On E-Pseudovarieties of Finite Regular Semigroups

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Declaration

The candidate hereby declares that the work contained in this thesis:

(i) is that of the candidate alone and has not been submitted previously, in whole or in part, in regard of any other academic award and has not been published in any form by any other person except where due reference is given; and

(ii) has been carried out since the official date of commencement of the research programme.

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J. D. Rodgers

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Peter Trotter and Tom Hall (quite independently) suggested studying congruences on lattices of e-varieties and e-pseudovarieties, research which forms the bulk of Chapter 3.

Much of the research was carried out while I was the recipient of an Australian Postgraduate Award.

Finally, I must thank my friends and family for supporting me over the past several years, especially my wife, Julie.

This thesis is dedicated to my father, J. G. Rodgers, who was not able to see it completed.

Contents

Introduction

Birkhoff [11] introduced the concept of a variety of algebras, a class of algebras all of the same type, closed under the taking of homomorphic images, subalgebras of the same type and direct products. Birkhoff showed that such classes were defined by sets of identities (for example, within the class of all semigroups, the collection of all commutative semigroups is a variety defined by the identity $xy = yx$). Conversely, any class of algebras, all of the same type, all of which satisfy the same set of identities is a variety. Thus began the study of universal algebra. Over the years, numerous classes of algebras were studied using the new tools of universal algebra - groups, rings, lattices and importantly for us, semigroups.

The survey by Evans [17] described the research into varieties of semigroups up to the early 1970's. In the mid-1970's a strong relationship was established between certain classes of finite semigroups and the theory of formal languages and automata that had been developed in the previous two decades. Since these classes consisted of finite semigroups, now only finite direct products were permitted. It is relatively easy to show that such classes of finite semigroups need not form varieties of semigroups and a new theory of pseudovarieties was developed, initially by Eilenberg and Schützenberger [16]. This first paper *On pseudovarieties* was part of a larger study on langauges and automata by Eilenberg [15].

The original work of Eilenberg and Schützenberger was based upon the discovery that pseudovarieties may be defined by sequences of identities (as opposed to sets of identities in the case of varieties). In the early 1980's, Reiterman [39] showed, using a topological argument, that pseudovarieties may be defined by sets of pseudoidentities. Shortly thereafter, Ash [3] proved that pseudovarieties consist of the finite members of certain classes of algebras called generalised varieties.

In the late 1980's and early 1990's existence varieties (or e-varieties) of regular semigroups were defined (independently) by Hall [21] and by Kaďourek and Szendrei [28]. E-varieties allow one to study classes of regular semigroups using universal algebraic techniques.

It seems natural to marry the two concepts of e-variety and pseudovariety in a new notion of e-pseudovariety. Mangold introduced e-pseudovarieties of finite regular semigroups in her doctoral thesis [31]. A class of finite regular semigroups forms an e-pseudovariety if it is closed under the taking of homomorphic images, regular subsemigroups and finite direct products.

In this thesis we further develop the theory of e-pseudovarieties, with particular emphasis on congruences on certain sublattices of the lattice of e-pseudovarieties of finite regular semigroups.

An overview of the thesis follows:

In Chapter 1, we discuss those concepts of semigroup theory, lattice theory and universal algebra that are used throughout the remainder of the thesis. In this chapter, we also survey those aspects of the theory of pseudovarieties and e-varieties that are important to this work. Finally, generalised varieties are discussed.

In Chapter 2, we introduce generalised e-varieties. We establish the important result that all e-pseudovarieties contained in $\mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$ consist of the finite members of some generalised e-variety Theorem 2.3.7. We are later able to refine this result in an interesting direction (Corollary 2.4.15).

The final results of Chapter 2 involve properties of certain sublattices of the lattice of e-pseudovarieties of finite regular semigroups. We demonstrate (in a result proved originally by Agliano and Nation for pseudovarieties [1]) that many important lattices of e-varieties are isomorphic to certain related lattices of e-pseudovarieties (Theorem 2.4.17). We then describe a complete lattice homomorphism between certain lattices of generalised e-varieties and lattices of e-pseudovarieties (Theorem 2.4.18).

Chapter 3 is the final chapter in this thesis. This chapter is heavily influenced by research carried out by Pastijn and Trotter [34]. We demonstrate that similar results exist for e-pseudovarieties to those that were obtained by Pastijn and Trotter for pseudovarieties.

Some results from Chapters 2 and 3 will appear in a paper in the Proceedings of the Sydney University Semigroups Conference of June 2005 [40]. Two other papers are currently in preparation, one based on additional results in Chapter 2 [41] and the other on additional results in Chapter 3 [42].

1

Semigroups, Lattices and Universal Algebra

E-pseudovarieties grew out of the study of e-varieties and pseudovarieties, which in turn are based on the notion of varieties of algebras. Varieties of algebras were initially studied by Birkhoff [11] in the 1930's. Pseudovarieties of finite algebras followed in the mid-1970's and existence varieties (e-varieties) of regular semigroups in the late 1980's and early 1990's.

Birkhoff recognised that lattices played an important part in the study of varieties of algebras. The natural ordering of classes of algebras leads directly to lattice structures. Semigroups, by virtue of their (relatively) simple structure, have long been studied within the setting of varieties.

In this preliminary chapter, we summarise the important results of semigroup theory, lattice theory and universal algebra which motivate the present thesis.

1.1 Regular Semigroups

1.1.1 Basic Definitions

A *semigroup* $\langle S; \circ \rangle$ is a pair consisting of a nonempty set S (the *underlying set*) together with an associative binary operation $\circ$ defined upon S. We adopt the usual practice of denoting a semigroup simply by its underlying set S and for elements $x, y \in S$, writing xy in place of $(x, y)\circ$. A semigroup with an identity element is called a *monoid*. It is usual in semigroup theory to denote by S^1 the monoid obtained by adjoining an identity element to the semigroup S. If S is already a monoid then an identity element is not adjoined and $S^1 = S$.

A semigroup S is said to be *regular* if for every $x \in S$ there exists an element $y \in S$ such that $xyx = x$. Two elements x and $x' \in S$ are said to be inverses of each other (in the sense of von Neumann) if $xx'x = x$ and $x'xx' = x'$. Every element $x = xyx$ in a regular semigroup has such an inverse: Define $x' = yxy$. A simple computation quickly verifies that $xx'x = x$ and $x'xx' = x'$. Of course, such an inverse need not be unique. The set of all inverses of x is denoted by $V(x)$. It is important to note that a regular semigroup need not have an identity element. Many important properties of regular semigroups were established in the early 1970's (see for example [19, 20]).

A semigroup S is said to be an *inverse semigroup* (see [35]) if for each $x \in S$ there exists a unique $x^{-1} \in S$ such that

$$xx^{-1}x = x \text{ and } x^{-1}xx^{-1} = x^{-1}.$$

Clearly all inverse semigroups are regular, however the comments above show that the converse need not be true.

It is quickly shown that any homomorphic image of a regular semigroup is regular as is the direct product of a family of regular semigroups.

Let S be a semigroup. An element $z \in S$ is said to be a *left* (respectively, *right*) *zero* if $(\forall a \in S)\, za = z$ (respectively $az = z$). An element z is a zero if it is both a left zero and a right zero. It is easily verified that a zero is unique, if it exists. Similarly, an element $e \in S$ is a *left* (respectively, *right*) *identity element* if $(\forall a \in S)\, ea = a$ (respectively $ae = a$).

Let S be a semigroup. An element $e \in S$ is said to be idempotent if $e^2 = ee = e$. Note that if S is regular and $x \in S$ then $(\forall x' \in V(x))$, xx' and $x'x$ are idempotent. The set of all idempotents in S is denoted by $E(S)$. The set $E(S)$ may be empty, although if S is regular then $E(x) \neq \varnothing$ since xx' and $x'x \in E(S)$ for each $x \in S$. A semigroup all of whose elements are idempotent is called a *band*. Note also that if S is finite then $E(S) \neq \varnothing$ since the monogenic semigroup $\langle x \rangle$ generated by x contains a subgroup.

On a regular semigroup S, we define a relation $\leq$ as follows:

$$a \leq b \Leftrightarrow a = eb = bf \text{ for some } e, f \in E(S).$$

It is well known that $\leq$ is a partial order on a regular semigroup S. The importance of this partial order justifies it being called the *natural partial order* on S. Restricting this partial order to the set of idempotents $E(S)$ in S yields:

$$e \leq f \Leftrightarrow e = ef = fe.$$

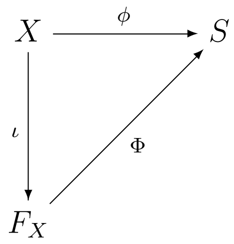
An idempotent $e \in E(S)$ is said to be a *primitive idempotent* if it is minimal among the nonzero idempotents of S with respect to the natural partial order.

1.1.2 Free Semigroups

Let X be a nonempty set. The set F_X of all nonempty words (that is, words of length ≥ 1) made up of symbols from the set X is a semigroup with the binary operation being concatenation of words. The semigroup constructed in this way is called the *free semigroup generated by* X. The free monoid F_X^1 is obtained by allowing the empty word (sometimes denoted by λ) to appear in the list of words.

The mapping $\iota : X \to F_X$ simply maps each element of X to its corresponding word of length 1 in F_X. The following theorem is fundamental in semigroup theory.

Theorem 1.1.1. *Let S be a semigroup, X be a nonempty set and $\phi : X \to S$ be* *a mapping. Then there exists a unique homomorphism $\Phi : F_X \to S$ such that the following diagram commutes.*



1.1.3 Ideals and Green's Relations

Let S be a semigroup. For A, $B \subseteq S$ we define the set AB as follows:

$$AB := \{ab \mid a \in A \text{ and } b \in B\}.$$

A set $I \subseteq S$ is a left (respectively, right) ideal of S if $SI \subseteq I$ (respectively $IS \subseteq I$). We say that I is a two sided ideal of S if it is both a left ideal of S and a right ideal of S. Clearly every ideal of S is a subsemigroup of S.

For a semigroup S, the smallest left ideal containing an element $a \in S$ is called the *principal ideal generated by a*. It is quickly verified that the set $Sa \cup \{a\}$ is the smallest left ideal of S containing a. We usually denote this set by S^1a. Similarly, $aS^1 = aS \cup \{a\}$ is the smallest right ideal of S which contains a and is the *principal right ideal generated by a*. The *principal two-sided ideal generated by a* is $S^1aS^1 = SaS \cup Sa \cup aS \cup \{a\}$. Closely related to these principal ideals are Green's relations. The relations $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{J}$ are defined as follows. For a, $b \in S$ define

$$\begin{aligned} a \mathscr{L} b &\Leftrightarrow S^1a = S^1b \\ a \mathscr{R} b &\Leftrightarrow aS^1 = bS^1 \\ a \mathscr{J} b &\Leftrightarrow S^1aS^1 = S^1bS^1 \end{aligned}$$

In the lattice of equivalence relations on S (see the following section for details on lattices), the meet and join of the relations $\mathscr{L}$ and $\mathscr{R}$ are defined as follows:

$$\begin{aligned} \mathscr{H} &= \mathscr{L} \wedge \mathscr{R} = \mathscr{L} \cap \mathscr{R} \\ \mathscr{D} &= \mathscr{L} \vee \mathscr{R} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L} \end{aligned}$$

The following result is well known. A proof can be found in [25, Prop. II.1.5].

Theorem 1.1.2. *If S is finite, then $\mathscr{D} = \mathscr{J}$.*

A semigroup S is said to be $\mathscr{K}$-trivial, $\mathscr{K} \in \{\mathscr{L}, \mathscr{R}, \mathscr{J}, \mathscr{H}, \mathscr{D}\}$, if for each $x \in S$, the $\mathscr{K}$-class containing x is a singleton.

The following classic result of semigroup theory is known as Lallement's lemma. This result gives us important information about the existence of idempotents in the preimage of a homomorphism between regular semigroups.

Theorem 1.1.3. *Let S be a regular semigroup, T a semigroup and $\phi : S \to T$ a homomorphism. The $S\phi$ is regular and for all $f \in E(S\phi)$ there exists $e \in E(S)$ such that $e\phi = f$.*

In the literature on semigroups, a great deal of attention has been given to completely regular semigroups. An element a in a semigroup S is said to be *completely regular* if there exists an $x \in V(a)$ such that $ax = xa$. It is a straightforward exercise to show that such an x in this case must be unique. Equivalently, H_a, the $\mathscr{H}$ class containing a, must be a subgroup of S. A semigroup S is said to be *completely regular* if every element of S is completely regular (see [36] for a recent survey). Such semigroups are often referred to (especially in the older literature, for example [14]) as *unions of groups*, as a semigroup S is completely regular if and only if S is the union of disjoint groups.

A semigroup S is said to be *simple* if it has no proper ideals. A completely regular simple semigroup is called a *completely simple* semigroup.

1.1.4 Congruences on Semigroups

A *congruence* on a semigroup S is a subset ρ of $S \times S$ with the following properties:

(i) ρ is an equivalence relation on S; and

(ii) if $(a_1, b_1) \in \rho$ and $(a_2, b_2) \in \rho$ then $(a_1a_2, b_1b_2) \in \rho$.

We will usually write $a\rho b$ instead of $(a, b) \in \rho$. We denote the set of all congruences on a semigroup S by $\mathrm{Con}(S)$. It is straightforward to demonstrate that $\mathrm{Con}(S)$ is a lattice (in fact, $\mathrm{Con}(S)$ is a sublattice of $\mathrm{Eq}(S)$, the lattice of all equivalence relations on S). The following sections discuss lattices in further detail.

For an equivalence relation α on S we denote by $a\alpha$ the set of elements α-equivalent to $a \in S$. That is, $a\alpha := \{x \in S \mid a\alpha x\}$. If ρ is a congruence on a semigroup S then the set S/ρ defined by

$$S/\rho := \{a\rho \mid a \in S\}$$

is a semigroup where the operation is:

$$a\rho b\rho := (ab)\rho.$$

A relation ρ on a semigroup S is said to be *fully invariant* if for all $x, y \in S$, $x\rho y \Rightarrow x\phi\rho y\phi$ for all endomorphisms $\phi : S \to S$. The set of all fully invariant congruences on S is a complete sublattice of the lattice of all congruences on S.

1.1.5 E-Solid and Locally Inverse Semigroups

The *core* of a semigroup S is denoted $C(S)$ and is defined to be the subsemigroup of S generated by $E(S)$. That is, $C(S) = \langle E(S)\rangle$. It can be shown that if S is regular then $C(S)$ is also regular.

A semigroup S is said to be *orthodox* if for all $e, f \in E(S)$, $ef \in E(S)$, that is, the set of idempotents in S form a subsemigroup of S. Equivalently, S is orthodox if $C(S) = E(S)$.

For a regular semigroup S, the *conjugate in S* of a subsemigroup T of S is defined to be the subsemigroup of S generated by $\{sts' : t \in T,\ s \in S\, s' \in V(S)\}$.

The conjugate of T in S is denoted by T_c. The conjugate in S of T_c is denoted T_{c^2} and recursively $T_{c^n} = (T_{c^{n-1}})_c$, $n \geq 2$.

If $T_c \subseteq T$ we say that T is *self-conjugate*. For $C = C(S)$ it is clear that $C \subseteq C_c$ and in general $C_{c^n} \subseteq C_{c^{n+1}}$, $n \geq 1$. The self-conjugate core of S is denoted $C_\infty(S)$ and is defined as follows:

$$C_\infty(S) = \bigcup_{n\geq 1} C_{c^n}$$

If S is regular then the self-conjugate core of S is a regular subsemigroup of S.

A semigroup S is said to be *E-solid* if for all $e,\ f,\ g \in E(S)$ such that $e\mathscr{L} f \mathscr{R} g$, there exists $h \in E(S)$ such that $e \mathscr{R} h \mathscr{L} g$. Note that S need not be regular.

Lemma 1.1.4. *Let S be a regular semigroup. Then the following are equivalent:*

(i) S is E-solid;

(ii) for all $e, f \in E(S)$, ef is in a subgroup of S;

(iii) $C(S)$ is completely regular; and

(iv) $C_\infty(S)$ is completely regular.

A semigroup S is said to be *locally inverse* if for all $e \in E(S)$, eSe is an inverse semigroup. Again, note that S need not be regular.

The following result of Johnston highlights the importance of E-solid and locally inverse regular semigroups. The family $\mathrm{Sub}(S)$ is the collection of all regular subsemigroups of S, ordered by (set) inclusion.

Theorem 1.1.5. [26, Theorems 3 and 4] *If S is a locally inverse regular semigroup or an E-solid regular semigroup then Sub(S) is a complete lattice.*

1.2 Lattice Theory

As we have already remarked, the set $\mathrm{Con}(S)$ of all congruences on a semigroup S forms a lattice. Lattices arise quite naturally in the study of semigroups and are of vital importance in the study of varieties, e-varieties, pseudovarieties and e-pseudovarieties. A major component of the present thesis is devoted to the examination of congruences on lattices of e-varieties and e-pseudovarieties.

1.2.1 Lattices and Semilattices

An algebra $\langle L; \wedge, \vee \rangle$ of type $(2, 2)$ is a *lattice* if it satisfies the following identities:

(i) $x \wedge x = x$ and $x \vee x = x$;

(ii) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$; and

(iii) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$.

A *meet semilattice* (*join semilattice*) is an algebra $\langle L; \wedge \rangle$ ($\langle L; \vee \rangle$) which satisfies the identities:

(i) $x \wedge x = x \quad (x \vee x = x)$;

(ii) $x \wedge y = y \wedge x \quad (x \vee y = y \vee x)$; and

(iii) $x \wedge (y \wedge z) = (x \wedge y) \wedge z \quad (x \vee (y \vee z) = (x \vee y) \vee z)$.

Since for every $x, y, z \in L$, $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $x \wedge x = x$ and $x \wedge y = y \wedge x$, every meet (or join) semilattice is a commutative band.

There exists a strong relationship between lattices and partial orders. For a lattice L, we define $\leq$ by:

$$x \leq y \Leftrightarrow x \wedge y = x.$$

A simple computation verifies that $\leq$ is a partial order on L. Conversely, if $\langle L; \leq \rangle$ is a partial order and every pair of elements $x, y \in L$ has a greatest lower bound $x \wedge y$ and a least upper bound $x \vee y$, then $\langle L; \wedge, \vee \rangle$ is a lattice.

An interval $[a, b]$ on a lattice L is a set defined as follows:

$$[a, b] := \{x \in L \mid a \leq x \text{ and } x \leq b\}.$$

A lattice L is said to be a *complete lattice* if every nonempty subset A of L has a meet $\bigwedge_{a \in A} a$ and a join $\bigvee_{a \in A} a$. The following result is well known.

Lemma 1.2.1. *A lattice L is a complete lattice if and only if L has a greatest element and each nonempty subset of L has a meet.*

If $L = \langle L; \vee \rangle$ is a join semilattice with least element 0, then a subset I of L is called an *ideal* if

(i) $0 \in I$;

(ii) $x, y \in I \Rightarrow x \vee y \in I$; and

(iii) $z \leq x \in I \Rightarrow z \in I$.

Dually, one may define a filter on a lattice L: A *(proper) filter* on a lattice L is a subset F of L with the following properties:

(i) $0 \notin F$;

(ii) $x, y \in F \Rightarrow x \wedge y \in F$; and

(iii) if $x \in F$ and $x \leq y \in F$ then $y \in F$.

Given a join semilattice L, the ideal generated by a set $X \subseteq L$ is defined to be the intersection of all ideals of L containing X. The principal ideal generated by an element $a \in L$ is simply the set $\{x \in L \,|\, x \leq a\}$.

Finally, we remark that a collection of sets $\mathscr{C}$ is said to be *directed* if for all $A, B \in \mathscr{C}$ there exists $C \in \mathscr{C}$ such that $A \subseteq C$ and $B \subseteq C$.

1.2.2 Congruences on Lattices

Congruences on lattices are defined similarly to congruences on semigroups. An equivalence relation ρ on a lattice L is a $\wedge$-congruence if

$$(\forall a \in L)\ x\rho y \Rightarrow (x \wedge a)\rho(y \wedge a).$$

Dually, a $\vee$-congruence is an equivalence relation ρ on L with the property

$$(\forall a \in L)\ \ x\rho y \Rightarrow (x \vee a)\,\rho\,(y \vee a)\,.$$

A congruence ρ on a complete lattice L is said to be a *complete congruence* if for families $\{x_\alpha\}_{\alpha\in A}$ and $\{y_\alpha\}_{\alpha\in A}$ of members of L,

$$\forall \alpha \in A\ x_\alpha \rho y_\alpha \Rightarrow \left(\bigvee_{\alpha\in A} x_\alpha\right) \rho \left(\bigvee_{\alpha\in A} y_\alpha\right) \text{ and } \left(\bigwedge_{\alpha\in A} x_\alpha\right) \rho \left(\bigwedge_{\alpha\in A} y_\alpha\right).$$

In light of the previous definitions, the terms *complete* $\vee$*-lattice* and *complete* $\wedge$*-lattice* are easily understood.

That congruence classes are intervals is a fundamental property of complete congruences on lattices. The following lemma demonstrates this.

Lemma 1.2.2. *For any complete congruence ρ on a complete lattice L, the ρ-class containing $a \in L$ is an interval $[a_\rho, a^\rho]$ where*

$$a_\rho = \bigwedge_{x\in a\rho} x \text{ and } a^\rho = \bigvee_{x\in a\rho} x.$$

The converse is also true, as stated below.

Lemma 1.2.3. *Let ρ be a congruence on a complete lattice L. If all of the congruence classes of ρ are intervals, then ρ is a complete congruence on L.*

We will find that the following result is occasionally useful.

Lemma 1.2.4. *Let L be a complete lattice and ρ a complete congruence on L. Suppose $a\rho \bigvee_{i\in I} a_i$ for some collection $\{a_i\}_{i\in I}$ of members of L. Then $a_\rho = \bigvee_{i\in I} (a_i)_\rho$.*

1.3 Varieties of Semigroups

1.3.1 Universal Algebra

The notion of a *variety of algebras* comes from universal algebra (see the books [12, 18]). It is important to note that while the definitions and theorems presented here are given for semigroups, the theory is easily extended to algebras of any finite type. Recall that a class $\mathcal{C}$ of semigroups is a *variety of semigroups* if it is closed under the taking of homomorphic images, subsemigroups and direct products (see Evans [17] for an excellent survey). In symbols we write:

$$\mathbb{H}(\mathcal{C}) \subseteq \mathcal{C},\ \mathbb{S}(\mathcal{C}) \subseteq \mathcal{C} \text{ and } \mathbb{P}(\mathcal{C}) \subseteq \mathcal{C}.$$

The notion of *subdirect product* is frequently utilised in universal algebra. Given a collection of semigroups $\{S_\alpha\}_{\alpha\in\Lambda}$, let T be isomorphic to a subsemigroup S' of $S = \Pi_{\alpha\in\Lambda}S_\alpha$ and let for each α, $\pi_\alpha : S \to S_\alpha$ be the projection homomorphism. Then T is a subdirect product of $\{S_\alpha\}_{\alpha\in\Lambda}$ if $S'\pi_\alpha = S_\alpha$ for each $\alpha \in \Lambda$. We denote by $\mathbb{P}_{\mathrm{s}}(\mathcal{C})$ the class of all semigroups isomorphic to subdirect products of collections of members of $\mathcal{C}$.

For a class $\mathcal{C}$ of semigroups, the variety generated by $\mathcal{C}$ is denoted $\langle\mathcal{C}\rangle_{\mathrm{v}}$ and is equal to the intersection of all semigroup varieties which contain $\mathcal{C}$. The following result is well known and says that the $\langle\mathcal{C}\rangle_{\mathrm{v}}$ consists precisely of those semigroups which are homomorphic images of subsemigroups of direct products of members of $\mathcal{C}$.

Theorem 1.3.1. $\langle\mathcal{C}\rangle_{\mathrm{v}} = \mathbb{HSP}(\mathcal{C})$.

Let X be a countable set and let $u, v \in F_X$, the free semigroup on X. We say that the (semigroup) identity $u = v$ is satisfied (or, equivalently, that S satisfies the identity $u = v$) in a semigroup S if $u\phi = v\phi$ for all homomorphisms $\phi : F_X \to S$. Let $\Sigma = [u_1 = v_1, \ldots, u_n = v_n]$ be a set of (semigroup) identities and let S be a semigroup. If S satisfies all of the identities $u_1 = v_1, \ldots, u_n = v_n$ then we say that S is defined by Σ. On occasion we may write $S \vDash \Sigma$.

A nonempty class $\mathcal{C}$ of semigroups is said to be an *equational class* defined by the set of identities Σ if every semigroup in $\mathcal{C}$ satisfies every identity in Σ and every semigroup that satisfies every identity in Σ is in $\mathcal{C}$. We write $\mathcal{C} = [\Sigma]$ if $\mathcal{C}$ is defined by Σ.

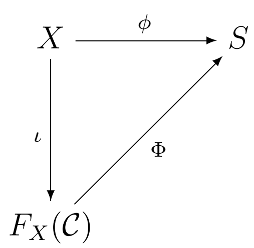
Birkhoff [11] demonstrated the equivalence of varieties and equational classes. We state Birkhoff's result for semigroups below.

Theorem 1.3.2. *A class of semigroups is a variety if and only if it is an equational class.*

Many important classes of semigroups form varieties of semigroups. We list several below, together with the identities that define them:

Symbol	**Description**	**Identity(ies)**
T	Trivial semigroups	$x = y$
LZ	Left zero semigroups	$xy = x$
RZ	Right zero semigroups	$xy = y$
B	Bands	$x^2 = x$
A	Commutative semigroups	$xy = yx$
SL	Semilattices	$x^2 = x,\ xy = yx$
RB	Rectangular bands	$x = x^2,\ xyz = xz$
N	Null semigroups	$xy = zt$
A_n	Abelian groups satisfying $x^n = 1$	$xy = yx,\ x^n y = y$

Let X be a nonempty set and $\mathcal{C}$ a class of semigroups. Let $F_X(\mathcal{C}) \in \mathcal{C}$ and $\iota : X \to F_X(\mathcal{C})$ be a mapping. We say that the pair $(F_X(\mathcal{C}), \iota)$ is a *free-$\mathcal{C}$ semigroup on X* if for each $S \in \mathcal{C}$ and each mapping $\phi : X \to S$, there is a unique homomorphism $\Phi : F_X(\mathcal{C}) \to S$ such that the following diagram commutes:



It can be shown that the free-$\mathcal{C}$ semigroup on X is unique, up to isomorphism.

As we shall see, many important classes of semigroups (including the classes of all finite semigroups and of all regular semigroups) do not contain free semigroups. Importantly, however, varieties of semigroups do contain free semigroups. The relationships between the free semigroups in a variety of semigroups and the identities which define that variety are outlined in the following results.

Theorem 1.3.3. *Let* V *be a variety of semigroups and X a nonempty set. Then the free-*V *semigroup (often referred to as the* V*-free semigroup) $F_X(\mathrm{V})$ exists.*

Birkhoff proved that the lattice of varieties $\mathscr{L}_{\mathrm{v}}(\mathrm{V})$ is antiisomorphic to a certain lattice of fully invariant congruences. The following result describes this.

Theorem 1.3.4. *Let* V *be a variety of semigroups and* $F_X(\mathrm{V})$ *be the free* V-*semigroup on the countably infinite set* X. *Let* W *be a subvariety of* V. *The relation* ρ_{W} *defined as follows*

$$\rho_{\mathrm{W}} = \{(u,v) \in F_X \times F_X \mid u\phi = v\phi \text{ for all homomorphisms } \phi : F_X \to S \in \mathrm{W}\}$$

is a fully invariant congruence on F_X *and is the least congruence on* F_X *such that* $F_X/\rho_{\mathrm{W}} \in \mathrm{W}$. *Conversely, if* ρ *is a fully invariant congruence on* F_X*, then*

$$\mathrm{W}_\rho = \{S \in \mathrm{V} \mid u\phi = v\phi \text{ for all } (u,v) \in \rho \text{ and all homomorphisms } \phi : F_X \to S\}$$

is a subvariety of V. *The mappings*

$$\mathrm{V} \mapsto \rho_{\mathrm{V}} \text{ and } \rho \mapsto \mathrm{V}_\rho$$

are mutually inverse antiisomorphisms between $\mathscr{L}_{\mathrm{v}}(\mathrm{V})$ *and the lattice of fully invariant congruences on* F_X.

1.3.2 Varieties of Unary Semigroups

The classes of completely regular semigroups and inverse semigroups are important in that to each element we can identify a particular inverse. Recall that a semigroup S is an inverse semigroup if for each $a \in S$ there exists a unique $a^{-1} \in S$ with the property that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$. A semigroup S is said to be completely regular if for each $a \in S$ there exists an element $x \in S$ with the property that $axa = a$ and $ax = xa$.

A unary semigroup is an algebra $\langle S; \cdot, ^{-1}\rangle$ of type $(2,1)$. Since we can identify with each element a in an inverse semigroup or completely regular semigroup a particular inverse, such semigroups form, in a most natural way, unary semigroups. The class RUS of all regular unary semigroups is a subvariety of the class of all algebras of type $(2,1)$ and is defined by the identities:

$$x(yz) = (xy)z,\ xx^{-1}x = x.$$

The classes CR and I of all completely regular semigroups and all inverse semigroups respectively form subvarieties of RUS and are defined by the following identities:

$$\begin{aligned} \mathrm{CR} &= [(x^{-1})^{-1} = x,\; xx^{-1} = x^{-1}x] \\ \mathrm{I} &= [(x^{-1})^{-1} = x, xx^{-1}y^{-1}y = y^{-1}yxx^{-1}] \end{aligned}$$

We list below some other important varieties of regular unary semigroups.

