvous procedure. With respect to the cable tip trajectory, Figure 6-7 reveals that the cable tip altitude is slightly higher during the first phase of the rendezvous procedure when aircraft manoeuvring is allowed. However, once the first rendezvous attempt has been achieved and until the final rendezvous occurs, the cable tip attains slightly higher altitude between the two target positions when the towing speed is constant.
There are noticeable discontinuities in the aircraft and cable reel accelerations at the instant of the first rendezvous attempt, as depicted in Figure 6-8. These discontinuities arise since the cable reel rate is constrained to be zero at the instant of the first rendezvous, although significantly less aircraft control and cable reel acceleration is then required in order for the subsequent rendezvous attempt to be executed. Therefore, the aircraft and cable reel accelerations do not remain continuous across the boundary between the first and second phases of the dual rendezvous procedure. When the aircraft is permitted to perform planar manoeuvres, the size of the discontinuity in the cable reel acceleration is notably lower than it otherwise is when the towing speed is constant.

When the towing speed is fixed, the cable reel acceleration is quasi-linear for approximately the first thirty seconds of the dual rendezvous procedure. From this point on and until the first rendezvous attempt is executed, the reel acceleration is highly non-linear, first increasing slowly before rapidly decreasing as the cable tip approaches the first target location, consistent with the zero reel rate requirement at this instant. However, when the aircraft is allowed to fly planar manoeuvres, the reel acceleration is quasi-linear during the first phase of the dual rendezvous procedure. The level of cable reel acceleration required for dual rendezvous is extensively lower when the aircraft is allowed to manoeuvre.

With respect to the acceleration required of the aircraft, the vertical acceleration is quasi-linear for most of the first phase, with an approximate 10 second period of constant vertical acceleration immediately before the first rendezvous attempt. Similarly, the horizontal acceleration required of the aircraft is quasi-linear for most of the first phase, before steadily
increasing as the cable tip approaches the first rendezvous location. As alluded to previously, almost no control is needed in the final phase of the dual rendezvous procedure when aircraft manoeuvres are permitted, whilst noticeable levels of cable reel acceleration are needed for the final rendezvous attempt to be carried out when the towing speed is constant.

The tension within the cable whilst performing dual rendezvous is outlined in Figure 6-9. When the aircraft undergoes planar manoeuvres during dual rendezvous, the tension within the cable rises steadily in a quasi-linear manner until the first rendezvous target is met. Immediately after the first rendezvous attempt, the tension within the cable discontinuously reduces by approximately 70 Newtons. Contrastingly, when dual rendezvous is performed at constant tow speed, after initially increasingly steadily, immediately before the first rendezvous attempt the tension within the cable increases rapidly by approximately 250 Newtons. The level of tension in the cable then reduces discontinuously by approximately 350 Newtons after the first rendezvous attempt has been executed. Thus, the cable undergoes an approximate five-fold reduction in tension at the instant of the first rendezvous attempt when the aircraft is permitted to manoeuvre, suggesting that aircraft-manoeuvring enables the ATC system to absorb most of the “shock” associated with multiple, sequential rendezvous operations. The tension within the cable during the final phase of the rendezvous procedure increases steadily when the towing speed is fixed due to the relatively high levels of reel acceleration occurring during this phase. However, when the aircraft is permitted to manoeuvre, the tension within the cable decreases slightly, in line with the marginal controls required during this period.

Figure 6-9: Cable Tension During Dual Rendezvous
6.3.5 Effect of Physical System Parameters on Constant Tow Speed Dual Rendezvous

The effect that the physical system parameters have on the dual instantaneous constant tow speed rendezvous problem is the subject of investigation in this sub-section. Once again, the physical parameters of the ATC system selected for thorough investigation are the mass of the payload $m_p$, the initial towing speed $U_0$ and the initial nominal length of the cable $l_0$. The parameters that govern the dual rendezvous problems studied in this sub-section are the same as those given in Table 6-1, unless otherwise specified. The domains considered for the various physical parameters under investigation, represent the widest possible breadth that still permit a solution to be obtained for the dual rendezvous problem posed in Section 6.3.2.

**Payload Mass**

The results demonstrating how the payload mass affects the ATC system when performing dual instantaneous rendezvous are given in Figure 6-10 through to Figure 6-14. The range of payload masses considered in this inquiry is $m_p \in [25, 250]$ [kg]. For the dual rendezvous problem depicted in Figure 6-1, the time required to reach each rendezvous location steadily decreases as the mass of the payload increases accordingly, as shown in Figure 6-10. This observation is consistent with the exponential-decay type relationships found during the investigation into how the payload mass affects single-phase rendezvous problems for the ATC system. Following on, Figure 6-10 also illustrates that the mass of the payload affects the times required for each rendezvous attempt in almost identical ways, as the relationship between the payload mass and each rendezvous time is approximately equal.

![Figure 6-10: Effect of Payload Mass on Rendezvous Time- (a) Phase 1 (b) Phase 2](image-url)
As shown in Figure 6-11, the discontinuity in the cable reel acceleration at the instant of the first rendezvous attempt rapidly increases in magnitude as the mass of the payload is correspondingly reduced. For payload masses greater than 125 kg, the reel acceleration required during the first and second phases of the dual rendezvous procedure is quasi-linear, steadily reducing in absolute magnitude and rate as the mass of the payload increases. However, when the payload mass falls to 35 kg, the reel acceleration required during the first phase is mostly quasi-linear, except for a fleeting period of non-linear increase and subsequent brief period of rapid discontinuous decrease, when the cable tip approaches the first rendezvous target. The magnitude and the corresponding length of time the reel acceleration is non-linearly reduced, increases and decreases respectively as the mass of the payload is reduced. Contrastingly, the length of time and magnitude of the initial non-linear increase, both increase as the mass of the payload decreases. During the second phase of the dual rendezvous operation, the reel acceleration is increasingly non-linear, both increasing in magnitude and applied for longer as the mass of the payload decreases.

The cable reel acceleration required for dual rendezvous is piecewise non-linear when the payload mass is less than 35 kg. Initially, the cable reel acceleration is quasi-linear during the first phase for approximately 20 seconds, the absolute magnitude and rate of which decreases as the mass of the payload increases. The reel acceleration then ceases for a period of time that increases as the mass of the payload decreases, before briefly resuming the quasi-linear profile for an additional (approximate) 15 seconds. Approximately 40 seconds into the dual rendezvous procedure, the now familiar sequence of non-linear increase and decrease in the reel acceleration is required for the first rendezvous manoeuvre to occur. In the final phase of the dual rendezvous attempt, the cable reel acceleration non-linearly increases in order for rendezvous with the final target location to occur, the rate at which increases as the mass of the payload decreases.

It can be observed from Figure 6-12 that an exponential-decay type relationship is shown to exist between the mass of the payload and the final value of the performance index for dual rendezvous, which again is consistent with the relationship found during the comparable investigation concerning single-phase rendezvous for the ATC system. Although heavier payloads are less manoeuvrable due to the increased inertial forces acting on them, this again does not translate into increased amounts of overall control required for dual rendezvous, since the cable tip is initially positioned closer to the targets the heavier the payload is. This phenomenon is clearly shown in Figure 6-12, which illustrates that the overall rate of applied control rapidly diminishes as the mass of the payload continues to increase beyond 100 kg.
As depicted in Figure 6-13, as the mass of the payload increases, initially the cable tip attains lower altitudes, since heavier payloads experience disproportionately more gravitational force than aerodynamic drag force, which results in the cable tip being positioned nominally closer to the ground. However, the action of the increasingly dominant control inputs cause lighter payloads to travel closer to the ground during the majority of the first phase of the dual rendezvous procedure. Once rendezvous with the first target has been achieved, in between each of the desired rendezvous locations, lighter payloads pass higher above the ground compared to heavy payloads, increasingly so as the mass of the payload is decreased.
Figure 6-13: Effect of Payload Mass on Cable Tip Trajectory for Dual Rendezvous

Figure 6-14: Effect of Payload Mass on Cable Tension for Dual Rendezvous

The effect that the payload mass has on the tension within the cable during constant tow speed dual rendezvous is displayed in Figure 6-14. Figure 6-14 shows an increasingly non-linear relationship existing between the payload mass and cable tension as the mass of the payload decreases. For payload masses greater than 100 kg, the tension within the cable steadily increases in a quasi-linear manner throughout both phases of the dual rendezvous manoeuvre, except at the instant of the first rendezvous attempt, where the tension decreases discontinuously. The magnitude of this tension drop decreases as the mass of the payload also decreases. When the payload mass is less than 100 kg, the tension within the cable non-linearly varies during both phases of the dual rendezvous procedure, increasingly so as the mass of the payload decreases further.
For payload masses between 25 kg and 100 kg, initially the tension within the cable slowly increases during the majority of the first phase, before decreasing marginally, then sharply rising as the cable tip approaches the first target position. The tension within the cable then steadily increases in the final phase as the cable tip approaches the final target, increasingly so as the mass of the payload reduces. As was observed during the investigation into how the payload mass affects single-phase constant tow speed rendezvous, the overall magnitude of the tension within the cable rapidly and non-linearly decreases as the payload mass correspondingly decreases. This is expected since lighter payloads experience disproportionately more aerodynamic drag force than gravitational force, which tends to reduce the tension level within the cable.

**Aircraft Towing Speed**

The results showing how the towing speed affects the ATC system when performing dual instantaneous rendezvous are given in Figure 6-15 through to Figure 6-19. The tow speed domain considered here is \( U_0 \in [40, 60] \) [m/s]. The time needed to reach each rendezvous target quasi-linearly decreases as the towing speed increases according to Figure 6-15. Again this observation is consistent with the quasi-linear relationships found during the investigation into how the towing speed affects single-phase rendezvous problems for the ATC system. Figure 6-15 indicates that like the payload mass, the towing speed affects each rendezvous time in essentially the same way. The quasi-linear relationships between the towing speed and rendezvous times are relatively strong, since a 50 % increase in the towing speed reduces the first and final rendezvous times by approximately 25 % and 27 % respectively.

![Figure 6-15: Effect of Aircraft Tow Speed on Rendezvous Time- (a) Phase 1 (b) Phase 2](image-url)
When dual rendezvous is carried out at towing speeds less than or equal 50 m/s, Figure 6-16 shows that the required cable reel acceleration is quasi-linear throughout the entire duration of both phases of the rendezvous operation. The absolute magnitude and rate at which the reel acceleration is applied decreases in line with reductions to the towing speed. However, for towing speeds up to 56 m/s, the cable reel acceleration takes up a quasi-sinusoidal variation after an initial quasi-linear profile, when the cable tip is in the vicinity of the first target location. The absolute magnitude of, and rate at which the reel acceleration first increases and then decreases during the first phase, increases non-linearly as the towing speed increases from 50 m/s to 56 m/s. Once the first rendezvous target has been met, the reel acceleration non-linearly decreases until the final rendezvous attempt occurs. During the final phase of the dual rendezvous operation, the rate at which the reel acceleration is decreased, increases as the towing speed increases from 50 m/s to 56 m/s.

For towing speeds greater than 56 m/s, initially the reel acceleration reduces significantly, before stabilizing momentarily, and then non-linearly varying in the aforementioned quasi-sinusoidal manner, once the cable tip is in the vicinity of the first target. After the first rendezvous target has been met, the reel acceleration non-linearly decreases until the final rendezvous attempt occurs. During the final phase of the operation, the rate at which the reel acceleration is reduced, increases as the towing speed increases from 56 m/s to 60 m/s. Overall, the discontinuous increase in the required cable reel acceleration at the instant of the first rendezvous attempt is significantly and progressively greater when the towing speed of the aircraft increases from 40 m/s to 60 m/s.

![Figure 6-16: Effect of Aircraft Tow Speed on Cable Control for Dual Rendezvous](image-url)
Consistent with what was observed during the investigations into how the towing speed affects single-phase rendezvous problems for the ATC system, Figure 6-17 indicates that a quadratic type relationship exists between the final value of the performance index and the towing speed. Hence, progressively higher overall levels of cable reel acceleration are required to achieve dual rendezvous as the towing speed increases, even though both rendezvous attempts occur progressively sooner when the dual rendezvous procedure is attempted at faster towing speeds.

Figure 6-17: Effect of Aircraft Tow Speed on Final Cost for Dual Rendezvous

Figure 6-18: Effect of Aircraft Tow Speed on Cable Tip Trajectory for Dual Rendezvous
Since the cable tip/payload will experience disproportionately more aerodynamic drag force than gravitational force as the towing speed increases, this results in the cable tip/payload being positioned nominally closer to the aircraft, a scenario illustrated initially in Figure 6-18. However, the increasingly prevalent control inputs needed at higher towing speeds, cause the cable tip/payload to travel closer to the ground during the majority of the first phase of the dual rendezvous procedure. Once rendezvous with the first target has been achieved, the cable tip trajectories associated with faster tow speeds travel higher above the ground, compared to those attributed to slower tow speeds, increasingly so as the speed of the aircraft increases.

![Figure 6-19: Effect of Aircraft Tow Speed on Cable Tension for Dual Rendezvous](image)

When the dual rendezvous procedure is carried out at towing speeds less than or equal 50 m/s, Figure 6-19 demonstrates that the tension within the cable steadily increases throughout the entire duration of both phases of the rendezvous operation. The overall magnitude of the tension within the cable decreases as the tow speed increases, yet the rate at which the cable tension increases, rises in line with increases to the towing speed. However, once the towing speed is increased to 56 m/s, after steadily increasing throughout most of the first phase, the tension within the cable begins to decrease, then continue escalating as the cable tip approaches the first target location. The absolute magnitude of, and rate at which the tension within the cable first decreases and then increases during the first phase, increases non-linearly (in the order of 200-250 Newtons) as the towing speed increases from 50 m/s to 56 m/s. During the final phase of the dual rendezvous procedure, the tension within the cable escalates rapidly for increasingly shorter periods of time, as the towing speed increases from 50 m/s to 56 m/s.
For towing speeds greater than 56 m/s, initially the tension levels within the cable rise significantly, before stabilizing momentarily, then non-linearly varying in the aforementioned quasi-sinusoidal manner once the cable tip is in the vicinity of the first target. This quasi-sinusoidal variation results in rapid changes in the order of ± 250 Newtons to occur in the tension level within the cable. When the first rendezvous target is reached, the tension within the cable non-linearly increases until the final rendezvous occurs. During the final phase of the dual rendezvous operation, the rate at which the cable tension rises, increases as the towing speed increases from 56 m/s to 60 m/s accordingly. Overall, the discontinuous reduction in the tension within the cable is significantly greater as the towing speed of the aircraft increases from 40 m/s to 60 m/s, whilst by virtue of equation (4.6.15), the overall tension level within the cable is lower when the towing speed increases.

**Initial Nominal Cable Length**

The results showing the effect that the initial nominal cable length has on the ATC system during dual instantaneous rendezvous are given in Figure 6-20 through to Figure 6-24. The cable lengths utilized to yield these results are $l_0 \in [100, 700] \text{ [m]}$. In the highly counter-intuitive manner that is consistent with what was deduced during the examination into how the initial cable length affects single instantaneous rendezvous, Figure 6-20 indicates that the time needed to reach each rendezvous point increases as the initial cable length increases, in a quasi-linear manner. These quasi-linear relationships are relatively weak, since a 600 % increase in the initial cable length, only reduces the first and final rendezvous times by approximately 12.5 % and 9.5 % respectively.

![Figure 6-20: Effect of Initial Cable Length on Rendezvous Time- (a) Phase 1 (b) Phase 2](image-url)
Regardless of the cable length initially, Figure 6-21 indicates that the reel acceleration required during the first phase is entirely quasi-linear, decreasing in absolute magnitude and rate as the initial cable length is increased. For initial cable lengths less than or equal to 300 metres, the cable reel acceleration required during the final phase of dual rendezvous is also entirely quasi-linear, although it decreases in absolute magnitude and rate in line with reductions to the initial cable length. For initial cable lengths greater than 300 metres, the cable reel acceleration required during the final phase is non-linear, increasing in absolute magnitude and rate as the initial cable length increases accordingly. Overall, the discontinuous increase in the cable reel acceleration required at the instant of the first rendezvous attempt, reduces significantly as the initial cable length is increased.

![Figure 6-21: Effect of Initial Cable Length on Cable Control for Dual Rendezvous](image)

Consistent with what was observed during the investigations into how the initial cable length affects single-phase rendezvous for the ATC system, Figure 6-22 indicates that a quasi-linear relationship exists between the final value of the performance index and the initial cable length. Overall, progressively lower levels of cable reel acceleration are required to achieve dual rendezvous as the initial cable length is increased, although this lower control input is needed for longer periods of time, when the dual rendezvous procedure is commenced with initially longer cables.
Figure 6-22: Effect of Initial Cable Length on Final Cost for Dual Rendezvous

Referring to Figure 6-23, the longer the cable is initially, the closer it is to the target and the shorter the distance it must travel in order to reach the first target position. As the initial cable length is varied, there is none of the overlap in the various tip trajectories that was observed during the previous investigations into how the payload mass and towing speed affect the dual rendezvous procedure (refer to Figure 6-13 and Figure 6-18). Instead, increasing the initial cable length has the effect of “flattening” the trajectory of the cable tip, resulting in the cable tip following a shallower trajectory towards the first target. In between the first and final rendezvous positions, the cable tip attains higher altitudes as the initial length of the cable is reduced.

Figure 6-23: Effect of Initial Cable Length on Cable Tip Trajectory for Dual Rendezvous
The tension within the cable decreases marginally at first when the initial cable length decreases, before significantly increasing as initial cable length decreases, as demonstrated by Figure 6-24. In line with the steady reduction in cable reel acceleration over the duration of the first phase, the tension levels within the cable steadily increase during the first phase of the dual rendezvous manoeuvre, regardless of the initial cable length. Once the first target location is reached, the tension within the cable reduces notably (approximately 50-150 Newtons), increasingly so as the initial cable length decreases, before non-linearly increasing once again throughout the final phase of the operation. During the final phase of the dual rendezvous procedure, the rate at which the tension levels in the cable rise, increases according to increases in the initial cable length, with increases in the order of 150 Newtons experienced for sufficiently long initial cable lengths.

![Figure 6-24: Effect of Initial Cable Length on Cable Tension for Dual Rendezvous](image)

### 6.3.6 Effect of Terrain Characteristics on Constant Tow Speed Dual Rendezvous

The impact that various terrain characteristics have on the constant tow speed dual rendezvous problem is investigated in this sub-section. The specific characteristics selected for inquiry are the maximum height $h$ and width $\lambda$ of the region of elevated terrain separating the two rendezvous locations. The additional parameters that govern the dual rendezvous problems studied in this sub-section are again the same as those given in Table 6-1. Similarly, the domains considered for the various terrain attributes are such that a solution to the dual rendezvous problem posed in Section 6.3.2 is still possible.
Terrain Height

The results elucidating the impact that variable terrain height has on the ATC system during dual rendezvous are given in Figure 6-25 through to Figure 6-29. The height of the terrain used to determine these results lies in the interval $h \in [0, 40]$ [m].

![Figure 6-25: Effect of Terrain Height on Rendezvous Time- (a) Phase 1 (b) Phase 2](image)

It can be seen from Figure 6-25 that the maximum height of the terrain separating each rendezvous location has a near negligible effect on the times required for the cable tip/payload to rendezvous with each target location. Surprisingly however, Figure 6-25 demonstrates that each rendezvous attempt occurs slightly sooner when the maximum height of the terrain increases, although these time savings are superficial.

![Figure 6-26: Effect of Terrain Height on Cable Control for Dual Rendezvous](image)
Contrastingly, Figure 6-26 indicates that the maximum terrain height has a strong impact on the required cable reel acceleration for dual rendezvous, particularly in the final phase of the procedure. Initially, the cable reel acceleration required by the ATC system reduces in absolute magnitude and rate as the maximum height of the terrain increases. However, after approximately 15 seconds into the first phase, this scenario is reversed, and increasingly more cable control is needed as the maximum height of the terrain increases, for reasons that will be discussed in due course. When the terrain height is less than 25 metres, the cable reel acceleration profile is approximately quasi-linear throughout the duration of the first phase, whilst an approximate quasi-linear control profile is employed during the final phase when the terrain height does not exceed 10 metres. When the maximum terrain height exceeds 20 metres, for the final 12.5 seconds (approximate) of the first phase, the cable reel acceleration profile is quasi-exponential, steadily increasing in magnitude as the height of the terrain increases accordingly. Similarly, during the final phase of dual rendezvous when the maximum terrain height exceeds 10 metres, the cable reel acceleration profile is also quasi-exponential; sharply increasing in magnitude as the height of the terrain also increases. At the instant of the first rendezvous attempt, the magnitude of the discontinuous change in cable reel acceleration increases as the maximum height of the terrain increases accordingly.

Arguably what is of most interest from Figure 6-26 is that once the terrain height reaches a critical altitude (35 metres), the cable reel acceleration during the first phase begins to be significantly affected by the height of the terrain. Hence, additional cable reel acceleration is required in the first phase of the procedure to ensure that the objectives of the final phase are successfully met. This observation indicates that when the terrain height separating each rendezvous point is sufficiently high, it is necessary to first induce and then utilize cable in-plane dynamical “swing” motion to prevent the cable tip/payload from colliding with regions of elevated terrain. Likewise, Figure 6-26 implies that the utilization of the cable planar “swing” motion is gradual; progressively more cable “swing” motion is required as the height of the terrain increases above the critical threshold level.

Expectedly, the total amount of cable reel acceleration required is significantly affected by the maximum height of the terrain that separates each rendezvous target. For dual rendezvous, Figure 6-27 illustrates that a quasi-exponential increase in the overall level of control is required as the maximum height of the terrain separating each rendezvous location increases from 0 metres to 40 metres respectively.
Figure 6-27: Effect of Terrain Height on Final Cost for Dual Rendezvous

Figure 6-28 reveals that the cable tip attains increasingly higher altitudes during the first phase of the dual rendezvous manoeuvre as the maximum terrain height increases. This further confirms the observation made previously regarding the requirement that the ATC system exploit the in-plane dynamical “swing” motion of the cable in the first phase to prevent the cable tip from colliding with the elevated terrain in the final phase. As expected, between each rendezvous location, the cable tip attains increasingly higher altitudes in order to successfully negotiate progressively higher terrain.

Figure 6-28: Effect of Terrain Height on Cable Tip Trajectory for Dual Rendezvous
In line with the required cable reel acceleration input, the tension levels within the cable during the dual rendezvous manoeuvre are given by Figure 6-29. The tension within the cable is initially higher for greater terrain heights, although after approximately 15 seconds into the first phase, the tension levels within the cable begin to increase more slowly when the height of terrain increases. For terrain heights less than 25 metres, the tension within the cable increases quasi-linearly throughout the duration of the first phase of the dual rendezvous operation. Similarly, the tension within the cable increases quasi-linearly throughout the duration of the final phase when the surface terrain height is 10 metres high or less.

For maximum terrain heights in excess of 20 metres, the tension in the cable increases quasi-exponentially for approximately 12.5 seconds before the conclusion of the first phase, the magnitude of which steadily increases in sync with increases to the height of the terrain. Likewise, during the final phase of the dual rendezvous procedure, the tension in the cable also increases quasi-exponentially when the maximum terrain height exceeds 10 metres, sharply increasing in magnitude as the height of the terrain increases further. Finally, at the instant of the first rendezvous attempt, the tension in the cable drops discontinuously, the magnitude of which increases in line with subsequent increases to the maximum height of the elevated terrain.
Terrain Width

The results revealing the impact that variable terrain width has on the ATC system during dual rendezvous are given in Figure 6-30 through to Figure 6-34. The width of the terrain used to determine these results lies in the interval $\lambda \in [500, 1500]$ [m].

![Graph showing the effect of terrain width on rendezvous time.](image)

**Figure 6-30: Effect of Terrain Width on Rendezvous Time- (a) Phase 1 (b) Phase 2**

Figure 6-30 illustrates that the width of the terrain separating each rendezvous location has a trivial effect on the time required to rendezvous with the first target, whilst having an irrefutably clear affect on the final rendezvous time. As the width of the terrain increases, Figure 6-30 indicates that the time required to reach the final rendezvous target increases linearly at an approximate rate of 0.02 s/m respectively.

Expectedly, Figure 6-31 demonstrates that the cable reel acceleration required in the final phase of the dual rendezvous procedure is resoundingly affected by the width of the terrain. In general, the cable reel acceleration required in the first phase is mostly unaffected by the width of the terrain, although there are exceptions to this generalization. Initially, the cable reel acceleration required to achieve dual rendezvous marginally increases in absolute magnitude and rate as the maximum width of the terrain decreases. The cable reel acceleration profile is quasi-linear for most of the first phase, varying quasi-sinusoidally with slowly increasing magnitude for approximately the last 12.5 seconds of the phase when the width of the terrain is reduced. During the final phase of the dual rendezvous procedure, the reel acceleration varies quasi-linearly when the terrain width is greater than 700 metres, incrementally transforming into a quasi-exponential profile as the desired target spacing is further reduced.
Figure 6-31 clearly shows how the absolute magnitude and gradient of the cable reel acceleration during the final phase rapidly increases (negative magnitude) for progressively closer rendezvous target locations. At the instant of the first rendezvous attempt, the magnitude of the discontinuous change in the required cable reel acceleration increases as the separation of the rendezvous targets decreases accordingly.

As observed previously during the investigation into how the maximum terrain height affects dual rendezvous, Figure 6-31 also indicates that the cable reel acceleration during the first phase is significantly affected by the width of the terrain once a critical threshold is encountered (550 metres). When the terrain width is reduced to 550 metres, in order to successfully rendezvous with the first target, significantly higher levels of positive cable reel acceleration are required for approximately the first 15 seconds of the dual rendezvous operation. Appreciably higher levels of negative reel acceleration are then needed for the next 25 seconds (approximate), followed by considerably less negative reel acceleration for approximately the last 5 seconds of the phase, until the first target has been met.

Once again, the ATC system is required to induce and utilize cable dynamical “swing” motion in the first phase in order to rendezvous the cable tip/payload with sufficiently close-spaced targets. Likewise, the utilization of the cable “swing” motion is gradual, as additional cable “swing” motion is progressively required when the separation of the rendezvous targets is reduced below the critical threshold level.