Symbol	**Description**	**Identities**
G	Groups	$(x^{-1})^{-1} = x, xx^{-1} = x^{-1}x,$
		$xx^{-1} = yy^{-1}$
Ab	Abelian groups	$(x^{-1})^{-1} = x,$
		$xy = yx, xx^{-1} = yy^{-1}$
SG	Clifford semigroups	$(x^{-1})^{-1} = x, xx^{-1} = x^{-1}x,$
		$xx^{-1}y^{-1}y = y^{-1}yxx^{-1}$
CS	Completely simple semigroups	$(x^{-1})^{-1} = x, xx^{-1} = x^{-1}x,$
		$x = xyx(xyx)^{-1}$
B	Bands	$x = x^2$
NB	Normal bands	$x = x^2, wxyw = wyxw$

The classic results of universal algebra stated for semigroups in the previous section can be restated for varieties of unary semigroups. In particular, the notion of a *free unary semigroup* is required. Let X be a nonempty set and let $P = X \cup \{(,)^{-1}\}$. The set U_X is defined to be the least subset of Σ_P which satisfies:

(i) $X \subseteq U_X$;

(ii) $u, v \in U_X \Rightarrow uv \in U_X$; and

(iii) $u \in U_X \Rightarrow (u)^{-1} \in U_X$.

With this description of a free unary semigroup it is possible to establish the familiar results of universal algebra for varieties of unary semigroups. The reader is directed to the books [35, 36] for further details.

1.4 Existence Varieties of Regular Semigroups

It is not the case that a subsemigroup of a regular semigroup need be regular. Consequently, the class RS of all regular semigroups is not closed under $\mathbb{S}$ and hence is not a variety of semigroups. Hall [21] and (independently) Kad̆ourek and Szendrei [28] devised the notion of an existence variety (or e-variety) of regular semigroups. A class $\mathcal{C}$ of regular semigroups is an e-variety (of regular semigroups) if it is closed under the taking of homomorphic images, *regular* subsemigroups and direct products.

In symbols we write:

$$\mathbb{H}(\mathcal{C}) \subseteq \mathcal{C},\ \mathbb{S}_{\mathrm{e}}(\mathcal{C}) \subseteq \mathcal{C}^1 \text{ and } \mathbb{P}(\mathcal{C}) \subseteq \mathcal{C}.$$

In the following sections, we will give details of several important e-varieties and describe two rather different approaches to the study of e-varieties. For further details about e-varieties, the reader is referred to the surveys by Jones [27] and Trotter [43].

1.4.1 Examples of E-Varieties

Given a variety V of semigroups, the class $\mathrm{V}^{\mathrm{Reg}} = \mathrm{V} \cap \mathrm{RS}$ of all regular members of V is clearly an e-variety since $\mathrm{V}^{\mathrm{Reg}}$ is obviously closed under $\mathbb{H}$ and $\mathbb{P}$. That $\mathrm{V}^{\mathrm{Reg}}$ is closed under $\mathbb{S}_{\mathrm{e}}$ is quickly verified.

Many important classes of regular semigroups can be shown to form e-varieties. The classes of all completely regular semigroups and all inverse semigroups (and many of their subclasses) when considered as classes of semigroups (as opposed to classes of unary semigroups) form e-varieties.

For a class $\mathcal{C}$ of regular semigroups, the class of regular semigroups whose core is in $\mathcal{C}$ is denoted $\mathcal{C}^{ig}$. That is,

$$\mathcal{C}^{ig} = \{S \in \mathrm{RS} \,|\, C(S) \in \mathcal{C}\}.$$

[1] The "e" in $\mathbb{S}_{\mathrm{e}}$ refers to "existence" (of inverses). Note that some authors use $\mathbb{S}_{\mathrm{reg}}$ or $\mathbb{S}_{\mathrm{r}}$ to denote this operator.

We point out that the superscript ig reminds us that the members of $\mathcal{C}^{ig}$ are those regular semigroups whose maximal idempotent generated subsemigroups are in $\mathcal{C}$.

Lemma 1.4.1. [21, Lemma 4.3.1] *For any e-variety* V, V^{ig} *is also an e-variety.*

It can be shown (see, for example [20, Theorem 3]) that a regular semigroup S is E-solid if and only if $C(S) \in \mathrm{CR}$, that is, the core of S is completely regular. Since CR is an e-variety, we have that the class of E-solid regular semigroups is also an e-variety and is equal to CR^{ig}. We denote the e-variety of all E-solid regular semigroups by ES.

Let C_2 be the Rees matrix semigroup (see [25, Chapter 3])

$$C_2 = \mathscr{M}^0\left(\langle 1\rangle; \mathbf{2}, \mathbf{2}; \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right).$$

where $\langle 1\rangle$ is the one element group, and $\mathbf{2} = \{0, 1\}$

Hall proved that the e-variety generated by C_2 is the e-variety of all strict, combinatorial, regular semigroups SC and that SC is the least e-variety of regular semigroups which is not E-solid [22, Theorems 3.3 and 3.5].

Lemma 1.4.2. [23, Corollary 4.3] *An e-variety* V *of regular semigroups is* E*-solid if and only if* V *does not contain* C_2.

A semigroup S is said to be *locally* in a class $\mathcal{C}$ of semigroups if for all $e \in E(S)$, $eSe \in \mathcal{C}$. So, for a class $\mathcal{C}$ of regular semigroups, the class $\mathcal{C}^{loc}$ is defined to be the class of all regular semigroups S with the property that for every $e \in E(S)$, $eSe \in \mathcal{C}$. That is,

$$\mathcal{C}^{loc} = \{S \in \mathrm{RS} \mid (\forall e \in E(S))\, eSe \in \mathcal{C}\}.$$

Lemma 1.4.3. [21, Lemma 4.6.1] *For any e-variety* V, V^{loc} *is also an e-variety.*

We will denote the e-variety I^{loc} of all locally inverse semigroups by LI. We will begin to appreciate the importance of E-solid and locally inverse semigroups in the study of e-varieties in the following result.

Theorem 1.4.4. [26, Theorem 5] *Suppose S is a regular semigroup which is not E-solid nor locally inverse. Then $\langle S\rangle_{\text{ev}}$ contains a regular semigroup T such that $Sub(T)$ is not a lattice.*

Recall the well known result of Tarski for varieties that $\langle \mathcal{C}\rangle_{\text{v}} = \mathbb{HSP}(\mathcal{C})$ for any class $\mathcal{C}$. This result remains true for e-varieties with an obvious modification and one important restriction:

Theorem 1.4.5. [45, Lemma 4.8] *For any class $\mathcal{C}$ of E-solid or locally inverse regular semigroups, $\langle \mathcal{C}\rangle_{\text{ev}} = \mathbb{HS}_{\text{e}}\mathbb{P}(\mathcal{C})$.*

The following table lists some of the important classes of regular semigroups which form e-varieties.

Symbol	**Description**
RS	Regular semigroups
ES	E-solid regular semigroups
LI	Locally inverse regular semigroups
O	Orthodox regular semigroups
CR	Completely regular semigroups
I	Inverse semigroups
T	Trivial semigroups

1.4.2 Regular Unary Semigroups

As we have already seen, a *regular unary semigroup* is an algebra $\langle S; \cdot, ' \rangle$ of type $(2,1)$ where $\langle S; \cdot\rangle$ is a regular semigroup and for every $x \in S$, $xx'x = x$ and $x'xx' = x'$. Important e-varieties such as the classes of all groups, inverse semigroups and completely regular semigroups may be studied within the framework of varieties of regular unary semigroups. That these particular regular semigroups are very naturally considered as regular unary semigroups follows, of course, because in each case there is an inverse with a well defined unique property. Given a regular semigroup S, each element $a \in S$ may have numerous inverses and so it is not possible to identify each regular semigroup with a particular regular unary semigroup. However, we do have the following:

Each regular semigroup may be associated with at least one regular unary semigroup in the following way: By the axiom of choice, there exists for each regular semigroup S a unary operation $' : S \to S$ with the property that for all $x \in S$, $xx'x = x$ and $x'xx' = x'$. Such an operation is called an *inverse unary operation.*

Given an e-variety V, Hall [21] defined a class V′ of regular unary semigroups as follows:

$$\mathrm{V}' = \{\langle S; \cdot, ' \rangle \in \mathrm{RUS} \,:\, \langle S; \cdot \rangle \in \mathrm{V}\}.$$

Theorem 1.4.6. [21, Theorem 2.1] *For each e-variety* V *of regular semigroups, the class* V′ *is a variety of regular unary semigroups.*

Given a set Σ of regular unary semigroup identities, we say that a regular semigroup $S = \langle S; \cdot \rangle$ satisfies Σ (and write $\langle S; \cdot \rangle \vDash \Sigma$) if the regular unary semigroup $\langle S; \cdot, ' \rangle$ satisfies Σ for every choice of inverse unary operation $'$ on $\langle S; \cdot \rangle$. Given a set Σ of regular unary semigroup identities, define:

$$\mathcal{E}(\Sigma) = \{\langle S; \cdot \rangle \in \mathrm{RS} \,:\, \langle S; \cdot \rangle \text{ satisfies } \Sigma\}.$$

The class $\mathcal{E}(\Sigma)$ is called an *equational class* and Σ is said to *determine* V. If Σ is a basis of Id(V′), where Id(V′) is the set of all regular unary semigroup identities satisfied by V′, we say that Σ *strongly determines* V. Hall proceeded to prove the following Birkhoff-type theorem.

Theorem 1.4.7. [21, Theorem 2.2] *Every e-variety* V *of regular semigroups is an equational class.*

The converse to the previous result is not true. Hall gave an example in [21, Example 2.4] of an equational class which contains all right zero semigroups and all left zero semigroups but fails to contain all direct products of such semigroups and thus is not an e-variety. Hall did however obtain the following analogue of Birkhoff's Theorem.

Theorem 1.4.8. **[23, Theorem 2.3]** *For an e-variety* V*, a set* Σ *of regular unary semigroup identities is a basis for* $\mathrm{Id}(\mathrm{V}')$ *if and only if*

$$\mathrm{V} = \{\langle S;\cdot\rangle \in \mathrm{RS} : \langle S;\cdot\rangle \vDash \Sigma \textit{ for some inverse unary operation } ' \textit{ on } S\}$$

and

$$\mathrm{V} = \{\langle S;\cdot\rangle \in \mathrm{RS} : \langle S;\cdot\rangle \vDash \Sigma \textit{ for every inverse unary operation } ' \textit{ on } S\}.$$

Let X be a nonempty set and let X' be a disjoint copy of X (see below for further details on this construction). Denote by $F_{X\cup X'}$ the free semigroup on $X \cup X'$. A pair (u, v), where $u, v \in F_{X\cup X'}$ is called a *biidentity* (although we will see in the next section that this definition can be refined). Write $u(x_1, x_1', \ldots, x_n, x_n')$ to indicate that the letters of u consist of symbols drawn from $\{x_1, x_1', \ldots, x_n, x_n'\} \subseteq X \cup X'$.

A regular semigroup S is said to satisfy the biidentity (u, v) if for all $s_1, s_1', \ldots, s_n, s_n' \in S$ such that for each $i \in \{1, \ldots, n\}$, $s_i' \in V(s_i)$ we have

$$u(s_1, s_1', \ldots, s_n, s_n') = v(s_1, s_1', \ldots, s_n, s_n').$$

It can be shown that the class of all regular semigroups which satisfy a given biidentity is an e-variety. While the converse is in general not true, for the special case of orthodox semigroups, Kad̆ourek and Szendrei were able to obtain the following result:

Theorem 1.4.9. **[28, Theorem 1.10]** *A class of orthodox semigroups is an e-variety if and only if it is defined by a set of biidentities.*

1.4.3 Bifree Objects and Biidentities

Let X be a nonempty set and let X' be a disjoint copy of X. That is, $X' = \{x' \mid x \in X\}$ where $X \cap X' = \varnothing$. Let $x \mapsto x'$ be a bijection of X onto X'. Let S be a regular semigroup and $\phi : X \cup X' \to S$ a mapping. We say that ϕ is a matched mapping if for all $x \in X$, $x'\phi \in V(x\phi)$.

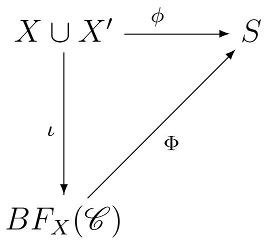
A *bifree object* on a nonempty set X in a class $\mathscr{C}$ of regular semigroups consists of a pair $(BF_X(\mathscr{C}), \iota)$ with the following properties:

(i) $BF_X(\mathscr{C}) \in \mathscr{C}$;

(ii) $\iota : X \cup X' \to BF_X(\mathscr{C})$ is a matched mapping; and

(iii) for each $S \in \mathscr{C}$ and for each matched mapping $\phi : X \cup X' \to S$, there is a unique homomorphism $\Phi : BF_X(\mathscr{C}) \to S$ such that $\iota\Phi = \phi$.

It is sometimes helpful to refer to a commutative diagram:



Yeh established the existence of bifree objects in certain classes of E-solid and locally inverse regular semigroups.

Theorem 1.4.10. [45, Theorem 4.12] *In any class $\mathscr{C}$ of E-solid or locally inverse semigroups closed under $\mathbb{S}_{\mathrm{e}}$ and $\mathbb{P}$ there is a bifree object on any non-empty set X.*

Importantly, in e-varieties which consist of locally inverse and E-solid regular semigroups, bifree objects on countably infinite sets behave like free objects in varieties.

Theorem 1.4.11. [30, Lemma 5.3] *Let* $\mathrm{V} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$*, and let X be a countably infinite set. Then* $\mathrm{V} = \langle BF_X(\mathrm{V}) \rangle_{\mathrm{ev}}$.

While various authors have established and utilised the existence of bifree objects, construction of concrete examples of these objects has proved somewhat more difficult. Auinger [5, 7] and Kadourek and Szendrei [29] have described identities for e-varieties of locally inverse and E-solid regular semigroups respectively. The methods used are different in the two cases. More recently, Churchill and Trotter [13] have devised a method of constructing bifree objects and biidentities for e-varieties of locally inverse and E-solid regular semigroups which we decribe below.

A *binary semigroup* is an algebra of type $(2,2)$ in which one of the binary operations is associative. Let $\overline{X} = X \cup X'$ where X is a countably infinite set and X' is a disjoint copy of X (as described above). Let $\langle F_{(2,2)}(\overline{X}); s\rangle$ be the free binary semigroup on $\overline{X}$ where s is a binary operation.

For any regular semigroup S and any $a, b \in S$ let $S(a,b)$ denote the set $bV(ab)a$. This set is often called the *sandwich set* of a and b. A *sandwich operation* s on a regular semigroup S is a mapping $s : S \times S \to S$ with the property that

$$s(a,b) \in S(a,b).$$

By the axiom of choice, every regular semigroup $\langle S; \cdot\rangle$ admits a sandwich operation and $\langle S; \cdot, s\rangle$ is a binary semigroup.

We create a set P by adjoining to the set $\overline{X}$ three distinct elements not in $\overline{X}$: "$s($", "," and ")". Let F be the least subset of Σ_P (the free semigroup on P) which satisfies

(i) $P \subseteq F$;

(ii) $u, v \in F \Rightarrow uv \in F$; and

(iii) $u, v \in F \Rightarrow s(u,v) \in F$.

For each $u \in F$, denote by $|u|$ the length of u.

Denote by $FG(X)$ the free group on the countably infinite set X and define a sandwich operation $s(a,b)$ on $FG(X)$ by $s(a,b) = 1$ for all $a, b \in FG(X)$. There is a binary semigroup homomorphism $F_{(2,2)}(\overline{X}) \to FG(X)$ which extends the natural injection $\overline{X} \to FG(X)$. Denote by $\overline{u}$ the image of $u \in F_{(2,2)}(\overline{X})$ under this homomorphism. Consequently, $\overline{s(u,v)} = 1$ for every $u, v \in F_{(2,2)}(\overline{X})$.

Let $R(X) = \{u \in F_{(2,2)}(\overline{X}) \mid \overline{u} = 1\}$ and let $W(X) = \bigcup_{i \geq 0} W_{2i+1}$ where

$$\begin{aligned} W_0 &= \overline{X} \\ W_1 &= \langle W_0 \rangle \\ &\vdots \\ W_{2i} &= \{s(a,b) \mid a, b \in W_{2i-1} \cap R(X)\} \cup W_{2i-1} \\ W_{2i+1} &= \langle W_{2i} \rangle. \end{aligned}$$

Churchill and Trotter [13] describe two partial binary semigroup congruences on $W(X)$, denoted ρ_{ES} and ρ_{LI}, with the property that $W(X)/\rho_{\mathrm{ES}}$ and $W(X)/\rho_{\mathrm{LI}}$ are isomorphic to the bifree objects on X in ES and LI respectively.

Let $S \in$ V where V = ES or V = LI. Churchill and Trotter have shown [13, Theorem 3.10] that each matched mapping $\phi : \overline{X} \to S$ extends uniquely to a semigroup homomorphism $\theta : W(X) \to S$. The homomorphism θ is called the V-extension of S.

We are now able to redefine the term *biidentity*: A biidentity is a pair (u, v) where $u, v \in W(X)$. As usual, we will write $u = v$ in place of (u, v). Let V denote the e-variety ES or LI and let $S \in$ V. The regular semigroup S is said to V-satisfy the biidentity $u = v$ if for any matched mapping $\phi : \overline{X} \to S$ and its V-extension $\theta : W(X) \to S$ we have $u\theta = v\theta$.

A biidentity is said to be V-satisfied by a class $\mathscr{C} \subseteq$ V if $u = v$ is satisfied by each member of $\mathscr{C}$. For any set Σ of biidentities, let $[\Sigma]_{\mathrm{V}}$ denote the class of regular semigroups in V that V-satisfy all biidentities in Σ.

Let $\mathscr{C} \subseteq$ V. Let

$$\rho_{\mathrm{V}}(\mathscr{C}) = \{(u, v) \in W(X) \times W(X) \mid u = v \text{ is V-satisfied by } \mathscr{C}\}.$$

In [13, Theorem 3.10] it is shown how to construct an object $(W(X)/\rho_{\mathrm{V}}, \iota_{\mathrm{V}})$ which is bifree in V on X where V = ES or V = LI. It can be shown that $\rho_{\mathrm{V}}(\mathscr{C})$ is a congruence on $W(X)$ and furthermore, that $\rho_{\mathrm{V}}(\mathrm{V}) = \rho_{\mathrm{V}}$.

The following result establishes the existence of bifree objects in e-varieties contained in ES and in LI.

Theorem 1.4.12. [13, Theorem 3.11(i)] *Let* V *denote the e-variety of all locally inverse or all* E*-solid regular semigroups and let* $\mathscr{C} \subseteq \mathrm{V}$ *be a class closed under* $\mathbb{S}_{\mathrm{e}}$ *and under* $\mathbb{P}$*. Then there exists a bifree object on any nonempty set* X*, and it is isomorphic to* $W(X)/\rho_{\mathrm{V}}(\mathscr{C})$*.*

For any subset ρ of $W(X) \times W(X)$ let

$$[\rho]_{\mathrm{V}} = [u = v \mid (u, v) \in \rho]_{\mathrm{V}}.$$

The next two results are analogues for e-varieties of Birkhoff's results for varieties. In particular, we have that e-varieties of locally inverse and E-solid regular semigroups are equational classes and that the lattice $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$, $\mathrm{V} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$, is antiisomporphic to a certain lattice of fully invariant congruences.

Theorem 1.4.13. [13, Theorem 3.11(ii)] *Let* $\mathscr{C} \subseteq \mathrm{V} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$*. Then* $\mathscr{C}$ *is an e-variety if and only if there exists a set* Σ *of biidentities such that* $\mathscr{C} = [\Sigma]_{\mathrm{V}}$*. In particular, if* $\mathscr{C}$ *is an e-variety then* $\mathscr{C} = [\rho_{\mathrm{V}}(\mathscr{C})]_{\mathrm{V}}$*.*

Theorem 1.4.14. [13, Theorem 3.11(iii)] *Let* V *denote the e-variety of all locally inverse or all* E*-solid regular semigroups. Define mappings between the lattice* $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$ *and the lattice of all* V*-biinvariant congruences of* $W(X)$ *by*

$$\mathrm{U} \mapsto \rho_{\mathrm{V}}(\mathrm{U}) \text{ and } \rho \mapsto [\rho]_{\mathrm{V}}.$$

Then these mappings are mutually inverse antiisomporphisms.

A congruence ρ *on* $W(X)$ *is* V*-biinvariant if and only if* $\rho_{\mathrm{V}} \subseteq \rho$ *and* ρ/ρ_{V} *is a fully invariant congruence on* $W(X)/\rho_{\mathrm{V}}$*. Therefore we may define mutually inverse antiisomorphisms between* $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$ *and* $FI[W(X)/\rho_{\mathrm{V}}]$*.*

1.5 Pseudovarieties of Finite Semigroups

A class of algebras (all of the same type) closed under the taking of homomorphic images, subalgebras and *finite* direct products is called a *pseudovariety.* While this definition allows for the inclusion of infinite algebras in a pseudovariety, it is more usual to restrict the definition to finite algebras only. Henceforth, we will assume that a pseudovariety is a class of finite algebras closed under $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}_{\mathrm{f}}$, where for a class $\mathcal{C}$ of algebras, $\mathbb{P}_{\mathrm{f}}(\mathcal{C})$ is the class of all finite direct products of members of $\mathcal{C}$.

The pseudovariety generated by a class $\mathcal{C}$ of finite algebras of a fixed type τ is denoted by $\langle \mathcal{C} \rangle_{\mathrm{pv}}$ and is equal to the intersection of all pseudovarieties of algebras of type τ which contain $\mathcal{C}$. The following Tarski-type theorem is well known.

Theorem 1.5.1. $\langle \mathcal{C} \rangle_{\mathrm{pv}} = \mathbb{HSP}_{\mathrm{f}}(\mathcal{C})$.

Given a variety V of algebras, the class $\mathbf{V} = \mathrm{V}^{\mathrm{Fin}}$ of all finite members of V is easily seen to be a pseudovariety. Pseudovarieties of this form are said to be *equational pseudovarieties* since they are defined by the identities of the variety of whose finite members they consist. However, not all pseudovarieties are of this form. It is easily shown that the class of all finite groups $\mathbf{G}$ is a pseudovariety of semigroups. However, there does not exist a variety V of semigroups such that $\mathbf{G} = \mathrm{V}^{\mathrm{Fin}}$. The smallest equational pseudovariety which contains $\mathbf{G}$ is $\mathbf{S}$, the pseudovariety of all finite semigroups. The class $\mathbf{N}$ of all finite nilpotent semigroups (recall that a semigroup S is nilpotent if it satisfies the identity

$$x_1 \dots x_n y = y x_1 \dots x_n = x_1 \dots x_n$$

for some $n \in \mathbb{N}$) also fails to be equational.

Many important classes of finite semigroups form pseudovarieties of finite semigroups. The list below details several important pseudovarieties of finite semigroups.

Symbol	**Description**
S	Finite semigroups
B	Finite bands
G	Finite groups
CR	Finite completely regular semigroups

Later we will discuss the concept of a pseudoidentity and describe several well known pseudovarieties in terms of pseudoidentities.

The class of all finite semigroups whose idempotents commute is easily shown to be a pseudovariety. Famously, Ash [4] showed that this pseudovariety is precisely the pseudovariety generated by the class of all finite inverse semigroups (the class of all finite inverse semigroups does not form a pseudovariety).

In the following sections we will give details of how some of the examples listed above may be described using various analogues to the notion of identity.

1.5.1 Sequences of Identities

We have seen that the pseudovariety $\mathbf{G}$ of all finite groups is not an equational pseudovariety. However, it is well known that $\mathbf{G}$ consists of all those finite semigroups which satisfy the identity $x^n y = yx^n = y$ for some $n \in \mathbb{N}$. Therefore one can consider $\mathbf{G}$ to be the union of a collection of equational pseudovarieties. Using the shorthand $u = 1$ to mean $uy = yu = y$ we can write

$$\mathbf{G} = \bigcup_{n \geq 1} [x^n = 1]^{\mathrm{Fin}}.$$

Note that the equational pseudovarieties of the form $[x^n = 1]^{\mathrm{Fin}}$, $n \in \mathbb{N}$ form a directed family. It is possible to show that every pseudovariety is the union of a directed family of equational pseudovarieties.

A class $\mathcal{C}$ of algebras (all of the same type) is said to be *ultimately defined* by a sequence $\{u_n = v_n\}_{n=0}^{\infty}$ of identities if $\mathcal{C}$ consists only of those algebras that satisfy all but finitely many of the identities from the sequence.

The following theorem was proved by Eilenberg and Schützenberger [16].

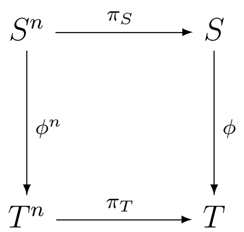
Theorem 1.5.2. [16, Theorem 1] *A class $\mathcal{C}$ of finite semigroups is a pseudovariety if and only if it is ultimately defined by a sequence of identities. Alternatively, $\mathcal{C}$ is a pseudovariety if and only if there exists a sequence $\{u_n = v_n\}_{n=0}^{\infty}$ of identities such that*

$$\mathcal{C} = \bigcup_{k \geq 1} [u_n = v_n \,:\, n \geq k]^{\mathrm{Fin}}.$$

1.5.2 Reiterman's Theorem

An alternative approach to the problem of describing pseudovarieties using identities has been given by Reiterman [39]. This approach has been widely utilised by those studying pseudovarieties and related structures (see, in particular, [2]). We outline Reiterman's theorem below.

Let $\mathcal{C}$ be a nonempty class of semigroups. An *n-ary implicit operation* in $\mathcal{C}$ is a family $\pi = \{\pi_S : S \in \mathcal{C}\}$ where $\pi_S : S^n \to S$ and for all homomorphisms $\phi : S \to T$ the following diagram commutes:



The set of all n-ary implicit operations on $\mathcal{C}$ is denoted $\overline{\Omega}_n\mathcal{C}$. A *pseudoidentity for* $\mathcal{C}$ is a pair (π, ρ) of members of $\overline{\Omega}_n\mathcal{C}$, for some n. An algebra $S \in \mathcal{C}$ is said to satisfy the pseudoidentity (π, ρ) if $\pi_S = \rho_S$. It is usual to write the pseudoidentity as an equality, $\pi = \rho$.

Theorem 1.5.3. [39, Theorem 3.1] *A class $\mathcal{C}$ of finite algebras is a pseudovariety if and only if it is defined by a set $[\![\Sigma]\!]$ of pseudoidentities.*

For the class $\mathbf{S}$ of finite semigroups, the operation

$$x \mapsto x^\omega$$

which maps x to the unique idempotent in $\langle x \rangle$ is a unary implicit operation. Consider the following examples of pseudovarieties (of finite semigroups) defined in terms of pseudoidentities of semigroups which utilise this important unary operation:

- The pseudovariety of all finite groups: $\mathbf{G} = [\![x^\omega y = yx^\omega = y]\!]$;
- The pseudovariety of all finite $\mathscr{L}$-trivial semigroups: $\mathbf{L} = [\![y(xy)^\omega = (xy)^\omega]\!]$;

- The pseudovariety of all finite $\mathscr{R}$-trivial semigroups: $\mathbf{R} = [\![(xy)^\omega x = (xy)^\omega]\!]$;
- The pseudovariety of all finite $\mathscr{J}$-trivial semigroups: $\mathbf{J} = [\![(xy)^\omega = (yx)^\omega, x^\omega x = x^\omega]\!]$; and
- The pseudovariety of all finite nilpotent semigroups: $\mathbf{N} = [\![x^\omega y = x^\omega = yx^\omega]\!]$.

The proof of Reiterman's theorem [39, Theorem 3.1] relies on topological arguments. Higgins [24] has provided an algebraic proof of the same result.

1.5.3 Generalised Varieties

Eilenberg and Schützenberger [16] showed that pseudovarieties are ultimately defined by sequences of identities, or equivalently that pseudovarieties are unions of directed families of equational pseudovarieties.

Ash [3] extended this idea by considering certain classes which are the unions of directed families of varieties. He was able to prove that the finite members of such classes form pseudovarieties and conversely, that every pseudovariety consists precisely of the finite members of such a class. Such classes are called *generalised varieties* and we recall Ash's principal results concerning them below.

Ash introduced the notion of a generalised variety in [3]. A *generalised variety* is a class $\mathcal{C}$ of algebras of the same type closed under the taking of homomorphic images, subalgebras, finite direct products and arbitrary powers of members of $\mathcal{C}$. In symbols we write:

$$\mathbb{H}(\mathcal{C}) \subseteq \mathcal{C},\ \mathbb{S}(\mathcal{C}) \subseteq \mathcal{C},\ \mathbb{P}_{\mathrm{f}}(\mathcal{C}) \subseteq \mathcal{C} \text{ and } \mathbb{P}_{\mathrm{ow}}(\mathcal{C}) \subseteq \mathcal{C}.$$

The generalised variety generated by a class of algebras $\mathcal{C}$ all of the same type is denoted $\langle \mathcal{C} \rangle_{\mathrm{gv}}$ and is equal to the intersection of all generalised varieties which contain $\mathcal{C}$. The following two results are due to Ash [3, Theorem 1].

Theorem 1.5.4. *For a nonempty class $\mathcal{C}$ of algebras, $\langle \mathcal{C} \rangle_{\mathrm{gv}} = \mathbb{HSP}_{\mathrm{f}}\mathbb{P}_{\mathrm{ow}}(\mathcal{C})$.*

As alluded to earlier, generalised varieties are simply unions of directed families of varieties.