![Figure 6-31: Effect of Terrain Width on Cable Control for Dual Rendezvous](image-url)
Figure 6-32: Effect of Terrain Width on Final Cost for Dual Rendezvous

Figure 6-32 demonstrates that progressively lower levels of cable reel acceleration are required to perform the dual rendezvous procedure as the width of the terrain is increased. Although the quasi exponential-decay type relationship between terrain width and final cost shown in Figure 6-32 is not strong, nevertheless the ATC system requires lower overall levels of control to achieve dual rendezvous when the desired target locations are spaced further apart. Intuitively, in terms of cable control considerations, the ATC system finds it increasingly more manageable to perform multiple, sequential rendezvous manoeuvres that are as widely spaced apart as possible, although the time required for subsequent rendezvousing increases steadily as the desired target spacing increases accordingly.

Figure 6-33: Effect of Terrain Width on Cable Tip Trajectory for Dual Rendezvous
Figure 6-33 discloses how the cable tip travels at increasingly higher altitudes during the first phase of the dual rendezvous manoeuvre as the separation between the rendezvous locations is increased. With respect to the final phase, as the width of the terrain increases, the cable tip travels increasingly further and attains ever higher altitude, although the basic trajectory is essentially the same and remains invariant to the effects of the terrain width. Figure 6-33 also clearly depicts the significant dynamical “swing” motion of the cable during the first phase of the dual rendezvous procedure when the desired rendezvous locations are closely spaced.

In sync with the associated cable reel acceleration inputs, the tension levels within the cable during dual rendezvous for variable target spacing are given by Figure 6-34. The tension within the cable is initially higher for greater terrain widths, although after approximately 15 seconds into the first phase, the tension levels within the cable begin to increase more slowly as the width of terrain is increased. Although the tension increases steadily throughout the first phase and in the final phase for sufficiently separated target points, the tension levels within the cable increases quasi-exponentially during the final phase when the rendezvous targets are separated by less than 600 metres. When the width of the terrain is 500 metres, the cable tension during the final phase rises quasi-exponentially by approximately 300 Newtons. Finally, at the first rendezvous point, the tension in the cable drops discontinuously by up to 150 Newtons (approximate), the magnitude of which increases according to reductions in the separation between the desired rendezvous locations.

![Figure 6-34: Effect of Terrain Width on Cable Tension for Dual Rendezvous](image-url)
6.3.7 Effect of Atmospheric Winds on Constant Tow Speed Dual Rendezvous

The effect of constant strength winds on the ATC system during constant tow speed dual rendezvous is now considered, the results of which appear in Figure 6-35 through to Figure 6-39. In this sub-section, the effects of both horizontal and vertical winds are investigated separately, as well as the case when each wind is concurrently encountered by the ATC system during dual rendezvous. The magnitude domain considered for the constant strength winds is $W_x, W_y \in [-12, 20]$ [m/s]. Since both the horizontal and vertical winds individually affect the control actuation, cable tip trajectory and cable tension in a similar way, the results shown in Figure 6-36, Figure 6-38 and Figure 6-39 are for the case when both winds are simultaneously encountered during the dual rendezvous procedure. Once again, in order to appropriately examine the effect that the combination constant strength wind (blue line) has on the final cost and time required for each rendezvous, the results concerning the individual contributions of the horizontal (red line) and vertical (green line) winds are included in Figure 6-35 and Figure 6-37.

![Figure 6-35: Effect of Constant Strength Wind on Rendezvous Time- (a) Phase 1 (b) Phase 2](image)

From Figure 6-35, it can be deduced that the time required for each rendezvous increases as the strength of downward acting head winds increases, whilst each rendezvous occurs faster as the magnitude of the upward acting tail winds increase. Quasi-linear relationships exist between both horizontal and combination winds and the time required for each rendezvous attempt, whilst a slight quadratic relationship exists between the vertical winds and each rendezvous time. Figure 6-35 clearly shows how the individual contribution of the horizontal wind disproportionately dominates the overall resultant effect that the combination wind has
on the time required for each rendezvous attempt, even though the magnitude of the individual horizontal and vertical wind components are equal. Consequentially, combined winds affect the time required for rendezvous in the same manner as horizontal winds do, although the extent to which is augmented (reduced) slightly due to the action of the vertical winds. Similar conclusions were reached when the effects of constant strength prevailing winds were studied during single-phase constant tow speed rendezvous problems.

Further analogous conclusions can also be drawn upon consideration of the results presented in Figure 6-37. Again the individual horizontal wind contribution can be seen to dominate the overall resultant effect that the combination wind has on the final value of the performance index. The overall control required for dual rendezvous quasi-quadratically decreases as the size of downward head winds decrease, with further, increased control input savings possible if rendezvous is performed in the presence of increasingly stronger upward tail winds. The overall cable reel acceleration needed for dual rendezvous quasi-quadratically increases as the direction of strong vertical winds shift from acting downward to acting upward.

The findings concerning the overall control required for dual rendezvous are further supported by the cable reel acceleration time histories outlined in Figure 6-36. Initially, higher levels of cable reel acceleration are required when dual rendezvous is attempted in the presence of downward acting head winds, compared to the corresponding cases when upwardly acting tail winds are encountered. For all upward acting tail winds, the cable reel acceleration required in the first phase is entirely quasi-linear, whilst for downward head winds, the required cable reel acceleration is initially quasi-linear, with a sinusoidal-type non-linear variation needed in the concluding stages of the phase. The corresponding magnitude of this sinusoidal-variation increases as the strength of the downward head winds increase. Immediately following the first rendezvous instance, the magnitude of discontinuous change in the cable reel acceleration increases as the nature of the prevailing wind changes from being an upward acting tail wind, to being a downward acting head wind.

For all downward acting head winds, the cable reel acceleration required in the final phase of dual rendezvous is entirely quasi-linear, decreasing in absolute magnitude and rate as the strength of the prevailing head wind decreases. For upward acting tail winds, the cable reel acceleration required in the final phase is mostly quasi-linear, increasing in absolute magnitude and rate as the strength of the prevailing tail wind increases. An additional piecewise linear segment is needed in the concluding stages of dual rendezvous procedures.
that experience upward acting tail winds, the absolute magnitude and rate of which decreases as the strength of the prevailing tail wind increases.

![Figure 6-36: Effect of Combination Constant Strength Wind on Cable Control](image)

Since the cable tip/payload will experience disproportionately more aerodynamic drag force than gravitational force when the prevailing combination wind acts as a head wind, the cable tip/payload will consequentially reside closer to the aircraft, a scenario initially depicted in Figure 6-38. As commonly observed throughout this dissertation however, the increasingly dominant control inputs required to successfully achieve dual rendezvous in the presence of strong head winds, compel the cable tip/payload to travel closer to the ground during the majority of the first phase. Once rendezvous with the first target has been achieved and until
the final rendezvous location has been reached, the cable tip travels higher above the ground when progressively stronger downward head winds are encountered, compared to comparable scenarios when strong upward tail winds prevail.

Figure 6-38: Effect of Combination Constant Strength Wind on Cable Tip Trajectory

Figure 6-39 illustrates that for all but the strongest of downward acting head winds, the tension within the cable during the first phase steadily increases, increasingly so initially, as the prevailing wind swings from being a head wind to being a tail wind. For head winds exceeding 2 m/s, the tension in the cable momentarily decreases in the mid-to-late stages of the first phase, before increasing rapidly when the cable tip approaches the first rendezvous location. Immediately subsequent to the first rendezvous attempt, the size of the discontinuous drop in cable tension decreases, as the direction of the prevailing wind changes from acting downward (head wind) to acting upward (tail wind). For all downward acting head winds, the tension within cable in the final phase increases steadily at a correspondingly reduced rate when the strength of the prevailing head wind decreases. Due to the nature of the cable reel acceleration, the tension in the cable also varies in a piecewise manner when dual rendezvous is performed in the presence of upward acting tail winds. Initially, the tension level in the cable increases at a progressively reduced rate as the strength of tail winds increase, however this scenario reverses late in the procedure; the tension in the cable increases at a increasingly rapid rate for progressively shorter time periods, as the strength of prevailing tail winds increase accordingly.
6.3.8 Dual Rendezvous With Three Dimensional Aircraft Manoeuvring

Whereas the previous dual rendezvous problems for the ATC system were two dimensional in nature, multiple rendezvous problems for the ATC system can be generalized further by incorporating full three-dimensional motion capabilities for the system. Once again, as an added level of complexity, the three-dimensional ATC system is required to perform the dual rendezvous manoeuvre whilst avoiding elevated regions of terrain. The dual rendezvous scenario with three-dimensional aircraft manoeuvring is depicted in Figure 6-40.
With full control actuation available, the three-dimensional dual rendezvous procedure depicted in Figure 6-40 again consists of two distinct phases. Dual rendezvous begins by deploying the cable so that the tip can rendezvous with the first desired location on the ground \( \left[ \{x_T, y_T, z_T\}_1 \right] \), and terminates once the final rendezvous attempt, located at the coordinates \( \left[ \{x_T, y_T, z_T\}_2 \right] \) has been completed, all the time ensuring that the cable tip does not collide with the surface terrain. The scenario begins with the ATC system in equilibrium with the same the physical parameters as given in Table 6-1, except slightly different boundary conditions and event conditions are used. Initially, no out-of-plane motion for the ATC system exists and the aircraft is positioned at the coordinates \( \{2750, 250, 600\} \). The first rendezvous attempt occurs at \( \{-1000, 0, 1.5\} \), whilst the final rendezvous attempt takes place at a location given by the coordinates \( \{1000, 200, 6.6\} \). The mathematical model of the ATC system used for this rendezvous problem is that given by equation (2.4.57).

The optimal control problem that characterizes the three-dimensional dual rendezvous procedure can now be formulated. The performance criterion, dynamic constraints, model “matching” procedure, initial conditions, non-linear path and box constraints that were used during the single three-dimensional rendezvous scenario are used again to formulate this newly proposed dual rendezvous problem. Only the event and final conditions are different, although they are very similar to those associated with the two-dimensional dual rendezvous problems previously outlined.

**Event Conditions**

Using equations (4.6.36) through to (4.6.38) for the coordinates of the cable tip, the event conditions governing the first rendezvous attempt of the three-dimensional dual rendezvous procedure are given by:

\[
\psi^e \left[ \dot{x}^e, x^e, i^e \right] = \left( x_c, y_c, z_c, 1 \right)_{e_{r}} - \left( x_{T_1}, y_{T_1}, z_{T_1}, 0 \right) = 0 \quad (6.3.4)
\]

**Final Conditions**

The final conditions for the three-dimensional dual rendezvous problem have the same form as those given are given by equation (6.3.3), although the cable tip is required to reach a different final target condition in this scenario. Again using equations (4.6.36) through (4.6.38) for the coordinates of the cable tip, the final conditions for dual rendezvous are:

\[
\psi^f \left[ \dot{x}^f, x^f, i^f \right] = \left( x_c, y_c, z_c, 1 \right)_{f_{r}} - \left( x_{T_2}, y_{T_2}, z_{T_2}, 0 \right) = 0 \quad (6.3.5)
\]
The results concerning the dual rendezvous problem with full three-dimensional aircraft manoeuvring are represented by Figure 6-41 through to Figure 6-48. Again the solid circular markers (●) represent the discrete values of the states and control found using DIRECT, the solid line (—) represents the propagated solution, whilst the transparent circles (○) represent the final target conditions. It can be seen from the results offered in Figure 6-41 through to Figure 6-45 that the discrete trajectories found using DIRECT are in close agreement with the propagated solution as desired. Similarly, it can also be concluded from Figure 6-43 and Figure 6-46 through to Figure 6-47 that all the final target states were attained by the ATC system, confirming that the objectives associated with the dual rendezvous manoeuvre were successfully met.

![Figure 6-41: In-Plane Cable Angular Dynamics During Dual Rendezvous- (a) Angle (b) Velocity](image1)

![Figure 6-42: Out-of-Plane Cable Angular Dynamics During Dual Rendezvous- (a) Angle (b) Velocity](image2)
During the initial phase of the dual rendezvous procedure, the in-plane and out-of-plane cable angular dynamics given by Figure 6-41 and Figure 6-42 can be seen to vary in accordance with those previously observed to occur in Section 4.6.5 when the single three-dimensional instantaneous rendezvous problem was examined. With reference to Figure 6-41, initially the in-plane cable dynamics are excited by the control actuation during the initial phase, although this excitation is small and results in the in-plane orientation angle both increasing and decreasing by approximately 5 degrees, at no more than 1.2 degrees per second. With respect to the out-of-plane cable dynamics, during the initial phase, Figure 6-42 indicates that in similar manner to the in-plane dynamics, the cable out-of-plane orientation angle initially increases by approximately 12 degrees, then decreases by approximately 5 degrees, when the cable tip is in the vicinity of the first target. In the final phase of the dual rendezvous procedure, the in-plane cable configuration angle increases slightly, whilst the out-of-plane configuration angle decreases, at an appreciably higher rate, as the cable tip travels towards the final target location.

The general nature of the results associated with the initial phase of the dual rendezvous operation shown in Figure 6-43 are analogous to those previously presented in Section 4.6.5, which concern single rendezvous for the ATC system with both planar and three-dimensional aircraft manoeuvring enabled.

![Figure 6-43: Cable Radial Dynamics During Dual Rendezvous- (a) Length (b) Length Rate (c) Control](image-url)
For successful initial rendezvous, the cable reel acceleration is linear, leading to a quadratic reel rate and cubic deployment profile for the cable. Similarly, during the final phase of the dual rendezvous procedure, linear cable reel acceleration is also required, resulting in quadratic reel rate and cubic deployment profiles for the cable, although the extent to which this occurs is marginal. Thus, the cable is deployed over the entire duration of both phases of dual rendezvous, significantly more so in the first phase than in the final phase. The use of minimum cable reel acceleration in the performance criterion ensures that smooth and conservative-valued radial dynamics are experienced by the cable during each phase of the dual rendezvous procedure. Figure 6-43 also clearly shows how the cable reel acceleration discontinuously increases at the instant of the first rendezvous attempt.

As observed previously during the examination of single rendezvous with three-dimensional aircraft manoeuvring, Figure 6-44 demonstrates that for the initial rendezvous attempt to successfully take place, the aircraft dives steadily towards the target position with decreasing horizontal, and increasing lateral and vertical velocities respectively. This dive manoeuvre is steepest early on, before becoming increasingly shallower as the cable tip approaches the first target location. Similarly, initially the three-dimensional velocity of the aircraft is manipulated to ensure that the aircraft is heading in direction of the first target, resulting in the aircraft performing a relatively quick starboard turn immediately after the dual rendezvous procedure commences.
In the latter stages of the first phase, the aircraft begins to initiate a portside turn towards the location of the first surface-based target. Once the first rendezvous has been performed, the aircraft continues to dive in the direction of the second target, although at a significantly lower descent rate and with reduced lateral velocity, thereby continuing to turn portside towards the final rendezvous target. The horizontal and vertical velocity components of the aircraft remain relatively constant throughout the duration of the final phase, whilst the aircraft lateral velocity component steadily decreases.

![Figure 6-45: Aircraft Control for Dual Rendezvous- (a) x-component (b) y-component (c) z-component](image)

The components of the aircraft acceleration representing the sequence of aircraft manoeuvres previously described for successful dual rendezvous are depicted in Figure 6-45. During the initial rendezvous phase, each aircraft acceleration component varies quasi-linearly, piecewise so for the horizontal component, after which the level of in-plane acceleration is then discontinuously lowered and the lateral component slightly increased for the final rendezvous procedure to occur. Figure 6-45 clearly shows the marginal levels of in-plane aircraft-related control that is required to carry out the final phase of the dual rendezvous procedure.

Figure 6-46 outlines the in-plane trajectory both the aircraft (top line) and cable tip (bottom line) follow during the dual rendezvous operation, along with the in-plane configuration of the cable at various instances. In the initial phase, the cable tip initially dives steadily towards the first target, before pulling-up marginally as the target approaches, consistent with the requirement of zero reel rate at rendezvous.
The lengthening of the cable during this phase of the operation and relatively constant length cable used in the final phase can also be clearly seen from Figure 6-46. The relatively steep dive the aircraft performs in the initial rendezvous phase is also evident, along with the much shallower dive performed during the final phase of the procedure. Although not readily seen in Figure 6-46, the cable tip climbs marginally during the initial stages of the final phase, before gradually descending towards the final target location.
Figure 6-47 clearly illustrates the three-dimensional starboard turning manoeuvre the aircraft initially executes to ensure that it is heading in the line-of-sight direction corresponding to the first desired target location. Once this turning manoeuvre is complete, the aircraft and cable tip continue to dive steadily towards the initial target, with the cable tip descending appreciably faster towards the target than the aircraft. Once the first rendezvous attempt has been performed, Figure 6-47 shows that the aircraft dives with a much lower descent rate, whilst simultaneously turning portside towards the location of the final rendezvous point. During the final phase, the cable tip trajectory in the $z$-$y$ plane illustrated in Figure 6-47 is essentially the same as that followed by the cable tip in the $z$-$x$ plane, since the cable tip initially climbs slightly, before gradually descending towards the final target point.

![Figure 6-48: Tension Within the Cable During Dual Rendezvous](image)

After rapidly decreasing initially by approximately 405 Newtons as the aircraft turns starboard, Figure 6-48 shows that the tension in the cable progressively increases in the initial phase to an approximate maximum value of 1610 Newtons, as the aircraft continues to turn portside and dive steadily towards the initial target location. Immediately after the initial rendezvous attempt, the tension in the cable discontinuously drops by approximately 115 Newtons, before continuing to slowly decrease further as the aircraft continues to turn portside and marginally dive towards the final rendezvous target location.
6.4 Concluding Remarks

In this Section, the first set of a series of multi-phase, non-linear optimal control problems for the ATC system were proposed and successfully demonstrated. These multi-phase, non-linear optimal control problems represent a specific set of the general class of multiple rendezvous operations for the ATC system, namely dual instantaneous rendezvous. Solutions and results concerning these dual rendezvous problems were successfully obtained, which demonstrated that the ATC system is capable of sequentially rendezvousing the cable tip with multiple surface-based locations. Using deployment and retrieval control of the cable and/or aircraft manoeuvring, it is possible to execute multiple rendezvous operations whilst avoiding collisions between the cable tip and regions of elevated terrain separating each rendezvous target.

A series of parametric studies were then undertaken to establish the impact that a variety of system and environmental parameters have on the ability of the ATC system to perform various dual rendezvous operations. These parametric studies allowed important insights into the nature of the dynamics and control of the ATC system to be available, which will assist the planning and development of more practical-orientated tasks such as payload capture and delivery that are formally considered later in this dissertation. The results and findings presented in this chapter extensively convey the characteristics concerning various multiple rendezvous problems for the ATC system. In the following chapter, the sophistication of the objectives, provisions and operating environment for the ATC system are significantly expanded. More specifically, the minutiae of various payload capture and delivery operations for the ATC system are formally considered in this work.
7 MULTI-PHASE OPTIMAL CONTROL OF AERIAL TOWED-CABLE SYSTEMS: PART B

7.1 Preface

This chapter is the final part of a two-part series devoted to the development of multi-phase, non-linear optimal control problems for the ATC system. The application of both payload capture and delivery operations for the ATC system, in both two and three-dimensions, is comprehensively explored in this section. In a similar manner to that employed previously in Section 5 and Section 6, numerous case study investigations will be performed to assess the impact that a wide range of system and environmental parameters have on the ability of the ATC system to perform payload transportation operations.

7.2 Payload Transportation Operations Using Aerial Towed-Cable Systems

This sub-section deals with the application of multi-phase optimal control theory to the practical ATC system application of payload transportation. The foremost intention of payload transportation is to precisely place the tip of the cable in the vicinity of a desired location on the ground, so that a payload can either be successfully captured or delivered to/from this surface-based location. The most appropriate manner to which these outcomes should be achieved is an endeavor the multi-phase optimal control methodology must confront. Payload transportation operations incorporating the various dynamic models of the ATC system will now be studied in depth, from which important implications will be elucidated regarding the feasibility and suitability of exploiting ATC systems for payload transportation operations.

7.2.1 Constant Tow Speed Payload Transportation

Figure 7-1 pictorially depicts the simplest possible payload transportation operation that the ATC system can perform, that which involves the aircraft flying steady and level at constant altitude and capturing or delivering a payload accordingly.
In this scenario, the aircraft flies steady and level at a constant speed and altitude and the cable is deployed in such manner as to rendezvous the tip of the cable with the desired surface location \([\{x_I, y_I\}]\) and either capture or deliver a payload, whilst at all times preventing the cable tip from colliding with the ground. The payload transportation operation terminates once the aircraft has flown to a desired location in the sky \([\{x_{II}, y_{II}\}]\) and the cable attains its final equilibrium configuration. Initially, the ATC system is in the equilibrium configuration governed by the physical parameters for the system, with the aircraft initially positioned at the following coordinates \([-2000, y_{AC}\]\. As in previous constant tow speed rendezvous problems, the dynamic mathematical model of the ATC system used for this investigation is that given by equation (2.4.38), along with the adjustment to the inertia of the system given by equation (5.2.1). The cable reel provides all the control actuation, whilst payload capture and delivery is modelled via an appropriate instantaneous change in mass for the payload at the instant rendezvous takes place. The physical system parameters that govern the payload transportation problems studied in this sub-section, and those that subsequently follow, are given in Table 7-1.
Table 7-1: System Parameters Governing Constant Tow Speed Payload Transportation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_o$</td>
<td>Aircraft Towing Speed</td>
<td>50 m/s</td>
</tr>
<tr>
<td>$y_{AC}$</td>
<td>Aircraft Altitude</td>
<td>600 m</td>
</tr>
<tr>
<td>$m_{i0}$</td>
<td>Initial Payload Mass</td>
<td>150 kg</td>
</tr>
<tr>
<td>$\Delta m_p$</td>
<td>Change in Payload Mass at Rendezvous</td>
<td>$\pm 25%$</td>
</tr>
<tr>
<td>$C_{D_p}$</td>
<td>Nominal Payload Drag Constant</td>
<td>0.5</td>
</tr>
<tr>
<td>$d_p$</td>
<td>Payload Diameter</td>
<td>0.8 m</td>
</tr>
<tr>
<td>$l_0$</td>
<td>Initial Cable Length</td>
<td>295.28 m</td>
</tr>
<tr>
<td>$l_f$</td>
<td>Final Cable Length</td>
<td>295.28 m</td>
</tr>
<tr>
<td>$\theta_0$</td>
<td>Initial Cable Orientation Angle</td>
<td>38.84º</td>
</tr>
<tr>
<td>$\rho_c$</td>
<td>Cable Mass Density</td>
<td>7850 kg/m$^3$</td>
</tr>
<tr>
<td>$d_c$</td>
<td>Cable Diameter</td>
<td>2.5 mm</td>
</tr>
<tr>
<td>$\rho_{air}$</td>
<td>Air Density</td>
<td>1.225 kg/m$^3$</td>
</tr>
<tr>
<td>$g$</td>
<td>Gravity Constant</td>
<td>9.81 m/s$^2$</td>
</tr>
<tr>
<td>$x_{T_i}$</td>
<td>x-coordinate of Rendezvous</td>
<td>0</td>
</tr>
<tr>
<td>$y_{T_i}$</td>
<td>y-coordinate of Rendezvous</td>
<td>2</td>
</tr>
<tr>
<td>$x_{T_f}$</td>
<td>Final x-coordinate of the Aircraft</td>
<td>4000</td>
</tr>
<tr>
<td>$y_{T_f}$</td>
<td>Final y-coordinate of the Aircraft</td>
<td>600</td>
</tr>
</tbody>
</table>

The multi-phase optimal control problem that characterizes the payload transportation operation shown in Figure 7-1 will now be articulated. The performance criterion, model “matching” procedure, initial conditions, event conditions and box constraints that were used to formulate constant tow speed dual rendezvous scenarios are retained here, except that the coordinates of the rendezvous location given in Table 7-1 are utilized, where appropriate. The dynamical constraints, final conditions and non-linear path constraints are unique in this case and will now be the subject of individual treatment.

### 7.2.1.1 Dynamic Constraints

The dynamic state-space equality constraints that are enforced within the optimal control problems governing constant tow speed payload transportation are given by equation (2.4.38). At the instant of rendezvous, the payload undergoes an instantaneous mass change $\Delta m_p$, whose magnitude is positive for payload capture operations and negative during payload delivery procedures.
The set-valued definition for the mass of the payload during payload transportation is:

\[
\begin{cases}
m_{P_k}, & t \leq t_e \\
m_{P_e} + m_{P_e} + \Delta m_P, & t = t_e \\
m_{P_e} + \Delta m_P, & t \geq t_e
\end{cases}
\]  

(7.2.1)

### 7.2.1.1.2 Final Conditions

Unlike the event conditions, the final boundary conditions governing constant tow speed payload transportation for the ATC system are designated by the aircraft coordinates, cable length, angular velocity and reel rate at the end of the manoeuvre. The final conditions for the constant tow speed payload transportation operations are:

\[
\psi_N [\dot{x}_N, x_N, t_N] = \begin{pmatrix} U_p t & y_{AC} & l \dot{\theta} \dot{i} \end{pmatrix}_{t=t_e} - \begin{pmatrix} x_{T_T} & y_{T_T} \end{pmatrix} = 0
\]

(7.2.2)

### 7.2.1.1.3 Non-Linear Path Constraints

Once again, a non-linear path constraint is utilized to ensure that at all times the cable tip/towed body is prevented from colliding with the ground. For all constant tow speed payload transportation operations considered in this section, the surface terrain is not elevated and the condition previously given by equation (4.6.5) is employed, along with equations (4.6.2) and (4.6.3) to determine the instantaneous position of the cable tip. Likewise, the collision tolerance \(y_{TOL}\) is again employed to ensure that the cable tip is always at least 2 metres above the ground.

### 7.2.2 Payload Transportation With In-plane Aircraft Manoeuvring

The undertaking of payload transportation operations with the inclusion of in-plane aircraft manoeuvring is graphically illustrated in Figure 7-2. This scenario is exactly the same as that previously presented in Figure 7-1, except that the aircraft is permitted to dynamically manoeuvre in-plane as a means to assist cable tip rendezvous and subsequent capture or delivery. The mathematical model of the ATC system used to study payload transportation operations of this nature is given by equation (2.4.35), governed by the physical parameters appearing in Table 7-1, along with the adjustment to the inertia of the system given by equation (5.2.1).
The multi-phase optimal control problem that characterizes payload transportation operations with two-dimensional aircraft manoeuvres will now be presented. The performance criterion, model “matching” procedure, initial conditions, event conditions and box constraints that were used during the dual rendezvous scenarios with aircraft manoeuvring are re-used here, along with the appropriate rendezvous location coordinates given in Table 7-1. Once more, only the dynamic constraints, final conditions and non-linear path constraints are different in this instance, although they are similar to those previously outlined for the constant tow speed payload transportation scenario.