Theorem 1.5.5. *A class of algebras* V *is a generalised variety if and only if* V *is the union of some directed family of varieties. Equivalently,* V *is a generalised variety if and only if there exists a filter F over E such that for all algebras A,* $A \in \mathrm{W} \Leftrightarrow \mathrm{Id}(A) \in F$*, where E is the set of all identities of the same type as the algebras in* V*.*

The following result linking generalised varieties and pseudovarieties is of great importance and we present an analogue of it in Chapter Two for e-pseudovarieties.

Theorem 1.5.6. [3, Theorem 2] *A class $\mathcal{C}$ of finite algebras is a pseudovariety if and only if it consists of the finite members of some generalised variety.*

By connecting strongly pseudovarieties and generalised varieties, Theorem 1.5.6 has made possible a great deal of research into the structure of lattices of pseudovarieties (see for example [33] and [34]).

Finally, we state a result of Ash which generalises Theorem 1.5.2.

Theorem 1.5.7. [3, Theorem 3] *Let C be a countable class of algebras. The following conditions on $K_0 \subseteq C$ are equivalent:*

(i) There is a generalised variety K with $K_0 = K \cap C$;

(ii) There is a chain $V_1 \subseteq V_2 \subseteq \cdots$ of varieties with $K_0 = (\cup V_n) \cap C$; and

(iii) There is a sequence $e_1, e_2, \ldots$ of identities such that for $A \in C$,

$$A \in K_0 \Leftrightarrow A \models e_n$$

for all but finitely many n.

2

E-Pseudovarieties of Finite Regular Semigroups

In this chapter we introduce e-pseudovarieties and develop methods for their study. E-pseudovarieties share the same relationship to e-varieties that pseudovarieties do to varieties. It is therefore unsurprising that e-pseudovarieties can be studied (at least initially) by exploring the connections between e-varieties and e-pseudovarieties in the same way that one would study the connections between varieties and pseudovarieties.

As we have seen in the preliminary chapter, there are several ways in which pseudovarieties can be characterised. The original researchers considered classes of finite algebras ultimately defined by sequences of identities. Reiterman introduced the notion of a pseudoidentity, and showed that all pseudovarieties are defined by sets of pseudoidentities. In the most recent literature, it is evident that this approach (via pseudoidentities) has been the most widely adopted.

Mangold [31] showed that e-pseudovarieties are ultimately defined by sequences of $\mathcal{RUS}$ identities. However, we are unaware of any regular analogue of the Reiterman results though it would seem plausible that such results could be developed.

The third method of describing pseudovarieties is via the notion of generalised varieties. It is this approach that we have followed in this thesis. Simple modifications of Ash's [3] main results allow us now to study e-pseudovarieties in a most natural way. Several authors (for example, Agliano and Nation [1], Pastijn [33] and Pastijn and Trotter [34]) have used the relationships between varieties, generalised varieties and pseudovarieties to obtain important results about pseudovarieties and the lattice of pseudovarieties. Having established regular analogues of Ash's fundamental results, our aim is to develop the theory of e-pseudovarieties in a manner akin to that used by the aforementioned authors in their studies of pseudovarieties.

We begin this chapter by reviewing the work presented by Mangold in her doctoral thesis. The notion of a generalised existence variety (or generalised e-variety) of regular semigroups is then introduced and some of the powerful techniques already in existence for the study of pseudovarieties are able to be brought to bear on the study of e-pseudovarieties.

2.1 E-Pseudovarieties

2.1.1 Preliminaries

The notion of an *e-pseudovariety* was introduced by M. Mangold in her doctoral thesis [31] as a finitary analogue of an e-variety. A class $\mathcal{C}$ of finite regular semigroups is an e-pseudovariety if it is closed under the taking of homomorphic images, regular subsemigroups and finite direct products. In symbols, we write:

$$\mathbb{H}(\mathcal{C}) \subseteq \mathcal{C},\ \mathbb{S}_{\mathrm{e}}(\mathcal{C}) \subseteq \mathcal{C},\ \mathbb{P}_{\mathrm{f}}(\mathcal{C}) \subseteq \mathcal{C}.$$

The collection of all e-pseudovarieties contained in $\mathbf{RS}$, the class of all finite regular semigroups, is a lattice. And since the intersection of any collection of e-pseudovarieties is again an e-pseudovariety and the class of all e-pseudovarieties has a maximum element, $\mathbf{RS}$, we have (by Lemma 1.2.1) that the collection of all e-pseudovarieties contained in $\mathbf{RS}$ is a complete lattice. For any e-pseudovariety $\mathbf{V}$ we denote by $\mathscr{L}_{\mathrm{epv}}(\mathbf{V})$ the complete lattice of e-pseudovarieties contained in $\mathbf{V}$.

An e-pseudovariety $\mathbf{V}$ is said to be generated by the nonempty class $\mathcal{C}$ if $\mathbf{V}$ is the smallest e-pseudovariety which contains $\mathcal{C}$. We write $\mathbf{V} = \langle \mathcal{C} \rangle_{\mathrm{epv}}$ to indicate this. Equivalently, $\langle \mathcal{C} \rangle_{\text{epv}}$ is the intersection of all those e-pseudovarieties which contain $\mathcal{C}$:

$$\langle \mathcal{C} \rangle_{\text{epv}} = \cap\{\mathbf{U} \in \mathscr{L}_{\text{epv}}(\mathbf{RS}) \,|\, \mathcal{C} \subseteq \mathbf{U}\}.$$

If $\mathcal{C}$ is finite, we say that $\mathbf{V} = \langle \mathcal{C} \rangle_{\text{epv}}$ is *finitely generated.* In the case that $\mathbf{V} = \langle \mathcal{C} \rangle_{\text{epv}}$ where $\mathcal{C} = \{S\}$, then we may write $\mathbf{V} = \langle S \rangle_{\text{epv}}$.

As with e-varieties (see Theorem 1.4.5), it is not true that $\langle \mathcal{C} \rangle_{\text{epv}} = \mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}_{\text{f}}(\mathcal{C})$ for all nonempty classes $\mathcal{C}$ of finite regular semigroups. However, Mangold established the following:

Theorem 2.1.1. [31, Proposition 4.1.1] *Let $\mathcal{C}$ be a class of finite regular semigroups. Then*

$$\langle \mathcal{C} \rangle_{\text{epv}} = \bigcup_{n=1}^{\infty} (\mathbb{H}\mathbb{S}_{\text{e}})^n \mathbb{P}_{\text{f}}(\mathcal{C}).$$

Denote by **ES** and **LI** the classes of finite E-solid and finite locally inverse regular semigroups respectively. The following result is familiar from the e-variety case.

Theorem 2.1.2. [31, Proposition 4.1.1] *If $\mathcal{C}$ is a class of finite regular semigroups contained in* **ES** *or* **LI** *then $\langle \mathcal{C} \rangle_{\text{epv}} = \mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}_{\text{f}}(\mathcal{C})$.*

Note again the importance of the classes of (finite) E-solid and locally inverse regular semigroups. These classes form a (relatively) impenetrable barrier, although several authors have begun to develop tools that make the study of e-varieties (and by extension, e-pseudovarieties) above ES and LI a possibility (see [13] for an overview of this research).

2.1.2 Examples of E-Pseudovarieties

It is easily seen that the finite members of an e-variety of regular semigroups form an e-pseudovariety, as do the regular members of a pseudovariety. Therefore examples of e-pseudovarieties may be quickly obtained. The following theorem restates this and establishes the notation that will be used throughout the remainder of this thesis.

Theorem 2.1.3. *Let* U *be an e-variety of regular semigroups and let* **V** *be a pseudovariety of finite semigroups. Then* $\mathrm{U}^{\mathrm{Fin}}$ *and* $\mathbf{V}^{\mathrm{Reg}}$ *are e-pseudovarieties of finite regular semigroups. Note that* $\mathbf{RS} = \mathrm{RS}^{\mathrm{Fin}} = \mathbf{S}^{\mathrm{Reg}}$.

However, it is not the case that all e-pseudovarieties consist of the finite members of an e-variety. Similarly, not all e-pseudovarieties consist of the regular members of a pseudovariety. Mangold gave an example of each [30, Examples 4.2.1 and 4.2.4].

In the table below, we list some of the most important e-pseudovarieties, several of which will feature prominently in the work to follow.

Symbol	Description
RS	Finite regular semigroups
T	Finite trivial semigroups
B	Finite bands
CR	Finite completely regular semigroups
G	Finite groups
O	Finite orthodox regular semigroups
I	Finite inverse semigroups
ES	Finite E-solid semigroups
LI	Finite locally inverse semigroups
LO	Finite locally orthodox regular semigroups
LES	Finite locally E-solid regular semigroups
LCR	Finite locally completely regular semigroups

The diagram Figure 2.1 shows where, in the lattice of e-pseudovarieties of finite regular semigroups, many of the e-pseudovarieties under consideration appear.

The following result concerning the lattice of e-pseudovarieties is probably well known. It will be needed later in this chapter.

Theorem 2.1.4. *The join of a family* $\{\mathbf{V}_\alpha : \alpha \in A\}$ *of e-pseudovarieties all contained in* **ES** *or* **LI** *consists of the homomorphic images of regular subsemigroups*

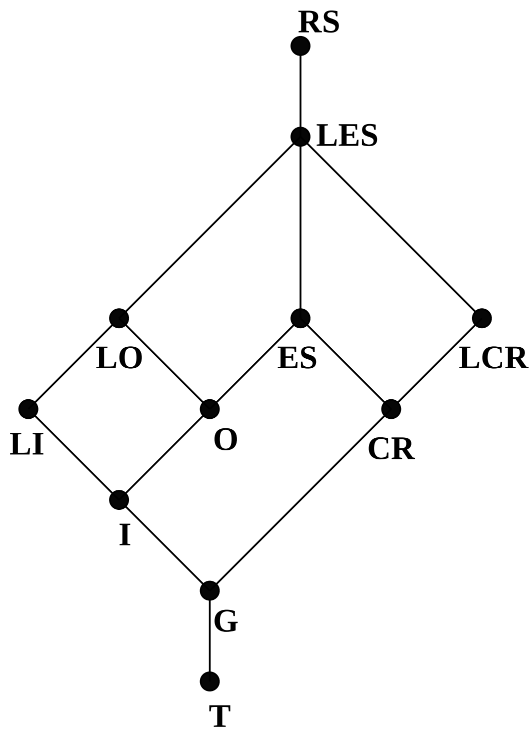


Figure 2.1: Partially ordered subset of $\mathscr{L}_{\text{epv}}(\mathbf{RS})$

of finite direct products of finite regular semigroups belonging to the union of the family $\{\mathbf{V}_\alpha : \alpha \in A\}$.

Proof. It is clear that the finite regular semigroups described in the statement of the theorem form an e-pseudovariety which contains all of the $\{\mathbf{V}_\alpha\}$. It is also clear that the join of the $\{\mathbf{V}_\alpha\}$ in the lattice of e-pseudovarieties must include all of these finite regular semigroups. □

2.2 Identities for E-Pseudovarieties

In this section we summarise some of the results obtained by Mangold. In the final chapter of her thesis, Mangold took the results of Hall [21] concerning identities for e-varieties and the results obtained by Eilenberg and Schützenberger [16] for describing pseudovarieties, combining them to prove that e-pseudovarieties are equational classes (in an appropriate sense).

Recall that with each e-variety V we may associate a variety V′ of regular unary semigroups. To any e-pseudovariety $\mathbf{V}$ of finite regular semigroups (not necessarily E-solid or locally inverse) we may associate a pseudovariety $\mathbf{V}'$ of finite regular unary semigroups where

$$\mathbf{V}' = \{\langle S; \cdot, ' \rangle \in \mathbf{RUS} \,:\, \langle S; \cdot \rangle \in \mathbf{V}\}$$

and **RUS** is the pseudovariety of all regular unary semigroups (see [31, Lemma 4.1.23]).

Given a sequence $\{u_n = v_n\}_{n=0}^{\infty}$ of regular unary semigroup identities we say that a class $\mathcal{C}$ of e-pseudovarieties is *ultimately defined* by this sequence of identities if $\mathcal{C}$ consists precisely of those finite regular semigroups which satisify all but finitely many of the identities of the sequence.

Theorem 2.2.1. [31, Theorem 4.3.1] *Every e-pseudovariety is ultimately defined by a sequence of* RUS *identities.*

In the alternative notation we have:

Lemma 2.2.2. *Let* $\mathbf{V}$ *be an e-pseudovariety. Then there exists a sequence* $\{u_n = v_n\}_{n=0}^{\infty}$ *of regular unary semigroup identities such that*

$$\mathbf{V} = \bigcup_{k \geq 1} [u_n = v_n \,:\, n \geq k]^{\mathrm{Fin}}.$$

While every e-pseudovariety is ultimately defined by a sequence of RUS identities, there are certain e-pseudovarieties which are defined rather more simply by a *set* of RUS identities. In particular, if S is a finite E-solid or locally inverse regular semigroup then the e-pseudovariety $\langle S \rangle_{\mathrm{epv}}$ generated by S is defined by a set of RUS identities [31, Corollary 4.3.5].

However, Mangold gave an example to show that:

Theorem 2.2.3. [31, Proposition 4.3.2] *Not every e-pseudovariety is defined by a set of* RUS *identities.*

The following result considers biidentities to be pairs (u, v) of members of $F_{X \cup X'}$ where X is a nonempty set and X' is a disjoint copy of X.

Theorem 2.2.4. [31, Proposition 4.3.3] *The class of all finite regular semigroups ultimately defined by a sequence of biiidentities is an e-pseudovariety.*

However, not every e-pseudovariety is ultimately defined by a sequence of biidentities, as the comments following [31, Proposition 4.3.3] argue.

The following results which end this section demonstrate that all $\mathbf{V} \in \mathscr{L}_{\mathrm{epv}}(\mathbf{ES}) \cup \mathscr{L}_{\mathrm{epv}}(\mathbf{LI})$ are defined by "equations". However, the nature of the equations will vary, depending on whether $\mathbf{V}$ is locally inverse or E-solid.

Theorem 2.2.5. [31, Theorem 4.4.3] *A class* $\mathcal{V} \subseteq \mathbf{ES}$ *is an e-pseudovariety if and only if it is ultimately defined by a sequence of semigroup identities.*

Let X be a nonempty set and let $\{(u_n, v_n)\}_{n=1}^{\infty}$ be a sequence of elements drawn from $BF_X(\mathrm{LI}) \times BF_X(\mathrm{LI})$. Following Mangold [31], we denote by $\mathrm{K}_{\mathrm{LI}}(\{(u_n, v_n)\}_{n=1}^{\infty})$ the class

$$\{T \in \mathbf{LI} \,|\, (\exists N \in \mathbb{N})(\forall n > N)(\forall \text{ homomorphisms } \phi : BF_X(\mathrm{LI}) \to T)\, u_n\phi = v_n\phi\}.$$

It can be shown that the class $\mathrm{K}_{\mathrm{LI}}(\{(u_n, v_n)\}_{n=1}^{\infty})$ is an e-pseudovariety contained in $\mathbf{LI}$. In fact, Mangold was able to prove that:

Theorem 2.2.6. [31, Theorem 4.4.4] $\mathcal{V} \subseteq \mathbf{LI}$ *is an e-pseudovariety if and only if* $\mathcal{V} = \mathrm{K}_{\mathrm{LI}}(\{(u_n, v_n)\}_{n=1}^{\infty})$ *where* $(u_n, v_n) \in BF_X(\mathrm{LI}) \times BF_X(\mathrm{LI})$ *for some nonempty set* X.

2.3 Generalised E-Varieties

Since it is not closed under $\mathbb{S}$, the class of all regular semigroups RS is not a generalised variety. However, by replacing $\mathbb{S}$ with $\mathbb{S}_{\mathrm{e}}$, the new notion of a generalised existence variety (or generalised e-variety) is obtained and RS may now be admitted. In the following section, generalised e-varieties are defined and results analogous to those obtained by C.J. Ash [3, Theorems 1, 2 and 3] are obtained. The remainder of this chapter is devoted to the study of e-pseudovarieties via generalised e-varieties.

2.3.1 Ash-Type Theorems for E-Pseudovarieties

A class $\mathcal{C}$ of regular semigroups closed under $\mathbb{H}$, $\mathbb{S}_e$, $\mathbb{P}_f$ and $\mathbb{P}_{ow}$ is called a *generalised e-variety*. It is clear that every e-variety of regular semigroups is also a generalised e-variety. The converse however, is not, in general, true.

Lemma 2.3.1. *For any class $\mathcal{C}$ of E-solid or locally inverse regular semigroups, the following are true:*

(i) $\mathbb{P}_f\mathbb{S}_e(\mathcal{C}) \subseteq \mathbb{S}_e\mathbb{P}_f(\mathcal{C})$;

(ii) $\mathbb{P}_{ow}\mathbb{S}_e(\mathcal{C}) \subseteq \mathbb{S}_e\mathbb{P}_{ow}(\mathcal{C})$;

(iii) $\mathbb{P}_f\mathbb{H}(\mathcal{C}) \subseteq \mathbb{H}\mathbb{P}_f(\mathcal{C})$; *and*

(iv) $\mathbb{P}_{ow}\mathbb{H}(\mathcal{C}) \subseteq \mathbb{H}\mathbb{P}_{ow}(\mathcal{C})$.

Proof. The proofs of statements (i)-(iv) can be obtained by obvious modifications of the proof of [45, Lemma 4.8]. □

Corollary 2.3.2. *If $\mathcal{C}$ is a nonempty class of E-solid or locally inverse regular semigroups, then the smallest generalised e-variety containing $\mathcal{C}$ is $\langle\mathcal{C}\rangle_{\mathrm{gev}} = \mathbb{H}\mathbb{S}_e\mathbb{P}_f\mathbb{P}_{ow}(\mathcal{C})$. In accordance with the usual practice, we call this the generalised e-variety generated by $\mathcal{C}$.*

Proof. Put $\mathrm{W} = \mathbb{H}\mathbb{S}_e\mathbb{P}_f\mathbb{P}_{ow}(\mathcal{C})$.

Since the four operators $\mathbb{H}$, $\mathbb{S}_e$, $\mathbb{P}_f$ and $\mathbb{P}_{ow}$ are all closure operators, $\mathcal{C} \subseteq \mathrm{W}$. Clearly, W is closed under $\mathbb{H}$. From $\mathbb{S}_e\mathbb{H}(\mathcal{C}) \subseteq \mathbb{H}\mathbb{S}_e(\mathcal{C})$ [45, Lemma 4.8] we have that W is closed under $\mathbb{S}_e$. Using Lemma 2.3.1 (iii) and (i) we can show that W is closed under $\mathbb{P}_f$. From (iv) and (ii) of Lemma 2.3.1 and the fact that $\mathbb{P}_{ow}\mathbb{P}_f(\mathcal{C}) = \mathbb{P}_f\mathbb{P}_{ow}(\mathcal{C})$ within any class of regular semigroups closed under isomorphisms, we have that W is closed under $\mathbb{P}_{ow}$. Thus W is a generalised e-variety.

Now any generalised e-variety containing $\mathcal{C}$ must also contain $\mathbb{HS}_e\mathbb{P}_f\mathbb{P}_{ow}(\mathcal{C})$ (being closed under $\mathbb{P}_{ow}$, $\mathbb{P}_f$, $\mathbb{S}_e$ and $\mathbb{H}$). So $W = \mathbb{HS}_e\mathbb{P}_f\mathbb{P}_{ow}(\mathcal{C})$ is the smallest generalised e-variety containing C. $\square$

Theorem 2.3.3. *For a class $\mathcal{C}$ of regular semigroups contained in ES or LI the following are equivalent:*

(i) $\mathcal{C}$ is a generalised e-variety;

(ii) $\mathcal{C} = \mathbb{HS}_e\mathbb{P}_f\mathbb{P}_{ow}(\mathcal{C})$; and

(iii) $\mathcal{C}$ is the union of a directed family of e-varieties.

Proof. (i) $\Rightarrow$ (ii) Let $\mathcal{C}$ be a generalised e-variety. Then $\mathcal{C} = \langle\mathcal{C}\rangle_{\text{gev}} = \mathbb{HS}_e\mathbb{P}_f\mathbb{P}_{ow}(\mathcal{C})$ by Corollary 2.3.2.
(ii) $\Rightarrow$ (iii) Let $\mathcal{C}$ be such that $\mathcal{C} = \mathbb{HS}_e\mathbb{P}_f\mathbb{P}_{ow}(\mathcal{C})$ and let $\mathcal{C}_0 \subseteq \mathcal{C}$ and $\mathcal{C}_1 \subseteq \mathcal{C}$ be finite. Then clearly $\langle\mathcal{C}_0\rangle_{\text{ev}} \subseteq \mathcal{C}$ and $\langle\mathcal{C}_1\rangle_{\text{ev}} \subseteq \mathcal{C}$ since any product of finitely many regular semigroups is isomorphic to a finite product of powers of those regular semigroups. Therefore $\langle\mathcal{C}_0 \cup \mathcal{C}_1\rangle_{\text{ev}} \subseteq \mathcal{C}$. The family $\Gamma = \{\langle\mathcal{C}_0\rangle_{\text{ev}} | \mathcal{C}_0 \subseteq \mathcal{C} \text{ is finite}\}$ is a directed family and so for any given $\langle\mathcal{C}_0\rangle_{\text{ev}}$, $\langle\mathcal{C}_1\rangle_{\text{ev}} \in \Gamma$ we have $\langle\mathcal{C}_0 \cup \mathcal{C}_1\rangle_{\text{ev}} \in \Gamma$ with $\langle\mathcal{C}_0\rangle_{\text{ev}} \subseteq \langle\mathcal{C}_0 \cup \mathcal{C}_1\rangle_{\text{ev}}$ and $\langle\mathcal{C}_1\rangle_{\text{ev}} \subseteq \langle\mathcal{C}_0 \cup \mathcal{C}_1\rangle_{\text{ev}}$. Finally we have $\mathcal{C} = \cup\Gamma$.
(iii) $\Rightarrow$ (i) Let $S \in \mathcal{C}$. Then S is in some e-variety $V \subseteq \mathcal{C}$ and so all homomorphic images of S are in V and so are in $\mathcal{C}$. Similarly, all regular subsemigroups and arbitrary powers of S are in $\mathcal{C}$. Let $S_1, \ldots, S_n \in \mathcal{C}$, with $S_i \in V_i \subseteq \mathcal{C}$, $i = 1, \ldots, n$. Since the family $\mathcal{C}$ is directed, there exists $V \subseteq \mathcal{C}$ such that $V_i \subseteq V$, $i = 1, \ldots, n$, and so $S_1 \times \cdots \times S_n \in V$. $\square$

We remarked earlier that while every e-variety is a generalised e-variety, the converse is not true. The following example demonstrates this.

Example 2.3.4. Let V_1 consist of all completely regular semigroups. These can be characterised as those E-solid regular semigroups having the property that every element belongs to a subgroup.

Let V_2 consist of all those E-solid regular semigroups having the property that the square of every element belongs to a subgroup.

Let V_3 consist of all those E-solid regular semigroups having the property that the fourth power of every element belongs to a subgroup.

In general, let V_n consist of those E-solid regular semigroups having the property that the m^{th} power of every element belongs to a subgroup, where $m = 2^{n-1}$.

Clearly, each V_n is closed under $\mathbb{H}$ and $\mathbb{P}$. In order to show that V_n is closed under $\mathbb{S}_{\mathrm{e}}$ we rely on the following well known argument (see [29, Page 472] for further details): Let a be an element in a subgroup H of a semigroup S. Let H_a be the $\mathscr{H}$-class containing a and let a^{-1} be the group inverse of a in $H \subseteq H_a$. If T is a regular subsemigroup of S and $b \in T$ is such that b^{-1} exists in S, then $b^{-1} \in T$.

Let $S \in \mathrm{V}_n$ and let T be a regular subsemigroup of S. Then for any $x \in T$, x^m is in a subgroup of S and so $(x^m)^{-1}$ exists in S and so $(x^m)^{-1} \in T$. Therefore the set $\{x^m \mid x \in T\}$ forms a subgroup of T and so V_n is closed under $\mathbb{S}_{\mathrm{e}}$.

Therefore each V_n is an e-variety and we have the following containments:

$$\mathrm{V}_1 \subseteq \mathrm{V}_2 \subseteq \mathrm{V}_3 \subseteq \cdots$$

(In fact, the containment above can be shown to be proper; for example, the 5-element Brandt combinatorial semigroup (see, for example, [14]) is in V_2 but not in V_1.)

Thus, the directed union of this collection of e-varieties is a generalised e-variety. However, the union is not an e-variety: Let $S_1 \in \mathrm{V}_1$ and $S_i \in \mathrm{V}_i \setminus \mathrm{V}_{i-1}$, $i \in \{2, 3, \ldots\}$ and consider the product

$$S_1 \times S_2 \times \cdots.$$

It is clear that this semigroup fails to be in $\mathrm{V}_1, \mathrm{V}_2, \ldots$ and so the directed union of these e-varieties is not closed under $\mathbb{P}$ and so is not an e-variety.

Example 2.3.4 describes a generalised e-variety consisting entirely of E-solid regular semigroups. Our next example of a generalised e-variety has only locally inverse regular semigroups as its members.

Example 2.3.5. Let V_1 consist of all generalised inverse semigroups. These can be characterised as those locally inverse regular semigroups having the property that the product of two idempotents is always an idempotent. So V_1 is determined (within the e-variety of all locally inverse regular semigroups) by the biidentity $(xx'yy')^2 = xx'yy'$.

Let V_2 consist of all those locally inverse regular semigroups satisfying the biidentity $(xx'yy')^4 = xx'yy'$.

Let V_3 consist of all those locally inverse regular semigroups satisfying the biidentity $(xx'yy')^{16} = xx'yy'$.

In general, let V_n consist of all those locally inverse regular semigroups satisfying the biidentity $(xx'yy')^m = xx'yy'$, where $m = 2^{(2^{n-1})}$.

Then each of these classes is an e-variety, since (see, for example, Auinger [6]) e-varieties of locally inverse semigroups correspond precisely to equational classes determined by biidentities.

Furthermore, it is clear that these e-varieties are all distinct and that they form an ascending chain of e-varieties. Since there is no finite bound to the index m, the union of these e-varieties is a generalised e-variety of locally inverse regular semigroups which is not itself an e-variety.

A further characterisation of generalised e-varieties is given below.

Lemma 2.3.6. *A class* V *of regular semigroups is a generalised e-variety if and only if it is the union of an ideal in* $\mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$.

Proof. Suppose V is the union of an ideal in $\mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$. Since an ideal in $\mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$ is a directed family of e-varieties, we have that V is a generalised e-variety.

Conversely, suppose V is a generalised e-variety. Then $V = \cup\Gamma$ where Γ is a directed family of e-varieties. Let I be the ideal in $\mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$ generated by Γ:

$$I = \{W \in \mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI}) \,:\, W \leq \bigvee_{i \in I} U_i,\ U_i \in \Gamma,\ I \text{ finite}\}.$$

Then $\cup I = \cup\Gamma = V$ and so V is the union of an ideal in $\mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$. □

Now we come to the key result linking generalised e-varieties and e-pseudovarieties.

Theorem 2.3.7. *A class* $\mathbf{C}$ *of finite regular semigroups contained in* $\mathbf{ES}$ *or* $\mathbf{LI}$ *is an e-pseudovariety if and only if* $\mathbf{C}$ *consists of the finite members of some generalised e-variety.*

Proof. Clearly the class of all finite members of a generalised e-variety is closed under $\mathbb{H}$, $\mathbb{S}_{\mathrm{e}}$ and $\mathbb{P}_{\mathrm{f}}$.

Conversely, suppose $\mathbf{C}$ is an e-pseudovariety. Let $K = \mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}_{\mathrm{f}}\mathbb{P}_{\mathrm{ow}}(\mathbf{C}) = \langle \mathbf{C} \rangle_{\mathrm{gev}}$ and let S be a finite regular semigroup in K. Then there exist finite regular semigroups $S_1, \dots, S_n \in \mathbf{C}$, sets (possibly infinite) $I_1, \dots, I_n$, a regular semigroup T and a surjective homomorphism $\phi : T \to S$ such that

$$S \xleftarrow{\phi} T \leq_{\mathrm{reg}} S_1^{I_1} \times \cdots \times S_n^{I_n}.$$

For each $s \in S$ choose $t_s = (f_{1_s}, \dots, f_{n_s}) \in T$ such that $t_s\phi = s$. Also, for each $s \in S$ we can choose a $t'_s = (f'_{1_s}, \dots, f'_{n_s}) \in V(t_s)$. Note that for each $s \in S$, $t'_s\phi \in V(s)$. Define a relation $\equiv_k$ on each I_k by

$$i \equiv_k j \Leftrightarrow (\forall s \in S)\ (i)f_{k_s} = (j)f_{k_s} \text{ and } (i)f'_{k_s} = (j)f'_{k_s}$$

Clearly $\equiv_k$ is an equivalence relation for each $k \in \{1, \dots, n\}$. For each element s of S, and for each $k \in \{1, \dots, n\}$, f_{k_s} induces a partition of I_k into at most $|S_k|$ equivalence classes. The inverse function f'_{k_s} induces the same partition. Hence $\equiv_k$ has at most $|S_K|^{|S|}$ equivalence classes. For each $k \in \{1, \dots, n\}$ let $J_k = I_k/\equiv_k$. Hence, $S_1^{J_1} \times \cdots \times S_n^{J_n}$ is finite.