The dynamic state-space equality constraints to be satisfied for payload transportation with in-plane aircraft manoeuvring are given by equation (2.4.35), with equation (7.2.1) again used to define the instantaneous change in payload mass at the instant of rendezvous. The non-linear path constraint employed in this circumstance is specified using equation (4.6.5) and the collision tolerance, with equations (4.6.26) and (4.6.31) used to appropriately determine the instantaneous position of the cable tip. The governing final conditions are again designated by the aircraft coordinates, cable length, angular velocity and reel rate at the final time, expressed mathematically as:

$$\psi_N[\dot{x}_N, x_N, t_N] = \left(\begin{array}{c} x \\ y \\ l \\ \dot{\theta} \\ \dot{i} \end{array}\right)_{T=\tau_N} - \left(\begin{array}{c} x_{T_N} \\ y_{T_N} \\ l_0 \\ 0 \\ 0 \end{array}\right) = 0$$  \hspace{1cm} (7.2.3)
The results that elucidate the nature of payload transportation using the ATC system with varying degrees of aircraft manoeuvring are shown in Figure 7-3 through to Figure 7-9. More specifically, these results concern the case when the ATC system is required to deliver 25% of the payload mass to the desired surface location. Since the disposition of the results concerning payload delivery and capture are essentially identical, to avoid unnecessary duplicity, only results pertaining to payload delivery are given in Figure 7-3 through to Figure 7-9. Additionally, the tension within the cable during capture is separately provided for in Figure 7-10, since the tension levels within the cable are significantly different during payload delivery and capture.

The results concerning payload transportation with full in-plane aircraft manoeuvring are given by the blue line (denoted “Plan” in each figure), whilst the results corresponding to the constant tow speed payload transportation problem are specified using the red line (denoted “Const” in each figure). The desired target conditions for the system during payload haulage are designated by transparent circles (○) where appropriate. For the record, the discrete optimal trajectories found using the modified version of DIRECT were found to be feasible and in close agreement with the propagated solution as obligated, although a direct comparison between the discrete and propagated solutions is not formally offered in Figure 7-3 through to Figure 7-10.

![Figure 7-3: Cable Angular Dynamics During Delivery- (a) Angle (b) Velocity](image)

It can be seen from Figure 7-3 that as observed during the previous investigations into both single and multi-phase optimal control problems for the ATC system, the cable in-plane configuration angle does not vary significantly during payload transportation, regardless of
whether the aircraft is permitted to manoeuvre or not. Figure 7-3 indicates that the cable configuration angle varies faster and by slightly increased amounts during the initial guidance phase of the payload transportation operation, increasingly so when the aircraft is permitted to manoeuvre. During the final phase, surprisingly the cable in-plane configuration is relatively unaffected by the instantaneous change in payload mass at the instant of rendezvous, particularly when aircraft manoeuvres are incorporated. It can also be noted from Figure 7-3 that the time required to complete the entire payload transportation operation is noticeably less (approximately 14%) when the tow speed is fixed, compared to the case when the aircraft is permitted to manoeuvre. To a lesser extent, the time required for rendezvous is lower when the tow speed is fixed (approximately 3%), compared to the case when the aircraft velocity is allowed to vary.

When the aircraft is permitted to manoeuvre, Figure 7-4 and Figure 7-5 demonstrate that the aircraft will steadily dive towards the rendezvous target, before climbing at a slightly lower rate until the desired location in the sky is attained. During the initial guidance phase, initially the aircraft dives with decreasing horizontal speed and increasing dive speed, before progressively reducing the dive rate, while still reducing the horizontal speed until rendezvous and payload capture/delivery is achieved. During the final phase, the horizontal velocity of the aircraft continues to reduce slowly as the vertical speed of the aircraft increases steadily, in an increasingly positive manner until the final target is met.
During the guidance phase, Figure 7-6 indicates that the cable is uniformly deployed in a quadratic manner, regardless of the level of aircraft manoeuvring. As observed during previous investigations into single and multi-phase rendezvous problems, significantly larger amounts of cable are deployed when payload transportation is performed at a constant towing speed. Consequentially, when the towing speed is fixed, cable deployment rates are appreciably higher during the guidance phase of the payload transportation procedure. During the final phase, the cable must be retrieved in order for the final cable length condition to be successfully met. Since the cable is significantly longer at the instant of capture/delivery when the towing speed is fixed, the cable is retrieved at an increased rate, compared to the analogous scenario associated with aircraft manoeuvring.
Figure 7-7 indicates that the cable tip will follow relatively the same path throughout the transportation operation when the towing speed is either fixed or free. The tip altitude is slightly higher during the guidance phase when aircraft manoeuvring is allowed, an occurrence consistently observed throughout this thesis during investigations into both single and multi-phase optimal control problems for the ATC system. However, once rendezvous has been achieved and a payload has been captured or delivered, the cable tip attains increasingly higher altitudes when the towing speed is constant. Comparing the results depicted in Figure 7-4 and Figure 7-7, it can be concluded that both the aircraft and cable tip trajectories have similar forms when the aircraft is free to fly general planar manoeuvres.

Figure 7-8: Control During Delivery- (a) Aircraft x-component Acceleration (b) Aircraft y-component Acceleration (c) Cable Reel Acceleration
Regardless of the level of aircraft manoeuvring employed during payload transportation, Figure 7-8 shows that quasi-linear cable reel acceleration is required during the initial guidance phase of the operation, increasing in absolute rate and magnitude as the degree of aircraft manoeuvring decreases. Regarding the aircraft acceleration during the guidance phase, the vertical component is also quasi-linear, while the horizontal component is largely non-linear, slowly increasing initially, before rapidly increasing as the cable tip approaches the rendezvous location. Relatively little control is required in the final phase when aircraft manoeuvres are enabled, with both the aircraft horizontal and cable reel accelerations quasi-linearly increasing in magnitude, while the aircraft vertical acceleration decreases in magnitude, also in quasi-linear manner. The cable reel acceleration varies non-linearly during the final phase of the constant tow speed operation, initially increasing quasi-linearly, before both decreasing and increasing in a quasi-quadratic manner late in the procedure. Overall, appreciably higher levels of cable reel acceleration are required when payload transportation is performed at a constant towing speed. This is in agreement with findings persistently made throughout this dissertation, when optimal control problems for both constant tow speed and aircraft-manoeuvrable ATC systems have been examined.

![Figure 7-9: Cable Tension During Delivery](image)

During the guidance phase of the payload transportation operation, Figure 7-9 and Figure 7-10 demonstrate that the tension levels within the cable increase quasi-linearly, irrespective of the level of aircraft manoeuvring. At the instant of payload delivery, when the tow speed is fixed, the tension level within the cable discontinuously falls by approximately 375 Newtons, compared to an approximate 380 Newton reduction in cable tension when the aircraft is permitted to manoeuvre, as shown in Figure 7-9.
As contrastingly illustrated in Figure 7-10, when a payload is captured immediately after rendezvous, the tension within the cable discontinuously increases by approximately 337 Newtons when the towing speed is fixed, as opposed to an approximate 245 Newton increase when the aircraft is allowed to manoeuvre. Throughout the duration of the final phase in each payload transportation operation, the tension within the cable reduces marginally when the aircraft is able to manoeuvre, whereas the cable tension rapidly reduces in a non-linear manner when the ATC system transports payloads at constant speeds. Upon reflection of the results outlined in Figure 7-9 and Figure 7-10, it can be deduced that the cable tension dynamics are affected in synonymous ways when payloads are either captured or delivered, except for an appropriate discontinuous net increase or decrease in magnitude at the instant of capture/delivery accordingly.

![Figure 7-10: Cable Tension During Capture](image)

### 7.2.3 Effect of Physical System Parameters on Payload Transportation

The impact that the system parameters have on the ATC system during the transportation of payloads is the subject of investigation in this sub-section. The physical parameters of the ATC system selected for investigation are the initial mass of the payload $m_i$, the change in payload mass at rendezvous $\Delta m_p$, the initial towing speed $U_o$ and the initial nominal cable length $l_o$. The physical parameters that govern the payload transportation problems studied in this sub-section are the same as those given in Table 7-1, except slightly different boundary conditions and event conditions are used. The ATC system model used for this study is that given by equation (2.4.35), with equation (5.2.1) used to correct the inertia of the system.
In all of the payload transportation problems that follow, the aircraft is initially positioned at the coordinates \( \{ -1000, y_{AC} \} \), rendezvous and capture/delivery occurs at \( \{ 500, y_{TOL} \} \), while the payload transportation operation terminates once the aircraft has the coordinates \( \{ 2000, y_{AC} \} \). Once again, the domains considered for the various physical parameters under investigation represent the widest possible breadth that still permit a solution to the payload transportation problem posed in Section 7.2.2.

**Initial Payload Mass**

The results demonstrating how the initial payload mass affects the ATC system whilst performing payload transportation are given in Figure 7-11 through to Figure 7-16. The initial payload mass domain considered is \( m_p \in [25, 250] \) [kg]. Surprisingly, regardless of whether a payload is captured or delivered, the initial payload mass affects the final time for each phase, the aircraft and cable tip trajectories, final value of the performance index and the discontinuous change in control at the instant of rendezvous in exactly the same manner. Only the discontinuous change in the cable tension at the instant of capture/delivery is unique.

![Figure 7-11: Effect of Initial Payload Mass on Final Phase Times- (a) Phase 1 (b) Phase 2](image)

As expected, Figure 7-11 outlines the same well-known trend between the initial mass of the payload and time required for rendezvous that was identified when single-phase instantaneous rendezvous problems for the ATC system were examined. Hence, a non-linear exponential-decay type relationship exists between the rendezvous time and initial payload mass, confirming that it takes progressively longer to perform the guidance phase of payload haulage operations for increasingly lighter payloads. Similarly, the aircraft takes progressively longer to fly to the final target position for increasingly lighter payloads, although the strength
of this relationship is weak. Thus, it can be concluded that the initial payload mass does not significantly affect the total time required to perform payload transportation operations using ATC systems.

The instantaneous change in the control inputs for the ATC system immediately after rendezvous has taken place is shown in Figure 7-12. The maximum absolute change in the aircraft horizontal acceleration at the instant of delivery or capture occurs when the payload initially weighs 100 kg, whilst the maximum change to the aircraft vertical and cable reel acceleration arises when the payload mass is 250 kg initially. The minimum absolute change in the aircraft horizontal and cable reel acceleration corresponds to the case when the payload initially weighs 25 kg, whilst the minimum absolute change in the aircraft vertical acceleration occurs when the payload is initially 50 kg.

![Figure 7-12: Effect of Initial Payload Mass on Change in Controls At Instant of Delivery/Capture- (a) Aircraft x-component Acceleration (b) Aircraft y-component Acceleration (c) Cable Reel Acceleration](image)

After initially decreasing as the mass of the payload increases, the change in the aircraft horizontal acceleration post delivery/capture steadily increases as the initial mass of the payload increases to 100 kg, before slightly reducing as the initial mass of the payload further increases. Similarly, the change in the aircraft vertical acceleration immediately after rendezvous initially decreases as the payload mass increases to 50 kg, before steadily increasing as the initial mass of the payload continues to increase. After steadily increasing up to the maximum value when the initial payload mass is 50 kg, the change in the cable reel acceleration immediately after delivery/capture rapidly reduces as the initial payload mass increases to 100 kg, before slowly reducing further as the initial mass of the payload increases to 250 kg. Although the magnitude of the absolute change in the control inputs shown in
Figure 7-12 appear to be small, they are appreciable as they are in the order of approximately 23% of the maximum absolute level of aircraft horizontal and cable reel acceleration required during payload transportation, and in the order of 17.5% of the maximum absolute level of aircraft vertical acceleration.

![Figure 7-13: Effect of Initial Payload Mass on Final Cost for Transportation](image)

In a similar manner to that uncovered when dual rendezvous problems at constant tow speeds were investigated, an exponential-decay type relationship exists between the initial mass of the payload and the final value of the performance index during payload transportation, as depicted in Figure 7-13. Thus progressively lower amounts of overall control are required to both capture and deliver payloads when the initial mass of such payloads is increased.

![Figure 7-14: Effect of Initial Payload Mass on Aircraft Trajectory for Transportation](image)
Dissimilar to what was observed when the impact of the payload mass on single rendezvous problems with aircraft manoeuvring was studied, Figure 7-14 demonstrates that the initial payload mass does significantly affect the trajectory of the aircraft during payload transportation operations. As the initial mass of the payload is increased, the aircraft tends to fly at a higher altitude throughout the entire duration of the payload transportation operation. Hence, the aircraft dives progressively steeper towards the rendezvous target and subsequently climbs at a faster rate towards the final target position, as the initial mass of the payload decreases. The physical reason for this is due to the disproportionate increase in aerodynamic drag acting on increasingly lighter payloads (compared to gravitational force), rendering the cable tip/payload closer to the aircraft and further away from the desired rendezvous location. However, the aircraft essentially flies the same nominal smooth trajectory regardless of the initial mass of the payload.

Contrastingly, it can be seen from Figure 7-15 that the cable tip/payload travels at higher altitudes over slightly longer distances during the payload transportation operation, as the initial mass of the payload is reduced. In a similar vein to the aircraft trajectory, the cable tip follows an equivalent nominally smooth trajectory regardless of the initial mass of the payload.
Figure 7-16 demonstrates that comparable quasi-linear relationships exist between the initial payload mass and the instantaneous change in the cable tension immediately after rendezvous. During both payload delivery and capture operations, the discontinuous change in the tension within the cable steadily increases as the initial mass of the payload increases, the nature of the change being positive during payload delivery operations and negative during payload capture procedures. Figure 7-16 also indicates that the initial payload mass has the strongest affect on the discontinuous change in cable tension during payload delivery compared to capture, as the relative gradient of the curve pertaining to delivery in Figure 7-16 is approximately twice as large as that corresponding to payload capture.

![Chart](image_url)

**Figure 7-16: Effect of Initial Payload Mass on Change in Cable Tension At Instant of Delivery/Capture**

To place the results outlined in Figure 7-16 into perspective, the magnitude of the absolute change in the cable tension after delivery is in the order of approximately 27 % of the maximum level ordinarily encountered by the cable (pure guidance, no capture/delivery) and after capture, in the order of approximately 17 % of the maximum level that ordinarily arises. This suggests that the initial mass of the payload has a considerable impact on the tension in the cable post capture/delivery. Although not shown here, the overall tension within the cable increases and decreases linearly during the initial guidance and final phase of payload transportation respectively, irrespective of the initial payload mass. The absolute magnitude of the tension within the cable decreases for payload masses that are initially low.
Change in Payload Mass

The results indicating how the magnitude of the instantaneous change in payload mass affects the dynamics of the ATC system during payload transportation are given in Figure 7-17 through to Figure 7-21. For payload delivery operations, the range considered for the payload mass change is $\Delta m_p \in [-90, 0] \%$, whilst for payload capture operations, the domain for the payload mass change is $\Delta m_p \in [0, 90] \%$ accordingly. Similar to the previous payload mass-related physical parameter case study, regardless of whether a payload is captured or delivered, the change in payload mass affects the final time of each phase, the aircraft and cable tip trajectories, final value of the performance index and the discontinuous change in the controls at the instant of rendezvous in essentially the same manner. Once again, only the discontinuous change in the cable tension at the instant of capture/delivery is distinctive for each payload transportation operation. Furthermore, it can be concluded from Figure 7-17 and Figure 7-19 that the time required to reach rendezvous and the final desired aircraft location, along with the overall level of control needed for both payload capture and delivery, is relatively unaffected by the magnitude of the mass change the payload undergoes immediately after rendezvous.

![Figure 7-17: Effect of Payload Mass Change on Final Phase Times- (a) Phase 1 (b) Phase 2](image)

Similar findings can also be inferred from the results portrayed in Figure 7-18, which signifies that the instantaneous change in the control inputs for the ATC system immediately after rendezvous has taken place, remain largely unaffected by the magnitude of the payload mass change that occurs at this instance. Nevertheless, the absolute magnitude of the change in the control inputs themselves are significant and comparable to the maximum levels of control.
required over the duration of the payload transportation procedure. An approximate 24 % increase occurs in the horizontal aircraft acceleration and an approximate 34 % increase and decrease is required of the vertical aircraft and cable reel accelerations respectively. Yet Figure 7-18 indicates that magnitude of the payload mass change has little effect, since the absolute magnitudes of the change for each control input remains relatively constant as magnitude of the payload mass change is varied.

![Figure 7-18: Effect of Payload Mass Change on Change in Controls At Instant of Delivery/Capture- (a) Aircraft x-component Acceleration (b) Aircraft y-component Acceleration (c) Cable Reel Acceleration](image)

![Figure 7-19: Effect of Payload Mass Change on Final Cost for Transportation](image)
Surprisingly, the magnitude of the payload mass change at rendezvous does not affect the aircraft trajectory during the payload transportation operation, regardless of whether a payload is captured or delivered. Similarly, the cable tip trajectory during payload capture remains relatively unaffected by the magnitude of the payload mass change at rendezvous. As Figure 7-20 attests, the magnitude of the payload mass change at rendezvous does not affect the cable tip trajectory during the initial guidance phase and has a marginal affect on the trajectory during the final phase. Figure 7-20 indicates that the cable tip will attain a slightly elevated altitude during the final phase of the payload transportation operation when it delivers a payload that has a mass that is increasingly heavier than that it commenced the transportation operation with.

![Figure 7-20: Effect of Payload Mass Change on Cable Tip Trajectory for Delivery](image)

Similar to the results found during the previous payload transportation study, Figure 7-21 indicates that quasi-linear relationships exist between the payload mass change and the corresponding instantaneous change in cable tension immediately after rendezvous. Likewise, during both payload delivery and capture operations, the discontinuous change in cable tension steadily increases as the payload mass change increases, again the nature of the change being positive during payload delivery operations and negative during payload capture procedures. Once again, Figure 7-21 indicates that the magnitude of the payload mass change has a stronger affect on the discontinuous change in cable tension during payload delivery compared to capture, since the relative gradient of the curve pertaining to delivery in Figure 7-21 is slightly larger than that corresponding to payload capture.
The magnitude of the absolute changes in the cable tension after delivery are in the order of approximately 118% of the maximum level ordinarily encountered by the cable, and after capture, in the order of approximately 75% of the maximum level that ordinarily arises. This suggests that the magnitude of the payload mass change post capture or delivery has a major impact on the tension in the cable thereafter. Although not included in the results presented in Figure 7-21, the overall tension within the cable increases and decreases linearly during the initial guidance and final phase of payload transportation respectively, regardless of the payload mass change. For the final phase of the payload haulage procedure, the absolute magnitude of the tension within the cable decreases during delivery and increases during capture, as the absolute size of the payload mass change increases accordingly.

![Graph showing effect of payload mass change on change in cable tension at instant of delivery/capture](image)

**Figure 7-21: Effect of Payload Mass Change on Change in Cable Tension At Instant of Delivery/Capture**

*Initial Towing Speed*

The manner in which the initial towing speed affects the ATC system whilst performing payload transportation operations is shown in Figure 7-22 through to Figure 7-27, with the range of initial tow speeds contemplated lying in the interval \( U_0 \in [40, 70] \) [m/s]. Similar to the previous payload mass-related parameter case studies, regardless of whether a payload is captured or delivered, changes to the initial tow speed affect each phase final time, aircraft and cable tip trajectories, final value of the performance index and the discontinuous change in control at the instant of rendezvous in essentially the same manner. Only the discontinuous change in the cable tension at the instant of capture or delivery are dissimilar.
Figure 7-22 proposes that a quasi-linear relationship exists between the initial towing speed and both the time required for rendezvous and the time taken to successfully complete the payload transportation operation. The relationship between the initial towing speed and the time required for rendezvous is particularly strong, as an almost doubling of the initial tow speed allows for an approximate 33 % time saving to be available. On the other hand, the relationship between the initial towing speed and the final transportation time required is not as strong, although an almost doubling of the initial tow speed enables an approximate 15 % final time saving to be possible.

The instantaneous change in the control inputs for the ATC system immediately post rendezvous is shown in Figure 7-23, indicating that the change in aircraft horizontal and vertical acceleration increase in a quasi-quadratic manner as the initial tow speed increases, whilst the change to the cable reel acceleration steadily decreases for increasing initial tow speeds. Thus, the change in the control inputs for the system immediately after rendezvous has taken place are manifestly affected by the velocity of the aircraft upon commencement of the payload transportation operation. The absolute magnitude of the change in the control inputs themselves are significant as they are in the order of approximately 23 % and 51 % of the maximum absolute level of aircraft horizontal and vertical acceleration required, and in the order of approximately 55 % of the maximum required cable reel acceleration. Figure 7-23 also indicates that a 75 % increase in the initial tow speed leads to an approximate 325 % discontinuous increase in both the horizontal and vertical aircraft acceleration, accompanied by an approximate 304 % absolute increase in the cable reel acceleration immediately after rendezvous has occurred.
By virtue of Figure 7-24 and similar to what was found during the parametric studies into both single-phase and multi-phase instantaneous rendezvous problems for the ATC system, a quadratic-type relationship exists between the initial towing speed and the final value of the performance index. This indicates that progressively higher levels of overall aircraft and cable reel actuation are required to perform payload transportation operations, as the initial towing speed of the aircraft is increased.

Figure 7-23: Effect of Initial Tow Speed on Change in Controls At Instant of Delivery/Capture- (a) Aircraft $x$-component Acceleration (b) Aircraft $y$-component Acceleration (c) Cable Reel Acceleration

Figure 7-24: Effect of Initial Tow Speed on Final Cost for Transportation
Unlike what was observed during the investigation into the effect that the initial towing speed has on single rendezvous problems for the aircraft-manoeuvrable ATC system, Figure 7-25 shows that the initial tow speed does significantly affect the trajectory of the aircraft during payload transportation operations. As the initial velocity of the aircraft is increased, the aircraft tends to fly at a higher altitude throughout the duration of the payload transportation operation. This results in the aircraft diving progressively steeper towards the rendezvous target and subsequently climbing at a higher rate towards the final target position when the initial tow speed is reduced accordingly. Nevertheless, the aircraft flies essentially the same nominal trajectory regardless of the initial towing speed.

![Figure 7-25: Effect of Initial Tow Speed on Aircraft Trajectory for Transportation](image)

Following on, it can be seen from Figure 7-26 that the cable tip/payload travels at higher altitudes and slightly longer distances during the initial guidance phase when the initial towing speed is increased, due to the nominal increase in aerodynamic drag experienced by the payload at higher speeds. Whereas in the final phase, this situation is reversed and the cable tip/payload travels at increasingly higher altitudes when the initial towing speed is reduced. However, Figure 7-26 clearly shows that the initial aircraft velocity has a far stronger impact during the initial guidance phase than it does during the final phase of payload transportation. As observed for the aircraft trajectory during payload transportation, the cable tip follows an equivalent nominally smooth trajectory irrespective of the magnitude of the initial aircraft velocity.
Contrary to the results found during previous payload transportation studies, Figure 7-27 indicates that quasi-quadratic relationships exist between the initial tow speed and the corresponding instantaneous change in cable tension immediately after rendezvous. At delivery, the change in cable tension increases as the initial tow speed increases, but at a significantly lower overall rate compared to those observed during previous payload transportation studies. However, at capture, the absolute magnitude of the tension change decreases as the initial tow speed increases, once again at a much lower absolute rate compared to those observed during previous payload transportation case studies. Counter-intuitively, it appears from Figure 7-27 that the “shock” associated with transporting payloads is initially tempered by the nominally low cable tensions that arise when payloads are towed at fast speeds. Figure 7-27 also indicates that the initial tow speed affects the discontinuous change in cable tension for both payload delivery and capture to the same degree, although marginally stronger during payload delivery compared to capture, a scenario that has consistently been observed during previous payload transportation studies.

![Figure 7-26: Effect of Initial Tow Speed on Cable Tip Trajectory for Transportation](image)

The magnitude of the absolute change in the cable tension after delivery is in the order of approximately 51% of the maximum level ordinarily encountered by the cable, and after capture, in the order of approximately 25% of the maximum level that ordinarily arises. This establishes that the velocity at which payload transportation commences at has a major impact on the tension in the cable post delivery or capture. The impact that the initial towing speed has on the magnitude of the absolute change in the cable tension is approximately twice as strong during payload delivery as it is during capture.
Although not included in this sub-section, the overall tension within the cable increases and decreases linearly during the initial guidance and final phase of payload transportation respectively, regardless of the initial towing speed. During both phases of the payload haulage procedure, the absolute magnitude of the tension within the cable decreases as the initial towing speed increases accordingly, regardless of whether a payload is delivered or captured.

![Figure 7-27: Effect of Initial Tow Speed on Change in Cable Tension At Instant of Delivery/Capture](image)

*Initial Nominal Cable Length*

The influence of the initial cable length on the ability of the ATC system to undertake payload transportation operations is given in Figure 7-28 through to Figure 7-33. The initial cable length domain considered in this study is given by \( l_0 \in [250, 700] \) [m]. Once again, regardless of whether a payload is captured or delivered, changes to the initial cable length affect each phase final time, the aircraft and cable tip trajectories, final value of the performance index and the discontinuous change in control at the instant of rendezvous in essentially the same manner, and only the change in the cable tension at the instant of capture or delivery is exclusive.

Figure 7-28 indicates that a piecewise-linear relationship exists between the initial cable length and both the time required for rendezvous and the time taken to successfully complete the payload transportation operation. The time required to perform the guidance phase increases as the initial cable length is increased, increasingly so when the cable is initially less than 500 metres long. However, the overall time needed to complete payload transportation reduces as the initial cable length is increased, particularly so when the cable is initially longer than 500 metres. Unlike the effect of the initial tow speed, the relationships between
the initial cable length and the time required to complete each phase are not particularly strong, as a 180 % increase in the initial cable length both increases the time required for rendezvous by approximately 7 % and reduces the overall procedure time by approximately 13 %.

![Figure 7-28: Effect of Initial Cable Length on Final Phase Times- (a) Phase 1 (b) Phase 2](image)

The effect of the initial cable length on the instantaneous change in the control inputs immediately post rendezvous is shown in Figure 7-29. Figure 7-29 depicts that the changes to the ATC system controls all decrease in a piecewise quasi-linear manner, as the initial cable length is increased. The absolute magnitude of the change in the control inputs are considerable and in the order of approximately 20 % and 29 % of the maximum absolute level of required aircraft horizontal and vertical acceleration, and in the order of approximately 33 % of the maximum required cable reel acceleration. Similarly, Figure 7-29 indicates that a 180 % increase in the initial cable length leads to an approximate 195 % and 395 % discontinuous decrease in the horizontal and vertical aircraft acceleration immediately post rendezvous, along with an approximate 341 % discontinuous decrease in the cable reel acceleration. As a result, the instantaneous change in the control inputs immediately after rendezvous are drastically affected by the cable length used initially during payload transportation operations.