Note that there is a (natural) homomorphism from $S_1^{I_1} \times \cdots \times S_n^{I_n}$ into $S_1^{J_1} \times \cdots \times S_n^{J_n}$ which is one-to-one on the subset $\{t_s, t'_s; s \in S\}$. Let T' be the least regular subsemigroup of T containing $\{t_s, t'_s; s \in S\}$. However, since $S_1^{J_1} \times \cdots \times S_n^{J_n}$ contains a copy of $\{t_s, t'_s; s \in S\}$ we have $T' \leq S_1^{J_1} \times \cdots \times S_n^{J_n}$ and T' is finite. Finally, the homomorphism $\phi : T \to S$ restricted to T' is surjective and so $S \in \mathbf{C}$. □

Note that since every e-variety is a generalised e-variety it follows from the first part of Theorem 2.3.7 that the finite members of an e-variety form an e-pseudovariety, as previously mentioned in Theorem 2.1.3

Ash was able [3, Theorem 3] to connect his work on generalised varieties with the earlier work of Eilenberg and Schutzenberger [16] on pseudovarieties. We demonstrate below that sequences of biidentities (in the sense of Theorem 1.4.13) can take the place of sequences of identities in Ash's original result. The proof is an adaptation of the proof presented by Ash.

Theorem 2.3.8. *Let* $\mathbf{V} \in \mathscr{L}_{\mathrm{epv}}(\mathbf{ES}) \cup \mathscr{L}_{\mathrm{epv}}(\mathbf{LI})$. *Then the following are equivalent:*

(i) There exists a generalised e-variety W *such that* $\mathbf{V} = \mathrm{W}^{\mathrm{Fin}}$*;*

(ii) There exists a chain of e-varieties $\mathrm{V}_1 \subseteq \mathrm{V}_2 \subseteq \cdots$ *such that* $\mathbf{V} = \left(\bigcup_{i=1}^{\infty} \mathrm{V}_i\right)^{\mathrm{Fin}}$*; and*

(iii) There exists a sequence of biidentities $p_1 = q_1, p_2 = q_2, \ldots$ *such that, for any* $S \in \mathbf{RS}$*,*

$$S \in \mathbf{V} \Leftrightarrow S \models p_n = q_n \textit{ for all but finitely many } n.$$

Proof. $(iii) \Rightarrow (ii)$ Let V_n be the e-variety defined by the biidentities $p_n = q_n, p_{n+1} = q_{n+1}, \ldots$. Then clearly $\mathrm{V}_1 \subseteq \mathrm{V}_2 \subseteq \cdots$ and $\mathbf{V} = \left(\bigcup_{i=1}^{\infty} \mathrm{V}_i\right)^{\mathrm{Fin}}$.

$(ii) \Rightarrow (i)$ This follows immediately from Theorem 2.3.3.

$(i) \Rightarrow (iii)$ Suppose W is a generalised e-variety such that $\mathbf{V} = \mathrm{W}^{\mathrm{Fin}}$. Let $\mathbf{V} = \{S_1, S_2, \ldots\}$ and $\mathbf{RS} \backslash \mathbf{V} = \{T_1, T_2, \ldots\}$. Note that $(\mathbf{RS} \backslash \mathbf{V}) \cap \mathrm{W} = \varnothing$ and consequently for each $i \geq 1$, $\mathbb{HS}_{\mathrm{e}}\mathbb{P}\left(\{S_1, \ldots, S_i\}\right) \cap (\mathbf{RS} \backslash \mathbf{V}) = \varnothing$.

Let Σ_i be the set of biidentities satisfied by $\{S_1, \ldots, S_i\}$ and let Σ'_j be the set of biidentities satisfied by T_j. For any i, j we can choose $p_{ij} = q_{ij} \in \Sigma_i \backslash \Sigma'_j$ and consequently produce a list of all biidentities

$$p_1 = q_1, p_2 = q_2, \ldots$$

where each $p_n = q_n$ is of the form $p_{ij} = q_{ij}$, $i > j$. Therefore for each j there are infinitely many $p_{ij} = q_{ij}$ and $T_j \not\models p_{ij} = q_{ij}$.

Now, let $S = S_k \in \mathbf{V}$ so that for each $i \geq k$, $S_k \models p_{ij} = q_{ij}$. Therefore $S \models p_n = q_n$ for all but finitely many n. □

2.3.2 The Lattice of Generalised E-Varieties

For any generalised e-variety V we denote by $\mathscr{L}_{\text{gev}}(\mathrm{V})$ the collection of all generalised e-varieties contained in V.

Theorem 2.3.9. *Let* V *be a generalised e-variety. Then* $\mathscr{L}_{\text{gev}}(\mathrm{V})$ *is a complete lattice.*

Proof. Since $\mathscr{L}_{\text{gev}}(\mathrm{V})$ is partially ordered by class containment, has a greatest element (namely V) and the intersection of every nonempty collection of members of $\mathscr{L}_{\text{gev}}(\mathrm{V})$ is again in $\mathscr{L}_{\text{gev}}(\mathrm{V})$, we have that $\mathscr{L}_{\text{gev}}(\mathrm{V})$ is a complete lattice. □

As with varieties and similarly described classes, it may be difficult to give a useful description of the join of a family of generalised e-varieties. We do, however, have the following result.

Theorem 2.3.10. *The join of a family* $\{\mathrm{V}_\alpha : \alpha \in A\}$ *of generalised e-varieties all contained in* ES *or* LI *consists of the homomorphic images of regular subsemigroups of finite direct products of regular semigroups belonging to the union of the family* $\{\mathrm{V}_\alpha : \alpha \in A\}$.

Proof. Let W be the class of all homomorphic images of regular subsemigroups of direct products of the form $S_{\alpha_1} \times S_{\alpha_2} \times \cdots \times S_{\alpha_n}$ where each S_{α_i} is in V_{α_i} and each $\alpha_i \in A$. Clearly, W is contained in the join of the family $\{\mathrm{V}_\alpha : \alpha \in A\}$.

Also, it is clear that each V_α is contained in W. So it remains to show that W is a generalised e-variety.

The fact that W is closed under $\mathbb{H}$, $\mathbb{S}_\mathrm{e}$ and $\mathbb{P}_\mathrm{f}$ comes by using Lemma 2.3.1 and [45, Lemma 4.8]. That W is closed under $\mathbb{P}_{\mathrm{ow}}$ comes from the fact that a power of a finite product of regular semigroups is isomorphic to a finite product of powers of those regular semigroups. □

The following result (together with the results of Theorem 2.3.7) shows that every e-pseudovariety can be considered as the join (within the lattice of all e-pseudovarieties) of a collection of e-pseudovarieties, all of which consist of the finite members of certain e-varieties.

Theorem 2.3.11. *Let* $\mathrm{W} \in \mathscr{L}_{\mathrm{gev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{gev}}(\mathrm{LI})$. *Then* $\mathrm{W}^{\mathrm{Fin}} = \bigvee^{\mathrm{epv}}_{S \in \mathrm{W}^{\mathrm{Fin}}} (\mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}(S))^{\mathrm{Fin}}$.

Proof. Since $\mathrm{W}^{\mathrm{Fin}}$ consists of the finite members of W and W is a generalised e-variety, it follows that $\mathrm{W}^{\mathrm{Fin}}$ is an e-pseudovariety. Let $S \in \mathrm{W}^{\mathrm{Fin}}$. Then

$$\begin{aligned} & S \in \mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}(S), \text{ and } S \text{ is finite} \\ \Rightarrow\ & S \in (\mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}(S))^{\mathrm{Fin}} \\ \Rightarrow\ & S \in \bigvee^{\mathrm{epv}}_{S \in \mathrm{W}^{\mathrm{Fin}}} (\mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}(S))^{\mathrm{Fin}}. \end{aligned}$$

Conversely, we show that $(\mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}(S))^{\mathrm{Fin}} \subseteq \mathrm{W}^{\mathrm{Fin}}$. Since $S \in \mathrm{W}$, and W is closed under $\mathbb{H}$, $\mathbb{S}_{\mathrm{e}}$, $\mathbb{P}_{\mathrm{f}}$ and $\mathbb{P}_{\mathrm{ow}}$,

$$\begin{aligned} \mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}(S) &= \mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}_{\mathrm{f}}\mathbb{P}_{\mathrm{ow}}(S) \subseteq \mathrm{W} \\ \Rightarrow (\mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}(S))^{\mathrm{Fin}} &\subseteq \mathrm{W}^{\mathrm{Fin}}. \end{aligned}$$

Since this inclusion holds true for every $S \in \mathrm{W}^{\mathrm{Fin}}$,

$$\bigvee^{\mathrm{epv}}_{S \in \mathrm{W}^{\mathrm{Fin}}} (\mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}(S))^{\mathrm{Fin}} \subseteq \mathrm{W}^{\mathrm{Fin}}.$$

□

2.4 Local Finiteness

2.4.1 Locally Finite Regular Semigroups

Given a nonempty subset A of a semigroup S, the subsemigroup generated by A is denoted $\langle A \rangle$ and is equal to the intersection of all subsemigoups of S which contain A. Equivalently, $\langle A \rangle$ is the smallest subsemigroup of S which contains A. If $S = \langle A \rangle$ for some finite $A \subseteq S$, then we say that S is *finitely generated*.

For regular semigroups however, the situation is slightly more complicated. Given a regular semigroup S and a nonempty subset A of S, the subsemigroup $\langle A \rangle$ need not be regular. However, Yeh [45] has shown that if certain restrictions are placed upon S and upon A we can generate $\langle A \rangle$ in the usual way and guarantee that it is regular.

Theorem 2.4.1. [45, Theorem 2.1] *Let S be a regular semigroup and A a non-empty subset of S such that for each $a \in A$, $A \cap V(a) \neq \varnothing$. If S is E-solid or locally inverse then there is a least regular subsemigroup T of S containing A.*

Implicit in [26] is an example of a regular semigroup which is neither E-solid nor locally inverse which fails to satisfy the conclusion of the above theorem. In fact this example can be constructed in any e-variety not consisting entirely of E-solid or locally inverse regular semigroups. Yeh also provides an example [45, Example 2.3] of a locally orthodox regular semigroup K in which the intersection T of all regular subsemigroups of K containing a set A which satisfies the requirements of Theorem 2.4.1 fails to be regular.

Recall that an algebra S is *locally finite* if every finitely generated subalgebra of S is finite. Because of the restrictions imposed by Theorem 2.4.1 we need to be careful when using the term *locally finite* in reference to regular semigroups. In particular, rather than considering arbitrary (finite) subsets A of a regular semigroup S and the (possibly regular) semigroups generated by such subsets, we must impose some restrictions upon A.

We will say that a nonempty subset A of a regular semigroup S is *inverse rich* if for each $a \in A$, $A \cap V(a) \neq \varnothing$. Consequently, we say that a regular semigroup S is *locally finite (in a regular sense)* if for every $A \subseteq S$, $\langle A \rangle_{\mathrm{lr}}$ is finite, where A is nonempty, finite and inverse rich. Where no confusion will arise, we may simply refer to a regular semigroup as being *locally finite.*

We prove the following result, although the status of the converse is not at present known.

Lemma 2.4.2. *Let S be a regular semigroup. If S is locally finite (in the regular sense) then S is locally finite as a semigroup.*

Proof. Consider two finite sets A' and A where

$$\varnothing \neq A' \subseteq A \subseteq S$$

and A is inverse rich. Then $\langle A' \rangle \subseteq \langle A \rangle_{\mathrm{lr}}$. Since S is locally finite in the regular sense we have that $\langle A \rangle_{\mathrm{lr}}$ is finite and so $\langle A' \rangle$ is also finite. Therefore S is locally finite as a semigroup. $\square$

We say that a regular semigroup S is *finitely generated (in a regular sense)* by $A \subseteq S$ if $S = \langle A \rangle_{\mathrm{lr}}$ and A is nonempty, finite and inverse rich. If it so happens that $S = \langle A \rangle$ where $A \subseteq S$ is nonempty, finite and inverse rich, we say that S is *finitely generated as a semigroup.* Where no confusion will arise we will simply state that a regular semigroup is finitely generated if it is finitely generated in the regular sense.

Lemma 2.4.3. *Suppose S is a regular semigroup and S is finitely generated as a semigoup. Then S is finitely generated as a regular semigroup.*

Proof. Let $A \subseteq S$ be nonempty, finite and inverse rich and suppose $S = \langle A \rangle$. Then $S = \langle A \rangle_{\mathrm{lr}}$ since S itself is regular. $\square$

We comment that the status of the converse of the previous lemma is unknown.

2.4.2 Locally Finite E-Varieties

Several authors (for example [1, 33]) have found useful relationships between lattices of locally finite varieties and lattices of pseudovarieties. We now explore this relationship for e-varieties and e-pseudovarieties. Recall that a variety V is locally finite if all finitely generated members of V are finite.

For e-varieties the definition is unchanged: We say that an e-variety V is *locally finite* if its finitely generated members are finite. Equivalently, a locally finite e-variety is one consisting entirely of locally finite regular semigroups. Similarly, we say that a generalised e-variety is locally finite if its finitely generated members are finite.

Let V be a generalised e-variety. Denote by $G(\mathrm{V})$ the union of all locally finite e-varieties contained in V. That is:

$$G(\mathrm{V}) = \bigcup \{\mathrm{W} \in \mathscr{L}_{\mathrm{gev}}(\mathrm{V}) \mid \mathrm{W} \text{ is a locally finite e-variety}\}.$$

We now establish some results about $G(\mathrm{V})$ that will be used in the remainder of this thesis. Several of the results given are analogues of known results from the literature on generalised varieties. However, we first need to prove some preliminary results.

The following is probably well known.

Lemma 2.4.4. *Let* U *and* V *be e-varieties of E-solid or locally inverse semigroups. Then the join* $\mathrm{U} \vee \mathrm{V}$ *in the lattice of all e-varieties of regular semigroups consists of all homomorphic images of regular subsemigroups of direct products of the form* $U \times V$ *where* $U \in \mathrm{U}$ *and* $V \in \mathrm{V}$.

Proof. Let W be the class of all homomorphic images of regular subsemigroups of direct products of the form $U \times V$ where $U \in \mathrm{U}$ and $V \in \mathrm{V}$.

Clearly U and V are contained in W. We show W is an e-variety.

Let $S \in \mathrm{W}$. So $S \in \mathbb{H}\mathbb{S}_{\mathrm{e}}(U \times V)$ for some $U \in \mathrm{U}$ and $V \in \mathrm{V}$. Suppose that ϕ is a homomorphism of S upon some regular semigroup T. Then $T \in \mathbb{H}\mathbb{H}\mathbb{S}_{\mathrm{e}}(U \times V) = \mathbb{H}\mathbb{S}_{\mathrm{e}}(U \times V)$. Also, let $A \leq_{\mathrm{reg}} S$. Then $A \in \mathbb{S}_{\mathrm{e}}\mathbb{H}\mathbb{S}_{\mathrm{e}}(U \times V) \subseteq \mathbb{H}\mathbb{S}_{\mathrm{e}}(U \times V)$ by [45, Lemma 4.8]. Thus W is closed under $\mathbb{H}$ and $\mathbb{S}_{\mathrm{e}}$.

By repeated application of Lemma 2.3.1 and [45, Lemma 4.8] it can also be shown that W is closed under $\mathbb{P}$. □

Note that for a finite collection $\{\mathrm{V}_1, \ldots, \mathrm{V}_n\}$ of e-varieties all of which are contained in ES or LI, their join $\overset{\mathrm{gev}}{\bigvee\limits_{1 \leq i \leq n}} \mathrm{V}_i$ within the lattice of generalised e-varieties is precisely their join within the lattice of e-varieties. This also follows from the fact that generalised e-varieties are directed unions of e-varieties.

Lemma 2.4.5. *Within the e-varieties* ES *and* LI*, the operators* $\mathbb{H}$*,* $\mathbb{S}_{\mathrm{e}}$ *and* $\mathbb{P}_{\mathrm{f}}$ *preserve local finiteness.*

Proof. Let S be a locally finite regular semigroup in ES or LI. Let U be a regular subsemigroup of S. Let A be a nonempty, finite, inverse rich subset of U, and therefore of S. Then $\langle A \rangle_{\mathrm{lr}}$ is a finite regular subsemigroup of S. But U is a regular subsemigroup of S which contains A. Therefore U contains $\langle A\rangle_{\mathrm{lr}}$. Thus U is locally finite.

Let ϕ be a homomorphism of S upon some regular semigroup T. Let B be a nonempty, finite, inverse rich subset of T. Let A be a finite set of pre-images in S for the elements in B. Let A' be a finite set of inverses for the elements in A. Put $\bar{A} = A \cup A'$. Then $\bar{A}$ is a nonempty, finite, inverse rich subset of S, whence $\langle \bar{A}\rangle_{\mathrm{lr}}$ is finite within S. Then $\langle \bar{A}\rangle_{\mathrm{lr}}\phi$ is a finite regular subsemigroup of T, and contains B. Thus T is locally finite.

Let $S_1, \ldots, S_n$ be locally finite regular semigroups in ES or LI. Put $S = S_1 \times \ldots \times S_n$. Let C be a nonempty, finite, inverse rich subset of S. Let $C_1, \ldots, C_n$ be subsets of $S_1, \ldots, S_n$ where $x_i \in C_i$ if and only if there exists $x \in C$ such that x_i is the i^{th} component of x. Then each C_i is finite, nonempty and inverse rich. So each $\langle C_i\rangle_{\mathrm{lr}}$ is a finite regular subsemigroup of S_i, whence the direct product $\langle C_1\rangle_{\mathrm{lr}} \times \ldots \times \langle C_n\rangle_{\mathrm{lr}}$ is a finite regular subsemigroup of S containing C. So S is locally finite. □

Lemma 2.4.6. *Let* W_1 *and* W_2 *be locally finite e-varieties where* $\mathrm{W}_1, \mathrm{W}_2 \in \mathscr{L}_{\mathrm{ev}}(\mathrm{ES})$ *or* $\mathrm{W}_1, \mathrm{W}_2 \in \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$. *Then* $\mathrm{W}_1 \vee \mathrm{W}_2$ *is locally finite.*

Proof. By Lemma 2.4.5, the operators $\mathbb{H}$, $\mathbb{S}_{\mathrm{e}}$ and $\mathbb{P}_{\mathrm{f}}$ preserve local finiteness. The result then follows immediately from Lemma 2.4.4. □

Corollary 2.4.7. *The join of a finite number of locally finite e-varieties is locally finite.*

Proof. Using Lemma 2.4.6 and mathematical induction, the proof follows. □

Theorem 2.4.8. *For any generalised e-variety* V *contained in* ES *or* LI, $G(\mathrm{V})$ *is a generalised e-variety.*

Proof. Since V is a generalised e-variety, it is the union of a directed family Γ of e-varieties. Without loss of generality, suppose Γ includes all possible joins of pairs of members of Γ. That is, $\mathrm{V}_1, \mathrm{V}_2 \in \Gamma \Rightarrow \mathrm{V}_1 \vee \mathrm{V}_2 \in \Gamma$. Let $\Gamma' = \{\mathrm{W} \in$

$\Gamma \,|\, \mathrm{W}$ is locally finite$\}$. Since (by Lemma 2.4.6) $\mathrm{W}_1, \mathrm{W}_2 \in \Gamma' \Rightarrow \mathrm{W}_1 \vee \mathrm{W}_2 \in \Gamma'$ we have that Γ' is directed and so $G(\mathrm{V}) = \bigcup \Gamma'$ is a generalised e-variety. □

The following results illustrate some fundamental properties of the lattice $\mathscr{L}_{\mathrm{gev}}(G(\mathrm{V}))$ for a generalised e-variety V contained in ES or LI. Note also that $G(\mathrm{V})$ is the greatest locally finite generalised e-variety contained in V.

Lemma 2.4.9. *The lattice of generalised e-varieties* $\mathscr{L}_{\mathrm{gev}}(G(\mathrm{V}))$ *consists entirely of locally finite generalised e-varieties.*

Proof. Let $\mathrm{W} \in \mathscr{L}_{\mathrm{gev}}(G(\mathrm{V}))$ and let $S \in \mathrm{W}$. Since S is also in $G(\mathrm{V})$, S is in some locally finite e-variety. Hence every $S \in \mathrm{W}$ is locally finite and so W is locally finite. □

Since every e-variety is also a generalised e-variety we can consider $G(\mathrm{V})$ where V is an e-variety.

Lemma 2.4.10. *If* $\mathrm{W} \in \mathscr{L}_{\mathrm{gev}}(G(\mathrm{V}))$*,* $\mathrm{V} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$ *and* $\mathbb{HS}_{\mathrm{e}}\mathbb{P}(\mathrm{W})$ *is locally finite then* $\mathrm{W}^{\mathrm{Fin}} = (\mathbb{HS}_{\mathrm{e}}\mathbb{P}(\mathrm{W}))^{\mathrm{Fin}}$.

Proof. Clearly $\mathrm{W} \subseteq \mathbb{HS}_{\mathrm{e}}\mathbb{P}(\mathrm{W})$ and so $\mathrm{W}^{\mathrm{Fin}} \subseteq (\mathbb{HS}_{\mathrm{e}}\mathbb{P}(\mathrm{W}))^{\mathrm{Fin}}$. To prove the reverse inclusion, suppose $S \in (\mathbb{HS}_{\mathrm{e}}\mathbb{P}(\mathrm{W}))^{\mathrm{Fin}}$. Then there exists a collection $\{S_\alpha\}_{\alpha \in A}$ of members of W, a regular semigroup T and a surjective homomorphism $\phi : T \to S$ such that

$$S \xleftarrow{\phi} T \leq_{\mathrm{reg}} \prod_{\alpha \in A} S_\alpha.$$

Let B be a finite set of pre-images in T of the elements of S. Let B' be a finite set of inverses for the elements in B. Put $\overline{B} = B \cup B'$. Then $\overline{B}$ is finite, nonempty and inverse rich. Put $U = \langle \overline{B} \rangle_{\mathrm{lr}}$. Since $\mathbb{HS}_{\mathrm{e}}\mathbb{P}(\mathrm{W})$ is locally finite, U is finite. Also, $U\phi = S$.

Now U is a regular subsemigroup of $\prod_{\alpha \in A} S_\alpha$. For each $\alpha \in A$, let π_α be the projection of $\prod_{\alpha \in A} S_\alpha$ upon S_α. Put $U_\alpha = U\pi_\alpha$. Then U_α is a regular subsemigroup of S_α, having cardinality less than that of U, and U is a regular subsemigroup of

$\prod_{\alpha \in A} U_\alpha$. But since there are only finitely many regular semigroups of cardinality at most $|U|$, there can only be finitely many different U_α (up to isomorphism). Thus

$$S \overset{\phi}{\longleftarrow} U \leq_{\mathrm{reg}} U_{\alpha_1}^{I_1} \times U_{\alpha_2}^{I_2} \times \cdots \times U_{\alpha_n}^{I_n}$$

so that $S \in \mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}_{\mathrm{f}}\mathbb{P}_{\mathrm{ow}}(\mathrm{W}) \subseteq \mathrm{W}$. Since S is finite, we have $S \in \mathrm{W}^{\mathrm{Fin}}$ as required.

□

We now give several related results that will allow us to restate the results of Theorem 2.3.7 in terms of locally finite generalised e-varieties only.

Lemma 2.4.11. *For any finite regular semigroup S contained in* **ES** *or* **LI***, the e-variety $\langle S \rangle_{\mathrm{ev}}$ generated by S is locally finite.*

Proof. Let $S \in \mathbf{ES}$ or $S \in \mathbf{LI}$ and let $B \in \mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}(S)$. Then there exists a regular semigroup A, a surjective homomorphism $\phi : A \to B$ and a nonempty set I such that

$$B \overset{\phi}{\longleftarrow} A \leq_{\mathrm{reg}} S^I.$$

Let Y be a nonempty, finite, inverse-rich subset of B. Then Y has a finite set of preimages in A. Let X' be a finite set of inverses for the elements of X and let $\overline{X} = X \cup X'$. Then each element of $\overline{X}$ can be regarded as a function $p : I \to S$ which induces a partition of I into finitely many (at most $|S|$) equivalence classes. Since $\overline{X}$ is finite, π of I which is the common refinement of the partitions induced by all of the elements of $\overline{X}$ has finitely many (at most $|S|^{|X|}$) equivalence classes.

The set of all functions from I to S which are constant on each equivalence class in π is clearly a finite regular subsemigroup of S^I containing $\overline{X}$. Hence $\langle \overline{X} \rangle_{\mathrm{lr}}$ is finite. Then the image under ϕ of $\langle \overline{X} \rangle_{\mathrm{lr}}$ is a finite regular subsemigroup of B containing Y. Hence $\langle Y \rangle_{\mathrm{lr}}$ is finite and so B is locally finite. □

Note that the locally finite e-variety generated by S is, in fact, the smallest generalised e-variety having S as a member. That is, $\langle S \rangle_{\mathrm{ev}} = \langle S \rangle_{\mathrm{gev}}$. This follows from the fact that

$$\mathbb{P}_{\mathrm{f}}\mathbb{P}_{\mathrm{ow}}(S) = \mathbb{P}_{\mathrm{ow}}(S) = \mathbb{P}(S)$$

for any semigroup S.

The following is an immediate consequence of Lemma 2.4.11.

Corollary 2.4.12. *Let $\mathcal{A}$ be a finite collection of finite regular semigroups all contained in either* **LI** *or* **ES***. Then $\langle \mathcal{A} \rangle_{\text{ev}}$ is locally finite.*

Proof. Let $\mathcal{A} = \{S_1, \ldots, S_n\}$. Then $\langle \mathcal{A} \rangle_{\text{ev}} = \mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(S_1 \times \cdots \times S_n)$ and the result follows from Lemma 2.4.11. □

Lemma 2.4.13. *Let $\mathbf{V} \in \mathscr{L}_{\text{epv}}(\mathbf{ES}) \cup \mathscr{L}_{\text{epv}}(\mathbf{LI})$. Then $\langle \mathbf{V} \rangle_{\text{gev}}$ is locally finite.*

Proof. Note that $\langle \mathbf{V} \rangle_{\text{gev}} = \bigcup \Gamma$ where

$$\Gamma = \{\langle \mathcal{C} \rangle_{\text{ev}} \,|\, \mathcal{C} \text{ is a finite collection of members of } \mathbf{V}\}.$$

By Corollary 2.4.12 each $\langle \mathcal{C} \rangle_{\text{ev}}$ is locally finite and so $\langle \mathbf{V} \rangle_{\text{gev}}$ is locally finite. □

Corollary 2.4.14. *The generalised e-variety $\langle \mathrm{V}^{\mathrm{Fin}} \rangle_{\text{gev}} \in \mathscr{L}_{\text{gev}}(G(\mathrm{V}))$.*

Proof. Since $\mathrm{V}^{\mathrm{Fin}}$ is an e-pseudovariety, it follows from Lemma 2.4.13 that $\langle \mathrm{V}^{\mathrm{Fin}} \rangle_{\text{gev}}$ is locally finite and hence is a member of $\mathscr{L}_{\text{gev}}(G(\mathrm{V}))$. □

So the generalised e-variety generated by an e-pseudovariety is locally finite and therefore we are now in a position to restate the Ash-type theorem (Theorem 2.3.7) in terms of locally finite e-varieties.

Corollary 2.4.15. *A class* **C** *of finite regular semigroups contained in* **ES** *or* **LI** *is an e-pseudovariety if and only if* **C** *consists of the finite members of a directed union of locally finite e-varieties.*

A close relationship between lattices of locally finite varieties and lattices of pseudovarieties was established by Agliano and Nation in [1]. We now demonstrate that similar results hold for lattices of locally finite e-varieties and lattices of e-pseudovarieties.

Theorem 2.4.16. *Let $\mathrm{V} \in \mathscr{L}_{\text{ev}}(\mathrm{LI}) \cup \mathscr{L}_{\text{ev}}(\mathrm{ES})$ be locally finite and let $\mathbf{W} \in \mathscr{L}_{\text{epv}}(\mathrm{V}^{\mathrm{Fin}})$. Then* *(i)* $\mathrm{V} = \langle \mathrm{V}^{\mathrm{Fin}} \rangle_{\mathrm{ev}}$*; and*

(ii) $\mathbf{W} = (\langle \mathbf{W} \rangle_{\mathrm{ev}})^{\mathrm{Fin}}$.

Proof. (i) Just as for varieties, locally finite e-varieties are generated by their finite members and so we have that $\langle \mathrm{V}^{\mathrm{Fin}} \rangle_{\mathrm{ev}} = \mathrm{V}$.

(ii) Since $\mathbf{W} \subseteq \langle \mathbf{W} \rangle_{\mathrm{ev}}$ and all members of $\mathbf{W}$ are finite, $\mathbf{W} \subseteq (\langle \mathbf{W} \rangle_{\mathrm{ev}})^{\mathrm{Fin}}$.

Now suppose $S \in (\langle \mathbf{W} \rangle_{ev})^{\mathrm{Fin}}$. Then S is a finite member of $\mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}(\mathbf{W})$. So there exists a family $\{S_\alpha : \alpha \in A\}$ of finite regular semigroups in $\mathbf{W}$, a regular subsemigroup T of the direct product of the family, and a homomorphism ϕ from T onto S. Now as in the proof of Lemma 2.4.10 we have

$$S \xleftarrow{\phi} U \leq_{\mathrm{reg}} U_{\alpha_1}^{I_1} \times U_{\alpha_2}^{I_2} \times \cdots \times U_{\alpha_n}^{I_n}$$

where U_{α_i} is a regular subsemigroup of S_{α_i} for some finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of A. Since the family $\{S_\alpha : \alpha \in A\}$ consists of finite regular semigroups, each U_{α_i} is finite. Since also U is finite, we have that $S \in \mathbf{W}$.

□

The following result was proved for varieties and pseudovarieties in [1, Lemma 1.4].