Figure 7-30 shows that progressively lower overall control actuation is required by the ATC system during payload transportation as the initial cable length increases, with an exponential-decay type relationship shown to occur between the initial cable length and final cost. Similar inverse relationships between the initial cable length and the overall amount of control
required by the system to successfully perform single and multi-phase rendezvous procedures were uncovered in Sections 5.2, 5.3 and 6.3 of this thesis.

![Figure 7-29: Effect of Initial Cable Length on Change in Controls At Instant of Delivery/Capture- (a) Aircraft x-component Acceleration (b) Aircraft y-component Acceleration (c) Cable Reel Acceleration](image)

Although the initial payload mass has the strongest effect on the aircraft trajectory, both Figure 7-31 and Figure 7-32 suggest that the initial cable length has a notable impact on the aircraft trajectory, and the strongest effect on the cable tip trajectory during payload transportation. As the initial cable length decreases, the aircraft is required fly significantly further and at a much higher altitude in order for payload transportation to take place. Furthermore, Figure 7-31 shows that the aircraft is required to dive progressively steeper...
towards the rendezvous target and subsequently climb at higher rates towards the final target position, when the initial cable length decreases accordingly.

Nevertheless, the aircraft flies essentially the same manoeuvre regardless of the cable length at the commencement of the payload transportation operation. Similarly, Figure 7-32 demonstrates that the cable tip is required to travel slightly further and visibly closer to the ground as the initial cable length is increased. Following on, the cable tip is also required to dive progressively steeper towards the rendezvous target and subsequently climb at higher rates towards the final target position when the initial cable length decreases accordingly. Finally, Figure 7-32 indicates that the cable tip follows basically the same nominal trajectory regardless of the initial cable length.

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Contrary to the results found during the previous payload transportation studies, Figure 7-33 indicates that during payload delivery procedures, the instantaneous change in cable tension immediately after rendezvous quasi-linearly decreases as the initial cable length increases. Likewise, during payload capture operations, the absolute magnitude of the tension change within the cable marginally decreases as the initial cable length increases. Once again, the nature of the discontinuous tension change is positive during payload delivery operations and negative during payload capture procedures. Expectedly, Figure 7-33 suggests that the “shock” associated with capturing/delivering payloads is mitigated when the initial cable length is long, even though the cable tension is nominally higher for initially longer cables.

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Figure 7-31: Effect of Initial Cable Length on Aircraft Trajectory for Transportation
Figure 7-32: Effect of Initial Cable Length on Cable Tip Trajectory for Transportation

Figure 7-33 indicates that the initial cable length affects the discontinuous change in cable tension during payload delivery more powerfully as compared to payload capture operations, a scenario that has been consistently observed during all previous payload transportation studies. The magnitude of the absolute change in the cable tension after delivery is in the order of approximately 29 % of the maximum level ordinarily encountered by the cable, and after capture, in the order of approximately 17 % of the maximum level that ordinarily arises. This indicates that the cable length at which payload transportation begins at has a noteworthy impact on the tension in the cable immediately after delivery or capture.

While absent from the results presented in Figure 7-33, the overall tension within the cable increases and decreases linearly during the initial guidance and final phase of payload transportation respectively, irrespective of the initial cable length. During both phases of the payload haulage procedure, the absolute magnitude of the tension level within the cable increases as the initial cable length is reduced, regardless of whether a payload is delivered or captured.
7.2.4 Effect of Atmospheric Winds on Payload Transportation

The effect of constant strength winds on the ATC system during payload transportation is now considered, the results of which appear in Figure 7-34 through to Figure 7-40. In this sub-section, the effects of both constant strength horizontal and vertical winds are investigated separately, as well as the case when each wind is simultaneously encountered by the ATC system during both payload capture and delivery operations. The magnitude domain considered for the constant strength winds is $W_x, W_y \in [-12,12] \text{ [m/s]}$. As both the horizontal and vertical winds individually affect the aircraft and cable tip trajectories in similar ways, the results shown in Figure 7-37 and Figure 7-38 are for the case when both winds are concurrently encountered during the payload transportation procedure. Likewise, in order to appropriately examine the effect that the combination constant strength wind (blue line) has on the final cost and phase times, the results concerning the individual contributions of the horizontal (red line) and vertical (green line) winds are included in Figure 7-34 through to Figure 7-36 and Figure 7-39 through to Figure 7-40. As during all previous case study investigations into payload transportation, regardless of whether a payload is captured or delivered, changes in the magnitude and direction wind affect each phase final time, the aircraft and cable tip trajectories, final value of the performance index and the discontinuous change in the controls at the instant of rendezvous in essentially the same manner. Only the change in the cable tension at the instant of capture or delivery is different during payload transportation.
Figure 7-34: Effect of Constant Strength Winds on Final Phase Times for Transportation- (a) Phase 1 (b) Phase 2

Whilst Figure 7-34 proposes that relatively strong quasi-linear relationships exist between the constant strength prevailing winds and the time required for rendezvous, the overall time required to complete payload delivery and capture operations is relatively unaffected by the strength and direction of the prevailing winds. The relationship between the horizontal winds and the time required for rendezvous is the strongest, whilst the vertical winds have the lowest effect on the time required to complete the initial guidance phase. The initial guidance phase takes approximately 5% longer to complete when the strength of the prevailing head wind increases to 12 m/s, approximately 2.5% shorter and longer when the strengths of the downward acting vertical and combination winds separately increase from 0 m/s to 12 m/s respectively. Alternatively, the initial guidance phase is approximately 8.4% shorter as the strength of prevailing tail winds increase, approximately 3.4% longer and 4.2% shorter when the strengths of the downward acting vertical and combination winds individually increase to 12 m/s respectively.

Figure 7-34 clearly shows how the individual contribution of the horizontal wind disproportionately dominates the overall resultant effect that the combination wind has on the time required for rendezvous to occur, even though the magnitude of each wind component is equal. Consequentially, combined winds affect the time required for rendezvous in the same manner as horizontal winds do, although the extent to which is reduced slightly due to the action of the vertical winds. Furthermore, the individual contribution of the vertical wind disproportionately dominates the overall resultant affect that the combination wind has on the total time required for payload transportation, bearing in mind that all prevailing winds affect
the total overall time in an extremely weak manner. Similar conclusions were reached when the effects of constant strength prevailing winds were studied for both single and multi-phase rendezvous problems for the ATC system.

![Graph showing the effect of constant strength winds on change in controls at capture/delivery.](image)

**Figure 7-35: Effect of Constant Strength Winds on Change in Controls at Capture/Delivery- (a) Aircraft x-component Acceleration (b) Aircraft y-component Acceleration (c) Cable Reel Acceleration**

The instantaneous change in the control inputs for the ATC system immediately post rendezvous is shown in Figure 7-35. Regardless of whether the prevailing wind acts horizontally or vertically, Figure 7-35 indicates that the change in aircraft vertical acceleration appreciably increases as the prevailing wind swings from being a head wind to a tail wind (or acting downward to acting upward), while the corresponding change in cable reel acceleration substantially decreases, each in an approximate quasi-linear manner. In each of these two cases, horizontal winds have the strongest effect on the change in aircraft vertical acceleration; vertical winds have the largest effect on the change in cable reel acceleration, whilst the impact of the combined winds emulates that associated with horizontal winds.

Figure 7-35 also outlines an approximate quasi-linear relationship between the vertical winds and the instantaneous change in the aircraft horizontal acceleration. The change in the aircraft horizontal acceleration considerably increases as the strength of the downward vertical wind decreases, and the corresponding magnitude of the upward acting wind increases accordingly. When either a horizontal or combination prevailing wind is encountered by the ATC system, the maximum instantaneous change in aircraft horizontal acceleration occurs when a head wind with a velocity of either 6 m/s or 4 m/s is encountered respectively. For horizontal winds both greater than and less than -6 m/s, the instantaneous change in the aircraft...
horizontal acceleration post rendezvous is considerably lower. For combination winds both greater than and less than \(-4\) m/s, the instantaneous change in the aircraft horizontal acceleration is only marginally lower. Once again, vertical winds have the strongest effect on the change in the aircraft horizontal acceleration, while the impact of combined winds mirrors that associated with horizontal winds.

The absolute magnitudes of the change in aircraft controls during payload capture and delivery in the presence of winds are significant, since the relevant values are typically in the order of approximately 24\% and 48\% of the maximum absolute levels of required horizontal and vertical aircraft acceleration. Similarly, the absolute magnitude of the change in cable reel acceleration is in the order of approximately 41\% of the maximum value that is ordinarily required. Hence, the change in the control inputs for the system immediately after rendezvous has taken place are significantly affected by both the strength and direction of prevailing winds encountered by the ATC system during payload transportation operations.

![Figure 7-36: Effect of Constant Strength Winds on Final Cost for Transportation](image)

The individual contribution of the horizontal wind can again be seen from Figure 7-36 to dominate the overall resultant effect that the combination wind has on the final cost associated with payload transportation, even though each component is equal in magnitude. The overall control required for payload transportation quasi-quadratically decreases as the strength of downward head winds decrease, with further control input savings possible if rendezvous is performed in the presence of increasingly stronger upward tail winds. The overall aircraft and cable reel acceleration needed for payload transportation quasi-quadratically increases as the direction of strong vertical winds shift from acting downward to acting upward.
Both Figure 7-37 and Figure 7-38 suggest that the nature of the prevailing winds have a noticeable impact on both the aircraft and cable tip trajectories during payload transportation. As the prevailing wind changes from being a head wind to a tail wind, tending to act upwards as opposed to acting downwards, the aircraft is required fly at a slightly higher altitude in order for payload transportation to successfully take place. Furthermore, Figure 7-37 shows that the aircraft is required to dive steeper towards the rendezvous target and subsequently climb faster towards the final target position when downward acting head winds are encountered by the ATC system. However, the aircraft flies essentially the same manoeuvre irrespective of the nature of the prevailing wind occurring during payload transportation.

Contrastingly, Figure 7-38 demonstrates that this situation is reversed when the impact that the prevailing winds have on the cable tip trajectory during payload transportation is considered. As the nature of the prevailing wind changes from being a tail wind to a head wind, tending to act downwards as opposed to acting upwards, the cable tip is consequentially required to travel at slightly higher altitudes. Following on, the cable tip dives progressively steeper towards the rendezvous target and subsequently climbs at high rates towards the final target position when downward head winds are encountered during payload transportation. The physical basis underpinning this behavior is due to the disproportionate increase in drag acting on the payload compared to gravitational force, occurring when prevailing head winds are encountered by the ATC system. This renders the cable tip/payload closer to the aircraft, causing it to travel slightly further during the payload transportation procedure. Finally, Figure 7-38 indicates that the cable tip follows the same nominal trajectory regardless of the direction and magnitude of the prevailing wind that occurs during payload haulage operations.
Figure 7-38: Effect of Combination Constant Strength Wind on Cable Tip Trajectory for Transportation

During payload delivery operations, Figure 7-39 illustrates that the discontinuous change in the cable tension immediately after rendezvous linearly increases as the strength of downward vertical winds decrease and the corresponding magnitude of upward vertical winds increases, although the strength of this relationship is relatively weak. Alternatively, the discontinuous change in the cable tension increases relatively strongly in a quasi-quadratic manner as the strength of downward acting head winds decrease and the corresponding magnitude of upward acting tail winds increase accordingly. Figure 7-39 also indicates that the horizontal winds have the strongest effect on the discontinuous change in the cable tension when such winds are tail winds, while constant strength vertical winds have the strongest effect on the cable tension change when such winds act downward. Expectedly, the impact of the combined winds on the discontinuous change in the cable tension during payload delivery imitates the impact associated with the horizontal prevailing winds.

Figure 7-39 suggests that variations in the magnitude and nature of the prevailing winds encountered by the ATC do have a notable impact on the tension in the cable immediately after delivery. The maximum magnitude of the absolute change in the cable tension after delivery is in the order of approximately 32% of the maximum level ordinarily encountered by the cable.
During payload capture operations, Figure 7-40 illustrates that the discontinuous change in the cable tension immediately after rendezvous decreases in an approximate quasi-quadratic manner as the strength of downward vertical winds decrease and the corresponding magnitude of upward vertical winds increases. Once again, in a similar manner to that observed during payload delivery, the strength of this relationship is relatively weak. Similarly, the change in the cable tension weakly decreases in a quasi-quadratic manner as the strength of head winds decrease and the corresponding magnitude of tail winds increase accordingly.

Contrary to the findings observed during payload delivery, Figure 7-40 indicates that the impact that combined winds have on the discontinuous change in the cable tension after capture is not analogous to the impact associated with horizontal winds. In much the same manner as the vertical winds, the discontinuous change in the cable tension immediately after rendezvous decreases in an approximate quasi-quadratic manner as the strength of downward acting head winds decrease and the corresponding magnitude of upward acting tail winds increase respectively. In direct contrast to the findings uncovered after delivery, the horizontal winds have the strongest effect on the change in the cable tension when such winds are heads winds, while vertical winds have the strongest effect on the cable tension change when such winds act upward. The maximum magnitude of the absolute changes in the cable tension shown in Figure 7-40 are in the order of approximately 21 % of the maximum levels ordinarily encountered by the cable. This suggests that the nature of the prevailing winds encountered by the system has an appreciable impact on the tension in the cable after capture, although the degree to which is not as significant as that associated with payload delivery.
7.2.5 Payload Transportation With Three Dimensional Aircraft Manoeuvring

Whereas the previous payload transportation problems for the ATC system were two dimensional in nature, payload transportation problems for the ATC system will be generalized further by incorporating the full three-dimensional motion capabilities for the system. Similarly, an added level of complexity is introduced into the scenario by requiring the ATC system to both capture and deliver payloads, whilst avoiding localized sections of elevated terrain during the one single transportation operation. A graphical depiction of this payload transportation problem with three-dimensional aircraft manoeuvring permitted is presented in Figure 7-41.

With full control actuation available, the three-dimensional payload transportation operation depicted in Figure 7-41 consists of three distinct phases. The transportation procedure begins by deploying the cable in such a manner so that its tip can rendezvous with the first desired location on the ground $\{x_T, y_T, z_T\}_I$ and deliver a payload without colliding with the prevailing terrain. Following on, the ATC system is then required to execute an additional rendezvous attempt, this time capturing a payload in the immediate vicinity of the second desired location on the ground $\{x_T, y_T, z_T\}_{II}$. The payload transportation operation terminates once the aircraft has flown to the desired location in the sky $\{x_T, y_T, z_T\}_{III}$. The scenario begins with the ATC system in equilibrium with the same the physical parameters as given in Table 7-1, except slightly different boundary conditions and event conditions are used.
Initially, no out-of-plane motion for the ATC system exists and the aircraft is positioned at the coordinates \((-2000, -200, 600)\). The first rendezvous attempt and delivery occurs at \((-1000, 0, 1.5)\), the final rendezvous attempt and capture occurs at \((0, 200, 5.5)\), whilst the payload transportation operation terminates once the aircraft has the coordinates \((3000, 300, 600)\).

The mathematical model of the ATC system used in this rendezvous problem is that given by equation (2.4.57). The optimal control problem that characterizes the three-dimensional payload transportation procedure can now be formulated. The performance criterion, dynamic constraints, model “matching” procedure, initial conditions, non-linear path constraints and box constraints that were used previously during the demonstration of the single three-dimensional rendezvous problem for the ATC system are re-utilized to formulate the payload transportation problem proposed in Figure 7-41.

Only the final conditions and appropriate event conditions are unique for this payload transportation operation, although they are similar to those previously outlined that governed the planar payload haulage procedures. Also, at the instant of each rendezvous attempt, the payload will undergo an appropriate instantaneous mass change \(\Delta m_p\), hence a relation similar to equation (7.2.1) is used to define the mass of the payload during this payload transportation procedure.
**Final Conditions**

The final boundary conditions governing the three-dimensional payload transportation problem are designated by the aircraft coordinates, cable length, angular velocities and cable reel rate at the final time. These boundary conditions are similar to those used to govern planar payload transportation operations, except in this instance, the aircraft is required to reach a different final target condition and both the in-plane and out-of-plane angular velocity for the cable should be zero at the conclusion of the procedure.

**Event Conditions**

The two rendezvous attempts shown in Figure 7-41 are governed by event conditions, which unlike the final boundary conditions, are specified in terms of the position of the cable tip and the reel rate of the cable. The event conditions for each rendezvous task are essentially the same and differ only in the relevant coordinates for each desired surface location. Using equations (4.6.36) through to (4.6.38) for the coordinates of the cable tip, the event conditions governing the rendezvous attempts for the three-dimensional payload transportation operation are mathematically given by:

\[
\psi_{k}^{\text{e}}(x_{c}^{k}, y_{c}^{k}, z_{c}^{k}, \dot{\theta}_{c}^{k}, t_{c}^{k}) = (x_{r_{k}}, y_{r_{k}}, z_{r_{k}}, 0) = 0, \quad k = I, II
\]  

(7.2.4)

The results concerning the payload transportation problem with full three-dimensional aircraft manoeuvring are represented by Figure 7-42 through to Figure 7-49. Again the solid circular markers (●) represent the discrete values of the states and controls found using DIRECT, the solid line (−) represents the propagated solution, whilst the transparent circles (○) represent the final target conditions. It can be seen from the results offered in Figure 7-42 through to Figure 7-46 that the discrete trajectories found using DIRECT are in close agreement with the propagated solution as desired. Similarly, it can also be concluded from Figure 7-42 through to Figure 7-44 and Figure 7-47 through to Figure 7-48 that all the final target states were attained by the ATC system, confirming that the desired payload transportation objectives were successfully met.

After a fleeting initial increase, Figure 7-42 shows that the in-plane cable configuration angle decreases slowly during the initial guidance phase, before increasing at a slightly faster rate once the delivery has taken place. The in-plane cable configuration angle then decreases once again, quickly at first, then slowly increases once capture has occurred and the aircraft makes its way to the final target position. The impact of payload delivery and capture is strongest on the in-plane dynamics of the cable, since Figure 7-42 illustrates that relatively sharp increases and decreases to the in-plane angular velocity occur at delivery and capture respectively.
Figure 7-42: In-Plane Cable Angular Dynamics During Transportation Manoeuvre (a) Angle (b) Velocity

Alternatively, Figure 7-43 demonstrates that the out-of-plane cable dynamics remain relatively unaffected by payload delivery and capture. Figure 7-43 indicates that the out-of-plane cable configuration angle generally increases steadily during the initial guidance phase as the aircraft travels towards the delivery point. In general, the level of out-of-plane motion for the cable then progressively reduces as the aircraft captures the payload and traverses directly towards the final target position.

Figure 7-43: Out-of-Plane Cable Angular Dynamics During Transportation Manoeuvre (a) Angle (b) Velocity
The general nature of the results associated with the initial guidance phase of the payload transportation operation shown in Figure 7-44, totally concur with those concerning instantaneous rendezvous for the ATC system with both planar and three-dimensional aircraft manoeuvring, previously presented in Section 4.6.5.

![Figure 7-44: Cable Radial Dynamics During Transportation Manoeuvre - (a) Length (b) Length Rate (c) Control](image)

For successful rendezvous, the cable reel acceleration is linear, leading to a quadratic reel rate and cubic deployment profile for the cable. During the penultimate phase of the transportation procedure after delivery has taken place, little variation in the cable reel acceleration is required, although it too is linear, resulting in less pronounced quadratic reel rate and cubic deployment profiles for the cable. In the final phase of the operation, the cable reel acceleration profile is once again entirely linear, although its gradient produces a quadratic reel rate that generates a cubic retrieval profile for the cable. Thus, the cable is deployed over the entire duration of the first two phases, significantly more so in the first phase than the second phase, whilst being retrieved in the final phase as required. Once again, the use of minimum cable reel acceleration in the performance criterion ensures that smooth and conservative-valued radial dynamics are experienced by the cable at all times. Figure 7-44 also clearly shows how the cable reel acceleration discontinuously increases and decreases at the instances of delivery and capture respectively, with the relevant change appreciably larger for delivery compared to capture as expected.
As observed previously during the examination of instantaneous rendezvous with three-dimensional aircraft manoeuvring, Figure 7-45 indicates that during the initial guidance phase, the aircraft dives steadily towards the target position with decreasing horizontal, and increasing lateral and vertical velocities respectively. This dive is steepest early on, before becoming increasingly shallower as the cable tip approaches the delivery location. Once again, variations to the three-dimensional velocity of the aircraft are initially required to ensure that the aircraft is heading in direction of the delivery point, resulting in the aircraft performing a relatively quick starboard turn immediately after the procedure commences. In the latter stages of the initial guidance phase, the aircraft begins to initiate a portside turn towards the location of the first target. Once the payload has been delivered, the aircraft continues to dive once again, this time in the direction of the second target location, at a much lower descent rate and lateral velocity, whilst continuing to turn portside towards the location of the capture point. Immediately after the second payload has been captured, the aircraft continues to perform the portside turn with increasing lateral velocity, whilst simultaneously climbing at a progressively higher rate until the aircraft reaches the desired final location in the sky. The horizontal velocity of the aircraft remains relatively constant throughout the duration of the final two phases of the payload transportation operation.
The components of the aircraft acceleration characterizing the aircraft manoeuvres previously described for successful payload transportation are depicted in Figure 7-46. During each of the phases constituting the payload transportation operation, each aircraft acceleration component varies quasi-linearly, piece-wise so for the horizontal acceleration component. The absolute rate of these variations is most significant for each component during the initial guidance phase. The level of applied aircraft acceleration is then progressively lowered as each phase of the payload transportation procedure is completed. Immediately after payload delivery takes place, a wholesale discontinuous decrease in magnitude occurs for each aircraft acceleration component, the largest of which corresponds to the horizontal component, followed in descending order by the vertical and lateral components respectively. Contrastingly, immediately after payload capture is performed, no discontinuous change in magnitude occurs for each aircraft acceleration component, suggesting that payload delivery has a much stronger effect on the acceleration of the aircraft than payload capture. As mentioned previously, this phenomenon is also true in the case of the cable reel acceleration; hence one can conclude that payload delivery has more of an impact than capture on the control inputs for sequential payload transportation.
Figure 7-47: In-Plane System Configuration During Transportation Manoeuvre

Figure 7-47 outlines the in-plane trajectory both the aircraft (top line) and cable tip (bottom line) follow during the payload transportation operation, along with the in-plane configuration of the cable at various times. In the initial guidance phase, the cable tip follows the now well-established smooth trajectory, initially diving steadily towards the delivery target, before pulling-up marginally as the target approaches, consistent with the requirement of zero deployment velocity at rendezvous. The lengthening of the cable during this phase of the operation and subsequent shortening in the final phase can also be clearly seen from Figure 7-47. The relatively steep dive the aircraft performs in the initial guidance phase is evident from Figure 7-47, along with the much shorter and shallower dive performed during the penultimate phase of the procedure. Although not readily apparent from Figure 7-47, in the middle phase of the operation, the cable tip follows a trajectory reminiscent of that observed when the ATC system is required to perform in-plane dual rendezvous; initially climbing slightly, before gradually descending towards the capture location. In the final phase of the payload transportation operation, both the aircraft and the cable tip travel along smooth trajectories corresponding to steady climbing manoeuvres, with the cable tip climbing at a slightly higher overall rate than the aircraft, as shown in Figure 7-47.
With respect to the out-of-plane configuration of the ATC system during the payload transportation operation, Figure 7-48 clearly illustrates the three-dimensional starboard turning manoeuvre the aircraft initially executes to ensure that it is heading in the line-of-sight direction that corresponds to the delivery location. Once this turning manoeuvre has taken place, the aircraft continues to dive steadily towards the delivery point. The out-of-plane cable tip trajectory during the initial guidance phase closely mirrors that flown by the aircraft during the same phase, although the cable tip dives towards the first target with an appreciably higher descent rate. Once the first payload has been delivered, the aircraft dives with a much lower descent rate, whilst simultaneously beginning a portside turning manoeuvre towards the location of the capture point. In the middle phase of the operation, the cable tip trajectory in the $z$-$y$ plane is essentially the same as that followed by the cable tip in the $z$-$x$ plane; initially climbing slightly, before gradually descending towards the capture location. In the final phase immediately after the payload has been captured, Figure 7-48 shows how the aircraft continues to perform an increasingly tighter portside turn, whilst simultaneously climbing at a progressively higher rate until the aircraft reaches the desired final location. The cable tip follows a similar trajectory to that of the aircraft, except it does not turn as tightly as the aircraft and climbs at a much higher rate.

After momentarily decreasing, Figure 7-49 shows that the tension within the cable progressively increases in the initial guidance phase of the manoeuvre to an approximate maximum value of 2595 Newtons, as the aircraft dives steadily towards the delivery location. Immediately after delivery, the tension in the cable discontinuously drops by approximately 325 Newtons, before continuing to decrease by a further 245 Newtons (approximate) as the
cable tip approaches the capture point. Once payload capture has been executed, the tension in the cable rapidly increases in the initial half of the final payload transportation phase, before slowly decreasing as the aircraft steadily climbs towards the final target location.

![Figure 7-49: Tension Within the Cable During Transportation Manoeuvre](image)

### 7.3 Concluding Remarks

In this chapter, the final set of multi-phase, non-linear optimal control problems for the ATC system were successfully introduced, representing a wide variety of payload transportation operations the ATC system is expertly capable of executing. Subsequent solutions and results pertaining to these payload transportation problems were successfully obtained. As a result, the possibility of robustly transporting payloads both to and from surface locations in a safe and accurate manner, using deployment and retrieval control of the cable and/or aircraft manoeuvering was categorically established. By undertaking of a series of parametric studies, the effects of a variety of system and environmental parameters on the ability of the ATC system to perform various payload transportation operations was explored. In accordance with the aims and terms of reference governing this section, the results presented in this chapter comprehensively expound the generic characteristics associated with various payload transportation problems for the ATC system. In the following section, the closed loop performance of the ATC system during rendezvous and payload transportation operations is briefly investigated. This is undertaken in order to better appreciate the robustness of the ATC system to external disturbances and examine the ability of the system to closely track the nominal optimal trajectories determined in this, and previous sections of this thesis.
8 FEEDBACK CONTROL FOR AERIAL TOWED-CABLE SYSTEMS

8.1 Preface

This section is dedicated to the development of a series of simple closed loop optimal feedback controllers for the ATC system. More specifically, the influence of time-varying wind gusts on the ability of the ATC system to follow the optimal open loop trajectories developed in Sections 4 through 7 is explored in this section. By investigating how various external disturbances affect the dynamics and control of the ATC system whilst performing rendezvous and payload transportation operations, a greater appreciation of the robustness of the system will be available.

8.2 Introduction

The various non-linear optimal controllers and trajectories developed for the ATC system in Sections 4 through to 7 are inherently open loop. If these nominal optimal control paths were implemented in practice, inevitable discretization errors and other uncertainties not modelled in the equations of motion are likely to cause the trajectories for the system to deviate from the desired optimal paths, if the real controller is not able to respond accordingly. Similarly, the ATC system may encounter external disturbances in the form of time-varying wind gusts en route during rendezvous and payload transportation operations. To ensure that the ATC system is able to recover from deviations about the nominal optimal trajectory and enable the real controller to respond appropriately, it is necessary to employ a particular means of feedback control for the system.

The design of practical controllers for non-linear, high degree of freedom systems such as ATC systems is not usually a straightforward task. To simplify the process, one often employs a strategy to fragment the entire trajectory optimization process into separate offline trajectory planning and online trajectory tracking tasks [172]. The offline task, such as those constituting Sections 4 through to 7 of this thesis, allows the trajectory of the system to be carefully designed, whereby all constraints are satisfied, the performance is optimized, and a compromise between contradictory requirements arising from various aspects of the operation is achieved. Thus, the offline portion of the trajectory optimization process usually involves
solving complete, fully non-linear optimal control problems for the system. To execute the online phase of the trajectory optimization task, a neighbouring optimal feedback control technique based on linearized system dynamics is often employed. As such, the online process usually involves carefully choosing exactly how best to accomplish the system and control objectives, since implementing neighbouring optimal feedback control is often arduous and time consuming for highly non-linear, large degree-of-freedom dynamical systems. This is due to the fact that employing neighbouring optimal feedback control has traditionally relied on solving the time-varying TPBVP constituting the matrix Riccati differential equation backwards in time [167].

An alternative approach is to treat the online tracking problem as a state space regulation problem about the reference optimal trajectory [172]. If one can synthesize effective control architectures using linearized system dynamics that nullify deviations from the reference optimal path, such an approach would be valuable since considerable effort has already been discharged in carefully designing and optimizing the reference trajectory. In light of this, limiting the online tracking problem to be as simple as possible is virtuous, particularly so since the time-varying nature of the reference trajectories result in time variant linearized system dynamics. Relatively few methods are available to design controllers that stabilize time-varying systems about reference trajectories [172], and those that do such as Gain Scheduling (see [173] for details) are often resource intensive and time consuming.

Gain scheduling involves designing feedback controllers at a series of points along the reference trajectory, then interpolating the appropriate gains to obtain their required magnitude over the entire nominal trajectory. Since the gain scheduling process needs to be repeated for different reference trajectories and the stability of the gain scheduler cannot be theoretically guaranteed, extensive resources and simulations are required to substantiate the control law. This increases development time and may be time consuming for highly non-linear, large degree-of-freedom dynamical systems such as ATC systems.

More recently, control laws based on approximations to the receding-horizon optimal control problem for non-linear dynamical systems have been proposed (see [167, 172, 174, 175] for various examples). These control laws are engineered to guarantee closed loop, asymptotic stability of linear time-varying systems, which are then used to regulate the original non-linear systems in the neighbourhood of desired reference trajectories. The basic idea behind receding-horizon optimal control is to solve a series of linear optimal control problems over a short, but moving time interval, with the current state as the initial condition and the
computed optimal control applied over this finite time interval. Once the interval has expired, the process is repeated and the optimal control is recomputed using updated values of the states, whilst the newly determined control is applied over the next finite time horizon. This is equivalent to solving a fixed-time optimal control problem online, which is practical provided that the time required to compute the optimal control is small compared to the horizon time. Thus, the receding horizon control scheme represents an intermediate step between a fully open loop and closed loop method [176].

Although potentially powerful and increasingly adopted in practice, the design and implementation of receding-horizon control methods is detailed and relatively complex. Similarly, the principal objective of this chapter is to provide an indication of the closed loop performance of the ATC system when undertaking rendezvous and payload transportation operations, not present a treatise on the subject. Fortunately, under mild assumptions, there does exist a class of design methods capable of regulating non-linear time-varying dynamical systems about reference trajectories, which are relatively easy to design and implement and whose closed loop stability is guaranteed. This powerful control design method is popularly known as Linear Quadratic Regulator (LQR) control and has been selected as the candidate control design technique to implement neighbouring optimal feedback controllers for the ATC system.

8.3 Neighbouring-Optimal Linear Quadratic Regulator Control

The majority of the difficulties and complexities associated with developing closed loop feedback controllers for the ATC system may be overcome by assuming that the deviations from the nominal optimal trajectories are not excessive and are within levels designated by linear perturbations. The presence of large aerodynamic drag forces that act on the payload and cable tend to quickly damp out any disturbances to the payload/cable motion, hence this assumption is generally upheld in most instances. As a result, the ATC system can be linearized about the nominal trajectories and the offline tracking problem becomes one associated with determining neighbouring-optimal linear feedback control, or in this particular case, linear quadratic feedback control.

The methodology of LQR control is based on minimizing the second variation of the cost function along each reference trajectory; hence the LQR control problem essentially involves the solution of a convex, least-squares optimization problem [177]. The resulting control
solution is known to have a number of useful properties; simple implementation and computation, provides stable closed loop performance and guaranteed levels of robustness [177]. However, a disadvantage associated with employing infinite horizon linear regulation for tracking control is that the time varying nature of the system dynamics are not accounted for in the calculation of the current value of the controls. This may degrade the performance of the controller, compared to the case for a control law designed using techniques that explicitly account for time-varying reference trajectories, such as receding-horizon control methods.

A formal mathematical formulation of the theory underlying the LQR control method will not be presented in this work, as such a task would significantly detract from the objectives, scope and findings of this thesis. Instead a brief outline of the general nature of LQR control problems will be provided, whilst more detailed theoretical material is available in texts by Kwakernaak and Sivan [178] and Anderson and Moore [179].

In order to apply LQR control, the system of $n$ state and $m$ control variables must be fully controllable and observable and the entire state vector $x(t)$ must be available for feedback. The first-order perturbations in the states $\delta x(t)$ and control variables $\delta u(t)$ have the following form in the state-space:

$$\delta \dot{x}(t) = \left[ A(t) \right] \delta x(t) + \left[ B(t) \right] \delta u(t), \quad t \geq t_0, \quad \delta x(t_0) = \delta \dot{x}_0 \quad (8.3.1)$$

where $A(t) \in R^{n \times n}$ and $B(t) \in R^{m \times m}$ are the system time-varying state and control influence matrices. The performance criterion $\delta J$ associated with the LQR control problem is:

$$\delta J = \delta x^T(t_f) \frac{\partial^2 M}{\partial x^2} \bigg|_{t=t_f} \delta x^T(t_f) + \delta \left\{ \int_{t_0}^{t_f} \delta x^T(t) \delta u^T(t) \right\} \begin{bmatrix} Q(t) & 0 \\ 0 & R(t) \end{bmatrix} \begin{bmatrix} \delta x(t) \\ \delta u(t) \end{bmatrix} dt \quad (8.3.2)$$

where $Q(t)$ is the state weighting matrix, $R(t)$ is the control weighting matrix and $S(t_f) = \frac{\partial^2 M}{\partial x^2} \bigg|_{t=t_f}$ is the final state weighting matrix. Each of these design matrices are symmetric, whose elements are chosen depending on the control objectives/requirements. The design weight matrices $Q(t)$ and $S(t_f)$ are assumed to be positive-semidefinite ($S(t_f) \geq 0, Q(t) \geq 0$), while $R(t)$ is assumed to be positive definite ($R(t) > 0$).
The appropriate conditions placed on the design weight matrices ensures that the performance criterion $\delta J$ is always bounded below by zero and an appropriate optimization problem results [180]. Since the integral component of the performance criterion has the form of a generalized energy function, minimizing the performance criterion ensures that the state and control variables will also be small [180]. Furthermore, as these state and control variables are in fact deviations from reference conditions, minimizing the performance criterion will ensure that the closed loop system closely tracks the nominal path, provided the deviations themselves are not too substantial.

Departing from time-dependent notation momentarily, the necessary condition that provides the optimal feedback control law is given by the solution to the following bilinear matrix differential (Riccati) equation:

$$\dot{S} = A^T S - S A + S B R^{-1} B^T S - Q, \quad t \leq t_f$$  \hspace{1cm} (8.3.3)

using the following governing boundary condition:

$$S(t_f) = \left. \frac{\partial^2 M}{\partial x^2} \right|_{x_{ref}}$$  \hspace{1cm} (8.3.4)

The optimal feedback gain $K(t)$ is determined from the solution to the matrix differential Riccati equation as follows:

$$K(t) = R^{-1} B^T S$$  \hspace{1cm} (8.3.5)

Finally, the time-varying optimal state feedback control law can be written as:

$$\delta u(t) = -K(t) \delta x(t)$$  

$$= -\left(R^{-1} B^T S\right) \delta x(t)$$  \hspace{1cm} (8.3.6)

The linear closed loop dynamics of the system, which are guaranteed to be asymptotically stable, can be found via substitution of equation (8.3.6) into equation (8.3.1), the result being:

$$\dot{x}(t) = [A - BK] x(t)$$  \hspace{1cm} (8.3.7)

In summary, the LQR design process begins by selecting the design parameter weight matrices $Q(t), R(t)$ and $S(t_f)$. The Riccati matrix differential equation given by equation (8.3.3) is then solved for $S(t)$, which is used to compute the optimal feedback gain $K(t)$. Simulations are then required to validate the closed loop performance of the system given by equation (8.3.7), which if not satisfactory, new design matrices should be selected and the
entire process repeated until the closed loop performance of the system is acceptable. The optimal feedback control law given by equation (8.3.6) is then applied to the system, which is usually performed using the non-linear state equations of the system to confirm the performance of the optimal feedback controller. The entire LQR design and implementation process is relatively expeditious and efficient if a commercial software package such as MATLAB® (which was used in this work) is utilized.

8.3.1 Linearization of Equations of Motion About Nominal Trajectories

Since the LQR control design method relies on the system being linearized about the nominal trajectories, an appropriate means is needed to linearize the non-linear ATC system motion equations. In general, two methods are available to linearize dynamical systems about time-varying trajectories. Once such method involves the expansion of the state and control variables using the equation (8.3.8) and equation (8.3.9):

\[
\begin{align*}
    x &= x_{REF} + \delta x \\
    u &= u_{REF} + \delta u
\end{align*}
\]  

(8.3.8)  

(8.3.9)

where the subscript “REF” denotes the desired reference condition and the “\( \delta \)” terms are the small perturbations from this condition. Each expression for the states and controls given by equation (8.3.8) and equation (8.3.9) is then directly substituted into the non-linear equations of motion. By neglecting any non-linear combination of the perturbation states and controls, the non-linear equations of motion can be simplified to equations of motion that are linear in the perturbation variables.

This linearization method is relatively straightforward for simple, low degree of freedom systems, but is arduous and cumbersome for complex, large degree of freedom systems such as ATC systems. Consequently, the linearization of complex equations of motion can be performed effortlessly by taking the partial derivative of each non-linear motion equation with respect to each state and control variable. The following general non-linear dynamical system:

\[
x(t) = f[x(t), u(t), t]
\]  

(8.3.10)

can be linearized into time-varying state-space form using this method as follows:

\[
\begin{align*}
    \dot{x}(t) &= \frac{\partial f}{\partial x} x(t) + \frac{\partial f}{\partial u} u(t) \\
    &= [A(t)] x(t) + [B(t)] u(t) \\
    &= \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_{1}(t) \\ b_{2}(t) \\ \vdots \\ b_{n}(t) \end{bmatrix} u(t)
\end{align*}
\]  

(8.3.11)
Since the partial derivatives (Jacobians) \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial u} \), equivalent to the state and control influence matrices respectively, will be time-varying in nature, their elements require computation at each time instant along the reference trajectory.

### 8.3.2 Controllability and Observability

As mentioned previously, the LQR control method can only be applied on dynamical systems that are fully observable and controllable. Whilst the ATC system is assumed to be fully observable in this work, ascertaining whether the ATC system is controllable is difficult to establish using standard techniques, due to the time-varying nature of the system. In general, a system is fully controllable if the rank of the controllability grammian is equal to the total number of states for a system. The controllability grammian is mathematically defined as:

\[
P = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}
\]

Since the state and control influence matrices are time varying, the controllability of the ATC is also time-varying, hence controllability can not be established for all time and only on defined time intervals. If the entries of the control influence matrix \( B(t) \) are \( n-1 \) differentiable, and the entries of the state influence matrix \( A(t) \) are \( n-2 \) differentiable, a sufficient condition for controllability is given by the matrix:

\[
\Phi(t) = \begin{bmatrix} \Phi_0(t) & \Phi_1(t) & \Phi_2(t) & \cdots & \Phi_{n-1}(t) \end{bmatrix}
\]

having a rank equal to \( n \) at least once during a time interval of interest [181]. The constituent elements (matrices) for the matrix \( \Phi(t) \) are given by [181]:

\[
\Phi_0(t) = B(t)
\]

\[
\Phi_i(t) = -A(t)\Phi_{i-1}(t) + \Phi_{i-1}(t) , \quad i = 1,2,\ldots,n-1
\]

Application of the controllability theorem given by equations (8.3.13) through to (8.3.15) indicates that the linear state space equations of motion for the ATC system are controllable, when the system executes rendezvous and payload transportation operations typical of those presented Section 4 through to 7 of this dissertation. Therefore, the LQR control method may be appropriately applied to the ATC system, the details and implementation of which are the subject of the following sub-section.
8.4 Linear Quadratic Regulator Control for Aerial Towed-Cable Systems

This sub-section concerns the application of the LQR control technique to the ATC system during the undertaking of various rendezvous and payload transportation operations. The specific influence of time-varying wind gusts on the ability of the system to track the optimal trajectories that represent the aforementioned manoeuvres is considered.

The time-varying wind gusts that constitute the nature of the chosen external disturbances for the ATC system are modelled as full-wave sinusoidal functions. Thus, the velocity of prevailing wind gusts $v_{\text{WIND}}$ has the following mathematical form:

$$v_{\text{WIND}} = W_x e_i + W_y e_j$$

$$= \begin{cases} 
G_x \sin \left( \frac{2\pi (t-t_g)}{\lambda_g} \right) e_i + G_y \sin \left( \frac{2\pi (t-t_g)}{\lambda_g} \right) e_j, & t \in \left[ t_g, t_g + \lambda_g \right] \\
0, & \text{otherwise}
\end{cases}$$

(8.4.1)

where:

$G_x$ is the magnitude of the horizontal gust component,

$G_y$ is the magnitude of the vertical gust component,

$t_g$ is the time at which each gust component commences,

$\lambda_g$ is the total length of time each gust component acts.

Following on, the time rate of change for the velocity of the prevailing wind gusts $\dot{v}_{\text{WIND}}$ as required within the equations of motion for the ATC system is:

$$\dot{v}_{\text{WIND}} = \dot{W}_x e_i + \dot{W}_y e_j$$

$$= \begin{cases} 
\left( \frac{2\pi G_x}{\lambda_g} \right) \cos \left( \frac{2\pi (t-t_g)}{\lambda_g} \right) e_i \\
\left( \frac{2\pi G_y}{\lambda_g} \right) \cos \left( \frac{2\pi (t-t_g)}{\lambda_g} \right) e_j, & t \in \left[ t_g, t_g + \lambda_g \right] \\
0, & \text{otherwise}
\end{cases}$$

(8.4.2)

The time-varying wind gusts are included in the aerodynamic drag model for the payload and ultimate dynamical equations of motion for the ATC system, in essentially the same manner as the constant strength prevailing winds, a process previously outlined in Section 5.2.4.
In each of the following figures presented in this sub-section, unless otherwise stated, the closed loop trajectories of the ATC system using the LQR control method are given by the blue line (denoted “CL” in the legend of each figure). The open loop trajectories of the system when gusts are encountered and no corrective feedback measures are provided are given by the green line (denoted “OL” in the legend of each figure). The nominal open loop optimal reference trajectories for the system are given by the red line (denoted “REF” in the legend of each figure). Finally, the transparent circle (○) represents a final target condition where appropriate. With respect to the particulars of the wind gusts, the gust magnitudes are quoted as a percentage of the initial towing speed, whilst the gust commencement times are given as a percentage of the time required for the relevant manoeuvre to take place. The gusts are timed to act so as exactly half of their duration occurs either side of the mid-point of the manoeuvre, unless otherwise specified.

### 8.4.1 Rendezvous With In-plane Aircraft Manoeuvring

The influence of time-varying wind gusts on the ability of the ATC system to track the optimal trajectories that represent single instantaneous rendezvous operations performed with aircraft manoeuvring will now be examined. The particular rendezvous scenario that serves as the basis of this investigation is the same as that previously introduced in Section 4.6.5. The values of additional physical and environmental parameters that govern the closed loop simulations investigated in this sub-section are given in Table 8-1, unless otherwise specified. Both the state and control design weighting matrices are selected as the identity matrix, whilst final state weighting matrices are not employed in any of the closed loop simulations that follow.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_x$</td>
<td>Magnitude of Horizontal Gust Component</td>
<td>5 %</td>
</tr>
<tr>
<td>$G_y$</td>
<td>Magnitude of Vertical Gust Component</td>
<td>5 %</td>
</tr>
<tr>
<td>$\lambda_g$</td>
<td>Gust Duration</td>
<td>5 secs</td>
</tr>
<tr>
<td>$t_g$</td>
<td>Gust Commencement Time</td>
<td>50 %</td>
</tr>
<tr>
<td>$Q$</td>
<td>State Weighting Matrix</td>
<td>$[I]$</td>
</tr>
<tr>
<td>$R$</td>
<td>Control Weighting Matrix</td>
<td>$[I]$</td>
</tr>
<tr>
<td>$u_{MIN}$</td>
<td>Minimum Allowable Control</td>
<td>-2.5 m/s$^2$</td>
</tr>
<tr>
<td>$u_{MAX}$</td>
<td>Maximum Allowable Control</td>
<td>2.5 m/s$^2$</td>
</tr>
</tbody>
</table>

Table 8-1: Additional Parameters Governing Closed Loop Rendezvous Simulations
Effect of Gust Magnitude and Direction

The results concerning specifically how the magnitude and direction of wind gusts affect the ATC system whilst performing rendezvous with in-plane aircraft manoeuvring are given in Figure 8-1 through to Figure 8-9. The effects of both horizontal and vertical wind gusts are investigated separately, as well as the case when each of these gusts is concurrently encountered by the system during rendezvous. The particular gust magnitudes considered here lie within the interval $G_x, G_y \in [-10, 10]$ [\%]. The results given in Figure 8-1 through to Figure 8-4 and Figure 8-9 are for the limiting cases when both downward head wind and upward tail wind gusts of maximum strength are encountered by the ATC system. Alternatively, Figure 8-5 through to Figure 8-8 briefly address the individual effect each type of gust has on the ATC system during rendezvous.

![Diagram](image.png)

Figure 8-1: Effect of Gust Magnitude and Direction on Cable Angular Dynamics During Rendezvous- (a) Angle (b) Velocity

The impact of the wind gust magnitude and direction on the angular dynamics of the cable during rendezvous is illustrated in Figure 8-1. The foremost conclusion that can be drawn from Figure 8-1 is that the cable angular dynamics are relatively unaffected by downward head wind and upward tail wind gusts during rendezvous. For wind gusts as strong as 10 \% of the initial towing speed (± 5 m/s), lasting for 5 seconds and occurring mid-way through the rendezvous manoeuvre, it can be seen from Figure 8-1 that the cable configuration angle will either increase (for downward head wind gusts) or decrease (for upward tail wind gusts) marginally in a quasi-sinusoidal manner as the gusts are encountered. The degree of deviation in the cable angular dynamics is slightly higher in the absence of corrective measures provided by the feedback controller. The feedback controller quickly damps out these
disturbances, nullifying the changes to the cable angular rate quicker than otherwise would be the case. Since the closed loop control inputs induced by the large magnitude wind gusts are clipped (see Figure 8-4) and no final state weighting matrix was employed in the LQR control formulation, the feedback controller is unable to fully restore the cable angular dynamics to the nominal reference trajectories. However, the residual deviation in the cable angular dynamics is sufficiently greater in the absence of feedback.

![Figure 8-2: Effect of Gust Magnitude and Direction on Cable Radial Dynamics During Rendezvous- (a) Length (b) Length Rate](image)

The effect of wind gust magnitude and direction on the radial dynamics of the cable during rendezvous is given in Figure 8-2. Like the cable configuration angle, the cable length varies in a quasi-sinusoidal manner over the duration of the manoeuvre when the gusts act. Figure 8-2 clearly shows how the deviations in the cable radial dynamics caused by both downward head wind and upward tail wind gusts are symmetrical about the nominal reference trajectory. During downward head wind gust activity, the cable length initially increases for the first 1.25 seconds, then decreases initially over the next 1.25 seconds, before increasing momentarily and then returning to the nominal reference condition over the next 5 seconds. The feedback controller essentially causes the opposite to happen to the cable radial dynamics when upward tail wind gusts are encountered. The overall degree of deviation in the cable radial dynamics progressively increases over the period of time the gusts act and approximately 2.5 seconds thereafter, before quickly returning to zero within the following 2.5 seconds (approximate).
Comparing the nature of the results in Figure 8-2 and Figure 8-3, it can be concluded that the magnitude and direction of the wind gusts affect both the aircraft and cable radial dynamics during rendezvous in essentially the same manner. Like the cable radial dynamics, Figure 8-3 indicates that the deviations to the velocity components of the aircraft caused by both downward head wind and upward tail wind gusts are symmetrical about the nominal reference trajectories. The absolute magnitude of the aircraft velocity component deviations are slightly lower compared to those associated with the cable reel rate, yet the nature of how each of these fluctuate under the action of the wind gusts is equivalent.
The control actuation needed by the ATC system to perform single instantaneous rendezvous manoeuvres whilst encountering wind gusts of variable magnitude and direction is provided in Figure 8-4. Pronounced sinusoidal variations in aircraft and cable reel acceleration are required by the system to mitigate the adverse effects of both downward head wind and upward tail wind gusts. Like the aircraft and cable radial dynamics previously examined, Figure 8-4 indicates that the deviations to the ATC system control inputs are symmetrical about the nominal reference trajectories. For wind gusts as strong as 10% of the initial towing speed, lasting for 5 seconds and occurring mid-way through the rendezvous manoeuvre, it can be seen that the feedback controller produces slightly higher overall levels of cable reel acceleration compared to aircraft acceleration, yet the closed loop variation of each actuator is essentially the same. The “clipping-off” of the controls once they reach the absolute minimum or maximum permitted value is clearly outlined in Figure 8-4, resulting in the closed loop control paths exhibiting unfavourable “bang-bang” behaviour. This suggests that the ATC system requires relatively high levels of control during rendezvous to nullify the effects of relatively strong wind gusts. Similarly, as discovered previously, the saturation of the controls prevents the feedback controller from fully restoring certain ATC system dynamics to their nominal reference conditions. However, the residual deviations are more significant in the absence of feedback control. Such a claim is suitably supported by the results contained within Figure 8-5 through to Figure 8-8, concerning the final cable tip position at rendezvous.

![Figure 8-5: Effect of Gust Magnitude and Direction on Cable Tip x-coordinate at Rendezvous Without Feedback Control](image-url)
The results indicating how the coordinates of the cable tip at rendezvous are affected by the magnitude and direction of the wind gusts in the absence of feedback control is given in Figure 8-5 and Figure 8-6. The corresponding results for the case when corrective feedback measures are utilized during rendezvous are given in Figure 8-7 and Figure 8-8. Although not immediately apparent, it can be surmised from Figure 8-5 and Figure 8-7 that the x-coordinate of the cable tip at rendezvous is moderately affected by the magnitude and direction of the wind gusts, regardless of whether or not feedback is employed. Although in percentage terms the overall effect of the wind gusts appears to be marginal, Figure 8-5 and Figure 8-7 attest that the deviations are appreciable, yet significantly lower when feedback control is used. The maximum absolute deviation in the x-coordinate of the cable tip at rendezvous reduces from approximately 0.3 % (3 m) to 0.075 % (0.75 m) when feedback control is adopted.

In the absence of feedback, it can be seen from Figure 8-5 that horizontal wind gusts have the strongest effect on the x-coordinate of the cable tip at rendezvous, whilst concurrent horizontal and vertical wind gusts have almost no effect. When feedback is employed, it can be seen from Figure 8-7 that head wind gusts and upward vertical gusts have the strongest effect on the x-coordinate of the cable tip at rendezvous, the degree to which is appreciable only as the magnitude of these gusts increase in excess of 7.5 % of the initial towing speed (3.75 m/s). When feedback is used, irrespective of their magnitude, concurrent horizontal and vertical wind gusts have relatively little effect on the x-coordinate of the cable tip at rendezvous.

![Figure 8-6: Effect of Gust Magnitude and Direction on Cable Tip y-coordinate at Rendezvous Without Feedback Control](image-url)
Figure 8-6 and Figure 8-8 indicate that the $y$-coordinate of the cable tip at rendezvous is strongly affected by the magnitude and direction of the wind gusts, in particular when feedback control is not provided. In the absence of feedback, both horizontal and vertical wind gusts have a pronounced effect on the altitude of the cable tip at rendezvous, causing deviations to occur of up to 86% (3.72 m) and 48% (2.96 m) respectively. In almost all instances, the effect of the combined wind gusts is notably lower than the corresponding effect due to either horizontal or vertical wind gusts, even when corrective feedback measures are utilized, as Figure 8-8 indicates.

![Figure 8-7: Effect of Gust Magnitude and Direction on Cable Tip $x$-coordinate at Rendezvous With Feedback Control](image)

![Figure 8-8: Effect of Gust Magnitude and Direction on Cable Tip $y$-coordinate at Rendezvous With Feedback Control](image)
However, comparing the results in Figure 8-6 and Figure 8-8 demonstrates that the deviations to the cable tip altitude at rendezvous are considerably reduced when feedback control is used, since the maximum deviation to the $y$-coordinate of the cable tip at rendezvous is in the order of 20% (0.4 m). Hence, one can conclude that without the corrective measures provided by the feedback controller, rendezvous may not be possible if the ATC system encounters sufficiently strong wind gusts. Figure 8-8 clearly shows how head wind gusts and upward vertical gusts have the strongest effect on the altitude of the cable tip at rendezvous, the degree to which is only appreciably large when the magnitude of these gusts increase beyond 7.5% of the initial towing speed. Once again, irrespective of their magnitude, concurrent horizontal and vertical wind gusts have relatively little effect on the $y$-coordinate of the cable tip at rendezvous, provided feedback is used.