Theorem 2.4.17. *Let* $\mathrm{V} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$ *be locally finite. Then* $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$ *is isomorphic to* $\mathscr{L}_{\mathrm{epv}}(\mathrm{V}^{\mathrm{Fin}})$.

Proof. Define two functions as follows:

$$\phi : \mathscr{L}_{\mathrm{ev}}(\mathrm{V}) \to \mathscr{L}_{\mathrm{epv}}(\mathrm{V}^{\mathrm{Fin}}), \qquad \mathrm{W}\phi = \mathrm{W}^{\mathrm{Fin}}$$

$$\theta : \mathscr{L}_{\mathrm{epv}}(\mathrm{V}^{\mathrm{Fin}}) \to \mathscr{L}_{\mathrm{ev}}(\mathrm{V}), \qquad \mathbf{W}\theta = \langle \mathbf{W} \rangle_{\mathrm{ev}}$$

By Theorem 2.4.16 we have that the functions ϕ and θ are one-to-one and mutually inverse. It is also clear that the functions are order preserving. Therefore the functions are lattice isomorphisms. □

The following theorem was proved for varieties and generalised varieties of completely regular semigroups by Pastijn [33, Lemma 6]. The proof presented here is an adaptation of that given by Pastijn.

Theorem 2.4.18. *Let* $\mathrm{V} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$ *be a locally finite e-variety. Then the map*

$$\phi : \mathscr{L}_{\mathrm{gev}}(G\mathrm{V}) \to \mathscr{L}_{\mathrm{epv}}(\mathrm{V}^{\mathrm{Fin}}); \quad \mathrm{W} \mapsto \mathrm{W}^{\mathrm{Fin}}$$

is a complete surjective lattice homomorphism.

Proof. Let $\{\mathrm{U}_i \,|\, i \in I\}$ be a collection of members of $\mathscr{L}_{\mathrm{gev}}(G(\mathrm{V}))$. It is obvious that $\left(\bigcap_{i\in I} \mathrm{U}_i\right)^{\mathrm{Fin}} = \bigcap_{i\in I}\left(\mathrm{U}_i^{\mathrm{Fin}}\right)$. We prove that $\left(\bigvee_{i\in I} \mathrm{U}_i\right)^{\mathrm{Fin}} = \bigvee_{i\in I}\left(\mathrm{U}_i^{\mathrm{Fin}}\right)$.

Let $S \in \left(\bigvee_{i\in I} \mathrm{U}_i\right)^{\mathrm{Fin}}$. Then by Theorem 2.3.10 there exists a regular semigroup T, a surjective homomorphism $\theta : T \to S$ and regular semigroups $S_{i_1}, \ldots, S_{i_n}$ where $S_{i_j} \in \mathrm{U}_{i_j}$, $i_j \in I$, $j = 1, \ldots, n$ such that

$$S \overset{\theta}{\longleftarrow} T \leq_{\mathrm{reg}} S_{i_1} \times \cdots \times S_{i_n}.$$

For each $s \in S$ choose $t_s = (s_{i_1}, \ldots, s_{i_n}) \in S_{i_1} \times \cdots \times S_{i_n}$ such that $t_s\theta = s$. We may also choose $t'_s = (s'_{i_1}, \ldots, s'_{i_n}) \in V(t_s)$.

Denote by $T_{i_1} \times \ldots \times T_{i_n}$ the least regular subsemigroup of $S_{i_1} \times \cdots \times S_{i_n}$ generated by $\{t_s, t'_s \,|\, s \in S\}$. Since each of the S_{i_j}, $j = 1, \ldots, n$ is locally finite, the T_{i_j}, $j = 1, \ldots, n$ are finite regular subsemigroups of S_{i_j}. Hence, $S \in \bigvee_{i\in I}\left(\mathrm{U}_i^{\mathrm{Fin}}\right)$.

Now, let $S \in \bigvee_{i\in I}\left(\mathrm{U}_i^{\mathrm{Fin}}\right)$. So by Theorem 2.1.4 there exists a regular semigroup T, a surjective homomorphism $\theta : T \to S$ and finite regular semigroups $S_{i_j} \in \mathrm{U}_{i_j}^{\mathrm{Fin}}$, $i_j \in I$, $j = 1, \ldots, n$ such that

$$S \overset{\theta}{\longleftarrow} T \leq_{\mathrm{reg}} S_{i_1} \times \cdots \times S_{i_n}.$$

However, each $S_{i_j} \in \mathrm{U}_{i_j}$, $j = 1, \ldots, n$ and so $S \in \bigvee_{i\in I} \mathrm{U}_i$. Since S is finite, $S \in \left(\bigvee_{i\in I} \mathrm{U}_i\right)^{\mathrm{Fin}}$. So $\left(\bigvee_{i\in I} \mathrm{U}_i\right)^{\mathrm{Fin}} = \bigvee_{i\in I}\left(\mathrm{U}_i^{\mathrm{Fin}}\right)$.

Now we show that ϕ is surjective: Let $\mathbf{W} \in \mathscr{L}_{\text{epv}}(\mathrm{V}^{\text{Fin}})$. By Theorem 2.3.7, $\mathbf{W} = (\langle \mathbf{W} \rangle_{\text{gev}})^{\text{Fin}}$ and by Lemma 2.4.13, $\langle \mathbf{W} \rangle_{\text{gev}}$ is locally finite. Therefore $\langle \mathbf{W} \rangle_{\text{gev}} \in \mathscr{L}_{\text{gev}}(G(\mathrm{V}))$ and $\langle \mathbf{W} \rangle_{\text{gev}}\phi = \mathbf{W}$. So ϕ is surjective. □

Corollary 2.4.19. *Let* $\mathrm{W} \in \mathscr{L}_{\text{gev}}(\mathrm{ES}) \cup \mathscr{L}_{\text{gev}}(\mathrm{LI})$. *Then* $\mathrm{W}^{\text{Fin}} = \left(\bigvee^{\text{gev}}_{S \in \mathrm{W}^{\text{Fin}}} \mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(S) \right)^{\text{Fin}}$.

Proof. By Theorem 2.3.11 we have that $\mathrm{W}^{\text{Fin}} = \bigvee^{\text{epv}}_{S \in \mathrm{W}^{\text{Fin}}} (\mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(S))^{\text{Fin}}$. Since each S is finite, $\mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(S)$ is a locally finite e-variety by Lemma 2.4.11. By the proof of Theorem 2.4.18, $\left(\bigvee^{\text{gev}}_{i \in I} \mathrm{U}_i \right)^{\text{Fin}} = \bigvee^{\text{epv}}_{i \in I} \mathrm{U}_i^{\text{Fin}}$ and so the result follows. □

The following result is proved for pseudovarieties and generalised varieties as [34, Lemma 4.1]. We will find this result useful in Chapter 3.

Lemma 2.4.20. *Let* $\mathbf{V} \in \mathscr{L}_{\text{epv}}(\mathbf{ES}) \cup \mathscr{L}_{\text{epv}}(\mathbf{LI})$ *and* $\mathrm{W} \in \mathscr{L}_{\text{gev}}(G(\mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(\mathbf{V})))$ *so that* W *is locally finite. Then*

$$\mathrm{W}^{\text{Fin}} = (\mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(\mathrm{W}))^{\text{Fin}}.$$

Proof. The proof follows directly from Lemma 2.4.10. □

3

Complete Congruences on Lattices of E-Varieties and E-Pseudovarieties

In this chapter we devise analogues for e-varieties and e-pseudovarieties of the results developed by Pastijn and Trotter in [34] for varieties and pseudovarieties. The theory of generalised e-varieties introduced in the previous chapter will be important in establishing these analogous results for e-pseudovarieties.

3.1 Introduction

Our aim is to demonstrate that the methods employed by Pastijn and Trotter in [34] to study congruences on lattices of varieties and pseudovarieties apply also to certain e-varieties and e-pseudovarieties. Methods for studying complete congruences on lattices of varieties, pseudovarieties and e-varieties have been developed by Auinger [8] and by Auinger, Hall, Reilly and Zhang [9]. Utilising the notion of a semidirect product, various properties of the lattice of e-pseudovarieties have been studied by Auinger and Trotter in [10]. However, our present approach is somewhat different. The paper [34] by Pastijn and Trotter extends many of the ideas of the earlier authors and motivates the present study. In particular, we present methods which allow all complete congruences on the lattices of e-varieties and e-pseudovarieties to be easily described, provided that the e-varieties or e-pseudovarieties consist of locally inverse or E-solid regular semigroups.

Suppose that ρ is a complete $\wedge$-congruence on a complete lattice (or complete $\wedge$-semilattice) L. Then for $a \in L$ we define $a_\rho = \wedge_{x \in a\rho} x$. Dually, if ρ is a complete $\vee$-congruence on a complete lattice (or complete $\vee$-semilattice) L, then for $a \in L$ we define $a^\rho = \vee_{x \in a\rho} x$. If ρ is a complete congruence on a complete lattice L then the congruence class containing $a \in L$, $a\rho = \{x \in L \,|\, a\rho x\}$ is an interval, $[a_\rho, a^\rho]$ where

$$a_\rho = \bigwedge_{x \in a\rho} x \text{ and } a^\rho = \bigvee_{x \in a\rho} x.$$

The following result will prove occasionally useful.

Lemma 3.1.1. *Let ρ be a complete $\wedge$-congruence on a complete lattice L. Then, $\{a_\rho \,|\, a \in L\}$ is a complete $\vee$-subsemilattice of L.*

Proof. Let $\{b_i \,|\, i \in I\} \subseteq \{a_\rho \,|\, a \in L\}$. It is clear that $(\vee_{i \in I} b_i)_\rho \leq \vee_{i \in I} b_i$. Now for all $b \in \{b_i \,|\, i \in I\}$,

$$b \leq \bigvee_{i \in I} b_i \Rightarrow b = b_\rho \leq \left(\bigvee_{i \in I} b_i \right)_\rho$$

and so $\vee_{i \in I} b_i \leq (\vee_{i \in I} b_i)_\rho$. Therefore $\vee_{i \in I} b_i = (\vee_{i \in I} b_i)_\rho$ and so $\{a_\rho \,|\, a \in L\}$ is a complete $\vee$-subsemilattice of L. □

The techniques described below allow us to describe (often, in several ways) the greatest lower bound a_ρ and the least upper bound a^ρ for each interval $[a_\rho, a^\rho]$ where ρ is a complete congruence on a lattice of e-varieties or e-pseudovarieties. We begin by examining complete $\cap$-congruences on lattice of e-varieties. The techniques examined in this section are quickly applied to complete $\cap$-congruences on certain lattices of e-pseudovarieties.

After examining complete $\cap$-congruences, we turn our attention to complete $\vee$-congruences. Again, our aim is to describe complete $\vee$-congruences on lattices of e-varieties by means of a simple (and quite natural) relation on the lattice. The techniques developed by Pastijn and Trotter are again able to be adapted to this new situation.

While the paper by Pastijn and Trotter shows that the use of divisor operators and congruence systems is somewhat less important than previously thought, we discuss these approaches to the study of complete congruences on lattices of e-varieties and e-pseudovarieties briefly.

3.2 Construction of Complete ∩-Congruences on Lattices of E-Varieties and E-Pseudovarieties

An important relation on lattices of varieties and pseudovarieties was introduced in [34]. In that paper, it was shown that from this relation, complete ∩-congruences can be quickly obtained, and furthermore, that all complete ∩-congruences are special cases of this relation. The original presentation was sufficiently general to allow us to quickly derive similar results for e-varieties and e-pseudovarieties. This is explored in the following sections.

3.2.1 A Fundamental Relation

Let V be an e-variety and let $\mathcal{A} \subseteq \mathrm{V}$. The relation $\theta_{\mathcal{A}}$ on $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$ is defined as follows:

$$\mathrm{U}_1 \theta_{\mathcal{A}} \mathrm{U}_2 \Leftrightarrow \mathrm{U}_1 \cap \mathcal{A} = \mathrm{U}_2 \cap \mathcal{A}.$$

It is clear that $\theta_{\mathcal{A}}$ is an equivalence relation on $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$ and it is also quickly demonstrated that $\theta_{\mathcal{A}}$ is a complete ∩-congruence on $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$, for any e-variety V of regular semigroups. That the above statements are true also for lattices of e-pseudovarieties of finite regular semigroups is equally evident.

The following straightforward results are adapted for e-varieties and e-pseudovarieties from [34, Theorem 3.1].

Lemma 3.2.1. *Let* V *be an e-variety and* $\mathcal{A} \subseteq \mathrm{V}$*. For any* $\mathrm{U} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{V})$*,* $\mathrm{U}_{\theta_{\mathcal{A}}} = \langle \mathrm{U} \cap \mathcal{A} \rangle_{\mathrm{ev}}$.

Proof. Suppose $\mathrm{W} \in \mathrm{U}\theta_{\mathcal{A}}$. Then,

$$\mathrm{U} \cap \mathcal{A} = \mathrm{W} \cap \mathcal{A} \subseteq \mathrm{W} \Rightarrow \langle \mathrm{U} \cap \mathcal{A} \rangle_{\mathrm{ev}} \subseteq \langle \mathrm{W} \rangle_{\mathrm{ev}} = \mathrm{W}$$

and so $\mathrm{U}_{\theta_{\mathcal{A}}} = \langle \mathrm{U} \cap \mathcal{A} \rangle_{\mathrm{ev}}$. □

Lemma 3.2.2. *Let* V *be an e-variety and* $\mathcal{A} \subseteq \mathrm{V}$. *Then, for all* $\mathrm{U}_1, \mathrm{U}_2 \in \mathscr{L}_{\mathrm{ev}}(\mathrm{V})$,

$$\mathrm{U}_1 \theta_{\mathcal{A}} \mathrm{U}_2 \Leftrightarrow \langle \mathrm{U}_1 \cap \mathcal{A} \rangle_{\mathrm{ev}} = \langle \mathrm{U}_2 \cap \mathcal{A} \rangle_{\mathrm{ev}}.$$

Proof. Suppose that $\mathrm{U}_1 \theta_{\mathcal{A}} \mathrm{U}_2$. Then

$$\begin{aligned} \mathrm{U}_1 \theta_{\mathcal{A}} \mathrm{U}_2 &\Leftrightarrow \mathrm{U}_1 \cap \mathcal{A} = \mathrm{U}_2 \cap \mathcal{A} \\ &\Rightarrow \langle \mathrm{U}_1 \cap \mathcal{A} \rangle_{\mathrm{ev}} = \langle \mathrm{U}_2 \cap \mathcal{A} \rangle_{\mathrm{ev}}. \end{aligned}$$

Conversely, suppose that $\langle \mathrm{U}_1 \cap \mathcal{A} \rangle_{\mathrm{ev}} = \langle \mathrm{U}_2 \cap \mathcal{A} \rangle_{\mathrm{ev}}$. Note that

$$\mathrm{U}_1 \cap \mathcal{A} \subseteq \langle \mathrm{U}_1 \cap \mathcal{A} \rangle_{\mathrm{ev}} \Rightarrow \mathrm{U}_1 \cap \mathcal{A} \subseteq \langle \mathrm{U}_1 \cap \mathcal{A} \rangle_{\mathrm{ev}} \cap \mathcal{A}$$

and that

$$\begin{aligned} \mathrm{U}_1 \cap \mathcal{A} \subseteq \mathrm{U}_1 &\Rightarrow \langle \mathrm{U}_1 \cap \mathcal{A} \rangle_{\mathrm{ev}} \subseteq \langle \mathrm{U}_1 \rangle_{\mathrm{ev}} = \mathrm{U}_1 \\ &\Rightarrow \langle \mathrm{U}_1 \cap \mathcal{A} \rangle_{\mathrm{ev}} \cap \mathcal{A} \subseteq \mathrm{U}_1 \cap \mathcal{A}. \end{aligned}$$

Therefore

$$\mathrm{U}_1 \cap \mathcal{A} = \langle \mathrm{U}_1 \cap \mathcal{A} \rangle_{\mathrm{ev}} \cap \mathcal{A}$$

and so

$$\mathrm{U}_1 \cap \mathcal{A} = \langle \mathrm{U}_1 \cap \mathcal{A} \rangle_{\mathrm{ev}} \cap \mathcal{A} = \langle \mathrm{U}_2 \cap \mathcal{A} \rangle_{\mathrm{ev}} \cap \mathcal{A} = \mathrm{U}_2 \cap \mathcal{A}.$$

Therefore $\mathrm{U}_1 \theta_{\mathcal{A}} \mathrm{U}_2$ and the proof is complete. □

Combining the previous results yields the following:

Corollary 3.2.3. *Let* V *be an e-variety and* $\mathcal{A} \subseteq \mathrm{V}$. *Then for all* $\mathrm{U}_1, \mathrm{U}_2 \in \mathscr{L}_{\mathrm{ev}}(\mathrm{V})$,

$$\mathrm{U}_1 \theta_{\mathcal{A}} \mathrm{U}_2 \Leftrightarrow (\mathrm{U}_1)_{\theta_{\mathcal{A}}} = (\mathrm{U}_2)_{\theta_{\mathcal{A}}}.$$

Note that in the previous results, there was no requirement that V be locally inverse or E-solid. Of course, if $\mathrm{V} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$ and $\mathcal{A} \subseteq \mathrm{V}$ then for any $\mathrm{U} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{V})$ we have that $\mathrm{U}_{\theta_{\mathcal{A}}} = \langle \mathrm{U} \cap \mathcal{A} \rangle_{\mathrm{ev}} = \mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}(\mathrm{U} \cap \mathcal{A})$.

The previous results are true also for e-pseudovarieties, as can readily be seen by straightforward modifications of the above proofs. Therefore if $\mathbf{V} \in \mathscr{L}_{\text{epv}}(\mathbf{ES}) \cup \mathscr{L}_{\text{epv}}(\mathbf{LI})$ and $\mathcal{A} \subseteq \mathbf{V}$ then for any $\mathbf{U} \in \mathscr{L}_{\text{epv}}(\mathbf{V})$ we have that $\mathbf{U}_{\theta_{\mathcal{A}}} = \mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}_{\text{f}}(\mathbf{U} \cap \mathcal{A})$.

The following result (in a more general context) appears in the proof of [34, Theorem 3.1]. We verify it here for e-varieties.

Lemma 3.2.4. *Let* $\text{V} \in \mathscr{L}_{\text{ev}}(\text{ES}) \cup \mathscr{L}_{\text{ev}}(\text{LI})$ *and* $\mathcal{A} \subseteq \text{V}$. *Then, for each* $S \in \mathcal{A}$, $(\mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(S))_{\theta_{\mathcal{A}}} = \mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(S)$.

Proof. Clearly, $\mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(S) \supseteq (\mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(S))_{\theta_{\mathcal{A}}}$. Conversely, since $S \in \mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(S)$, we have

$$S \in \mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(S) \cap \mathcal{A} \subseteq \mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(\mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(S) \cap \mathcal{A}) = (\mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(S))_{\theta_{\mathcal{A}}}.$$

Consequently, $\{S\} \subseteq (\mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(S))_{\theta_{\mathcal{A}}} \Rightarrow \mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(S) \subseteq \mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}((\mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(S))_{\theta_{\mathcal{A}}}) = (\mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(S))_{\theta_{\mathcal{A}}}$ and so $(\mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(S))_{\theta_{\mathcal{A}}} = \mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(S)$. □

The question that we wish to answer is: Are all complete ∩-congruences on lattices of e-varieties and e-pseudovarieties of the form $\theta_{\mathcal{A}}$ for some suitable class $\mathcal{A}$? The answer, as we shall see below, is a qualified "yes".

The following concept of monogenic operators will assist us when describing complete congruences on lattices of e-varieties of regular semigroups.

3.2.2 Monogenic Operators

We recall below the notion of a monogenic operator on a class of algebras. This notion was used successfully in [34] to help find and describe complete conguences on certain classes of algebras.

An operator $\mathbb{K}$ on a class C is a *closure operator* if for all X, Y $\subseteq$ C,

(i) $\text{X} \subseteq \mathbb{K}(\text{X})$;

(ii) $\mathbb{K}^2(\text{X}) = \mathbb{K}(\text{X})$; and

(iii) $\text{X} \subseteq \text{Y} \Rightarrow \mathbb{K}(\text{X}) \subseteq \mathbb{K}(\text{Y})$.

Let C be a class of algebras closed under $\mathbb{K}$ and let $\mathscr{L}(\mathrm{C})$ be the lattice of all $\mathbb{K}$-closed subclasses of C. If for each $\mathrm{X} \in \mathscr{L}(\mathrm{C})$ there exists an $x \in \mathrm{X}$ such that $\mathbb{K}(x) = \mathrm{X}$ we say that $\mathbb{K}$ is a *monogenic operator for* C.

For example, consider the operator $\mathbb{HSP}$ acting upon a variety V of semigroups. It is well known that $\mathbb{HSP}$ is a closure operator for a variety V and since every variety W of semigroups contained in $\mathscr{L}_{\mathrm{v}}(\mathrm{V})$ is generated by a free semigroup $F_X(\mathrm{V}) \in \mathrm{W}$ on a countably infinite set X, we have that $\mathbb{HSP}$ is a monogenic operator for V.

In any situation in which no confusion should arise, we may simply state that the closure operator $\mathbb{K}$ is *monogenic* (without referring to a particular class C).

We now establish an important fact about monogenic operators and existence varieties.

Lemma 3.2.5. *The operator* $\mathbb{HS}_{\mathrm{e}}\mathbb{P}$ *is monogenic for* ES *and for* LI*.*

Proof. By [45, Theorem 4.12] we have that each e-variety V which consists entirely of E-solid or locally inverse semigroups is generated by a bi-free object $BF_X(\mathrm{V})$ on a countably infinite set X and by [45, Lemma 4.8] we have that $\mathbb{HS}_{\mathrm{e}}\mathbb{P}(\mathcal{C})$ is the least e-variety containing $\mathcal{C}$, provided that $\mathcal{C}$ consists entirely of E-solid or locally inverse regular semigroups. □

It is not the case, however, that $\mathbb{HSP}_{\mathrm{f}}$ is monogenic for **S** nor is $\mathbb{HS}_{\mathrm{e}}\mathbb{P}_{\mathrm{f}}$ monogenic for the e-pseudovarieties **ES** or **LI**. The usual counterexample given is that of finite bands: The class **B** of all finite bands is not of the form $\mathbb{HSP}_{\mathrm{f}}(B)$ ($\mathbb{HS}_{\mathrm{e}}\mathbb{P}_{\mathrm{f}}(B)$) for some finite band B since each finite band generates a proper sub-pseudovariety (sub-e-pseudovariety) of **B**.

For completeness we list all class operators which will be used in this chapter. Many of these have already featured prominently in the first two chapters. Let $\mathcal{C}$ be a class of regular semigroups:

$\mathbb{H}(\mathcal{C})$	-	the class of all homomorphic images of members of $\mathcal{C}$;
$\mathbb{S}_{\mathrm{e}}(\mathcal{C})$	-	the class of all regular subsemigroups of members of $\mathcal{C}$;
$\mathbb{P}(\mathcal{C})$	-	the class of all direct products of members of $\mathcal{C}$;
$\mathbb{P}_{\mathrm{f}}(\mathcal{C})$	-	the class of all finite direct products of members of $\mathcal{C}$;
$\mathbb{P}_{\mathrm{ow}}(\mathcal{C})$	-	the class of all powers of members of $\mathcal{C}$;
$\mathbb{P}_{\mathrm{s}}(\mathcal{C})$	-	the class of all subdirect products of members of $\mathcal{C}$;
$\mathbb{P}_{\mathrm{sf}}(\mathcal{C})$	-	the class of all subdirect products of a finite number of members of $\mathcal{C}$;
$\mathbb{I}(\mathcal{C})$	-	the class of all isomorphic copies of members of $\mathcal{C}$.

Throughout this chapter, we assume that for each $\mathbb{K} \in \{\mathbb{H}, \mathbb{S}_{\mathrm{e}}, \mathbb{P}, \mathbb{P}_{\mathrm{f}}, \mathbb{P}_{\mathrm{ow}}, \mathbb{P}_{\mathrm{s}}, \mathbb{P}_{\mathrm{sf}}\}$, $\mathbb{I}(\mathcal{C}) \subseteq \mathbb{K}(\mathcal{C})$.

3.2.3 Complete ∩-Congruences on Lattices of E-Varieties

For an e-variety V and $\mathcal{A} \subseteq \mathrm{V}$ we have already established that $\theta_{\mathcal{A}}$ is a complete ∩-congruence on $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$. We now prove that all complete ∩-congruences on $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$ are of the form $\theta_{\mathcal{A}}$ for some class $\mathcal{A} \subseteq \mathrm{V}$, if $\mathrm{V} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$.

For an e-variety V and a complete ∩-congruence θ on $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$ we say that $\mathcal{A} \subseteq \mathrm{V}$ *determines* θ if $\theta = \theta_{\mathcal{A}}$. The following result, whose proof follows directly from [34, Theorem 3.2], restated here for e-varieties, indicates that for each e-variety V and complete ∩-congruence θ on $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$, we should be looking for the largest class $\mathcal{A}$ which determines θ.

Lemma 3.2.6. *Let* V *be an e-variety, let* $\mathcal{A} \subseteq \mathrm{V}$ *and let* $\{\mathcal{A}_i \mid i \in I\}$ *be a family of subclasses of* V *such that* $\theta_{\mathcal{A}} = \theta_{\mathcal{A}_i}$ *for all* $i \in I$*. Then* $\theta_{\mathcal{A}} = \theta_{\cup_{i \in I} \mathcal{A}_i}$.

Proof. Suppose $\mathrm{U}_1 \theta_{\mathcal{A}} \mathrm{U}_2$. Then

$$\begin{aligned} \mathrm{U}_1 \theta_{\mathcal{A}_i} \mathrm{U}_2 &\Leftrightarrow \mathrm{U}_1 \cap \mathcal{A}_i = \mathrm{U}_2 \cap \mathcal{A}_i \text{ for each } i \in I \\ &\Leftrightarrow \textstyle\bigcup_{i \in I} (\mathrm{U}_1 \cap \mathcal{A}_i) = \bigcup_{i \in I} (\mathrm{U}_2 \cap \mathcal{A}_i) \\ &\Leftrightarrow \mathrm{U}_1 \cap (\textstyle\bigcup_{i \in I} \mathcal{A}_i) = \mathrm{U}_2 \cap (\bigcup_{i \in I} \mathcal{A}_i) \\ &\Leftrightarrow \mathrm{U}_1 \theta_{\cup_{i \in I} \mathcal{A}_i} \mathrm{U}_2. \end{aligned}$$

Therefore $\theta_{\mathcal{A}} = \theta_{\cup_{i \in I} \mathcal{A}_i}$ and so the proof is complete. □

Lemma 3.2.7. *Let* $\mathrm{V} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$ *and let* $\mathcal{A} \subseteq \mathrm{V}$ *satisfy the properties*

(i) $\mathbb{I}(\mathcal{A}) \subseteq \mathcal{A}$*; and*

(ii) $\mathrm{T} \subseteq \mathcal{A}$.

Then $\mathbb{P}(\mathcal{A})$ *and* $\mathbb{P}_{\mathrm{s}}(\mathcal{A})$ *satisfy properties (i) and (ii). Furthermore,* $\theta_{\mathcal{A}} = \theta_{\mathbb{P}(\mathcal{A})} = \theta_{\mathbb{P}_{\mathrm{s}}(\mathcal{A})}$.

Proof. The proof is similar to that given in [34, Theorem 3.5]. That $\mathbb{P}(\mathcal{A})$ and $\mathbb{P}_{\mathrm{s}}(\mathcal{A})$ satisfy properties (i) and (ii) follows directly from the definitions of $\mathbb{P}$ and $\mathbb{P}_{\mathrm{s}}$. Let $\mathrm{W} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{V})$. Clearly, $\mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}(\mathrm{W} \cap \mathcal{A}) \subseteq \mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}(\mathrm{W} \cap \mathbb{P}(\mathcal{A}))$. To establish the reverse containment, suppose that $S \in \mathrm{W} \cap \mathbb{P}(\mathcal{A})$. So $S = \prod_{j \in J} S_j$ where each $S_j \in \mathcal{A}$. Since $S \in \mathrm{W}$ we have that for each $j \in J$, $S_j \in \mathrm{W}$ and so $S \in \mathbb{P}(\mathrm{W} \cap \mathcal{A})$. Therefore $\mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}(\mathrm{W} \cap \mathbb{P}(\mathcal{A})) \subseteq \mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}(\mathrm{W} \cap \mathcal{A})$ and so $\mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}(\mathrm{W} \cap \mathbb{P}(\mathcal{A})) = \mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}(\mathrm{W} \cap \mathcal{A})$. Now, suppose $\mathrm{U}_1 \theta_{\mathcal{A}} \mathrm{U}_2$. Then (utilising the comment following Corollary 3.2.3) we have

$$\begin{aligned} \mathrm{U}_1 \cap \mathcal{A} = \mathrm{U}_2 \cap \mathcal{A} &\Leftrightarrow \mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}(\mathrm{U}_1 \cap \mathcal{A}) = \mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}(\mathrm{U}_2 \cap \mathcal{A}) \\ &\Leftrightarrow \mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}(\mathrm{U}_1 \cap \mathbb{P}(\mathcal{A})) = \mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}(\mathrm{U}_2 \cap \mathbb{P}(\mathcal{A})) \\ &\Leftrightarrow \mathrm{U}_1 \cap \mathbb{P}(\mathcal{A}) = \mathrm{U}_2 \cap \mathbb{P}(\mathcal{A}). \end{aligned}$$

Therefore $\mathrm{U}_1 \theta_{\mathbb{P}(\mathcal{A})} \mathrm{U}_2$. The proof that $\theta_{\mathcal{A}} = \theta_{\mathbb{P}_{\mathrm{s}}(\mathcal{A})}$ is similar. □

We are now in a position to describe all complete $\cap$-congruences on $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$ where $\mathrm{V} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$.