![Figure 8-9: Effect of Gust Magnitude and Direction on Cable Tension During Rendezvous](image)

The effect of wind gust magnitude and direction on the tension within the cable during rendezvous is given in Figure 8-9. Without feedback, the tension in the cable varies quasi-sinusoidally both during and after the gusts are encountered, both increasing and decreasing in magnitude from reference levels by up to 35% when wind gusts are encountered. When feedback control is utilized, the overall degree of variation in the tension within the cable is sufficiently higher, largely due to the pronounced “bang-bang” nature of the closed loop control inputs. The feedback control causes increases in cable tension of up to 56% when upward tail wind gusts occur, compared to decreases of up to 45% when downward head wind gusts are encountered. Only when feedback control is adopted, are the deviations to the cable tension caused by the combined wind gusts symmetrical about the nominal reference trajectory.
Effect of Gust Duration

The results outlining how the duration of the wind gusts affect the ATC system as it performs rendezvous operations with aircraft manoeuvring are given in Figure 8-10 through to Figure 8-16. The gust duration considered as part of this investigation lie within the interval $\lambda_g \in [0,10]$ [secs]. The results in Figure 8-10 through to Figure 8-12, quoted as percentages of the maximum allowable levels of control, concern the limiting case when a downward head wind of 5 % of the initial towing speed is encountered by the system. Figure 8-13 and Figure 8-16 briefly explore the individual impact each type of gust has on the ATC system during rendezvous. In Figure 8-13 and Figure 8-16, the results produced by both the horizontal and vertical gusts are when each of these gusts are 5 % of the initial towing speed, whilst the results concerning the combined winds are when the vector addition of each component sums to 5 %, acting in a downward sense (head wind gust).

![Figure 8-10: Effect of Gust Duration on Maximum Aircraft Horizontal Acceleration Required for Rendezvous](image)

Figure 8-10 through to Figure 8-12 outlines how the duration of the combined wind gusts affect the maximum level of aircraft and reel acceleration provided by the feedback controller for rendezvous to occur. Since the nature of the required closed loop control is essentially the same as that previously illustrated in Figure 8-4 regardless of the gust duration, the results contained within Figure 8-10 through to Figure 8-12 are utilized to provide an indication of the overall level of feedback needed as the duration of the wind gusts is varied. Following on, quasi-quadratic relationships exist between the duration of the gusts and the levels of feedback control required by the ATC system. As a result, it can be seen that progressively lower levels of closed loop aircraft vertical and cable reel acceleration are required as the
duration of the wind gusts increases to 9 seconds, with control savings in the order of 45 %
possible. Slightly increased amounts are required thereafter once the gust duration exceeds 9
seconds. Similarly, progressively lower levels of aircraft horizontal acceleration are required
as the duration of the wind gusts increases to 8 seconds (control savings in the order of 27.5
% possible), with slight increased amounts required when gusts act for longer periods of time.

Figure 8-11: Effect of Gust Duration on Maximum Aircraft Vertical Acceleration Required for
Rendezvous

Figure 8-12: Effect of Gust Duration on Maximum Cable Reel Acceleration Required for Rendezvous

The results indicating how the coordinates of the cable tip at rendezvous are affected by the
duration of wind gusts in the absence of feedback control is given in Figure 8-13 and Figure
8-14, whilst the results when feedback is utilized are given in Figure 8-15 and Figure 8-16.
Once again, it can be inferred from Figure 8-13 and Figure 8-15 that the \( x \)-coordinate of the cable tip at rendezvous is reasonably affected by the duration of the wind gusts, appreciably more so when feedback is not employed. Although the relevant percentages are small, Figure 8-13 and Figure 8-15 demonstrate that the deviations are palpable since the maximum absolute deviation in the \( x \)-coordinate of the cable tip at rendezvous is 0.175 % (1.75 m) without feedback, reducing to approximately 0.043 % (0.43 m) when feedback is employed. In the absence of feedback, it can be seen from Figure 8-13 that the various natured wind gusts generally have the most pronounced effect on the \( x \)-coordinate of the cable tip at rendezvous when their duration is relatively short, although there are exceptions to this generalization. Combined and individual vertical wind gusts affect the ATC system during rendezvous the most when they act for approximately 1 second, whilst horizontal wind gusts have the strongest effect when they act for approximately 8 seconds.

![Figure 8-13: Effect of Gust Duration on Cable Tip \( x \)-coordinate at Rendezvous Without Feedback Control](image)

When feedback is employed, it can be seen from Figure 8-15 that the various wind gusts have a progressively more pronounced effect on the \( x \)-coordinate of the cable tip at rendezvous as their duration increases. The duration of the horizontal wind gusts has the strongest effect on \( x \)-coordinate of the cable tip, followed by the combined wind gusts and then the vertical wind gusts, whose duration does not significantly affect the final \( x \)-coordinate of the cable tip. Figure 8-14 and Figure 8-16 demonstrate that the \( y \)-coordinate of the cable tip at rendezvous is strongly affected by the duration of the wind gusts, in particular when feedback control is not provided to the ATC system.
Without feedback, the various wind gusts generally have the most pronounced effect on the altitude of the cable tip at rendezvous when their duration is moderate. Horizontal and vertical wind gusts have the strongest effect when they act for 5 and 4 seconds respectively, causing deviations to occur of up to 29.95 % (2.6 m) and 25.91 % (2.52 m) respectively. Combined wind gusts affect the altitude of the cable tip at rendezvous the most when they act for only 1 second, causing a maximum absolute deviation of 19.28 % (2.39 m). As outlined in Figure 8-16 however, these deviations notably reduce in absolute magnitude to approximately 8.4-11.55 % (2.17-2.23 m) when corrective measures provided by the feedback controller are utilized.

![Figure 8-14: Effect of Gust Duration on Cable Tip y-coordinate at Rendezvous Without Feedback Control](image)

In essentially the same manner as the $x$-coordinate, the various wind gusts have a progressively more pronounced effect on the closed loop $y$-coordinate of the cable tip at rendezvous as their duration increases. The duration of the horizontal wind gusts has the strongest effect on the $y$-coordinate of the cable tip at rendezvous, followed by the combined wind gusts and then the vertical wind gusts, whose duration once again does not significantly affect the final $y$-coordinate of the cable tip.
Effect of Gust Timing

The results elucidating how the gust commencement time affects the ATC system during rendezvous operations are given in Figure 8-17 through to Figure 8-20. The gust commencement times used for this case study are governed by the domain $t_g \in [0, 90]$ [%]. The duration of each wind gust occurring as part of this investigation is limited to 2.5 seconds, with both the horizontal and vertical gusts having a magnitude of 5 % of the initial towing speed, whilst the combined wind gusts act in a downward sense (head wind gust) and have a magnitude of 5 %. The nature of the results in this sub-section is limited to examining how the final position of the cable tip is affected by the commencement time of the various gusts.

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**Figure 8-15:** Effect of Gust Duration on Cable Tip $x$-coordinate at Rendezvous With Feedback Control

**Figure 8-16:** Effect of Gust Duration on Cable Tip $y$-coordinate at Rendezvous With Feedback Control
The results indicating how the coordinates of the cable tip at rendezvous are impacted by the gust commencement time in the absence of feedback control are given in Figure 8-17 and Figure 8-18, whilst the appropriate results linked to the case when corrective feedback is used are given in Figure 8-19 and Figure 8-20. Likewise during previous investigations into the effects of wind gusts during rendezvous, it can be deduced from Figure 8-17 through to Figure 8-20 that the position of the cable tip at rendezvous is noticeably affected by the time at which the various gusts begin to act, increasingly so in the absence of corrections attributed to feedback control. In the absence of feedback, Figure 8-17 demonstrates that both the horizontal and combined wind gusts have the most prominent effect on the $x$-coordinate of the cable tip at rendezvous, when they begin acting mid-way through the rendezvous manoeuvre. Alternatively, vertical wind gusts have the strongest effect when they begin acting immediately after the rendezvous procedure begins.

Contrastingly, Figure 8-18 indicates that the horizontal and combined wind gusts have the most pronounced effect on the $y$-coordinate of the cable tip at rendezvous when they begin acting shortly after the rendezvous manoeuvre has been initiated, whilst vertical wind gusts have the strongest effect mid-way through the procedure. The maximum absolute deviation in the $x$-coordinate and $y$-coordinate of the cable tip at rendezvous is approximately 0.075 % (7.5 m) and 24.6 % (2.49 m) without feedback, reducing to approximately 0.042 % (0.42 m) and 11.15 % (2.22 m) with feedback, indicating that rendezvous may not be possible in these instances if feedback control is not incorporated.
Figure 8-18: Effect of Gust Timing on Cable Tip y-coordinate at Rendezvous Without Feedback Control

Figure 8-19 and Figure 8-20 indicate how the position of the cable tip at rendezvous is relatively unaffected by the various wind gusts when they commence acting in the initial two-thirds of the rendezvous manoeuvre. The effect of combined and vertical gusts becomes progressively more pronounced as they act further on in the rendezvous manoeuvre, whilst the impact of the horizontal gusts is noticeably lower when they begin to act increasingly later.

Figure 8-19: Effect of Gust Timing on Cable Tip x-coordinate at Rendezvous With Feedback Control
When closed loop control is provided to the ATC system during rendezvous, horizontal wind gusts have the strongest effect on the position of the cable at rendezvous when acting in the initial two-thirds of the manoeuvre. Contrastingly, both the vertical and combined winds dominate when they occur later on in the rendezvous attempt. The vertical wind gust produces the largest deviation in the cable tip position when it occurs with 20 % (5.65 secs) of the rendezvous manoeuvre still left to be completed.

### 8.4.2 Constant Tow Speed Dual Rendezvous

The effect of time-varying wind gusts on the performance of the ATC system whilst tracking the optimal trajectories representing constant tow speed dual rendezvous operations is the subject of examination in this sub-section. The particular dual rendezvous scenario that constitutes this investigation is the same as that previously introduced in Section 6.3.1. The same additional parameters given in Table 8-1 that governed the closed loop simulations previously investigated are again employed, unless otherwise specified. Similarly, both the state and control design weighting matrices are chosen as the identity matrix and no final state weighting matrices were again used in the LQR methodology.
The results outlining how the magnitude and direction of the wind gusts affects the ATC system whilst performing constant tow speed dual rendezvous manoeuvres are given in Figure 8-21 through to Figure 8-26. The effects of both horizontal and vertical wind gusts were again investigated separately, as well as the case when each of these gusts was concurrently encountered by the system. The particular gust magnitudes considered here again lie within the interval $G_x, G_y \in [-10, 10]$ [%]. The results given in Figure 8-21 and Figure 8-22 are for the limiting cases when both downward head wind and upward tail wind gusts of maximum strength are encountered by the ATC system during dual rendezvous. Figure 8-23 through to Figure 8-26 then briefly address the specific effect each gust type has on the ATC system during dual rendezvous.

The impact of the wind gust magnitude and direction on the angular dynamics of the cable during constant tow speed dual rendezvous is illustrated in Figure 8-21. Unlike the preceding single rendezvous problem, it can be seen from Figure 8-21 that the cable angular dynamics are strongly affected by the wind gusts during dual rendezvous, particularly so in the absence of feedback control. Once again, the cable in-plane configuration angle will either increase (for downward head wind gusts) or decrease (for upward tail wind gusts) marginally in a quasi-sinusoidal manner as the gusts are encountered. After the gusts have finished acting and immediately after the first rendezvous manoeuvre is attempted, the degree of deviation to the cable angular dynamics is significantly higher in the absence of corrective measures provided by the feedback controller.
The additional disturbing effect on the cable angular dynamics due to the zero reel rate condition at the initial rendezvous attempt causes the open loop cable angular dynamics to deviate significantly from the nominal reference trajectories. As will be shown later in this section, this has profound implications on the ability of the open loop ATC system to carry out the final rendezvous component of the dual rendezvous procedure when relatively strong wind gusts are encountered. Encouragingly, the feedback controller is able to damp out these disturbances, nullifying the changes to the cable angular dynamics and returning the ATC system towards the nominal path. Although the feedback controller is once again unable to fully restore the cable angular dynamics to the nominal reference trajectories, Figure 8-21 clearly shows the residual deviation is sufficiently greater in the absence of feedback control.

Figure 8-22: Effect of Gust Magnitude and Direction on Cable Tip Trajectory During Dual Rendezvous

The effect of wind gust magnitude and direction on the cable tip trajectory during the constant tow speed dual rendezvous manoeuvre is outlined in Figure 8-22. Although each gust occurs in the latter stages of the initial phase of dual rendezvous, Figure 8-22 clearly shows how the cable tip trajectory remains relatively unaffected by the action of the wind gusts during this phase. The adverse effects of the gusts only appreciably manifest during the final phase of the dual rendezvous procedure, since in the absence of feedback, Figure 8-22 clearly shows that the cable tip/payload will deviate considerably from the nominal optimal path. In fact, Figure 8-22 indicates that the cable tip/payload will collide with the ground during the final phase when significantly strong wind gusts are encountered by the system in the initial phase, which has profound safety implications. Thankfully, the feedback controller is successfully able to prevent these large-scale disturbances from occurring; by ensuring that cable tip always closely follows the nominal optimal path throughout the final phase of the dual
rendezvous procedure. Thus, one can confidently conclude that without corrective feedback control, successfully and safely performing the final rendezvous attempt would not be possible if the ATC system encounters sufficiently strong wind gusts, even if these gusts occur in the initial phase of the dual rendezvous manoeuvre.

The results detailing how the coordinates of the cable tip at each rendezvous instant are impacted by the wind gust magnitude and direction in the absence of feedback control are given in Figure 8-23 and Figure 8-24. The appropriate results linked to the case when corrective feedback measures are used throughout dual rendezvous are given in Figure 8-25 and Figure 8-26. Irrespective of whether or not feedback control is utilized, it is evident from Figure 8-23 through to Figure 8-26 that the cable tip position at the initial rendezvous instant is significantly more sensitive to the magnitude and direction of the wind gusts than the cable tip position corresponding to the final rendezvous attempt. In all cases, the deviations to the cable tip position at the final rendezvous instant are considerably (orders of magnitude) lower than the deviations occurring at the initial rendezvous moment. Similar to single rendezvous operations performed with variable aircraft manoeuvres, it can be seen from Figure 8-23 through to Figure 8-26 that the cable tip position at each rendezvous instant is noticeably affected by the magnitude and direction of the wind gusts, the degree of which is considerably higher when feedback control is not utilized.

Figure 8-23: Effect of Gust Magnitude and Direction on Cable Tip x-coordinate at Each Rendezvous Without Feedback Control
Without feedback, Figure 8-23 demonstrates that both horizontal and combined wind gusts have the most prominent effect on the $x$-coordinate of the cable tip at the initial rendezvous instant, increasingly so when they manifest as progressively stronger upward tail wind gusts. Vertical gusts have the strongest effect when they act downward and their strength increases accordingly. Figure 8-23 also demonstrates how the $x$-coordinate of the cable tip at the final rendezvous is relatively unaffected by the magnitude and direction of the wind gusts. However, low magnitude horizontal wind gusts have the strongest effect, whilst low magnitude combined winds have the weakest impact.

In contrast, Figure 8-24 indicates that vertical wind gusts have the most pronounced effect on the $y$-coordinate of the cable tip at the initial rendezvous attempt, in particular when these gusts are strong and upward acting. Low magnitude combined winds have the weakest impact on the $y$-coordinate of the cable tip at each rendezvous instance, whilst low magnitude horizontal wind gusts have the strongest effect on the $y$-coordinate of the cable tip at the final rendezvous moment. The maximum absolute deviation in the $x$-coordinate and $y$-coordinate of the cable tip at the initial rendezvous attempt is approximately 0.075 % (7.5 m) and 54.08 % (3.08 m) without feedback, reducing to approximately 0.037 % (0.37 m) and 19 % (2.38 m) with feedback. At the final rendezvous instant, the corresponding maximum absolute deviation in the $x$-coordinate and $y$-coordinate of the cable tip is 0.77 % (76.7 m) and 1798.6 % (35.97 m) without feedback and 0.0058 % (0.06 m) and 3.88 % (2.08 m) when feedback control is used.
Consideration of these results truly reflects the absolute necessity of employing a feedback controller, such as the LQR-based variants proposed in this chapter, if successful execution of dual rendezvous operations is to be achieved. In closed loop simulations, Figure 8-25 and Figure 8-26 indicate that the position of the cable tip at the initial rendezvous moment is most affected by combination wind gusts, the degree to which becomes increasingly significant when these gusts are strong and act as upward tail wind gusts. Strong upward vertical wind gusts and horizontal head wind gusts also produce relatively large deviations to the position of the cable tip at the initial rendezvous location.
Following on, it can be seen from Figure 8-25 and Figure 8-26 that the position of the cable tip at the final rendezvous instant is most affected by horizontal wind gusts, the degree to which is significant only as the magnitude of these gusts increase beyond 7.5 % of the initial towing speed (3.75 m/s). Strong upward vertical wind gusts and combined upward tail wind gusts also cause relatively appreciable deviations to the cable tip position at the final rendezvous instant.

**Effect of Gust Duration**

The results outlining how the duration of the wind gusts affect the ATC system as it performs dual rendezvous operations at constant towing speed are given in Figure 8-27 through to Figure 8-33. The gust duration considered as part of this investigation again lie within the interval $\lambda_g \in [0,10]$ [secs]. The results in Figure 8-27 through to Figure 8-29, quoted as percentages of the maximum allowable levels of control, concern the limiting case when a downward head wind of 5 % of the initial towing speed is encountered by the ATC system during dual rendezvous. Figure 8-30 and Figure 8-33 briefly explores the individual impact each type of gust has on the ATC system during dual rendezvous. In Figure 8-30 and Figure 8-33, the results produced by both the horizontal and vertical gusts are when each of these gusts are 5 % of the initial towing speed, whilst the results regarding the combined wind gusts are when the addition of each component sums to 5 %, acting in a downward sense.

![Figure 8-27: Effect of Gust Duration on Maximum Aircraft Horizontal Acceleration Required for Dual Rendezvous](image)

Figure 8-27 through to Figure 8-29 outline how the duration of combined wind gusts affect the maximum level of aircraft and cable reel acceleration provided by the feedback controller for dual rendezvous to occur.
Once again, the results contained within Figure 8-27 through to Figure 8-29 provide an overall indication of the levels of closed loop control needed to successfully complete dual rendezvous as the duration of the gusts is varied. Although quasi-quadratic relationships exist between the duration of the gusts and the levels of closed loop aircraft acceleration, it can be seen from Figure 8-29 that the corresponding level of closed loop cable reel acceleration is relatively unaffected by the duration of the gusts. Following on, progressively lower levels of closed loop aircraft horizontal and vertical acceleration are required as the duration of the wind gusts increases to 10 seconds, with control savings in the order of 40% possible. Unlike the single instantaneous rendezvous, increased amounts of feedback control are not required for sufficiently longer acting wind gusts. In general, it can be inferred from Figure 8-29 that slightly lower overall levels of closed loop reel acceleration control are required as the duration of the gusts increases, although the magnitudes involved here are marginal.

![Figure 8-28: Effect of Gust Duration on Maximum Aircraft Vertical Acceleration Required for Dual Rendezvous](image)

The results indicating how the position of the cable tip at each rendezvous instant are affected by the duration of the gusts are given in Figure 8-30 through to Figure 8-33. Figure 8-30 and Figure 8-31 refer to the respective scenario when feedback control is not adopted, whilst the appropriate results associated with corrective feedback measures are given in Figure 8-32 and Figure 8-33. Once again, regardless of whether feedback control is utilized or not, it is clear from Figure 8-30 through to Figure 8-33 that the duration of the wind gusts have a more pronounced effect on the cable tip position at the initial rendezvous instant than at the final rendezvous attempt.
In all cases, the deviations to the position of the cable tip at the final rendezvous instant are several orders of magnitude lower than the deviations occurring at the initial rendezvous moment. Similarly, it can be seen from Figure 8-30 through to Figure 8-33 that the position of the cable tip at each rendezvous instant is markedly affected by the duration of the wind gusts, the degree of which is patently stronger when corrective feedback control is not employed. Moreover, the degree of variation in the position of the cable tip at each rendezvous instant, whether it is an increase or decrease, is generally larger in absolute magnitude as the wind gusts act for progressively longer periods, irrespective of whether feedback is used or not.
Without feedback, Figure 8-30 and Figure 8-31 demonstrate that the position of the cable tip at each rendezvous moment is most prominently affected by wind gusts that act for approximately 8 seconds. The \( x \)-coordinate of the cable tip at the initial rendezvous attempt suffers the largest deviation when horizontal wind gusts are encountered, whilst at the final rendezvous instant, the largest deviation occurs when vertical wind gusts are prevalent. Alternatively, Figure 8-31 shows that the \( y \)-coordinate of the cable tip at initial and final rendezvous is most affected when combination and vertical wind gusts are encountered respectively. The maximum absolute deviation in the \( x \)-coordinate and \( y \)-coordinate of the cable tip at the initial rendezvous attempt is approximately 0.109 \% (10.9 m) and 45.3 \% (2.91 m) in the absence of feedback control, reducing to approximately 0.033 \% (0.33 m) and 17.4 \% (2.35 m) when feedback is utilized. At the final rendezvous, the maximum absolute deviation in the \( x \)-coordinate and \( y \)-coordinate of the cable tip is 0.76 \% (7.6 m) and 1786.5 \% (35.7 m) without feedback and 0.0019 \% (0.019 m) and 1.3 \% (2.03 m) when feedback control is used.

![Figure 8-31: Effect of Gust Duration on Cable Tip \( y \)-coordinate at Each Rendezvous Without Feedback Control](image)

When feedback control is deployed during dual rendezvous, it can be seen from Figure 8-32 and Figure 8-33 that horizontal wind gusts cause the most significant deviations to the position of the cable tip at each rendezvous instant, increasingly so as their duration progressively increases. The combined wind gusts cause the next most significant level of deviations, followed by the vertical wind gusts, whose effect is marginal and relatively constant as the wind gust duration is increased. Similar trends in the closed loop cable tip rendezvous coordinates were also observed to occur when the effect of gust duration on single instantaneous rendezvous problems was previously examined.
Effect of Gust Timing

The results detailing how the gust commencement time affects the ATC system during constant tow speed dual rendezvous are given in Figure 8-34 through to Figure 8-37. The gust commencement times used here lie in the interval $t_g \in [0, 90] \ [%]$. The duration of each wind gust occurring as part of this investigation is limited to 2.5 seconds, with both the horizontal and vertical gusts having a magnitude of 5 % of the initial towing speed, whilst the combined gusts act in a downward sense (head wind gust) and have a magnitude of 5 %. The nature of the results in this sub-section is limited to examining how the final position of the cable tip at each rendezvous attempt is affected by the commencement time of the various gusts.
The results illustrating how the coordinates of the cable tip at each rendezvous instant are impacted by the gust commencement time in the absence of feedback control are given in Figure 8-34 and Figure 8-35. The appropriate results for the case when corrective feedback control is used are given in Figure 8-36 and Figure 8-37. When feedback control is not utilized, the deviations to the cable tip position at the initial rendezvous moment are the largest when gusts begin acting in the final third of the dual rendezvous procedure. Figure 8-34 and Figure 8-36 indicate that vertical wind gusts that commence acting with 30 % (19.22 secs) left of the manoeuvre still to complete, cause the largest disturbance to the *x*-coordinate of the cable tip, whilst horizontal gusts that commence with 40 % (25.63 secs) left of the
procedure to complete cause the largest disturbance to the altitude of the cable tip. Contrastingly, the deviations to the cable tip position at the final rendezvous instant are most pronounced when gusts begin acting in the initial quarter of the dual rendezvous procedure. Figure 8-34 and Figure 8-36 suggest that combined upward head wind gusts that commence acting with 80% (51.26 secs) left of the dual rendezvous manoeuvre still to complete, cause the largest disturbance to the cable tip position at the final rendezvous location.

**Figure 8-36: Effect of Gust Timing on Cable Tip x-coordinate at Each Rendezvous With Feedback Control**

**Figure 8-37: Effect of Gust Timing on Cable Tip y-coordinate at Each Rendezvous With Feedback Control**
The maximum absolute deviation in the $x$-coordinate and $y$-coordinate of the cable tip at the initial rendezvous attempt is approximately 0.13 % (1.3 m) and 53.42 % (3.07 m) in the absence of feedback control, reducing to approximately 0.02 % (0.2 m) and 5.71 % (2.11 m) when feedback is utilized. At the final rendezvous, the maximum absolute deviation in the $x$-coordinate and $y$-coordinate of the cable tip is 0.77 % (7.7 m) and 1802.8 % (36.06 m) without feedback and 0.0071 % (0.071 m) and 4.8 % (2.096 m) with feedback control.

When feedback control is employed, the deviations to the cable tip position at the initial rendezvous instant only become relatively pronounced when gusts begin acting after the first third of the dual rendezvous manoeuvre has been completed, and have the strongest impact when they commence acting in the final third of the procedure. Conversely, the deviations to the position of the cable tip at the final rendezvous instant are relatively unaffected by the action of the gusts when they commence acting before the halfway point of the dual rendezvous procedure. In general, the wind gusts have more of a pronounced effect on the deviations to the cable tip final position, when the gusts commence acting progressively later on in the dual rendezvous manoeuvre.

### 8.4.3 Payload Transportation With In-plane Aircraft Manoeuvring

The impact of time-varying wind gusts on the performance of the ATC system whilst tracking the optimal trajectories that represent payload transportation operations for the ATC is the subject of enquiry in this sub-section. The particular payload transportation scenario that constitutes this investigation is the same as that previously introduced in Section 7.2.2, thus payload delivery operations performed with in-plane aircraft manoeuvring are considered. This does not significantly detract from the scope of the work proposed in this sub-section, since it was routinely observed throughout Section 7.2.3 and 7.2.4 that during delivery operations, the ATC system exhibits appreciably higher levels of sensitivity to system and environmental parameters, compared to comparable capture procedures. The same parameters given in Table 8-1 that governed previous closed loop simulations are again employed, unless otherwise specified. Similarly, both the state and control design weighting matrices were again chosen as the identity matrix, whilst no final state weighting matrices were used.
Effect of Gust Magnitude and Direction

The results outlining how the wind gust magnitude and direction affects the ATC system whilst performing payload delivery with aircraft manoeuvring are given in Figure 8-38 through to Figure 8-42. The effects of both horizontal and vertical wind gusts were again investigated separately, as well as the case when each of these gusts was concurrently encountered by the ATC system during delivery. The particular gust magnitudes considered here again lie within the interval \( G_x, G_y \in [-10, 10] \) [%]. The results given in Figure 8-38 through to Figure 8-42 address the specific effects each gust type has on the level of closed loop control required for payload delivery, along with the impact of the gusts on the final aircraft and cable tip position.