Theorem 3.2.8. *Let* $\mathrm{V} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$ *and* θ *a complete* $\cap$*-congruence on* $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$*. Let* $\mathcal{V} \subseteq \mathrm{V}$ *be as follows:*

$$\mathcal{V} = \{S \in \mathrm{V} \mid (\mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}\,(S))_\theta = \mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}\,(S)\}.$$

Then $\theta = \theta_{\mathcal{V}}$ *and* $\mathcal{V}$ *is the largest subclass of* V *with this property.*

Proof. Suppose $\mathrm{U}_1 \theta \mathrm{U}_2$. Then $(\mathrm{U}_1)_\theta = (\mathrm{U}_2)_\theta$. Since $\mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}$ is monogenic for V there exist S_1 and $S_2 \in \mathrm{V}$ such that $(\mathrm{U}_1)_\theta = \mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}(S_1)$ and $(\mathrm{U}_2)_\theta = \mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}(S_2)$. Therefore

$S_1, S_2 \in \mathcal{V}$. We now have the following:

$$\begin{aligned} \mathrm{U}_1\theta\mathrm{U}_2 &\Leftrightarrow (\mathrm{U}_1)_\theta = (\mathrm{U}_2)_\theta \\ &\Leftrightarrow \mathbb{HS}_\mathrm{e}\mathbb{P}(S_1) = \mathbb{HS}_\mathrm{e}\mathbb{P}(S_2) \\ &\Leftrightarrow (\mathbb{HS}_\mathrm{e}\mathbb{P}(S_1))_{\theta_\mathcal{V}} = (\mathbb{HS}_\mathrm{e}\mathbb{P}(S_2))_{\theta_\mathcal{V}} \text{ (by Lemma 3.2.4)} \\ &\Leftrightarrow \mathbb{HS}_\mathrm{e}\mathbb{P}(S_1)\theta_\mathcal{V}\mathbb{HS}_\mathrm{e}\mathbb{P}(S_2) \\ &\Leftrightarrow (\mathrm{U}_1)_\theta\theta_\mathcal{V}(\mathrm{U}_2)_\theta \\ &\Leftrightarrow \mathrm{U}_1\theta_\mathcal{V}\mathrm{U}_2. \end{aligned}$$

So $\mathcal{V}$ determines θ. The proof that $\mathcal{V}$ is the largest class which determines θ given in [34, Theorem 3.1 (ii)] is restated here for e-varieties. Suppose $\theta = \theta_\mathcal{U}$. Let $S \in \mathcal{U}$. Then

$$S \in \mathbb{HS}_\mathrm{e}\mathbb{P}(S) \cap \mathcal{U} \subseteq \mathbb{HS}_\mathrm{e}\mathbb{P}(\mathbb{HS}_\mathrm{e}\mathbb{P}(S) \cap \mathcal{U}) = (\mathbb{HS}_\mathrm{e}\mathbb{P}(S))_\theta$$

by the remark following Corollary 3.2.3 and so $\mathbb{HS}_\mathrm{e}\mathbb{P}(S) \subseteq \mathbb{HS}_\mathrm{e}\mathbb{P}((\mathbb{HS}_\mathrm{e}\mathbb{P}(S))_\theta) = (\mathbb{HS}_\mathrm{e}\mathbb{P}(S))_\theta$. Since $(\mathbb{HS}_\mathrm{e}\mathbb{P}(S))_\theta \subseteq \mathbb{HS}_\mathrm{e}\mathbb{P}(S)$ we have that $\mathbb{HS}_\mathrm{e}\mathbb{P}(S) = (\mathbb{HS}_\mathrm{e}\mathbb{P}(S))_\theta$ for all $S \in \mathcal{U}$. Therefore $\mathcal{U} \subseteq \mathcal{V}$ and the proof is complete. □

Corollary 3.2.9. *Let* $\mathrm{V} \in \mathscr{L}_\mathrm{ev}(\mathrm{ES}) \cup \mathscr{L}_\mathrm{ev}(\mathrm{LI})$ *and* θ *a complete* $\cap$*-congruence on* $\mathscr{L}_\mathrm{ev}(\mathrm{V})$*. For any* $\mathrm{U} \in \mathscr{L}_\mathrm{ev}(\mathrm{V})$*, we have that*

$$\mathrm{U}_\theta = \bigvee \{\mathbb{HS}_\mathrm{e}\mathbb{P}(S) \mid S \in \mathrm{U} \text{ and } (\mathbb{HS}_\mathrm{e}\mathbb{P}(S))_\theta = \mathbb{HS}_\mathrm{e}\mathbb{P}(S)\}.$$

Proof. Since (by Theorem 3.2.8) $\theta = \theta_\mathcal{V}$ where

$$\mathcal{V} = \{S \in \mathrm{V} \mid (\mathbb{HS}_\mathrm{e}\mathbb{P}(S))_\theta = \mathbb{HS}_\mathrm{e}\mathbb{P}(S)\}$$

we have that

$$\mathrm{U}_\theta = \mathrm{U}_{\theta_\mathcal{V}} = \mathbb{HS}_\mathrm{e}\mathbb{P}(\mathrm{U} \cap \mathcal{V}) = \bigvee_{S \in \mathrm{U}\cap\mathcal{V}} \mathbb{HS}_\mathrm{e}\mathbb{P}(S) = \bigvee_{S \in \mathrm{U}\cap\mathcal{V}} (\mathbb{HS}_\mathrm{e}\mathbb{P}(S))_\theta$$

thus completing the proof. □

3.2.4 Complete ∩-Congruences on Lattices of E-Pseudovarieties

Let $\mathbf{V}$ be an e-pseudovariety which consists entirely of finite E-solid or locally inverse regular semigroups and let θ be a complete $\cap$-congruence on $\mathscr{L}_\mathrm{epv}(\mathbf{V})$.

Suppose that for each $\mathbf{W} \in \mathscr{L}_{\text{epv}}(\mathbf{V})$ we have that

$$\mathbf{W}_\theta = \bigvee\{\mathbb{H}\mathbb{S}_\text{e}\mathbb{P}_\text{f}(S) \mid S \in \mathbf{W} \text{ and } (\mathbb{H}\mathbb{S}_\text{e}\mathbb{P}_\text{f}(S))_\theta = \mathbb{H}\mathbb{S}_\text{e}\mathbb{P}_\text{f}(S)\}.$$

Then the following result, analogous to [34, Theorem 3.6], may be obtained.

Theorem 3.2.10. *Let* $\mathbf{V} \in \mathscr{L}_{\text{epv}}(\mathbf{ES}) \cup \mathscr{L}_{\text{epv}}(\mathbf{LI})$ *satisfy the condition above and let* θ *be a complete* $\cap$*-congruence on* $\mathscr{L}_{\text{epv}}(\mathbf{V})$*. Then* $\theta = \theta_{\mathcal{V}}$ *where*

$$\mathcal{V} = \{S \in \mathbf{V} \mid (\mathbb{H}\mathbb{S}_\text{e}\mathbb{P}_\text{f}\,(S))_\theta = \mathbb{H}\mathbb{S}_\text{e}\mathbb{P}_\text{f}\,(S)\}.$$

Proof. Suppose that for each $\mathbf{W} \in \mathscr{L}_{\text{epv}}(\mathbf{V})$ we have that

$$\mathbf{W}_\theta = \bigvee\{\mathbb{H}\mathbb{S}_\text{e}\mathbb{P}_\text{f}(S) \mid S \in \mathbf{W} \text{ and } (\mathbb{H}\mathbb{S}_\text{e}\mathbb{P}_\text{f}(S))_\theta = \mathbb{H}\mathbb{S}_\text{e}\mathbb{P}_\text{f}(S)\}.$$

Suppose $\mathcal{V} = \{S \in \mathbf{V} \mid (\mathbb{H}\mathbb{S}_\text{e}\mathbb{P}_\text{f}\,(S))_\theta = \mathbb{H}\mathbb{S}_\text{e}\mathbb{P}_\text{f}\,(S)\}$, $\mathbf{U}_1, \mathbf{U}_2 \in \mathscr{L}_{\text{epv}}(\mathbf{V})$ and let

$$\begin{aligned} \mathcal{U}_1 &= \{S \in \mathbf{U}_1 \mid (\mathbb{H}\mathbb{S}_\text{e}\mathbb{P}_\text{f}(S))_\theta = \mathbb{H}\mathbb{S}_\text{e}\mathbb{P}_\text{f}(S)\}; \text{ and} \\ \mathcal{U}_2 &= \{S \in \mathbf{U}_2 \mid (\mathbb{H}\mathbb{S}_\text{e}\mathbb{P}_\text{f}(S))_\theta = \mathbb{H}\mathbb{S}_\text{e}\mathbb{P}_\text{f}(S)\}. \end{aligned}$$

So

$$\begin{aligned} \mathbf{U}_1\theta\mathbf{U}_2 &\Leftrightarrow (\mathbf{U}_1)_\theta = (\mathbf{U}_2)_\theta \\ &\Leftrightarrow \bigvee\{\mathbb{H}\mathbb{S}_\text{e}\mathbb{P}_\text{f}(S) \mid S \in \mathcal{U}_1\} = \bigvee\{\mathbb{H}\mathbb{S}_\text{e}\mathbb{P}_\text{f}(S) \mid S \in \mathcal{U}_2\} \\ &\Leftrightarrow \mathbb{H}\mathbb{S}_\text{e}\mathbb{P}_\text{f}(\mathbf{U}_1 \cap \mathcal{V}) = \mathbb{H}\mathbb{S}_\text{e}\mathbb{P}_\text{f}(\mathbf{U}_2 \cap \mathcal{V}) \\ &\Leftrightarrow \mathbf{U}_1 \cap \mathcal{V} = \mathbf{U}_2 \cap \mathcal{V} \\ &\Leftrightarrow \mathbf{U}_1\theta_{\mathcal{V}}\mathbf{U}_2. \end{aligned}$$

Therefore $\theta = \theta_{\mathcal{V}}$. □

Corollary 3.2.11. *Let* $\mathcal{V}$ *be as described in the previous theorem. Then*

(i) $\mathbb{I}(\mathcal{V}) \subseteq \mathcal{V}$*;*

(ii) $\mathbb{P}_{\text{sf}}(\mathcal{V}) \subseteq \mathcal{V}$*;*

(iii) $\mathbf{T} \subseteq \mathcal{V}$*; and*

(iv) if $\mathcal{U} \subseteq \mathbf{V}$ *has the property that* $\theta_{\mathcal{U}} = \theta$*, then* $\mathcal{U} \subseteq \mathcal{V}$*.*

Proof. (i)-(iii) follow directly from the definitions. To prove (iv), recall that $\mathbb{H}\mathbb{S}_\text{e}\mathbb{P}_\text{f}(S) = (\mathbb{H}\mathbb{S}_\text{e}\mathbb{P}_\text{f}(S))_{\theta_{\mathcal{U}}}$ for all $S \in \mathcal{U}$. Since $\theta = \theta_{\mathcal{U}}$ we have $\mathbb{H}\mathbb{S}_\text{e}\mathbb{P}_\text{f}(S) = (\mathbb{H}\mathbb{S}_\text{e}\mathbb{P}_\text{f}(S))_\theta$ and so $S \in \mathcal{V}$. □

3.3 Complete $\bigvee$-Congruences on Lattices of E-Varieties

Having demonstrated in the previous section that complete $\cap$-conguences on e-varieties of regular semigroups which are contained in $\mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$ may be described in a straightforward way using a simple relation on these lattices, we turn our attention to complete $\vee$-congruences on lattices of e-varieties. As in the earlier work, the methods used by Pastijn and Trotter in [34] are adapted in a natural way.

3.3.1 Complete $\vee$-Congruences Via the Fundamental Relation

The following results summarise for e-varieties [34, Theorems 3.4 and 3.11]. The proofs follow those of Pastijn and Trotter. We present them here for completeness.

Theorem 3.3.1. *Let θ be a complete $\cap$-congruence on $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$ where $\mathrm{V} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$ and let $\{\mathrm{V}_i \,|\, i \in I\}$ be a collection of e-varieties contained in V. Then θ is a complete congruence on $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$ if and only if*

$$\bigvee_{i\in I}(\mathrm{V}_i)_\theta = \left(\bigvee_{i\in I}\mathrm{V}_i\right)_\theta .$$

Proof. Since θ is a complete $\cap$-congruence on $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$, we have (by Lemma 3.1.1) that the collection of e-varieties $\{\mathrm{W}_\theta \,|\, \mathrm{W} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{V})\}$ forms a complete $\vee$-sub-semilattice of $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$. In particular, we have that $\bigvee_{i\in I}(\mathrm{V}_i)_\theta = (\bigvee_{i\in I}(V_i)_\theta)_\theta$.

Now, since θ is a complete congruence,

$$\begin{aligned}\mathrm{V}_i\theta(\mathrm{V}_i)_\theta &\Rightarrow \textstyle\bigvee_{i\in I}\mathrm{V}_i\theta\bigvee_{i\in I}(\mathrm{V}_i)_\theta \\ &\Rightarrow (\textstyle\bigvee_{i\in I}\mathrm{V}_i)_\theta = (\bigvee_{i\in I}(\mathrm{V}_i)_\theta)_\theta\end{aligned}$$

and so $\bigvee_{i\in I}(\mathrm{V}_i)_\theta = (\bigvee_{i\in I}\mathrm{V}_i)_\theta$ as required.

Suppose now that θ is a complete $\cap$-congruence and that $\bigvee_{i\in I}(\mathrm{V}_i)_\theta = (\bigvee_{i\in I}\mathrm{V}_i)_\theta$ for every collection $\{\mathrm{V}_i \,|\, i \in I\}$ of e-varieties contained in $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$. Let $\{\mathrm{U}_i \,|\, i \in I\}$ be a collection of e-varieties contained in $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$ such that $\mathrm{U}_i\theta\mathrm{V}_i$ for all $i \in I$.

Note that $\bigvee_{i\in I}(\mathrm{U}_i)_\theta = (\bigvee_{i\in I}\mathrm{U}_i)_\theta$ and so

$$\begin{aligned}\textstyle\bigvee_{i\in I}\mathrm{U}_i\theta\,(\bigvee_{i\in I}\mathrm{U}_i)_\theta &= \textstyle\bigvee_{i\in I}(\mathrm{U}_i)_\theta \\ &= \textstyle\bigvee_{i\in I}(\mathrm{V}_i)_\theta \\ &= \textstyle(\bigvee_{i\in I}\mathrm{V}_i)_\theta\,\theta\bigvee_{i\in I}\mathrm{V}_i.\end{aligned}$$

Therefore $\bigvee_{i\in I}\mathrm{U}_i\theta\bigvee_{i\in I}\mathrm{V}_i$ and θ is a complete congruence. □

Theorem 3.3.2. *Let θ be a complete $\cap$-congruence on $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$ where $\mathrm{V}\in\mathscr{L}_{\mathrm{ev}}(\mathrm{ES})\cup\mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$ and let $\{\mathrm{V}_i\,|\,i\in I\}$ be a collection of e-varieties contained in V. If $\theta=\theta_{\mathcal{A}}$ for some $\mathcal{A}\subseteq\mathrm{V}$ then θ is a complete congruence on $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$ if and only if*

$$\mathbb{HS}_{\mathrm{e}}\mathbb{P}\left(\left(\bigvee_{i\in I}\mathrm{V}_i\right)\cap\mathcal{A}\right)=\mathbb{HS}_{\mathrm{e}}\mathbb{P}\left(\left(\bigcup_{i\in I}\mathrm{V}_i\right)\cap\mathcal{A}\right).$$

Proof. Let θ be a complete $\cap$-congruence and suppose that $\theta=\theta_{\mathcal{A}}$ for some $\mathcal{A}\subseteq\mathrm{V}$. Then

$$\begin{aligned}\mathbb{HS}_{\mathrm{e}}\mathbb{P}\left((\vee_{i\in I}\mathrm{V}_i)\cap\mathcal{A}\right) &= (\vee_{i\in I}\mathrm{V}_i)_{\theta_{\mathcal{A}}} \\ &= (\vee_{i\in I}\mathrm{V}_i)_{\theta}\,.\end{aligned}$$

Also,

$$\begin{aligned}\mathbb{HS}_{\mathrm{e}}\mathbb{P}\left((\cup_{i\in I}\mathrm{V}_i)\cap\mathcal{A}\right) &= \mathbb{HS}_{\mathrm{e}}\mathbb{P}\left(\cup_{i\in I}\left(\mathrm{V}_i\cap\mathcal{A}\right)\right) \\ &= \vee_{i\in I}\mathbb{HS}_{\mathrm{e}}\mathbb{P}\left(\mathrm{V}_i\cap\mathcal{A}\right) \\ &= \vee_{i\in I}\left(\mathrm{V}_i\right)_\theta\,.\end{aligned}$$

Now, by Theorem 3.3.1 we have that θ is a complete congruence on $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$ if and only if

$$\mathbb{HS}_{\mathrm{e}}\mathbb{P}\left(\left(\bigvee_{i\in I}\mathrm{V}_i\right)\cap\mathcal{A}\right)=\mathbb{HS}_{\mathrm{e}}\mathbb{P}\left(\left(\bigcup_{i\in I}\mathrm{V}_i\right)\cap\mathcal{A}\right)$$

as required. □

The following result, whose proof is straightforward, is analogous to [34, Corollary 3.12].

Corollary 3.3.3. *Let $\mathrm{V}\in\mathscr{L}_{\mathrm{ev}}(\mathrm{ES})\cup\mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$ and let $\mathcal{A}\subseteq\mathrm{V}$ be such that*

(i) $\mathbb{I}(\mathcal{A})\subseteq\mathcal{A}$*; and*

(ii) $\mathrm{T}\subseteq\mathcal{A}$.

If $\theta_{\mathcal{A}}$ is a complete congruence on $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$ and $\mathrm{U}\in\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$ then

$$\mathrm{U}^{\theta_{\mathcal{A}}}=\{S\in\mathrm{V}\,|\,\langle S\rangle_{\mathrm{ev}}\cap\mathcal{A}\subseteq\mathrm{U}\}.$$

3.3.2 Regular Divisor Operators

Recall that a semigroup T is said to be a divisor of S if T is the homomorphic image of a subsemigroup of S. We can extend this definition to regular semigroups: Let S be a regular semigroup. We say that T *regularly divides* S if T is a homomorphic image of a regular subsemigroup of S. It follows, of course, that T is regular.

This idea may be formulated as a class operator. For any class $\mathcal{C}$ consisting entirely of E-solid or locally inverse semigroups and any $S, T \in \mathrm{V}$ where $\mathrm{V} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$ we say that the operator $\mathbb{D}_{\mathrm{e}}$ defined on $\mathcal{C}$ is a regular divisor operator if:

(i) $\mathrm{T} \subseteq \mathbb{D}_{\mathrm{e}}(\mathcal{C}) \subseteq \mathbb{HS}_{\mathrm{e}}(\mathcal{C})$;

(ii) $\mathbb{D}_{\mathrm{e}}(\mathcal{C})$ is closed under $\mathbb{I}$;

(iii) $\mathbb{D}_{\mathrm{e}}(\mathcal{C}) = \bigcup_{S \in \mathcal{C}} \mathbb{D}_{\mathrm{e}}(S)$;

(iv) if $S \in \mathbb{D}_{\mathrm{e}}(T)$ then $S \in \mathbb{D}_{\mathrm{e}}(S)$; and

(v) if $S \in \mathbb{HS}_{\mathrm{e}}(T)$ then $\mathbb{D}_{\mathrm{e}}(S) \subseteq \mathbb{HS}_{\mathrm{e}}\mathbb{D}_{\mathrm{e}}(T)$.

The description of the regular divisor operator above applies also to classes of finite regular semigroups, after replacing T with $\mathbf{T}$, the class of all finite trivial semigroups.

Let $\mathrm{V} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$ and let $\theta_{\mathbb{D}_{\mathrm{e}}(\mathrm{V})}$ denote the relation on $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$ defined by:

$$\mathrm{U}_1 \theta_{\mathbb{D}_{\mathrm{e}}(\mathrm{V})} \mathrm{U}_2 \Leftrightarrow \mathrm{U}_1 \cap \mathbb{D}_{\mathrm{e}}(\mathrm{V}) = \mathrm{U}_2 \cap \mathbb{D}_{\mathrm{e}}(\mathrm{V}).$$

Clearly, $\theta_{\mathbb{D}_{\mathrm{e}}(\mathrm{V})}$ is a complete $\cap$-congruence on $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$ and we have $\mathbb{D}_{\mathrm{e}}(\mathrm{U}) = \mathrm{U} \cap \mathbb{D}_{\mathrm{e}}(\mathrm{V})$ where $\mathrm{U} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{V})$. Pastijn and Trotter [34, Theorem 3.7] established an analogous result for varieties of algebras. For existence varieties we have the following result:

Lemma 3.3.4. *Let* $\mathrm{V} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{LI}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{ES})$ *and let* $\mathrm{U}_1, \mathrm{U}_2 \in \mathscr{L}_{\mathrm{ev}}(\mathrm{V})$. *Then*

$$\begin{aligned} \mathrm{U}_1 \theta_{\mathbb{D}_{\mathrm{e}}(\mathrm{V})} \mathrm{U}_2 &\Leftrightarrow \mathbb{D}_{\mathrm{e}}(\mathrm{U}_1) = \mathbb{D}_{\mathrm{e}}(\mathrm{U}_2) \\ &\Leftrightarrow \mathbb{HS}_{\mathrm{e}}\mathbb{PD}_{\mathrm{e}}(\mathrm{U}_1) = \mathbb{HS}_{\mathrm{e}}\mathbb{PD}_{\mathrm{e}}(\mathrm{U}_2). \end{aligned}$$

Proof. Following Pastijn and Trotter we have that:

$$\begin{aligned}\mathbb{D}_{\mathrm{e}}(\mathrm{U}_1) &= \bigcup_{S\in \mathrm{U}_1} (\mathbb{D}_{\mathrm{e}}(\mathrm{V}) \cap \mathbb{HS}_{\mathrm{e}}(S)) \\ &= \mathbb{D}_{\mathrm{e}}(\mathrm{V}) \cap (\bigcup_{S\in \mathrm{U}_1} \mathbb{HS}_{\mathrm{e}}(S)) \\ &= \mathbb{D}_{\mathrm{e}}(\mathrm{V}) \cap \mathrm{U}_1\end{aligned}$$

and so $\mathrm{U}_1\theta_{\mathbb{D}_{\mathrm{e}}(\mathrm{V})}\mathrm{U}_2 \Leftrightarrow \mathbb{D}_{\mathrm{e}}(\mathrm{U}_1) = \mathbb{D}_{\mathrm{e}}(\mathrm{U}_2)$.

Consequently

$$\begin{aligned}\mathrm{U}_1\theta_{\mathbb{D}_{\mathrm{e}}(\mathrm{V})}\mathrm{U}_2 &\Leftrightarrow \mathrm{U}_1 \cap \mathbb{D}_{\mathrm{e}}(\mathrm{V}) = \mathrm{U}_2 \cap \mathbb{D}_{\mathrm{e}}(\mathrm{V}) \\ &\Leftrightarrow \mathbb{HS}_{\mathrm{e}}\mathbb{P}(\mathrm{U}_1 \cap \mathbb{D}_{\mathrm{e}}(\mathrm{V})) = \mathbb{HS}_{\mathrm{e}}\mathbb{P}(\mathrm{U}_2 \cap \mathbb{D}_{\mathrm{e}}(\mathrm{V})) \\ &\Leftrightarrow \mathbb{HS}_{\mathrm{e}}\mathbb{P}\mathbb{D}_{\mathrm{e}}(\mathrm{U}_1) = \mathbb{HS}_{\mathrm{e}}\mathbb{P}\mathbb{D}_{\mathrm{e}}(\mathrm{U}_2)\end{aligned}$$

and the proof is complete. □

Let $\mathrm{V} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{LI}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{ES})$ and let $\mathcal{U} \subseteq \mathrm{V}$ be such that

(i) $\mathbb{I}(\mathcal{U}) \subseteq \mathcal{U}$; and

(ii) $\mathrm{T} \subseteq \mathcal{U}$.

For each $\mathcal{A} \subseteq \mathrm{V}$ define $\mathbb{D}_{\mathrm{e}}^{\mathcal{U}}(\mathcal{A}) = \mathbb{HS}_{\mathrm{e}}(\mathcal{A}) \cap \mathcal{U}$. We then have the following:

Lemma 3.3.5. *If $\mathcal{U} \subseteq \mathrm{V}$ then $\mathbb{D}_{\mathrm{e}}^{\mathcal{U}}$ is a regular divisor operator and $\mathcal{U} = \mathbb{D}_{\mathrm{e}}^{\mathcal{U}}(\mathrm{V})$.*

Proof. Let us check the requirements (i)-(v) for $\mathbb{D}_{\mathrm{e}}^{\mathcal{U}}$ to be a divisor operator:

(i) Since $\mathrm{T} \subseteq \mathcal{U}$ and $\mathrm{T} \subseteq \mathbb{HS}_{\mathrm{e}}(\mathcal{A})$ for all $\mathcal{A} \subseteq \mathrm{V}$ we have that $\mathrm{T} \subseteq \mathbb{HS}_{\mathrm{e}}(\mathcal{A}) \cap \mathcal{U} = \mathbb{D}_{\mathrm{e}}^{\mathcal{U}}(\mathcal{A}) \subseteq \mathbb{HS}_{\mathrm{e}}(\mathcal{A})$.

(ii) By definition, $\mathbb{HS}_{\mathrm{e}}(\mathcal{A})$ is closed under $\mathbb{I}$ as is $\mathcal{U}$. Therefore $\mathbb{D}_{\mathrm{e}}^{\mathcal{U}}(\mathrm{A})$ is closed under $\mathbb{I}$.

(iii) Note that

$$\begin{aligned}\mathbb{HS}_{\mathrm{e}}(\mathcal{A}) \cap \mathcal{U} &= \bigcup_{S\in\mathcal{A}} \mathbb{HS}_{\mathrm{e}}(S) \cap \mathcal{U} \\ &= \bigcup_{S\in\mathcal{A}} (\mathbb{HS}_{\mathrm{e}}(S) \cap \mathcal{U}) \\ &= \bigcup_{S\in\mathcal{A}} \mathbb{D}_{\mathrm{e}}^{\mathcal{U}}(S).\end{aligned}$$

(iv) Suppose $S \in \mathbb{HS}_e(T) \cap \mathcal{U}$. Then $S \in \mathcal{U}$ and $S \in \mathbb{HS}_e(T) \Rightarrow S \in \mathbb{HS}_e(S)$ and so $S \in \mathbb{HS}_e(S) \cap \mathcal{U} = \mathbb{D}_e^{\mathcal{U}}(S)$.

(v) Suppose $S \in \mathbb{HS}_e(T)$. Then,

$$\mathbb{D}_e^{\mathcal{U}}(S) = \mathbb{HS}_e(S) \cap \mathcal{U} \subseteq \mathbb{HS}_e(T) \cap \mathcal{U} = \mathbb{D}_e^{\mathcal{U}}(T) \subseteq \mathbb{HS}_e\mathbb{D}_e^{\mathcal{U}}(T).$$

To prove the final statement of the theorem, let $S \in \mathcal{U}$. Then,

$$S \in \mathbb{HS}_e(S) \cap \mathcal{U} \subseteq \mathbb{HS}_e(\mathrm{V}) \cap \mathcal{U} = \mathbb{D}_e^{\mathcal{U}}(\mathrm{V}).$$

Conversely, $S \in \mathbb{D}_e^{\mathcal{U}}(\mathrm{V}) \Rightarrow S \in \mathbb{HS}_e(\mathrm{V}) \cap \mathcal{U} \Rightarrow S \in \mathcal{U}$. □

Corollary 3.3.6. *For* $\mathcal{U} \in \mathrm{V}$, $\theta_{\mathcal{U}} = \theta_{\mathbb{D}_e^{\mathcal{U}}(\mathrm{V})}$.

Proof. The result follows immediately since $\mathcal{U} = \mathbb{D}_e^{\mathcal{U}}(\mathrm{V})$. □

Theorem 3.3.7. *Every complete* $\cap$*-congruence on* $\mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$ *is determined by a regular divisor operator.*

Proof. Let θ be a complete $\cap$-congruence on $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$, $\mathrm{V} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$. Since by Theorem 3.2.8 we have that $\theta = \theta_{\mathcal{U}}$ for some $\mathcal{U} \subseteq \mathrm{V}$ and since $\theta_{\mathcal{U}} = \theta_{\mathbb{D}_e^{\mathcal{U}}(\mathrm{V})}$ we have established that θ is determined by $\mathbb{D}_e^{\mathcal{U}}(\mathrm{V})$. □

This result remains true for complete $\cap$-congruences on sublattices of $\mathscr{L}_{\mathrm{epv}}(\mathbf{LI}) \cup \mathscr{L}_{\mathrm{epv}}(\mathbf{ES})$.

Theorem 3.3.8. *Every complete* $\cap$*-congruence on* $\mathscr{L}_{\mathrm{epv}}(\mathbf{ES}) \cup \mathscr{L}_{\mathrm{epv}}(\mathbf{LI})$ *is determined by a regular divisor operator.*

Proof. Note that by Theorem 3.2.10 we have that every complete $\cap$-congruence θ on $\mathscr{L}_{\mathrm{epv}}(\mathbf{V})$ is of the form $\theta_{\mathcal{V}}$ for some $\mathcal{V} \subseteq \mathbf{V}$. Since $\theta_{\mathcal{V}} = \theta_{\mathbb{D}_e^{\mathcal{V}}(\mathbf{V})}$ the result follows. □

The following result will be useful in proving the final result of this section.

Lemma 3.3.9. *For* $\mathrm{V} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$ *and* $\mathcal{U} \subseteq \mathrm{V}$, $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$ *is closed under* $\mathbb{D}_e^{\mathcal{U}}$. *Also, if* $\mathcal{W}_1 \subseteq \mathcal{W}_2 \subseteq \mathrm{V}$ *then* $\mathbb{D}_e^{\mathcal{U}}(\mathcal{W}_1) \subseteq \mathbb{D}_e^{\mathcal{U}}(\mathcal{W}_2)$.

Proof. Let $\mathrm{W} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{V})$. Then

$$\begin{aligned} \mathbb{D}_{\mathrm{e}}^{\mathcal{U}}(\mathrm{W}) &= \mathbb{HS}_{\mathrm{e}}(\mathrm{W}) \cap \mathcal{U} \\ &\subseteq \mathrm{W} \cap \mathcal{U} \subseteq \mathrm{W}. \end{aligned}$$

If $\mathcal{W}_1 \subseteq \mathcal{W}_2 \subseteq \mathrm{V}$ then

$$\begin{aligned} \mathbb{D}_{\mathrm{e}}^{\mathcal{U}}(\mathcal{W}_1) &= \mathbb{HS}_{\mathrm{e}}(\mathcal{W}_1) \cap \mathcal{U} \\ &\subseteq \mathbb{HS}_{\mathrm{e}}(\mathcal{W}_2) \cap \mathcal{U} \\ &= \mathbb{D}_{\mathrm{e}}^{\mathcal{U}}(\mathcal{W}_2). \end{aligned}$$

□

Lemma 3.3.10. *Let* $\mathrm{V} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$ *and let* $\mathcal{A} \subseteq \mathrm{V}$ *be such that*

(i) $\mathbb{I}(\mathcal{A}) \subseteq \mathcal{A}$*; and*

(ii) $\mathrm{T} \subseteq \mathcal{A}$.

Then

$$\mathbb{HS}_{\mathrm{e}}\mathbb{PD}_{\mathrm{e}}^{\mathcal{A}}\mathbb{P}(\mathcal{C}) = \mathbb{HS}_{\mathrm{e}}\mathbb{PD}_{\mathrm{e}}^{\mathcal{A}}\mathbb{P}_{\mathrm{ow}}(\mathcal{C}) \Leftrightarrow \mathbb{D}_{\mathrm{e}}^{\mathcal{A}}\mathbb{P}(\mathcal{C}) \subseteq \mathbb{HS}_{\mathrm{e}}\mathbb{PD}_{\mathrm{e}}^{\mathcal{A}}\mathbb{P}_{\mathrm{ow}}(\mathcal{C}).$$

Proof. Note that for all $\mathcal{C} \subseteq \mathrm{V}$ we have that

$$\begin{aligned} \mathbb{HS}_{\mathrm{e}}\mathbb{PD}_{\mathrm{e}}^{\mathcal{A}}\mathbb{P}_{\mathrm{ow}}(\mathcal{C}) &= \mathbb{HS}_{\mathrm{e}}\mathbb{P}\left(\mathbb{HS}_{\mathrm{e}}\mathbb{P}_{\mathrm{ow}}(\mathcal{C}) \cap \mathcal{A}\right) \\ &\subseteq \mathbb{HS}_{\mathrm{e}}\mathbb{P}\left(\mathbb{HS}_{\mathrm{e}}\mathbb{P}(\mathcal{C}) \cap \mathcal{A}\right) \\ &= \mathbb{HS}_{\mathrm{e}}\mathbb{PD}_{\mathrm{e}}^{\mathcal{A}}\mathbb{P}(\mathcal{C}). \end{aligned}$$

Suppose that $\mathbb{D}_{\mathrm{e}}^{\mathcal{A}}\mathbb{P}(\mathcal{C}) \subseteq \mathbb{HS}_{\mathrm{e}}\mathbb{PD}_{\mathrm{e}}^{\mathcal{A}}\mathbb{P}_{\mathrm{ow}}(\mathcal{C})$. Then

$$\mathbb{HS}_{\mathrm{e}}\mathbb{PD}_{\mathrm{e}}^{\mathcal{A}}\mathbb{P}(\mathcal{C}) \subseteq \mathbb{HS}_{\mathrm{e}}\mathbb{P}(\mathbb{HS}_{\mathrm{e}}\mathbb{PD}_{\mathrm{e}}^{\mathcal{A}}\mathbb{P}_{\mathrm{ow}}(\mathcal{C})) \Rightarrow \mathbb{HS}_{\mathrm{e}}\mathbb{PD}_{\mathrm{e}}^{\mathcal{A}}\mathbb{P}(\mathcal{C}) \subseteq \mathbb{HS}_{\mathrm{e}}\mathbb{PD}_{\mathrm{e}}^{\mathcal{A}}\mathbb{P}_{\mathrm{ow}}(\mathcal{C}).$$

Now, suppose that $\mathbb{HS}_{\mathrm{e}}\mathbb{PD}_{\mathrm{e}}^{\mathcal{A}}\mathbb{P}(\mathcal{C}) \subseteq \mathbb{HS}_{\mathrm{e}}\mathbb{PD}_{\mathrm{e}}^{\mathcal{A}}\mathbb{P}_{\mathrm{ow}}(\mathcal{C})$. Then,

$$\mathbb{D}_{\mathrm{e}}^{\mathcal{A}}\mathbb{P}(\mathcal{C}) \subseteq \mathbb{HS}_{\mathrm{e}}\mathbb{PD}_{\mathrm{e}}^{\mathcal{A}}\mathbb{P}(\mathcal{C}) \subseteq \mathbb{HS}_{\mathrm{e}}\mathbb{PD}_{\mathrm{e}}^{\mathcal{A}}\mathbb{P}_{\mathrm{ow}}(\mathcal{C})$$

and the proof is complete. □

An analogue of the following result has been proved for varieties by Pastijn and Trotter [34, Theorem 3.11 and Corollary 3.12]. This result will be important in proving the fundamental result of this section, Theorem 3.3.12.

Theorem 3.3.11. *Let* $\mathrm{V} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$ *and let* $\mathcal{A} \subseteq \mathrm{V}$ *be such that*

(i) $\mathbb{I}(\mathcal{A}) \subseteq \mathcal{A}$*; and*

(ii) $\mathrm{T} \subseteq \mathcal{A}$.

Then the following are equivalent:

(i) $\theta_{\mathcal{A}}$ *is a complete congruence;*

(ii) for each $\mathrm{W} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{V})$, $\{\mathrm{U} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{V}) \,|\, \mathrm{U} \cap \mathcal{A} \subseteq \mathrm{W}\}$ *has a greatest element; and*

(iii) for each $\mathrm{W} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{V})$, $\{S \in \mathrm{V} \,|\, \mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}(S) \cap \mathcal{A} \subseteq \mathrm{W}\}$ *is closed under* $\mathbb{P}$.

Furthermore, if $\theta_{\mathcal{A}}$ *is a complete congruence, then for each* $\mathrm{W} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{V})$,

$$\mathrm{W}^{\theta_{\mathcal{A}}} = \bigvee\{\mathrm{U} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{V}) \,|\, \mathrm{U} \cap \mathcal{A} \subseteq \mathrm{W}\} = \{S \in \mathrm{V} \,|\, \mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}(S) \cap \mathcal{A} \subseteq \mathrm{W}\}.$$

Proof. The equivalence of statements (i)-(iii) may be obtained by obvious modifications of the proof of [34, Theorem 3.11]. The proof of the final statement is given for varieties as [34, Corollary 3.12] and may be modified in an obvious way to demonstrate the present result. Note that the final equality is a restatement of Corollary 3.3.3. □

We are now able to prove the most important result of this section. This result is analogous to [34, Theorem 3.13]. The proof (which we include here) follows that given by Pastijn and Trotter.

Theorem 3.3.12. *Let* $\mathrm{V} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$ *and let* $\mathcal{A} \subseteq \mathrm{V}$ *be such that*

(i) $\mathbb{I}(\mathcal{A}) \subseteq \mathcal{A}$*; and*

(ii) $\mathrm{T} \subseteq \mathcal{A}$.

Then $\theta_{\mathcal{A}}$ *is a complete congruence on* $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$ *if and only if*

$$\mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}\mathbb{D}_{\mathrm{e}}^{\mathcal{A}}\mathbb{P}(\mathcal{C}) = \mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}\mathbb{D}_{\mathrm{e}}^{\mathcal{A}}\mathbb{P}_{\mathrm{ow}}(\mathcal{C})$$

for all $\mathcal{C} \subseteq \mathrm{V}$.

Proof. By Lemma 3.3.10 it suffices to show that $\theta_{\mathcal{A}}$ is a complete congruence on $\mathscr{L}_{\text{ev}}(\mathrm{V})$ if and only if $\mathbb{D}_{\text{e}}^{\mathcal{A}}\mathbb{P}(\mathcal{C}) \subseteq \mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}\mathbb{D}_{\text{e}}^{\mathcal{A}}\mathbb{P}_{\text{ow}}(\mathcal{C})$. Suppose $\theta_{\mathcal{A}}$ is a complete congruence. Let $\mathrm{W} = \mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}\mathbb{D}_{\text{e}}^{\mathcal{A}}\mathbb{P}_{\text{ow}}(\mathcal{C})$ so that by Theorem 3.3.11, $\mathrm{W}^{\theta_{\mathcal{A}}} = \{S \in \mathrm{V} \,|\, \mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(S) \cap \mathcal{A} \subseteq \mathrm{W}\}$. Now, for each $S \in \mathcal{C}$, $\mathbb{D}_{\text{e}}^{\mathcal{A}}\mathbb{P}(S) = \mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(S) \cap \mathcal{A} \subseteq \mathrm{W}$ which implies that $S \in \mathrm{W}^{\theta_{\mathcal{A}}}$ and so $\mathcal{C} \subseteq \mathrm{W}^{\theta_{\mathcal{A}}}$. So

$$\begin{aligned} \mathbb{D}_{\text{e}}^{\mathcal{A}}\mathbb{P}(\mathcal{C}) &= \mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(\mathcal{C}) \cap \mathcal{A} \\ &\subseteq \mathrm{W} = \mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}\mathbb{D}_{\text{e}}^{\mathcal{A}}\mathbb{P}_{\text{ow}}(\mathcal{C}). \end{aligned}$$

Now suppose that $\mathbb{D}_{\text{e}}^{\mathcal{A}}\mathbb{P}(\mathcal{C}) \subseteq \mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}\mathbb{D}_{\text{e}}^{\mathcal{A}}\mathbb{P}_{\text{ow}}(\mathcal{C})$ for each $\mathcal{C} \subseteq \mathrm{V}$. Let $\mathrm{W} \in \mathscr{L}_{\text{ev}}(\mathrm{V})$ and let $\mathcal{W} = \{S \in \mathrm{V} \,|\, \mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(S) \cap \mathcal{A} \subseteq \mathrm{W}\}$. Clearly, since for each $S \in \mathcal{W}$, $\mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(S) \cap \mathcal{A} \subseteq \mathrm{W}$ we have that $\mathbb{D}_{\text{e}}^{\mathcal{A}}\mathbb{P}_{\text{ow}}(\mathcal{W}) = \mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}_{\text{ow}}(\mathcal{W}) \cap \mathcal{A} \subseteq \mathrm{W}$.

Since W is an e-variety, $\mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}\mathbb{D}_{\text{e}}^{\mathcal{A}}\mathbb{P}_{\text{ow}}(\mathcal{W}) \subseteq \mathrm{W}$ and so by the assumption, $\mathbb{D}_{\text{e}}^{\mathcal{A}}\mathbb{P}(\mathcal{W}) \subseteq \mathrm{W}$. Let $\{S_i\}_{i \in I}$ be a family of members of $\mathcal{W}$. Since $\mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}\left(\prod_{i \in I} S_i\right) \cap \mathcal{A} \subseteq \mathrm{W}$, we have that $\prod_{i \in I} S_i$ is in $\mathcal{W}$ and so $\mathcal{W}$ is closed under $\mathbb{P}$. By Theorem 3.3.11 we have that $\theta_{\mathcal{A}}$ is a complete congruence. □

Of course, we can replace $\mathbb{D}_{\text{e}}^{\mathcal{A}}$ in the previous theorem with a general regular divisor operator $\mathbb{D}_{\text{e}}$ and obtain the following result:

Corollary 3.3.13. *Let* $\mathbb{D}_{\text{e}}$ *be a regular divisor operator on* $\mathrm{V} \in \mathscr{L}_{\text{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\text{ev}}(\mathrm{LI})$. *Then,* $\theta_{\mathbb{D}_{\text{e}}}$ *is a complete congruence on* $\mathscr{L}_{\text{ev}}(\mathrm{V})$ *if and only if*

$$\mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}\mathbb{D}_{\text{e}}\mathbb{P}(\mathcal{C}) = \mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}\mathbb{D}_{\text{e}}\mathbb{P}_{\text{ow}}(\mathcal{C})$$

for all $\mathcal{C} \subseteq \mathrm{V}$.

3.4 Complete Congruences on Lattices of E-Pseudovarieties

The description of all complete congruences on lattices of e-pseudovarieties which consist entirely of E-solid or locally inverse semigroups requires a slightly different approach to the one used to study complete congruences on lattices of e-varieties.

We will utilise the important facts about generalised e-varieties developed in Chapter 2 to aid us in exploring complete congruences on lattices of e-pseudovarieties.

3.4.1 Complete Congruences Induced by Generalised E-Varieties

In an earlier section we found that complete $\cap$-congruences on certain lattices of e-pseudovarieties are determined by a class $\mathcal{A}$. Our aim now is to demonstrate that all complete congruences on lattices of e-pseudovarieties contained in $\mathscr{L}_{\text{epv}}(\mathbf{LI})$ or $\mathscr{L}_{\text{epv}}(\mathbf{ES})$ may be thus described.

Let $\mathbf{V} \in \mathscr{L}_{\text{epv}}(\mathbf{LI}) \cup \mathscr{L}_{\text{epv}}(\mathbf{ES})$ and let $\mathrm{V} = \langle \mathbf{V} \rangle_{\text{ev}} = \mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(\mathbf{V})$. Recall that $G(\mathrm{V}) = \cup\{\mathrm{W} \in \mathscr{L}_{\text{ev}}(\mathrm{V}) \mid \mathrm{W} \text{ is locally finite}\}$ is a generalised e-variety. Given a relation θ on $\mathscr{L}_{\text{epv}}(\mathbf{V})$ we define a relation θ_g on $\mathscr{L}_{\text{gev}}(G(\mathrm{V}))$ as follows:

$$\mathrm{U}_1 \theta_g \mathrm{U}_2 \Leftrightarrow \mathrm{U}_1^{\text{Fin}} \theta \mathrm{U}_2^{\text{Fin}}.$$

We begin by proving some fundamental results about θ_g. These results are derived from related results about congruences on pseudovarieties and generalised varieties obtained by Pastijn and Trotter [34].

Lemma 3.4.1. *If θ is a complete congruence on $\mathscr{L}_{\text{epv}}(\mathbf{V})$ then θ_{g} is a complete congruence on $\mathscr{L}_{\text{gev}}(G(\mathrm{V}))$.*

Proof. Let $\{\mathrm{U}_i \mid i \in I\}$ and $\{\mathrm{V}_i \mid i \in I\}$ be families of members of $G(\mathrm{V})$ such that $(\forall i \in I)\ \mathrm{U}_i \theta_{\text{g}} \mathrm{V}_i$ and so $(\forall i \in I)\ (\mathrm{U}_i)^{\text{Fin}} \theta (\mathrm{V}_i)^{\text{Fin}}$. Therefore we have:

$$\begin{aligned}
(\forall i \in I)\ (\mathrm{U}_i)^{\text{Fin}} \theta (\mathrm{V}_i)^{\text{Fin}} &\Rightarrow \left(\cap_{i\in I} \mathrm{U}_i^{\text{Fin}}\right) \theta \left(\cap_{i\in I} \mathrm{V}_i^{\text{Fin}}\right) \\
&\Rightarrow \left(\cap_{i\in I} \mathrm{U}_i\right)^{\text{Fin}} \theta \left(\cap_{i\in I} \mathrm{V}_i\right)^{\text{Fin}} \\
&\Rightarrow \cap_{i\in I} \mathrm{U}_i \theta_{\text{g}} \cap_{i\in I} \mathrm{V}_i
\end{aligned}$$

and so θ_g is a complete $\cap$-congruence. In order to show that θ_{g} is a complete $\vee$-congruence, we utilise Theorem 2.4.18:

$$\begin{aligned}
(\forall i \in I)\ \mathrm{U}_i^{\text{Fin}} \theta \mathrm{V}_i^{\text{Fin}} &\Rightarrow \left(\vee_{i\in I} \mathrm{U}_i^{\text{Fin}}\right) \theta \left(\vee_{i\in I} \mathrm{V}_i^{\text{Fin}}\right) \\
&\Rightarrow \left(\vee_{i\in I} \mathrm{U}_i\right)^{\text{Fin}} \theta \left(\vee_{i\in I} \mathrm{V}_i\right)^{\text{Fin}} \\
&\Rightarrow \vee_{i\in I} \mathrm{U}_i \theta_{\text{g}} \vee_{i\in I} \mathrm{V}_i
\end{aligned}$$

and so θ_{g} is a complete $\bigvee$-congruence and therefore is a complete congruence. $\square$

Let $\mathbf{V} \in \mathscr{L}_{\mathrm{epv}}(\mathbf{LI}) \cup \mathscr{L}_{\mathrm{ev}}(\mathbf{ES})$ and let θ be a complete congruence on $\mathscr{L}_{\mathrm{epv}}(\mathbf{V})$. Let $\mathrm{W} \in \mathscr{L}_{\mathrm{ev}}(\mathbb{HS}_{\mathrm{e}}\mathbb{P}(\mathbf{V}))$ be locally finite. We define a relation θ_{W} on $\mathscr{L}_{\mathrm{ev}}(\mathrm{W})$ as follows:

$$\mathrm{U}_1 \theta_{\mathrm{W}} \mathrm{U}_2 \Leftrightarrow (\mathrm{U}_1, \mathrm{U}_2) \in \theta_{\mathrm{g}} \cap (\mathscr{L}_{\mathrm{ev}}(\mathrm{W}) \times \mathscr{L}_{\mathrm{ev}}(\mathrm{W})).$$

Clearly, if θ is a complete $\cap$-congruence on $\mathscr{L}_{\mathrm{epv}}(\mathbf{V})$, $\mathbf{V} \in \mathscr{L}_{\mathrm{epv}}(\mathbf{LI}) \cup \mathscr{L}_{\mathrm{ev}}(\mathbf{ES})$, then θ_{W} is a complete $\cap$-congruence on $\mathscr{L}_{\mathrm{ev}}(\mathrm{W})$. Suppose that θ is a complete congruence on $\mathbf{V}$. We have established that θ_{g} is a complete congruence on $\mathscr{L}_{\mathrm{gev}}(G(V))$ and since $\mathscr{L}_{\mathrm{ev}}(\mathrm{W})$ is a sublattice of $\mathscr{L}_{\mathrm{gev}}(G(V))$ (since W is, by definition, locally finite) we have that θ_{W} is a congruence on $\mathscr{L}_{\mathrm{ev}}(\mathrm{W})$. We now show that θ_{W} is a complete congruence on $\mathscr{L}_{\mathrm{ev}}(\mathrm{W})$. The proof follows that of [34, Lemma 4.3].

Theorem 3.4.2. *Let θ be a complete congruence on $\mathscr{L}_{\mathrm{epv}}(\mathbf{V})$, $\mathbf{V} \in \mathscr{L}_{\mathrm{epv}}(\mathbf{ES}) \cup \mathscr{L}_{\mathrm{epv}}(\mathbf{LI})$ and let $\mathrm{W} \in \mathscr{L}_{\mathrm{ev}}(\mathbb{HS}_{\mathrm{e}}\mathbb{P}(\mathbf{V}))$ be locally finite. Then θ_{W} is a complete congruence on $\mathscr{L}_{\mathrm{ev}}(\mathrm{W})$.*

Proof. Let $\mathrm{U} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{W})$ and let $\mathrm{X} \in \mathrm{U}\theta_{\mathrm{W}}$. Then $\mathrm{X} \subseteq \mathrm{U}^{\theta_{\mathrm{g}}} \cap \mathrm{W} \subseteq \mathbb{HS}_{\mathrm{e}}\mathbb{P}(\mathrm{U}^{\theta_{\mathrm{g}}} \cap \mathrm{W}) \subseteq \mathrm{W}$. We wish to demonstrate that $\mathbb{HS}_{\mathrm{e}}\mathbb{P}(\mathrm{U}^{\theta_{\mathrm{g}}} \cap \mathrm{W}) \in \mathrm{U}\theta_{\mathrm{W}}$. Observe that by Lemma 2.4.20 we have that $\left(\mathbb{HS}_{\mathrm{e}}\mathbb{P}(\mathrm{U}^{\theta_{\mathrm{g}}} \cap \mathrm{W})\right)^{\mathrm{Fin}} = \left(\mathrm{U}^{\theta_{\mathrm{g}}} \cap \mathrm{W}\right)^{\mathrm{Fin}}$ and so $\mathbb{HS}_{\mathrm{e}}\mathbb{P}(\mathrm{U}^{\theta_{\mathrm{g}}} \cap \mathrm{W})\theta_{\mathrm{g}}\mathrm{U}^{\theta_{\mathrm{g}}} \cap \mathrm{W}$.

Now, since $\mathrm{U} \subseteq \mathrm{W}$ and $\mathrm{U}\theta_{\mathrm{g}}\mathrm{U}^{\theta_{\mathrm{g}}}$ we have that $\mathrm{U}\cap\mathrm{W}\theta_{\mathrm{g}}\mathrm{U}^{\theta_{\mathrm{g}}}\cap\mathrm{W}$ and so $\mathrm{U}\theta_{\mathrm{g}}\mathrm{U}^{\theta_{\mathrm{g}}}\cap\mathrm{W}$. By transitivity we have that $\mathrm{U}\theta_{\mathrm{g}}\mathbb{HS}_{\mathrm{e}}\mathbb{P}(\mathrm{U}^{\theta_{\mathrm{g}}} \cap \mathrm{W})$. Therefore $\mathrm{U}\theta_{\mathrm{W}}\mathbb{HS}_{\mathrm{e}}\mathbb{P}(\mathrm{U}^{\theta_{\mathrm{g}}} \cap \mathrm{W})$ and so $\mathrm{U}^{\theta_{\mathrm{W}}} = \mathbb{HS}_{\mathrm{e}}\mathbb{P}(\mathrm{U}^{\theta_{\mathrm{g}}} \cap \mathrm{W})$. Since each congruence class $\mathrm{U}\theta_{\mathrm{W}}$ is an interval we have that θ_{W} is a complete congruence. $\square$

The principal result obtained by Pastijn and Trotter [34, Theorem 4.4] links complete congruences on lattices of pseudovarieties with complete congruences on lattices of varieties, via complete congruences on generalised varieties. We prove that the same linking holds for complete congruences on lattices of e-pseudovarieties and complete congruences on lattices of e-varieties. Having previously established that every complete congruence θ on $\mathscr{L}_{\mathrm{epv}}(\mathbf{V})$, where $\mathbf{V} \in \mathscr{L}_{\mathrm{epv}}(\mathbf{ES}) \cup \mathscr{L}_{\mathrm{epv}}(\mathbf{LI})$, gives rise to a complete congruence θ_{V} on $\mathscr{L}_{\mathrm{ev}}(\mathrm{V})$ where each $\mathrm{V} \in \mathscr{L}_{\mathrm{ev}}(\mathbb{HS}_{\mathrm{e}}\mathbb{P}(\mathbf{V}))$ is locally finite, we now investigate the converse, that from each complete congruence on a certain lattice of e-varieties, a complete congruence on a lattice of e-pseudovarieties can be obtained.

We will break this task into two parts. To begin with, we will show that from a complete congruence θ_{W} on $\mathscr{L}_{\mathrm{ev}}(\mathrm{W})$, where $\mathrm{W} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{V})$ is locally finite and $\mathrm{V} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{ES}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{LI})$, we can obtain a complete congruence θ_{g} on $\mathscr{L}_{\mathrm{gev}}(G(\mathrm{V}))$.

Theorem 3.4.3. *Let* $\mathrm{W} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{V})$ *where* $\mathrm{V} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{LI}) \cup \mathscr{L}_{\mathrm{ev}}(\mathrm{ES})$ *be locally finite and let* θ_{W} *be a complete congruence on* $\mathscr{L}_{\mathrm{ev}}(\mathrm{W})$ *with the property that for any* $\mathrm{U} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{W})$, $\theta_{\mathrm{U}} = \theta_{\mathrm{W}} \cap \mathscr{L}_{\mathrm{ev}}(\mathrm{U}) \times \mathscr{L}_{\mathrm{ev}}(\mathrm{U})$. *Define on* $\mathscr{L}_{\mathrm{gev}}(G(\mathrm{V}))$ *the relation* θ_g *by:*

$$\mathrm{X}_1 \theta_g \mathrm{X}_2 \Leftrightarrow \left(\bigvee_{S \in \mathrm{X}_1^{\mathrm{Fin}}} (\langle S \rangle_{\mathrm{ev}})_{\theta_{\langle S \rangle_{\mathrm{ev}}}} \right)^{\mathrm{Fin}} = \left(\bigvee_{S \in \mathrm{X}_2^{\mathrm{Fin}}} (\langle S \rangle_{\mathrm{ev}})_{\theta_{\langle S \rangle_{\mathrm{ev}}}} \right)^{\mathrm{Fin}}.$$

Then θ_{g} *is a complete congruence on* $\mathscr{L}_{\mathrm{gev}}(G(\mathrm{V}))$ *and* $\theta_{\mathrm{W}} = \theta_{\mathrm{g}} \cap \mathscr{L}_{\mathrm{ev}}(\mathrm{W}) \times \mathscr{L}_{\mathrm{ev}}(\mathrm{W})$.

Proof. Clearly θ_{g} is an equivalence relation on $\mathscr{L}_{\mathrm{gev}}(G(\mathrm{V}))$. In order to show that θ_{g} is a complete congruence on $\mathscr{L}_{\mathrm{gev}}(G(\mathrm{V}))$ it suffices to show that for each family $\{\mathrm{U}_i \mid i \in I\}$ of members of $\mathscr{L}_{\mathrm{gev}}(G(\mathrm{V}))$,

$$\left(\bigvee_{i \in I}^{\mathrm{gev}} \mathrm{U}_i \right)_{\theta_{\mathrm{g}}} = \left(\bigvee_{i \in I}^{\mathrm{gev}} (\mathrm{U}_i)_{\theta_{\mathrm{g}}} \right)_{\theta_{\mathrm{g}}}$$

and that for all U, $\mathrm{U}' \in \mathscr{L}_{\mathrm{gev}}(G(\mathrm{V}))$,

$$\mathrm{U} \subseteq \mathrm{U}' \Rightarrow \mathrm{U}_{\theta_{\mathrm{g}}} \subseteq \mathrm{U}'_{\theta_{\mathrm{g}}}.$$

Following [34, Theorem 4.4] we suppose that

$$\mathrm{U}_{\theta_{\mathrm{g}}} = \mathbb{HS}_{\mathrm{e}}\mathbb{P}_{\mathrm{f}}\mathbb{P}_{\mathrm{ow}} \left(\left(\bigvee_{S \in \mathrm{U}^{\mathrm{Fin}}}^{\mathrm{gev}} (\langle S \rangle_{\mathrm{ev}})_{\theta_{\langle S \rangle_{\mathrm{ev}}}} \right)^{\mathrm{Fin}} \right)$$

and show that this supposition is in fact correct.