![Figure 8-38: Effect of Gust Magnitude and Direction on Maximum Aircraft Horizontal Acceleration Required for Payload Delivery](image)

Figure 8-38 through to Figure 8-40 illustrate how the magnitude and direction of combined wind gusts affect the maximum level of aircraft and cable reel acceleration required for payload deliver to occur. In all cases, it can be seen that the maximum level of feedback actuation increases quasi-linearly as the magnitude of the gusts increase up to approximately 5-7 % of the initial towing speed, regardless of the direction the gusts act in. The maximum level of closed loop control is impacted in approximately the same manner when either upward tail wind gusts or downward head wind gusts occur. Figure 8-38 through to Figure 8-40 indicate that the controls become saturated when the ATC system encounters sufficiently strong gusts during payload delivery (in excess of 5-7 %), particularly for the cable reel mechanism. Since the results contained within Figure 8-38 through to Figure 8-40 are an indirect measure of the overall level of closed loop control needed to successfully complete
payload delivery, it can be generally surmised that progressively higher levels of feedback control are required as the magnitude of each variety of wind gust increases. The overall level of closed loop cable reel acceleration is most affected by the presence of the wind gusts, followed by the aircraft horizontal acceleration and then the aircraft vertical acceleration, which is least of all affected by the action of the gusts, albeit marginally so. For the record, the effect of gust magnitude and direction on the maximum level of feedback control required for both single and dual rendezvous operations, is essentially the same as the results contained within Figure 8-38 through to Figure 8-40 that concern payload delivery.

![Figure 8-39: Effect of Gust Magnitude and Direction on Maximum Aircraft Vertical Acceleration Required for Payload Delivery](image1)

![Figure 8-40: Effect of Gust Magnitude and Direction on Maximum Cable Reel Acceleration Required for Payload Delivery](image2)
The results revealing how the final position of the aircraft is affected by the magnitude and direction of the wind gusts with corrective feedback control applied are given in Figure 8-41. Since the commencement time of the wind gusts is such that they occur shortly after delivery has taken place, the dynamics of the aircraft during the initial guidance phase remain unaffected by the action of the wind gusts and are not shown here. Similarly, although Figure 8-41 indicates that the final position of the aircraft is virtually unaffected by the magnitude and direction of the wind gusts when feedback is applied, the same observation was made in the instance when feedback was not applied. Since the impact of the gusts is markedly (4 orders of magnitude) lower when feedback was used, the impact of variable wind gust magnitude and direction on the final aircraft position in the absence of feedback control is not formally explored. It is important to note that the actual values of the results given in Figure 8-41 may not be entirely accurate since their order of magnitude is comparable to the integration error associated with the integration routine in MATLAB® that was used to implement the LQR control (ODE45). Nevertheless, it can be seen from Figure 8-41 that horizontal tail winds having a magnitude of 10% of the initial towing speed cause the largest deviation to the final position of the aircraft, this remaining the case when feedback control is not adopted. When feedback is not utilized, the maximum absolute deviation in the final x-coordinate and y-coordinate of the aircraft is 0.0081% (0.32 m) and 0.0816% (0.49 m) respectively, yet Figure 8-41 demonstrates that these deviations are practically zero when feedback is employed. Thus, deviations to final aircraft position are not prominent, despite the use of feedback control and irrespective of the gust magnitude and direction.

Figure 8-41: Effect of Gust Magnitude and Direction on Final Aircraft Position With Feedback Control - (a) x-coordinate (b) y-coordinate
The results indicating how the final cable tip position is affected by the magnitude and direction of the wind gusts when corrective feedback measures are provided during payload delivery are given in Figure 8-42. Like the aircraft dynamics, the cable tip dynamics during the initial guidance phase are unaffected by the action of the wind gusts and are not provided in this sub-section. When feedback control is not employed, the impact of the gusts is approximately twice as strong as their corresponding impact when feedback measures are adopted. Although Figure 8-42 highlights that the final position of the cable tip is relatively unaffected by the magnitude and direction of the wind gusts with feedback applied, it is evident that strong horizontal head winds cause the largest deviations to the final cable tip position. Moreover, it can be deduced from Figure 8-42 that the effect of each type of wind gust on the final cable tip position is essentially the same, if the absolute magnitude of the gusts do not exceed approximately 7% of the initial towing speed. When feedback is not utilized, the maximum absolute deviation in the final $x$-coordinate and $y$-coordinate of the cable tip is 0.063% (2.46 m) and 0.32% (1.04 m) respectively, reducing to 0.037% (1.44 m) and 0.176% (0.57 m) respectively when corrective feedback measures are employed.

![Figure 8-42: Effect of Gust Magnitude and Direction on Final Cable Tip Position With Feedback Control-](image)

(a) $x$-coordinate  (b) $y$-coordinate

However, it is important to bear in mind that the objective of the final phase of the transportation operation is for the aircraft to reach the desired location in the sky and the cable to be in equilibrium with a length equal to its initial length. Hence, deviations to the final cable tip position are not critical to the success of the transportation operation during that phase. What is important is that deviations to the final aircraft position, cable length and length rate, along with the cable angular rate, remain sufficiently small.

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Although not entirely included here, when feedback control is utilized, it can be shown that these deviations are, for all practical purposes, zero at the conclusion of the delivery procedure, regardless of the magnitude and direction in which each of the wind gusts act.

Effect of Gust Duration

The results outlining how the duration of the wind gusts affect the ATC system as it performs payload delivery operations with in-plane aircraft manoeuvring are given in Figure 8-43 through to Figure 8-47. The gust duration considered as part of this investigation again lie within the interval $\lambda_g \in [0, 10]$ [secs]. The results in Figure 8-43 through to Figure 8-45, quoted as percentages of the maximum allowable levels of control, concern the limiting case for downward head wind gusts of 5 % of the initial towing speed. Figure 8-46 and Figure 8-47 briefly explore the individual impact each type of gust has on the ATC system during delivery, for the scenario when feedback control is employed. The results in Figure 8-46 and Figure 8-47 concerning the horizontal and vertical gusts are when each of these gusts are 5 % of the initial towing speed, whilst the results pertaining to the combined gusts are when the vector addition of each component sums to 5 %, acting in a downward sense.

Figure 8-43: Effect of Gust Duration on Maximum Aircraft Horizontal Acceleration Required for Payload Delivery

Figure 8-43 through to Figure 8-45 detail how the duration of combined wind gusts affect the maximum level of closed loop aircraft and reel acceleration required for payload delivery. Once again, the maximum levels of actuation provided the ATC system vary with respect to the gust duration in essentially the same manner. Irrespective of which actuator is considered, it can be seen from Figure 8-43 through to Figure 8-45 that the absolute maximum level of closed loop control occurs when the duration of combined wind gust is 2 seconds. The overall
level of feedback control then progressively decreases non-linearly in a quasi-quadratic manner as the duration of the gusts increases. Hence, for a 400 % increase in the duration of the gust, overall closed loop control savings in the order of 60-65 % are possible.

![Graph](image)

**Figure 8-44: Effect of Gust Duration on Maximum Aircraft Vertical Acceleration Required for Payload Delivery**

The results illustrated in Figure 8-43 through to Figure 8-45 are in general agreement with the relevant results concerning how the duration of the wind gusts affect the levels of closed loop control required for single and dual rendezvous operations. Finally, the overall level of closed loop cable reel acceleration for payload delivery is most affected by the duration of the wind gusts, followed by the aircraft vertical acceleration and then the aircraft horizontal acceleration, although the relative degree of difference between each actuator is low.

The results outlining the weak effect of gust duration on the final aircraft position when corrective feedback control measures are utilized is given in Figure 8-46. With respect to the impact that the gust duration has when feedback is not adopted, it can be shown that it is again approximately 4 orders of magnitude higher than otherwise the case when feedback is used. When corrective feedback is adopted, Figure 8-46 shows that deviations to the final aircraft position remain unaffected by the duration of the wind gusts. Horizontal tail wind gusts acting for 8 seconds cause the largest deviation to the final position of the aircraft. These two observations also remain true in the instance when feedback control is not utilized. When feedback is not used, the maximum absolute deviation in the final x-coordinate and y-coordinate of the aircraft is again 0.0081 % (0.32 m) and 0.082 % (0.49 m) respectively, reducing to zero when feedback is employed, as shown in Figure 8-46.
The results elucidating the impact that the gust duration has on the final position of the cable tip when corrective feedback control is applied during payload delivery are given in Figure 8-47. The relevant results concerning the cable tip position at the instant of delivery, as well as the corresponding cases when feedback control is not employed are again not included. Once again, in the absence of closed loop control, the duration of the gusts have an approximate impact on the final cable tip position that is twice as strong as the impact corresponding to the relevant scenario when feedback measures are adopted. Whilst Figure 8-47 demonstrates that the final cable tip position is relatively unaffected by the duration of
the wind gusts when feedback control is applied, unlike the previous sub-section, it is apparent that upward vertical head wind gusts cause the largest deviations to the final cable tip position, increasingly so as their duration increases accordingly. Alternatively, the impact that combined wind gusts have on the final cable tip position is progressively reduced as their duration increases, whilst the duration of horizontal wind gusts has almost no effect on the final cable tip position. When feedback is not employed, the maximum absolute deviation in the final \( x \)-coordinate and \( y \)-coordinate of the cable tip is 0.061 % (2.38 m) and 0.32 % (1.04 m) respectively, reducing to 0.0366 % (1.42 m) and 0.17 % (0.55 m) respectively when corrective feedback control is adopted.

Figure 8-47: Effect of Gust Duration on Final Cable Tip Position With Feedback Control- (a) \( x \)-coordinate (b) \( y \)-coordinate

Once again, it must be stressed that the deviations to the final cable tip position are not critical to the success of the transportation operation in the final phase. The deviations to the final aircraft position, cable length and length rate, and cable angular rate are the most important concern. When feedback control is used, it can again be shown that these deviations are sufficiently close to zero at the end of the procedure as required, regardless of the duration of the wind gusts. Since the next sub-section deals with the effect of the gust commencement time on the performance of ATC system during payload delivery, an appreciation of the important role that the wind gusts play during the initial guidance phase will be available for the first time. This will compliment the significant insights previously uncovered into the various effects that the gusts have during the final phase of the delivery operation.
The results concerning how the gust commencement time affects the ATC system during payload delivery operations are given in Figure 8-48 through to Figure 8-51. The gust commencement times used for this case study are governed by the domain $t_g \in [0, 90]$ [%]. The duration of each wind gust occurring as part of this investigation is limited to 2.5 seconds, with both horizontal and vertical gusts having a magnitude of 5 % of the initial towing speed, whilst the combined winds act in a downward (head wind) sense and have a magnitude of 5 %. The results in this sub-section are limited to analysing how the position of the aircraft and cable tip at delivery and at the finale of the delivery operation are affected by the commencement time of the various gusts, in the case when feedback control is used.

![Figure 8-48: Effect of Gust Timing on Aircraft Position at Delivery With Feedback Control- (a) x-coordinate (b) y-coordinate](image)

The results outlining how the aircraft position at the instant of delivery is affected by the gust commencement time when feedback control is employed during the delivery operation is given in Figure 8-48. It can be seen from Figure 8-48 that the aircraft position at the instant of delivery is relatively unaffected by the gust commencement time. Overall, deviations to the aircraft position at delivery are diminutive regardless of when the wind gusts commence acting. Relatively speaking, deviations to the aircraft position at the point it delivers the payload are only apparent when the gusts commence acting 30 % of the way into the payload transportation operation, or approximately 6.7 secs before delivery occurs. When the wind gusts commence acting at this instant, the combined wind gusts have the lowest affect on the aircraft position, whilst the horizontal tail wind and upward vertical wind gusts cause the largest deviations to occur in the aircraft x-coordinate and y-coordinate respectively.
Once again, it should not be forgotten that the actual values of the results given in Figure 8-48 may not be entirely accurate as their order of magnitude is comparable to the associated integration errors, yet this is not particularly problematic since an extremely small deviation magnitude is desirable. Moreover, when feedback is not employed, the maximum absolute deviation in the $x$-coordinate and $y$-coordinate of the aircraft at delivery is 0.0079 % (0.16 m) and 0.003 % (0.013 m) respectively, reducing to 0.003 % (0.06 m) and 0.0012 % (0.005 m) respectively when feedback control is adopted.

Figure 8-49: Effect of Gust Timing on Cable Tip Position at Delivery With Feedback Control- (a) $x$-coordinate (b) $y$-coordinate

The results demonstrating how the cable tip position at the instant of delivery is affected by the gust commencement time with feedback control is given in Figure 8-49. Although Figure 8-49 indicates that the $x$-coordinate of the cable tip at delivery is relatively unaffected by the gust commencement time and the deviations are comparable to the integration errors, the time at which the gusts begin acting has a relatively strong impact on the cable tip altitude at delivery. Deviations to the cable tip altitude at delivery are most pronounced when gusts commence acting in the first third of the payload transportation operation, with the maximum deviation arising when horizontal tail wind gusts commence acting 6.7 secs before delivery occurs. Comparing the results contained within Figure 8-48 and Figure 8-49, it can be concluded that the cable tip position at delivery is most affected by the action of the wind gusts compared to the aircraft position. Ignoring the $x$-coordinate momentarily, the maximum absolute deviation in the cable tip altitude at delivery is 5.29 % (0.11 m) when feedback is not adopted, lowering to 1.97 % (0.039 m) when feedback control is provided. Thus, it is likely that successful delivery would occur even in the absence of feedback control.
The results outlining how the final aircraft position is affected by the gust commencement time when feedback control is utilized are provided in Figure 8-50. As observed during previous investigations into the effects that various wind gusts have during payload delivery, it can be concluded from Figure 8-50 that the final position of the aircraft is unaffected by the gust commencement time. The variation in the deviation of the aircraft final position is negligible as the gust commencement time is altered, with the absolute magnitude of the deviations themselves trivial and sufficiently close to zero for all practical purposes.

Figure 8-51: Effect of Gust Timing on Final Cable Tip Position With Feedback Control- (a) $x$-coordinate (b) $y$-coordinate
The results depicting how the final cable tip position is impacted by the gust commencement time when feedback control is used are outlined in Figure 8-51. On the whole, deviations to final cable tip position are small and not particularly pronounced when gusts commence acting anytime throughout the payload transportation operation. The variation in the deviation of the cable tip final position is noticeably larger as the gusts commence acting progressively later on in the delivery operation, especially so in the final third of the procedure. The maximum deviation to the final tip position arises when horizontal tail wind gusts commence acting 13.65 secs before the payload delivery operation terminates. In the absence of closed loop control, the maximum absolute deviation in the final $x$-coordinate and $y$-coordinate of the cable tip is 0.045 % (1.75 m) and 0.76 % (2.46 m) respectively. These reduce to 0.037 % (1.44 m) and 0.18 % (0.57 m) respectively when feedback control measures are provided to the ATC system. Although not formally included, when feedback control is used, it can again be shown that the deviations to the final cable length, length rate and cable angular rate are suitably close to zero at the end of the payload delivery procedure for it to be successful, irrespective of the gust commencement times considered in this enquiry.

8.5 Concluding Remarks

In this chapter, a series of simple closed loop optimal feedback controllers, designed using the LQR control methodology, were developed for the ATC system to ensure that various rendezvous and payload transportation operations could be successfully performed in the presence of external disturbances. The particular external disturbances considered in the resulting investigations were time-dependent wind gusts. The influence of various wind gust properties on the ability of the ATC system to closely track the optimal open loop trajectories previously developed in this thesis was examined through a series of parametric studies. These inquiries provided an initial indication of the robustness of the system to wind gusts it could potentially encounter en route during the undertaking of numerous rendezvous and payload transportation operations.
9 CONCLUSIONS AND RECOMMENDATIONS

9.1 Key Findings

The primary purpose of this thesis was to develop innovative, non-linear optimal control strategies for manoeuvring a cable towed beneath an aircraft, in order to transport payloads both to and from surface locations. To appropriately meet this objective, it was necessary to develop accurate and efficient modelling capabilities to implement and evaluate these control systems. Thus, the equations of motion for numerous models of the ATC system, each possessing varying degrees of sophistication, were methodically developed and widely used throughout this thesis.

Furthermore, a systematic framework for matching the dynamical motion of simple ATC system models to those of more sophisticated models was also proposed within this thesis. Two approaches to the model “matching” problem were developed, each based on the proposition that if the equilibrium configurations of the simple and complex ATC system models were made to be equal, then the representativeness of the simple model in a dynamic sense would approach that of the complex model. The essential difference between each approach was that one was based solely on static equilibrium considerations, whilst the other was able to integrate dynamic inertial effects into the model “matching” methodology. By prescribing changes to the cable length and payload drag coefficient of the simple model, it was possible to ensure that the dynamic motion of the cable tip given by this model closely tracked the tip motion specified by the complex models.

The advantages associated with using either of the proposed model “matching” procedures were shown to be valuable and significant. Most encouragingly however, these rewards were possible without undue complexity and did not require a significant outlay of computational resources. Model “matching” was shown to be unquestionably necessary for accurate simulations of the ATC system when using simple single-link models. Similarly, it was shown that the development of representative controllers for the ATC system would not be possible if simple models were utilized and a model “matching” procedure was not deployed. As a result, either model “matching” technique was utilized during the synthesis and demonstration of non-linear optimal controllers for the ATC system, exactly which was found to be dependent on the system parameters and the physical scenario in question.
A series of single-phase, non-linear optimal control problems for the ATC system were formally proposed in this thesis. These were then converted into non-linear programming problems using direct transcription and solved using a sparse, non-linear quasi-Newton algorithm. The possibility of achieving accurate, instantaneous rendezvous of the cable tip with desired surface locations on the ground, both in two and three-dimensions, was successfully demonstrated in this work through the use of deployment and retrieval control of the cable and/or aircraft manoeuvring. Overall, it was found that non-linear optimal control problems for the ATC system should be discretized using the Hermite Simpson transcription method. Similarly, it was found that these optimal control problems should be formulated using minimum control performance indices. This ensured smooth radial dynamics for the system, leading to modest control inputs and potentially low power requirements needed by the reel mechanism. Solutions to such problems were rapidly determined with low associated computational overheads.

To successfully perform single instantaneous rendezvous, it was found that initially the cable should be deployed quickly, upon which the deployment rate should then be slowed just as quick as the cable tip approached the target location. If possible, the aircraft would tend to dive with an increasing rate of descent and reduced horizontal velocity in order to successfully carry out rendezvous. Generally, it was easier to achieve rendezvous in instances where the aircraft was permitted to perform manoeuvres, as this caused the total control effort to be dispersed uniformly across all actuators, resulting in a lower overall control workload. As the aircraft degree of freedom was reduced, progressively higher levels of control actuation were required by the cable reel mechanism. As a result, for successful rendezvous, the length of the cable and the absolute speed to which it should be deployed or retrieved was reduced when the level of sophistication in the aircraft manoeuvres was increased.

By undertaking a series of parametric studies addressing how various system and environmental parameters affect the ATC system during rendezvous operations, a better appreciation of the dynamics and control of the system was obtained. Regardless of whether the aircraft was permitted to manoeuvre, it took progressively longer and significantly higher levels of control to rendezvous the cable tip with a desired location on the ground, as the mass of the payload was reduced. Faster towing speeds permitted rendezvous to occur quicker, albeit at the cost of significantly increased levels of overall control. Similarly, when the tow speed was fixed, longer cables reduced the time, but increased the level of control required for rendezvous.
However, when the aircraft was permitted to manoeuvre, the time at which rendezvous occurred was largely unaffected by the initial cable length, but longer cables required significantly lower levels of control for rendezvous. The initial length of the cable was found to have the most pronounced effect on both the aircraft and cable tip trajectories during rendezvous. If allowed to do so, the aircraft would tend to fly further and dive steeper with a slower horizontal velocity when the initial cable length, towing speed and payload mass were all reduced. Exceptionally lower overall control was required to perform a given rendezvous operation when the aircraft was permitted to manoeuvre, compared to when actuation was limited to cable reel acceleration.

The effect of atmospheric winds on the ATC system whilst performing instantaneous rendezvous procedures was found to be more pronounced when the aircraft was restricted to flying steady and level at constant altitude. For these constant tow speed rendezvous operations, the stronger the horizontal tail wind or downward acting vertical wind, the lower the time and overall level of control required for rendezvous. These overall trends remained the same when the aircraft was permitted to manoeuvre, although sufficiently strong tail winds incurred additional increased control liabilities. Head winds were found to adversely affect all aspects of the dynamics and control of the ATC system during rendezvous, increasingly so as their strength increased accordingly. When simultaneously encountered by the ATC system during rendezvous, it was found that the horizontal component of equal strength horizontal-vertical prevailing winds was the most dominant, having a similar, yet less pronounced effect on the ATC system than when it occurred individually in isolation.

The development of various multi-phase, non-linear optimal control problems for the ATC system were proposed and successfully demonstrated in this dissertation. Multiple rendezvous operations for the ATC system were the first class of problems that were explored. Once again, using just deployment and retrieval control of the cable and/or aircraft manoeuvring, it was successfully demonstrated that the ATC system is capable of sequentially rendezvousing the cable tip with multiple surface locations, whilst avoiding collisions between the cable tip and regions of elevated terrain separating each rendezvous target. A series of parametric studies were again undertaken to establish the impact that various parameters had on the ability of the ATC system to perform various dual rendezvous operations. These parametric studies allowed important insights to be available, which assisted the planning and development of more practical-orientated tasks such as payload capture and delivery.
To successfully perform dual instantaneous rendezvous, it was found that initially the cable should be deployed quickly, then deployment should be slowed just as quick when the cable tip is in the vicinity of the first target location. To then rendezvous with subsequent targets that weren’t separated by elevated terrain, marginal amounts of cable reel acceleration were required. If possible, the aircraft would initially dive towards the first target with an increasing descent rate and reduced horizontal velocity, before pulling up and increasing its horizontal speed when the cable tip approached the first target. The aircraft then performed a shallow dive until the subsequent rendezvous attempt had been completed. Like single rendezvous, it was found to be easier to perform dual rendezvous with aircraft manoeuvring, although the time needed to perform each rendezvous slightly increased. Aircraft manoeuvres were also found to better equip the ATC system in nullifying the adverse dynamic excitation associated with each rendezvous instance.

The mass of the payload, towing speed and initial cable length were all found to have an appreciable impact on the ATC system whilst undertaking dual rendezvous operations at constant towing speeds. It took progressively longer and significantly more control to successfully perform dual rendezvous with increasingly lighter payloads. Faster towing speeds also permitted dual rendezvous to occur quicker, but required significantly higher levels of overall control. The time required to perform dual rendezvous manoeuvres was found to increase as the initial cable length increased, yet considerably lower overall levels of control actuation were required. The mass of the payload, tow speed and initial cable length were all found to significantly affect the drop in cable tension at the instant of the initial rendezvous attempt, all increasing the size of the tension change, for increasing values of each parameter. Similarly, each parameter was found to have a marked effect on the size and nature of the change in the cable reel acceleration immediately after the initial rendezvous attempt.

When each rendezvous target was separated by a region of elevated terrain, the maximum height and target separation distance were both found to considerably affect the ability of the ATC system to perform dual rendezvous at constant tow speeds. The time required to perform each rendezvous attempt was found to be relatively unaffected by the maximum terrain height, although the overall required level of control was found to significantly increase as the terrain height increased accordingly. As the target separation distance was increased, so too did the final rendezvous time. Progressively lower overall levels of cable reel acceleration were necessary to perform multiple, sequential rendezvous manoeuvres that were increasingly further apart, although the time required to perform each rendezvous attempt increased steadily.
The nature of the terrain separating each rendezvous target was found to mostly affect the ATC system only during the final phase of the dual rendezvous manoeuvre. However, there were exceptions to this generalization, as certain terrain heights and separation distances were found to notably affect the reel acceleration and resulting dynamics of the cable during the initial rendezvous phase. When the maximum terrain height was sufficiently high and the target separation sufficiently close, the ATC system was observed to first induce, then utilize cable dynamical “swing” motion in order to prevent the cable tip from colliding with regions of elevated terrain. The impact of atmospheric winds on the ATC system whilst performing dual rendezvous operations at constant towing speeds was also found to be important. Constant strength horizontal head winds and downward vertical winds were found to adversely affect all aspects of the dynamics, control and performance of the ATC system during dual rendezvous operations, increasingly so as the strength of these winds increased. Alternatively, horizontal tail and upward vertical winds were found to favorably affect the ATC system during the undertaking of dual rendezvous operations, increasingly so as these winds became progressively stronger.

Robustly transporting payloads both to and from surface locations in a safe and accurate manner, using deployment and retrieval control of the cable and/or aircraft manoeuvring was also categorically established in this thesis. By performing a series of parametric studies, the effects of a variety of parameters on the ability of the ATC system to execute various payload transportation operations was explored. Examining how these factors affected the performance of the ATC system during payload transportation greatly expanded knowledge of critical concepts related to the dynamics and control ATC systems hitherto undiscovered.

To successfully capture or deliver payloads, the initial guidance phase of the payload transportation procedure was found to be the same as that used by the ATC system for single and multiple rendezvous. After the initial guidance phase was completed, it was found that uniform retrieval of the cable was then required, the rate of which increased as the level of aircraft manoeuvring decreased respectively. If possible, the aircraft would tend to dive towards the desired surface location with an increased rate of descent and reduced horizontal velocity, then climb progressively steeper with a further reduced horizontal speed to successfully carry out payload transportation. Generally, it was found to be easier to transport payloads when the aircraft was permitted to manoeuvre, although the time required to perform the manoeuvre was noticeably longer. Overall, the level of control actuation was highest during the initial guidance phase, yet significant cable reel acceleration was required in the final phase when payload transportation operations were performed at constant tow speeds.
Regardless of whether payload delivery or capture was performed, for all practical purposes, the configuration of the cable, aircraft and cable radial dynamics, along with the ATC system control actuation were identical throughout each operation. Only the variation of the tension within the cable was found to be unique. Generally, the tension in the cable was higher when payload transportation was performed at constant towing speeds, especially once a payload was captured or delivered. During payload delivery operations, the cable suffered a discontinuous drop in tension immediately after rendezvous, whereas the cable experienced a discontinuous rise in tension when a payload was captured immediately after rendezvous. When undergoing payload delivery procedures, the ATC system was routinely found to be more sensitive to changes in the physical and environmental parameters, compared to when comparable capture operations were performed.