Note that for any $\mathrm{U}' \in \mathrm{U}\theta_{\mathrm{g}}$ we have $\mathrm{U}'\theta_{\mathrm{g}}\mathrm{U}$ and so

$$\begin{aligned}\mathrm{U}_{\theta_{\mathrm{g}}} &= \mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}_{\mathrm{f}}\mathbb{P}_{\mathrm{ow}}\left(\left(\bigvee_{S\in\mathrm{U}^{\mathrm{Fin}}}^{\mathrm{gev}}(\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}\right)^{\mathrm{Fin}}\right)\\ &= \mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}_{\mathrm{f}}\mathbb{P}_{\mathrm{ow}}\left(\left(\bigvee_{S\in\mathrm{U}'^{\mathrm{Fin}}}^{\mathrm{gev}}(\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}\right)^{\mathrm{Fin}}\right)\\ &\subseteq \mathrm{U}'\end{aligned}$$

and consequently that $\mathrm{U}_{\theta_{\mathrm{g}}} \subseteq \mathrm{U}$. Note that this gives us that

$$\left(\bigvee_{S\in\mathrm{U}_{\theta_{\mathrm{g}}}^{\mathrm{Fin}}}^{\mathrm{gev}}(\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}\right)^{\mathrm{Fin}} \subseteq \left(\bigvee_{S\in\mathrm{U}^{\mathrm{Fin}}}^{\mathrm{gev}}(\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}\right)^{\mathrm{Fin}}.$$

We now show that the reverse inequality is true. That is,

$$\left(\bigvee_{S\in\mathrm{U}^{\mathrm{Fin}}}^{\mathrm{gev}}(\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}\right)^{\mathrm{Fin}} \subseteq \left(\bigvee_{S\in\mathrm{U}_{\theta_{\mathrm{g}}}^{\mathrm{Fin}}}^{\mathrm{gev}}(\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}\right)^{\mathrm{Fin}}.$$

Suppose that $T \in \left(\bigvee_{S\in\mathrm{U}^{\mathrm{Fin}}}^{\mathrm{gev}}(\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}\right)^{\mathrm{Fin}}$. Of course, T is a finite regular semigroup drawn from the e-pseudovariety which consists of the finite members of $\bigvee_{S\in\mathrm{U}^{\mathrm{Fin}}}^{\mathrm{gev}}(\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}$. Therefore there exists a finite collection of members of $\mathrm{U}^{\mathrm{Fin}}$, $\{S_i \mid 1 \le i \le n\}$ such that

$$T \in \bigvee_{1\le i\le n}^{\mathrm{gev}}(\langle S_i\rangle_{\mathrm{ev}})_{\theta_{\langle S_i\rangle_{\mathrm{ev}}}} \subseteq \bigvee_{1\le i\le n}^{\mathrm{ev}}(\langle S_i\rangle_{\mathrm{ev}})_{\theta_{\langle S_i\rangle_{\mathrm{ev}}}}.$$

Set $A = S_1 \times S_2 \times \cdots \times S_n$ and note that

$$\begin{aligned}\bigvee_{1\le i\le n}^{\mathrm{ev}}(\langle S_i\rangle_{\mathrm{ev}})_{\theta_{\langle S_i\rangle_{\mathrm{ev}}}} &= \left(\bigvee_{1\le i\le n}^{\mathrm{ev}}(\langle S_i\rangle_{\mathrm{ev}})\right)_{\theta_{\langle A\rangle_{\mathrm{ev}}}}\\ &= (\langle A\rangle_{\mathrm{ev}})_{\theta_{\langle A\rangle_{\mathrm{ev}}}}.\end{aligned}$$

Since $A \in \mathrm{U}^{\mathrm{Fin}}$ we have that

$$\left((\langle A\rangle_{\mathrm{ev}})_{\theta_{\langle A\rangle_{\mathrm{ev}}}}\right)^{\mathrm{Fin}} \subseteq \left(\mathrm{U}_{\theta_{\mathrm{g}}}\right)^{\mathrm{Fin}}.$$

Let BF denote the class of all finitely generated bifree objects in $(\langle A\rangle_{\mathrm{ev}})_{\theta_{\langle A\rangle_{\mathrm{ev}}}}$. Since $(\langle A\rangle_{\mathrm{ev}})_{\theta_{\langle A\rangle_{\mathrm{ev}}}}$ is locally finite the members of BF are all finite and furthermore,

$\mathrm{BF} \subseteq \left(\mathrm{U}_{\theta_{\mathrm{g}}}\right)^{\mathrm{Fin}}$. So

$$\begin{aligned}
(\langle A\rangle_{\mathrm{ev}})_{\theta_{\langle A\rangle_{\mathrm{ev}}}} &= \left(\bigvee_{1\le i\le n}^{\mathrm{ev}} \langle S_i\rangle_{\mathrm{ev}}\right)_{\theta_{\langle A\rangle_{\mathrm{ev}}}} \\
&= \left(\bigvee_{S\in\mathrm{BF}}^{\mathrm{ev}} \langle S\rangle_{\mathrm{ev}}\right)_{\theta_{\langle A\rangle_{\mathrm{ev}}}} \\
&= \bigvee_{S\in\mathrm{BF}}^{\mathrm{ev}} (\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle A\rangle_{\mathrm{ev}}}} \\
&= \bigvee_{S\in\mathrm{BF}}^{\mathrm{ev}} (\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}} \\
&= \left\langle \bigvee_{S\in\mathrm{BF}}^{\mathrm{gev}} (\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}} \right\rangle_{\mathrm{ev}}.
\end{aligned}$$

By Lemma 2.4.20 we have that

$$\begin{aligned}
\left((\langle A\rangle_{\mathrm{ev}})_{\theta_{\langle A\rangle_{\mathrm{ev}}}}\right)^{\mathrm{Fin}} &= \left(\bigvee_{S\in\mathrm{BF}}^{\mathrm{gev}} (\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}\right)^{\mathrm{Fin}} \\
&\subseteq \left(\bigvee_{S\in\mathrm{U}_{\theta_{\mathrm{g}}}^{\mathrm{Fin}}}^{\mathrm{gev}} (\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}\right)^{\mathrm{Fin}}.
\end{aligned}$$

Therefore $T \in \left(\bigvee_{S\in\mathrm{U}_{\theta_{\mathrm{g}}}^{\mathrm{Fin}}}^{\mathrm{gev}} (\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}\right)^{\mathrm{Fin}}$ and so

$$\left(\bigvee_{S\in\mathrm{U}_{\theta_{\mathrm{g}}}^{\mathrm{Fin}}}^{\mathrm{gev}} (\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}\right)^{\mathrm{Fin}} = \left(\bigvee_{S\in\mathrm{U}^{\mathrm{Fin}}}^{\mathrm{gev}} (\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}\right)^{\mathrm{Fin}}.$$

Therefore $\mathrm{U}_{\theta_{\mathrm{g}}}\theta\mathrm{U}$ and the hypothesis is correct.

We now show that θ_{g} is a complete $\vee$-congruence. Note that if $\left(\left(\bigvee_{i\in I}^{\mathrm{gev}} \mathrm{U}_i\right)_{\theta_{\mathrm{g}}}\right)^{\mathrm{Fin}} = \left(\left(\bigvee_{i\in I}^{\mathrm{gev}} (\mathrm{U}_i)_{\theta_{\mathrm{g}}}\right)_{\theta_{\mathrm{g}}}\right)^{\mathrm{Fin}}$ then

$$\left\langle \left(\bigvee_{S\in\left(\left(\bigvee_{i\in I}^{\mathrm{gev}}\mathrm{U}_i\right)_{\theta_{\mathrm{g}}}\right)^{\mathrm{Fin}}}^{\mathrm{gev}} (\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}\right)^{\mathrm{Fin}} \right\rangle_{\mathrm{gev}} = \left\langle \left(\bigvee_{S\in\left(\left(\bigvee_{i\in I}^{\mathrm{gev}}(\mathrm{U}_i)_{\theta_{\mathrm{g}}}\right)_{\theta_{\mathrm{g}}}\right)^{\mathrm{Fin}}}^{\mathrm{gev}} (\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}\right)^{\mathrm{Fin}} \right\rangle_{\mathrm{gev}}$$

which in turn implies that

$$\left(\bigvee_{i\in I}^{\mathrm{gev}} \mathrm{U}_i\right)_{\theta_{\mathrm{g}}} = \left(\bigvee_{i\in I}^{\mathrm{gev}} (\mathrm{U}_i)_{\theta_{\mathrm{g}}}\right)_{\theta_{\mathrm{g}}}.$$

Suppose $T \in \left(\left(\bigvee_{i\in I}^{\mathrm{gev}} \mathrm{U}_i\right)_{\theta_{\mathrm{g}}}\right)^{\mathrm{Fin}}$. We have then that

$$T \in \left[\mathbb{H}\mathbb{S}_{\mathrm{e}}\mathbb{P}_{\mathrm{f}}\mathbb{P}_{\mathrm{ow}}\left(\bigvee_{S\in\left(\bigvee_{i\in I}^{\mathrm{gev}} \mathrm{U}_i\right)^{\mathrm{Fin}}} (\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}\right)\right]^{\mathrm{Fin}}.$$

Note that there exists $A \in (\bigvee_{i\in I}^{\mathrm{gev}} \mathrm{U}_i)^{\mathrm{Fin}}$ such that $T \in (\langle A\rangle_{\mathrm{ev}})_{\theta_{\langle A\rangle_{\mathrm{ev}}}}$. Now, since $A \in (\bigvee_{i\in I}^{\mathrm{gev}} \mathrm{U}_i)^{\mathrm{Fin}}$ there exists a finite set $\{A_j \mid j \in 1,\dots,m\}$ of finite regular semigroups from $\bigcup_{i\in I} \mathrm{U}_i^{\mathrm{Fin}}$ such that $A \in \bigvee_{1\leq j\leq m}^{\mathrm{ev}} \langle A_j\rangle_{\mathrm{ev}}$ and so

$$(\langle A\rangle_{\mathrm{ev}})_{\theta_{\langle A\rangle_{\mathrm{ev}}}} \subseteq \bigvee_{1\leq j\leq m}^{\mathrm{ev}} (\langle A_j\rangle_{\mathrm{ev}})_{\theta_{\langle A_j\rangle_{\mathrm{ev}}}}.$$

Since

$$\begin{aligned} T \in (\langle A\rangle_{\mathrm{ev}})_{\theta_{\langle A\rangle_{\mathrm{ev}}}} &\subseteq \bigvee_{1\leq j\leq m}^{\mathrm{ev}} (\langle A_j\rangle_{\mathrm{ev}})_{\theta_{\langle A_j\rangle_{\mathrm{ev}}}} \\ &\subseteq \bigvee_{S\in \mathrm{U}^{\mathrm{Fin}}}^{\mathrm{gev}} (\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}} \end{aligned}$$

we have that

$$\begin{aligned} T \in \left(\bigvee_{i\in I}^{\mathrm{gev}}\left(\bigvee_{S\in \mathrm{U}_i^{\mathrm{Fin}}}^{\mathrm{gev}} (\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}\right)\right)^{\mathrm{Fin}} &= \bigvee_{i\in I}^{\mathrm{epv}}\left(\bigvee_{S\in \mathrm{U}_i^{\mathrm{Fin}}} (\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}\right)^{\mathrm{Fin}} \\ &= \bigvee_{i\in I}^{\mathrm{epv}}\left((\mathrm{U}_i)_{\theta_{\mathrm{g}}}\right)^{\mathrm{Fin}}. \end{aligned}$$

So $\left(\left(\bigvee_{i\in I}^{\mathrm{gev}} \mathrm{U}_i\right)_{\theta_{\mathrm{g}}}\right)^{\mathrm{Fin}} \subseteq \bigvee_{i\in I}^{\mathrm{epv}}\left((\mathrm{U}_i)_{\theta_{\mathrm{g}}}\right)$ and therefore

$$\left(\left(\bigvee_{i\in I}^{\mathrm{gev}} \mathrm{U}_i\right)_{\theta_{\mathrm{g}}}\right)^{\mathrm{Fin}} = \bigvee_{i\in I}^{\mathrm{epv}}\left((\mathrm{U}_i)_{\theta_{\mathrm{g}}}\right).$$

Similarly,

$$\begin{aligned} \left(\left(\bigvee_{i\in I}^{\mathrm{gev}} (\mathrm{U}_i)_{\theta_{\mathrm{g}}}\right)_{\theta_{\mathrm{g}}}\right)^{\mathrm{Fin}} &= \bigvee_{i\in I}^{\mathrm{epv}}\left(\left((\mathrm{U}_i)_{\theta_{\mathrm{g}}}\right)_{\theta_{\mathrm{g}}}\right)^{\mathrm{Fin}} \\ &= \bigvee_{i\in I}^{\mathrm{epv}}\left((\mathrm{U}_i)_{\theta_{\mathrm{g}}}\right)^{\mathrm{Fin}} \\ &= \left(\left(\bigvee_{i\in I}^{\mathrm{gev}} \mathrm{U}_i\right)_{\theta_{\mathrm{g}}}\right)^{\mathrm{Fin}} \end{aligned}$$

and so

$$\left(\bigvee_{i\in I}^{\mathrm{gev}} \mathrm{U}_i\right)_{\theta_{\mathrm{g}}} = \left(\bigvee_{i\in I}^{\mathrm{gev}} (\mathrm{U}_i)_{\theta_{\mathrm{g}}}\right)_{\theta_{\mathrm{g}}}.$$

Now, suppose that $\mathrm{U}, \mathrm{U}' \in \mathscr{L}_{\mathrm{gev}}(G(\mathrm{V}))$ and $\mathrm{U} \subseteq \mathrm{U}'$. Then

$$\begin{aligned} \mathrm{U}_{\theta_{\mathrm{g}}} &= \mathbb{HS}_{\mathrm{e}}\mathbb{P}_{\mathrm{f}}\mathbb{P}_{\mathrm{ow}}\left(\left(\bigvee_{S\in \mathrm{U}^{\mathrm{Fin}}}^{\mathrm{gev}} (\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}\right)^{\mathrm{Fin}}\right) \\ &\subseteq \mathbb{HS}_{\mathrm{e}}\mathbb{P}_{\mathrm{f}}\mathbb{P}_{\mathrm{ow}}\left(\left(\bigvee_{S\in \mathrm{U}'^{\mathrm{Fin}}}^{\mathrm{gev}} (\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}\right)^{\mathrm{Fin}}\right) \\ &= \mathrm{U}'_{\theta_{\mathrm{g}}} \end{aligned}$$

and so θ_{g} is a complete $\vee$-congruence on $\mathscr{L}_{\mathrm{gev}}(G(\mathrm{V}))$. Therefore θ_{g} is a complete congruence on $\mathscr{L}_{\mathrm{gev}}(G(\mathrm{V}))$. □

Theorem 3.4.4. *Let* $\mathbf{V} \in \mathscr{L}_{\mathrm{epv}}(\mathbf{LI}) \cup \mathscr{L}_{\mathrm{epv}}(\mathbf{ES})$ *and let* θ_{g} *be a complete congruence on* $\mathscr{L}_{\mathrm{gev}}(G(\langle \mathbf{V}\rangle_{\mathrm{ev}}))$. *Suppose that for all* $\mathrm{V}_1, \mathrm{V}_2 \in \mathscr{L}_{\mathrm{gev}}(G(\mathrm{V}))$ *we have* $\mathrm{V}_1^{\mathrm{Fin}} = \mathrm{V}_2^{\mathrm{Fin}} \Rightarrow V_1\theta_{\mathrm{g}}V_2$. *Then there exists a complete congruence* θ *on* $\mathscr{L}_{\mathrm{epv}}(\mathbf{V})$ *given by*

$$\mathbf{U}_1\theta\mathbf{U}_2 \Leftrightarrow (\exists \mathrm{V}_1, \mathrm{V}_2 \in \mathscr{L}_{\mathrm{gev}}(G(\langle \mathbf{V}\rangle_{\mathrm{ev}}))) \text{ such that } \mathrm{V}_1^{\mathrm{Fin}} = \mathbf{U}_1,\ \mathrm{V}_2^{\mathrm{Fin}} = \mathbf{U}_2 \text{ and } \mathrm{V}_1\theta_{\mathrm{g}}\mathrm{V}_2.$$

Proof. The result follows immediately from Theorem 2.4.18. □

The following result is analogous to [34, Corollary 4.5].

Corollary 3.4.5. *Let* $\mathbf{V} \in \mathscr{L}_{\mathrm{epv}}(\mathbf{LI}) \cup \mathscr{L}_{\mathrm{epv}}(\mathbf{ES})$ *and let* θ *be a complete congruence on* $\mathscr{L}_{\mathrm{epv}}(\mathbf{V})$. *Let* $\mathrm{V} = \langle \mathbf{V}\rangle_{\mathrm{ev}}$ *and let* $\mathrm{U} \in \mathscr{L}_{\mathrm{ev}}(\mathrm{V})$ *be a locally finite e-variety. Then*

$$\mathrm{U}_{\theta_{\mathrm{g}}} = \mathrm{U}_{\theta_{\mathrm{V}}} \Leftrightarrow \mathrm{U}_{\theta_{\mathrm{V}}} \text{ is finitely generated.}$$

Proof. Since $\theta_{\mathrm{V}} \subseteq \theta_{\mathrm{g}}$ we have that $\mathrm{U}_{\theta_{\mathrm{g}}} \subseteq \mathrm{U}_{\theta_{\mathrm{V}}}$ and $\mathrm{U}_{\theta_{\mathrm{V}}} = \langle \mathrm{U}_{\theta_{\mathrm{g}}}\rangle_{\mathrm{ev}}$. So $\mathrm{U}_{\theta_{\mathrm{g}}} = \mathrm{U}_{\theta_{\mathrm{V}}} \Leftrightarrow \mathrm{U}_{\theta_{\mathrm{g}}}$ is an e-variety.

Suppose that $\mathrm{U}_{\theta_{\mathrm{V}}}$ is not finitely generated. That is, there does not exist $S \in (\mathrm{U}_{\theta_{\mathrm{V}}})^{\mathrm{Fin}}$ such that $\mathrm{U}_{\theta_{\mathrm{V}}} = \langle S\rangle_{\mathrm{ev}}$. Therefore $\langle \mathrm{U}_{\theta_{\mathrm{V}}}^{\mathrm{Fin}}\rangle_{\mathrm{gev}} \subset \mathrm{U}_{\theta_{\mathrm{V}}}$. However, $\left(\langle \mathrm{U}_{\theta_{\mathrm{V}}}^{\mathrm{Fin}}\rangle_{\mathrm{gev}}\right)^{\mathrm{Fin}} = \mathrm{U}_{\theta_{\mathrm{V}}}^{\mathrm{Fin}}$ which implies that $\langle \mathrm{U}_{\theta_{\mathrm{V}}}^{\mathrm{Fin}}\rangle_{\mathrm{gev}}\theta_{\mathrm{g}}\mathrm{U}_{\theta_{\mathrm{V}}}$. Also $\mathrm{U}_{\theta_{\mathrm{V}}}\theta_{\mathrm{g}}\mathrm{U}$ and so by transitivity, $\langle \mathrm{U}_{\theta_{\mathrm{V}}}^{\mathrm{Fin}}\rangle_{\mathrm{gev}}\theta_{\mathrm{g}}\mathrm{U}$. But, $\mathrm{U}_{\theta_{\mathrm{g}}}\theta_{\mathrm{g}}\mathrm{U} \Rightarrow \mathrm{U}_{\theta_{\mathrm{g}}} \subseteq \langle \mathrm{U}_{\theta_{\mathrm{V}}}^{\mathrm{Fin}}\rangle_{\mathrm{gev}} \subset \mathrm{U}_{\theta_{\mathrm{V}}}$. The contrapositive result has that $\mathrm{U}_{\theta_{\mathrm{V}}} \subseteq \mathrm{U}_{\theta_{\mathrm{g}}}$ (that is, $\mathrm{U}_{\theta_{\mathrm{V}}} = \mathrm{U}_{\theta_{\mathrm{g}}}$) which implies that $\mathrm{U}_{\theta_{\mathrm{V}}}$ is finitely generated.

The converse follows by noting that if $\mathrm{U}_{\theta_{\mathrm{V}}}$ is finitely generated then there exists an $S \in \mathrm{U}_{\theta_{\mathrm{V}}}^{\mathrm{Fin}}$ such that $\mathrm{U}_{\theta_{\mathrm{V}}} = \langle S\rangle_{\mathrm{ev}}$. However,

$$\langle S\rangle_{\mathrm{ev}} = (\langle S\rangle_{\mathrm{ev}})_{\theta_{\mathrm{V}}} = (\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}$$

and so $\left((\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}\right)^{\mathrm{Fin}} \subseteq \mathrm{U}_{\theta_{\mathrm{g}}}$. Therefore $S \in \mathrm{U}_{\theta_{\mathrm{g}}}$ and so

$$\mathrm{U}_{\theta_{\mathrm{V}}} = \langle S\rangle_{\mathrm{ev}} = \langle S\rangle_{\mathrm{gev}} \subseteq \mathrm{U}_{\theta_{\mathrm{g}}}$$

thus completing the proof. □

Theorem 3.4.6. *Let $\mathbf{V} \in \mathscr{L}_{\mathrm{epv}}(\mathbf{ES}) \cup \mathscr{L}_{\mathrm{epv}}(\mathbf{LI})$ and let θ be a complete congruence on $\mathscr{L}_{\mathrm{epv}}(\mathbf{V})$. Then there exists $\mathcal{A} \subseteq \bigcup_{S\in\mathbf{V}}\langle S\rangle_{\mathrm{ev}}$ such that $\mathcal{A}_S = \mathcal{A} \cap \langle S\rangle_{\mathrm{ev}}$ determines $\theta_{\langle S\rangle_{\mathrm{ev}}}$ and satisfies:*

(i) $\mathbb{I}(\mathcal{A}_S) \subseteq \mathcal{A}_S$;

(ii) $\mathrm{T} \subseteq \mathcal{A}_S$ *and*

(iii) $\mathbb{HS}_{\mathrm{e}}\mathbb{PD}_{\mathrm{e}}^{\mathcal{A}_S}\mathbb{P}(\mathcal{C}) = \mathbb{HS}_{\mathrm{e}}\mathbb{PD}_{\mathrm{e}}^{\mathcal{A}_S}\mathbb{P}_{\mathrm{ow}}(\mathcal{C})$ *for all* $\mathcal{C} \subseteq \mathbf{V}$.

Proof. Following Pastijn and Trotter [34, Lemma 4.7] we let

$$\mathcal{A} = \left\{S \in \mathbf{V} \mid \langle S\rangle_{\mathrm{ev}} = (\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}\right\}.$$

Clearly $\mathcal{A} \subseteq \bigcup_{S\in\mathbf{V}}\langle S\rangle_{\mathrm{ev}}$ and we have that $\mathcal{A}_S$ determines $\theta_{\langle S\rangle_{\mathrm{ev}}}$ for all $S \in \mathbf{V}$ and satisfies conditions (i) through (iii). □

Corollary 3.4.7. *Let $\mathbf{V} \in \mathscr{L}_{\mathrm{epv}}(\mathbf{ES}) \cup \mathscr{L}_{\mathrm{epv}}(\mathbf{LI})$, let θ be a complete congruence on $\mathscr{L}_{\mathrm{epv}}(\mathbf{V})$ and let $\mathrm{W} \in \mathscr{L}_{\mathrm{gev}}(G(\langle\mathbf{V}\rangle_{\mathrm{ev}}))$. Then*

$$\left(\bigvee_{S\in\mathrm{W}^{\mathrm{Fin}}} (\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}\right)^{\mathrm{Fin}} = \left(\langle\mathcal{A} \cap \langle\mathrm{W}^{\mathrm{Fin}}\rangle_{\mathrm{gev}}\rangle_{\mathrm{gev}}\right)^{\mathrm{Fin}}$$

where $\mathcal{A}$ is the class described in the previous theorem.

Proof. Suppose $T \in \left(\bigvee_{S\in\mathrm{W}^{\mathrm{Fin}}} (\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}\right)^{\mathrm{Fin}}$. There exists $T' \in \mathrm{W}^{\mathrm{Fin}}$ such that $T \in \left((\langle T'\rangle_{\mathrm{ev}})_{\theta_{\langle T'\rangle_{\mathrm{ev}}}}\right)^{\mathrm{Fin}}$. Let $B = BF_X\left((\langle T'\rangle_{\mathrm{ev}})_{\theta_{\langle T'\rangle_{\mathrm{ev}}}}\right)$ be the bifree object on the countably infinite set X in the e-variety $(\langle T'\rangle_{\mathrm{ev}})_{\theta_{\langle T'\rangle_{\mathrm{ev}}}}$. Then

$$\begin{aligned}\langle B\rangle_{\mathrm{ev}} \subseteq \langle T'\rangle_{\mathrm{ev}} \Rightarrow (\langle B\rangle_{\mathrm{ev}})_{\theta_{\langle B\rangle_{\mathrm{ev}}}} &= (\langle B\rangle_{\mathrm{ev}})_{\theta_{\langle T'\rangle_{\mathrm{ev}}}} \\ &= \left((\langle T'\rangle_{\mathrm{ev}})_{\theta_{\langle T'\rangle_{\mathrm{ev}}}}\right)_{\theta_{\langle T'\rangle_{\mathrm{ev}}}} \\ &= \langle B\rangle_{\mathrm{ev}}.\end{aligned}$$

Now, since $T \in \langle B\rangle_{\mathrm{ev}}$ and

$$\langle B\rangle_{\mathrm{ev}} = \bigvee^{\mathrm{ev}}_{S\in\langle B\rangle_{\mathrm{ev}}\cap\mathcal{A}} \langle S\rangle_{\mathrm{ev}} \subseteq \bigvee^{\mathrm{ev}}_{S\in\langle T'\rangle_{\mathrm{ev}}\cap\mathcal{A}} \langle S\rangle_{\mathrm{ev}}$$

we have that $T \in \bigvee^{\mathrm{ev}}_{S\in\langle T'\rangle_{\mathrm{ev}}\cap\mathcal{A}} \langle S\rangle_{\mathrm{ev}}$. But since T and T' are finite, we have that $T \in \left(\langle \mathcal{A} \cap \langle \mathrm{W}^{\mathrm{Fin}}\rangle_{\mathrm{gev}}\rangle_{\mathrm{gev}}\right)^{\mathrm{Fin}}$ and so

$$\left(\bigvee_{S\in\mathrm{W}^{\mathrm{Fin}}} (\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}\right)^{\mathrm{Fin}} \subseteq \left(\langle \mathcal{A} \cap \langle \mathrm{W}^{\mathrm{Fin}}\rangle_{\mathrm{gev}}\rangle_{\mathrm{gev}}\right)^{\mathrm{Fin}}.$$

To establish the converse, suppose $T \in \left(\langle \mathcal{A} \cap \langle \mathrm{W}^{\mathrm{Fin}}\rangle_{\mathrm{gev}}\rangle_{\mathrm{gev}}\right)^{\mathrm{Fin}}$. So there exists a finite collection $T_1, \ldots, T_n$ of members of $\mathcal{A} \cap \langle \mathrm{W}^{\mathrm{Fin}}\rangle_{\mathrm{gev}}$ such that

$$T \in \langle T_1, \ldots, T_n\rangle_{\mathrm{gev}}.$$

Proceeding a level further, each T_i, $i \in \{1, \ldots, n\}$ is contained in $\langle S_i\rangle_{\mathrm{gev}}$ where $S_i = C_{i_1} \times \cdots \times C_{i_k}$ where each $C_{i_j} \in \mathrm{W}^{\mathrm{Fin}}$ for some finite k. Since each $T_i \in \mathcal{A}$ we have

$$\begin{aligned}\langle T_i\rangle_{\mathrm{ev}} = (\langle T_i\rangle_{\mathrm{ev}})_{\theta_{\langle S_i\rangle_{\mathrm{ev}}}} &= \bigvee^{\mathrm{ev}}_{1\le j\le k} \left(\langle C_{i_j}\rangle_{\mathrm{ev}}\right)_{\theta_{\langle S_i\rangle_{\mathrm{ev}}}} \\ &= \bigvee^{\mathrm{ev}}_{1\le j\le k} \left(\langle C_{i_j}\rangle_{\mathrm{ev}}\right)_{\theta_{\langle C_{i_j}\rangle_{\mathrm{ev}}}} \\ &= \bigvee^{\mathrm{gev}}_{1\le j\le k} \left(\langle C_{i_j}\rangle_{\mathrm{ev}}\right)_{\theta_{\langle C_{i_j}\rangle_{\mathrm{ev}}}}\end{aligned}$$

and so $T \in \langle \bigvee^{\mathrm{gev}}_{S\in\mathrm{W}^{\mathrm{Fin}}} (\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}\rangle_{\mathrm{gev}} = \bigvee^{\mathrm{gev}}_{S\in\mathrm{W}^{\mathrm{Fin}}} (\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}$. Since T is finite we have established that

$$\left(\langle \mathcal{A} \cap \langle \mathrm{W}^{\mathrm{Fin}}\rangle_{\mathrm{gev}}\rangle_{\mathrm{gev}}\right)^{\mathrm{Fin}} \subseteq \left(\bigvee_{S\in\mathrm{W}^{\mathrm{Fin}}} (\langle S\rangle_{\mathrm{ev}})_{\theta_{\langle S\rangle_{\mathrm{ev}}}}\right)^{\mathrm{Fin}}$$

and so the proof is complete. □

The following theorem gives a characterisation of all complete congruences on lattices of e-pseudovarieties which consist entirely of E-solid or locally inverse regular semigroups.

Theorem 3.4.8. *Let* $\mathbf{V} \in \mathscr{L}_{\text{epv}}(\mathbf{ES}) \cup \mathscr{L}_{\text{epv}}(\mathbf{LI})$ *and let* $\mathcal{A} \subseteq \bigcup_{S \in \mathbf{V}} \mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}(S)$. *For each* $S \in \mathbf{V}$, *let* $\mathcal{A}_S = \langle S \rangle_{\text{ev}} \cap \mathcal{A}$ *and suppose that*

(i) $\mathbb{I}(\mathcal{A}_S) \subseteq \mathcal{A}_S$;

(ii) $\mathrm{T} \subseteq \mathcal{A}_S$; *and*

(iii) $\mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}\mathbb{D}_{\text{e}}^{\mathcal{A}_S}\mathbb{P}(\mathcal{C}) = \mathbb{H}\mathbb{S}_{\text{e}}\mathbb{P}\mathbb{D}_{\text{e}}^{\mathcal{A}_S}\mathbb{P}_{\text{ow}}(\mathcal{C})$ *for all* $\mathcal{C} \subseteq \mathbf{V}$.

Define θ *on* $\mathscr{L}_{\text{epv}}(\mathbf{V})$ *by*

$$\mathbf{U}_1 \theta \mathbf{U}_2 \Leftrightarrow \left(\langle \mathcal{A} \cap \langle \mathbf{U}_1 \rangle_{\text{gev}} \rangle_{\text{gev}}\right)^{\text{Fin}} = \left(\langle \mathcal{A} \cap \langle \mathbf{U}_2 \rangle_{\text{gev}} \rangle_{\text{gev}}\right)^{\text{Fin}}.$$

Then θ *is a complete congruence on* $\mathscr{L}_{\text{epv}}(\mathbf{V})$. *Furthermore, every complete congruence on* $\mathscr{L}_{\text{epv}}(\mathbf{V})$ *is of this form.*

Proof. That θ is a complete congruence on $\mathscr{L}_{\text{epv}}(\mathbf{V})$ is quickly established. The converse follows directly from Corollary 3.4.7. □

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