For a given percentage change in payload mass immediately after rendezvous, it was found that it took progressively longer, with significantly higher levels of control, to successfully transport payloads that were initially increasingly lighter. Faster initial towing speeds also permitted transportation to occur quicker, although this was again only possible through significantly increased levels of overall control. The time required to perform the initial guidance phase was found to increase slightly as the initial cable length was increased, while the overall manoeuvre execution time was notably reduced when initially longer cables were utilized. Considerably lower overall levels of control actuation were required when payload transportation commenced with increasingly longer cables. The time required to perform each phase, along with the overall level of control, were found to be relatively unaffected by the magnitude of the mass captured or delivered immediately after rendezvous.

The change in control actuation and cable tension once a payload was captured or delivered was found to be significantly affected by changes to the initial payload mass, cable length and towing speed. Surprisingly, the change in control actuation immediately after rendezvous was not sensitive to changes in the magnitude of the mass captured or delivered. To transport payloads, changes to the controls in the order of ±20-50 % were required immediately after rendezvous. The change in tension within the cable immediately after rendezvous was found to increase as the initial payload mass and delivered/captured mass increased accordingly. Increases to the initial towing speed were found to increase and decrease the change in tension after payload delivery and capture respectively. Alternatively, increases to the initial cable length were found to reduce the tension change after delivery, yet had a marginal effect after capture. Immediately after payload capture or delivery, the cable experienced absolute tension changes in the order of 17-118 %, for the range of parameter variations considered.
Both the initial payload mass and cable length were found to have the largest effect on the aircraft trajectory during payload transportation. The cable tip during the initial guidance phase was strongly affected by changes to the initial payload mass, cable length and towing speed, whilst the cable tip trajectory during the final phase was most affected by the initial payload mass and cable length only.

The impact of atmospheric winds on the ATC system whilst performing payload transportation procedures was found to be appreciable. The overall time required to complete payload delivery and capture operations was relatively unaffected by the strength and direction of prevailing winds, as opposed to the time required for rendezvous, which was notably longer as the strength of horizontal head winds and downward vertical winds was increased. Alternatively, the time required for rendezvous was increasing shorter when progressively stronger horizontal tail winds and upward vertical winds were encountered. Horizontal winds were found to have the strongest overall effect on the time required to perform the initial guidance phase. Progressively higher levels of overall control were required to execute payload transportation in the presence of increasingly strong head winds and downward acting winds. Both the aircraft and cable tip trajectories were relatively unaffected by the action of winds during payload transportation. Tail winds and downward acting vertical winds were found to have the strongest effect on the change in the cable tension after delivery, whilst head winds and upward acting vertical winds were found to have the strongest effect on the change in the cable tension after capture.

To ensure that the various aforementioned rendezvous and payload transportation operations could be successfully performed in the presence of external disturbances, a series of simple closed loop optimal feedback controllers were successfully developed for the ATC system. These neighbouring optimal feedback controllers were designed using the LQR control methodology, assuming that potential deviations from the nominal optimal trajectories were not excessive and within levels designated by linear perturbations. The particular external disturbances considered within the closed loop simulations were time-dependent wind gusts. The influence that various properties of these wind gusts had on the ability of the ATC system to track the nominal open loop trajectories constituting rendezvous and payload transportation operations for the system were closely examined through a succession of parametric studies. The results obtained and findings deduced from these studies provided an initial indication of the robustness of the ATC system to wind gusts it could encounter whilst undertaking various rendezvous and payload transportation operations.
For single instantaneous rendezvous problems performed with planar aircraft manoeuvres, it was found that the cable angular dynamics were relatively unaffected by magnitude of downward head wind and upward tail wind gusts. Quasi-sinusoidal variations in the cable in-plane angle and angular rate were found to occur both whilst the gusts were acting and shortly thereafter. The degree of deviation in the cable angular dynamics was slightly higher in the absence of corrective feedback measures, although marginally so. Large magnitude wind gusts caused the controls to become saturated, which together with the absence of a final state-weighting matrix, prevented the feedback controller from fully restoring the cable angular dynamics to the nominal reference trajectory. The residual deviation in the cable angular dynamics was sufficiently small to still permit successful rendezvous to occur. The magnitude and direction of the wind gusts were found to affect both the aircraft and cable radial dynamics during rendezvous in essentially the same manner. The absolute magnitude of the aircraft velocity component deviations were slightly lower compared to those associated with the cable reel rate. In each case, the wind gusts caused sinusoidal deviations to the aircraft and cable radial dynamics during the time they acted, as well as for a short period of time after they ceased acting.

It was observed that pronounced sinusoidal variations in aircraft and cable reel acceleration were required by the ATC system to mitigate the adverse effects of the wind gusts during rendezvous. The closed loop variation of each control actuator in response to the wind gusts was found to be essentially the same, with saturation of the controls found to be the case for sufficiently strong wind gusts. This lead to the closed loop control paths exhibiting unfavourable “bang-bang” behaviour, indicating that the ATC system required relatively high levels of control during rendezvous to nullify the effects of relatively strong wind gusts. Similarly, the tension in the cable varied quasi-sinusoidally both during and after the wind gusts acted, the overall degree of which was sufficiently higher when feedback control was employed, largely due to the pronounced “bang-bang” nature of the feedback control.

It was postulated that without the corrective measures provided by the feedback controller, rendezvous might not be possible if the ATC system encountered sufficiently strong wind gusts. Horizontal wind gusts were found to cause the largest deviation to the position of the cable tip at rendezvous, demonstrably so in the absence of closed loop control. Appreciable tip deviations were only observed to occur when the magnitude of gusts increased beyond 7.5 % of the initial towing speed. In percentage terms, wind gusts produced the largest disturbance to the altitude of the cable tip compared to the x-coordinate, although the deviation to the x-coordinate of the tip was larger in dimensional terms.
Following on, quasi-quadratic relationships were shown to exist between the duration of the gusts and the levels of closed loop control required by the system during rendezvous. It was shown that progressively lower levels of aircraft vertical and cable reel acceleration were required of the feedback controller as the duration of the wind gusts increased from 1 second to 9 seconds, with control savings in the order of 45 % possible. Slightly increased amounts were required thereafter if the gust duration exceeded 9 seconds. Similarly, progressively lower levels of aircraft horizontal acceleration were required when the duration of the wind gusts increased to 8 seconds, with control savings in the order of 27.5 % available, although slightly increased levels were then required when the gusts acted for progressively longer.

The degree to which the duration of the wind gusts caused deviations to the position of the cable tip at the instant of rendezvous was considerably greater when corrective feedback control measures were not provided to the ATC system. In the absence of feedback, it was found that the various wind gusts generally had the most pronounced effect on the cable tip position at rendezvous when their duration was relatively short to medium, although there were exceptions to this generalization. When feedback control was utilized, the various wind gusts caused progressively larger disturbances to the position of the cable tip at rendezvous as their duration increased. The duration of the horizontal wind gusts had the strongest effect on altitude of the cable tip at rendezvous. Once again, in percentage terms the wind gust duration had a significantly stronger effect on the altitude of the cable tip compared to the $x$-coordinate, yet the deviation to the $x$-coordinate of the tip was notably larger in dimensional terms, irrespective of whether feedback control was used.

The position of the cable tip at rendezvous was noticeably affected by the wind gust commencement time, increasingly so in the absence of corrections attributed to feedback control. In the absence of feedback, horizontal and combined wind gusts had the most prominent effect on the $x$-coordinate of the cable tip at rendezvous when they began acting mid-way through the rendezvous manoeuvre. On the other hand, vertical wind gusts had the strongest effect on the cable tip altitude at rendezvous when they began acting immediately after the rendezvous procedure began. When closed loop control was employed, the position of the cable tip at rendezvous was relatively unaffected by the various wind gusts when they commenced acting in the initial two-thirds of the rendezvous procedure. The impact of combined and vertical wind gusts was progressively more pronounced when they acted later on in the rendezvous manoeuvre, whilst the effect of the horizontal gusts was strongest when they commenced acting in the initial two-thirds of the manoeuvre.
Unlike the findings uncovered during the investigation into the impact of wind gusts on single instantaneous rendezvous operations, it was found that the cable angular dynamics were strongly affected by wind gusts occurring during dual rendezvous, particularly so in the absence of feedback control. Once again, the cable angular dynamics were found to vary quasi-sinusoidally both during and shortly after the gusts were encountered by the ATC system. After the gusts had finished acting and immediately after the first rendezvous manoeuvre was attempted, the degree of deviation in the cable angular dynamics was significantly higher in the absence of feedback control. The additional unsettling disturbance to the cable angular dynamics associated with the zero reel rate condition at the initial rendezvous attempt, caused the open loop cable angular dynamics to deviate significantly away from the nominal reference trajectory. This was found to have profound implications on the ability of the open loop ATC system to carry out the final rendezvous component of the dual rendezvous procedure when relatively strong wind gusts were encountered. Reassuringly, the corrective action of the feedback controller was able to damp out these disturbances, nullifying the changes to the cable angular dynamics and returning the ATC system to the nominal path. Once again however, the feedback controller was unable to fully restore the cable angular dynamics to the nominal reference trajectory, although the residual deviation was sufficiently greater when feedback control was not implemented.

Even though each considered gust occurred late in the initial phase of the dual rendezvous procedure, it was observed that the cable tip trajectory was relatively unaffected by the action of the wind gusts during that phase. The adverse effects of the wind gusts only appreciably manifested during the final phase of the procedure. More concerning, it was found that the cable tip would collide with the ground during the final phase of the operation if significantly strong wind gusts were encountered by the open loop ATC system in the initial phase. This profoundly and adversely affected the safe implementation of the dual rendezvous procedure. Fortunately, the feedback controller was successfully able prevent these large-scale disturbances from occurring, by ensuring that cable tip always closely followed the nominal optimal path throughout the final phase of the procedure. Thus, it can be confidently concluded that without the corrective measures provided by the feedback controller, successful and safe execution of the final rendezvous attempt would not be possible in the presence of sufficiently strong wind gusts. This would remain the case even if these gusts occurred in the initial phase of the dual rendezvous manoeuvre.
Similar to single rendezvous operations performed with variable aircraft manoeuvres, it was observed that the cable tip position at each rendezvous instant was noticeably affected by the magnitude and direction of the wind gusts, the degree of which was considerably higher when feedback control was not employed. Irrespective of whether or not feedback control was utilized, it was found that the position of the cable tip at the initial rendezvous instant was significantly more sensitive to the magnitude and direction of the wind gusts, than the corresponding position of the cable tip at the final rendezvous attempt. In all cases, the deviations to the position of the cable tip at the final rendezvous instant were orders of magnitude lower than the deviations that occurred at the initial rendezvous moment.

Appreciable cable tip deviations were observed to occur when the magnitude of gusts increased beyond 7.5% of the initial towing speed. With respect to the overall levels of required feedback control, quasi-quadratic relationships were shown to exist between the duration of the gusts and the maximum levels of closed loop aircraft acceleration. The corresponding level of closed loop cable reel acceleration was observed to be relatively unaffected by the duration of the gusts. Thus, progressively lower levels of aircraft horizontal and vertical acceleration were required of the feedback controller as the duration of the wind gusts increased, with control savings in the order of 40% possible. It was deduced that slightly lower levels of closed loop cable reel acceleration were required to perform dual rendezvous as the duration of the gusts increased, although the magnitudes involved were found to be marginal.

Regardless of whether feedback control was utilized or not, it was clearly shown that the duration of the wind gusts had a more pronounced effect on the position of the cable tip at the initial rendezvous instant than at the final rendezvous attempt. Moreover, the degree of variation in the position of the cable tip at each rendezvous instant was generally larger in absolute magnitude when the wind gusts acted for progressively longer periods, irrespective of whether feedback was used or not. Similarly, it was found that the position of the cable tip at each rendezvous instant was appreciably affected by the duration of the wind gusts, the degree of which was patently stronger when corrective feedback control was not employed.

Without feedback, the position of the cable tip at each rendezvous moment was most prominently affected by wind gusts that acted for approximately 8 seconds. When feedback control was used, it was uncovered that the wind gusts caused incrementally larger deviations to the position of the cable tip at each rendezvous instant, increasingly so as their duration progressively increased. Under the influence of feedback control, horizontal wind gusts were found to cause the largest deviations to the cable tip at each rendezvous instant.
When feedback control was not implemented, the deviations to the position of the cable tip at the initial rendezvous moment were the largest when gusts began acting in the final third of the dual rendezvous procedure. Contrastingly, the deviations to the position of the cable tip at the final rendezvous instant were most pronounced when gusts began acting in the initial quarter of the procedure. When feedback control was employed, the deviations to the position of the cable tip at the initial rendezvous instant only became relatively pronounced when the gusts began acting after the first third of the dual rendezvous manoeuvre had been completed.

The wind gusts had the strongest impact when they commenced acting in the final third of the dual rendezvous procedure. Conversely, the deviations to the position of the cable tip at the final rendezvous instant were relatively unaffected by the action of the gusts when they commenced acting before the halfway point of the operation. In general, the wind gusts had more of an effect on the deviations to the final cable tip position when the wind gusts commenced acting progressively later on in the dual rendezvous manoeuvre.

When time-varying wind gusts were encountered by the ATC system during payload delivery operations, it was found that the maximum level of closed loop actuation increased quasi-linearly as the magnitude of the gusts increased up to approximately 5-7 % of the initial towing speed, regardless of the direction the gusts acted in. The maximum level of feedback control was impacted in approximately the same manner when either upward tail wind gusts or downward head wind gusts occurred. The controls became saturated when the ATC system encountered sufficiently strong gusts during payload delivery (greater than 5-7 % of the initial towing speed), particularly for the cable reel mechanism. In general, it was surmised that progressively higher levels of feedback actuation were required as the magnitude of the wind gusts increased, with the levels of cable reel acceleration most affected by the presence of the gusts, albeit marginally so.

Generally, when wind gusts were encountered by the ATC during the final phase of the payload delivery operation, deviations to final aircraft position were not found to be prominent, irrespective of whether feedback control was implemented and regardless of the magnitude and direction of the gusts. The impact of the gusts on the final aircraft position was observed to be approximately 4 orders of magnitude lower when feedback control was used. However, in the absence of closed loop ATC system control, the relevant deviations were not sufficiently large to warrant particular concern. Following on, it was found that the magnitude of the deviations to the closed loop final aircraft position from the nominal optimal position were so small that they were comparable to the level of numerical error associated with the integration routine.
With respect to the final position of the cable tip, when feedback control was not employed, the impact of the gusts was approximately twice as strong as their corresponding impact when feedback measures were adopted, although the final position of the cable tip was relatively unaffected by the magnitude and direction of the wind gusts. Strong horizontal head wind gusts caused the largest deviation to the final cable tip position when feedback control was utilized, however the degree of which was marginal and not particularly concerning. However, the degree of the deviations to the final cable tip position was shown not to be critical to the success of the transportation operation during the final phase. It was the deviations to the final aircraft position, cable length and length rate, along with the cable angular rate that were of most interest. Provided that feedback control was implemented, it was found that these deviations were zero at the conclusion of the delivery procedure, regardless of the magnitude and direction of the wind gusts, within the relevant domains considered.

Like both single and dual instantaneous rendezvous operations, it was observed that the maximum levels of feedback actuation provided the ATC system during delivery were affected by the gust duration in essentially the same manner. Irrespective of the particular actuator, it was uncovered that the absolute maximum level of closed loop control occurred when the duration of wind gusts was 2 seconds. The overall level of feedback control then progressively decreased non-linearly in a quasi-quadratic manner as the duration of the gusts increased, with overall closed loop control savings in the order of 60-65 % possible when the gust duration was increased by 400 %. The results concerning how the duration of the wind gusts affected the levels of closed loop control for payload delivery were found to be in general agreement with those associated with single and dual instantaneous rendezvous operations. The overall level of closed loop cable reel acceleration was most affected by the duration of the wind gusts, although the degree of the relative difference between each actuator was minimal.

Deviations to final aircraft position were observed to remain relatively invariant to the duration of the gusts, regardless of whether feedback control was implemented. The impact of the gust duration on the final aircraft position was observed to be approximately 4 orders of magnitude lower when feedback control was used, although the relevant deviations were again comparable to the level of numerical error associated with the chosen integration routine. In the absence of closed loop control, the duration of the gusts were found to cause deviations to the final tip position that were twice as large as those corresponding to the scenario when feedback measures were adopted. Although the final position of the cable tip
was relatively unaffected by the duration of the wind gusts under the influence of feedback control, it was apparent that upward vertical wind gusts caused the largest deviation to the final cable tip position, increasingly so as their duration increased accordingly. However, the size of these deviations were not sufficiently large enough to jeopardize the success of the payload transportation operation. Similarly, the degree of the deviation to the final aircraft position, cable length, length rate and cable angular rate were all found to be sufficiently close to zero at the conclusion of the delivery procedure, irrespective of the wind gust duration. This certified the success of the transportation operation within the final phase, provided that feedback control was implemented.

It was uncovered that the position of the aircraft at the instant of delivery was relatively unaffected by gust commencement time, since deviations to aircraft position at delivery were diminutive, irrespective of when the wind gusts commenced acting. Deviations to the aircraft position at delivery were most apparent when the gusts commenced acting approximately 6.7 secs before delivery occurred, although the relevant deviations were again comparable to the level of numerical error associated with the integration processes. It was found that the cable tip position at delivery was significantly more affected by the action of the wind gusts than the aircraft position, in particular the cable tip altitude at delivery when feedback control was not adopted. Deviations to the cable tip altitude at delivery were most pronounced when the gusts commenced acting in the first third of the payload transportation operation, with the maximum deviation arising when horizontal tail wind gusts began acting 6.7 secs before delivery occurred.

Regardless of whether or not feedback control was implemented, the final position of the aircraft was unaffected by the gust commencement time. In general, deviations to final cable tip position were small and not potentially troubling when the gusts commenced acting anytime throughout the payload transportation operation, especially when closed loop control was provided to the system. The variation in the deviation of the final cable tip position was progressively larger as the gusts commenced acting progressively later on in the delivery operation, especially so in the final third of the procedure. When feedback control was utilized, it was again shown that the deviations to the final cable length, length rate and cable angular rate were all suitably close to zero at the end of the payload delivery procedure, irrespective of the considered gust commencement times.
9.2 Recommendations

This thesis has formally proposed and developed a number of new concepts and methodologies in the twin fields of aerial towed-tether dynamics and control. While the results and findings contained within this dissertation have unequivocally demonstrated the theoretical feasibility of using ATC systems to safely and accurately transport payloads to and from remote locations without landing, a series of additional milestones still need to be accomplished. Appropriately meeting these milestones will ensure that the full implementation of this technology is possible in a practical real-world system. Accordingly, the following set of recommendations represent important and outstanding areas of research that would serve to further enhance the major findings of this thesis.

9.2.1 Aerial Towed-Cable System Modelling

Additional research in the general area of ATC system modeling would be beneficial during the planning and experimental design phases of potential rendezvous and transportation exhibition operations. Possible avenues of research include:

- Dynamically incorporating the motion of the aircraft within the ATC system models to gain more of an appreciation of the required aircraft thrust, angle of attack and bank angle for rendezvous and payload transportation.

- Provide full dynamic coupling between the motion of aerodynamic rigid-body payloads and the cable tip within the ATC system model.

- Modelling the cable reel/winch as an electric motor, to investigate more closely the cable reel mechanism dynamics (response time, power requirements, noise, saturation, etc.) during both rendezvous and payload transportation operations.

- Investigate the dynamics of deployment and retrieval of the cable using the flexible multi-link ATC system model, capable of simulating variable length cables. This would involve the intimate examination of how the cable reel rates required during the proposed rendezvous and payload transportation operations, affect the stability of the cable and the tension within it. This could be extended to the actual closed loop deployment and retrieval control of the cable. The preferred and most representative method of handling cable deployment and retrieval is that outlined and widely utilized by Williams and Trivailo [6] and Trivailo et al. [127].
• Undertake a parametric study into the capture and delivery operations presented in this thesis using the flexible multi-link cable model, to more thoroughly analyze how the discontinuous tension change in the cable is affected by various system and operational parameters.

9.2.2 Aerial Towed-Cable System Model “Matching”

Although each of the two model “matching” procedures proposed in this thesis were shown to perform consummately in their current condition, the fidelity of each could benefit from a number of enhancements. Two possible advancements are common to both model “matching” procedures, the first involving the formal extension of each technique so that the three-dimensional dynamical motion of the ATC system can be appropriately accounted for within each procedure. Secondly, if appropriate data was available, be it laboratory or flight test data, it would be highly advantageous to incorporate real experimental results into each “matching” procedure, in order to further validate and improve the representativeness of the simple single-link ATC system models. Individually, each model “matching” procedure could profit from the following improvements:

**Method 1**

• Investigate alternative approaches based on mathematical principles to determine the best estimate of the parameter adjustments \( p_1 \) and \( p_2 \) to use within the “matching” procedure.

• Explore the possibility of employing a mathematically rigorous process for selecting the values of the towing speed and cable length used to calculate the parameter adjustments.

**Method 2**

• Develop a routine to map how the “equivalent” radial velocity of the cable relates to the cable deployment rate for each complex multi-link model. This would further enhance the mathematical completeness of the model “matching” algorithm.

• Explore the possibility of training a series of artificial neural networks to approximate the multi-dimensional relationship that exists between the physical parameters of the ATC system and its “equivalent” kinematics.

• Employ an optimization procedure to determine the optimum value of the parameters \( K_1 \) and \( K_2 \) to ensure the closest possible model “match”, given the set of physical parameters governing the ATC system.


9.2.3 Non-linear Optimal Controllers for the Aerial Towed-Cable System

Although comprehensively developed within this thesis, the maturity and application of non-linear, open loop optimal control systems for the ATC system could be refined via the following measures:

- Explore non-linear optimal control problems for the ATC system where rendezvous is not limited to stationary targets, but those that are capable of dynamically manoeuvring.

- Examine additional numerical techniques that enable rapid surface interpolation and approximation, such as those based of efficient B-spline methods. This will improve and accelerate the implementation of the path constraint used within the optimal control methodology, to ensure that the cable tip is prevented from colliding with the ground during three-dimensional rendezvous and payload transportation operations.

- Include aircraft performance constraints directly within the optimal control framework for rendezvous and payload transportation problems. This could be achieved through the use of a generalized non-liner path constraint that calculates the aircraft thrust, angle of attack and bank angle indirectly from the aircraft acceleration components. Alternatively, if there is dynamic coupling between the motion of the aircraft and the cable/payload combination, the optimal control problem could be formulated using a performance index containing the aircraft thrust, angle of attack and bank angle. Minimizing this performance index will ensure that aircraft control requirements will also be a minimum, thereby guaranteeing the resulting aircraft trajectories to be physically realizable. Although preliminary results [102, 103, 115] indicate that in the majority of cases the aircraft does not operate sufficiently close to the constraints, incorporating aircraft performance constraints into the optimal control problem is an important conservative measure for safe implementation.

- Investigate the possibility of utilizing active aerodynamic control surfaces on the payload to mitigate the adverse oscillations and unwanted dynamic excitations induced in the cable immediately after rendezvous and payload delivery/capture.
9.2.4 Closed Loop Feedback Controllers for the Aerial Towed-Cable System

The neighbouring optimal feedback controllers for the ATC system and the closed loop simulations presented in this thesis only provide an initial indication of the closed loop performance of the ATC system when undertaking rendezvous and payload transportation operations. As a result, there remains a sufficiently wide scope for further research in this field, including the following specific areas:

- Investigate the impact that modelling uncertainties and initial state disturbances have on the ability of the ATC system to track the nominal optimal paths corresponding to rendezvous and payload transportation operations.

- Consider using an internal optimization routine to calculate the optimum values of the design state and control weighting matrices. Similarly, investigate the effect of employing a final state design weighting matrix within the LQR control methodology, particularly for rendezvous operations.

- Restrict the use of neighbouring optimal feedback control for the aircraft itself and not the entire ATC system, assuming as Williams et al. [104] have that if the aircraft trajectory can be adequately controlled, then the cable tip/payload should be able to track desired trajectories. The actual validity of this assumption should also be examined via a direct comparison between the differing implementations.

- Explore additional means of online neighbouring feedback control for the ATC system, such as a Receding Horizon Control method similar to those proposed by Lu [172] and Williams [167]. This will permit the examination of the effect that accounting for the time-varying system dynamics in the calculation of the feedback control has on the closed loop system when tracking optimal paths.

- Consider evaluating the performance of the LQR feedback controllers during rendezvous and payload transportation operations by using the more sophisticated flexible multi-link ATC system model during the final implementation of the online closed loop control.
9.3 Potential Further Applications

The results and findings uncovered throughout this dissertation are anticipated to assist the physical implementation of payload transportation applications for ATC systems in the near-term. The following ATC system applications promise significant benefits and could be implemented relatively simply through a series of scaled-down experimental demonstrations, before progressing to full-scale systems. Examples of viable near-term applications that could exploit the expertise contained within this thesis are as follows:

- Enhanced aerial wildfire fighting capabilities through the use of a cable towed beneath an aircraft to capture large volumes of water and precisely deliver them to critical fire locations. The control systems developed in this thesis would allow the cable tip to capture a quantity of water/fire retardant solution housed within a pod, simulated as a discontinuous increase in payload mass and accompanying increase in drag. The aircraft could then fly as fast and safely as possible towards the fire front and precisely deliver the water/fire retardant solution to a specific location along the front, that deemed of most assistance to the ultimate containment of the fire. The actual delivery instant would be modelled as a relatively instantaneous discharge of payload mass and reduction in aerodynamic drag at the cable tip. A generalized path constraint could be employed to appropriately constrain the aircraft not to fly too close to the actual fire front, whilst strong, highly variable winds and localized gusts in the vicinity of the fire front could also be simulated using methods outlined in this dissertation.

- Rapid and highly detailed geophysical surveys of both surface and subterranean environments using aerial-towed remote sensing and monitoring instrumentation. Since this thesis demonstrated that the cable tip could be forced to closely track surface-based terrain without making direct contact, it is envisaged that new remote mineral exploration, mine sweeping and oceanographic survey experiments could be proposed and demonstrated using technology developed in this thesis.

- Precise, safe and expedient transportation of provisions both to and from isolated, hostile or otherwise inaccessible locations, all without the need for landing. The general of class of payload transportation operations for the ATC system were successfully proposed in this thesis. These scenarios could be extended and tailored to more specific applications such as disaster relief and search and rescue operations, involving distressed personnel lost/trapped in isolated areas or hostile battlefield environments.
10 REFERENCES


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