Appendix A

Magnetic Field Calculations

A.1 Magnetic Field of a Wire Segment

The magnetic field of a segment of wire was introduced in Chapter 2.6.1. This appendix deals with the derivation of the magnetic field calculation solely in terms of the vectors $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{m}$. The relevant figure, Figure A.1 is reproduced below.

![Diagram of a wire segment with vectors](image)

**Figure A.1:** Line segment defined by vectors $\mathbf{r}$ and $\mathbf{v}$, such that current flows along $\mathbf{m}$ ($\mathbf{m} = \mathbf{v} - \mathbf{r}$). Constant current $I$ is flowing in the wire.

The magnetic field at point $P$ points out of the page. The direction of the magnetic field is contained in the sine functions. We can represent the value of a sine with its cross product interpretations:

$$\sin \theta = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}||\mathbf{b}|},$$

(A.1)
where $\mathbf{a}$ and $\mathbf{b}$ are two arbitrary vectors connected toe to toe, and $\theta$ is the angle between them.

To prevent you having to find it yourself, I will repeat equation 2.28 below.

\[
B = \frac{\mu_0}{4\pi} \frac{I}{z} (\sin \theta_2 - \sin \theta_1). \tag{A.2}
\]

Looking at the sine arguments in equation (A.2), the directional information for $\mathbf{B}$ is given by these. Using the relation of equation (A.1) for the vector interpretation of the sines, we can determine the direction of $\mathbf{B}$ at point $P$ by substituting these into equation (A.2) and removing the magnitude operators from the numerator. In order to do this properly, we need to be aware of the intended direction of $\sin \theta_1$. The contribution must point out of the page at point $P$, so, since

\[
\sin \theta_1 = \frac{\mathbf{n} \times \mathbf{r}}{||\mathbf{n}|| ||\mathbf{r}||}.
\]

Since $\mathbf{n} \times \mathbf{r}$ points into the page, we need to use a minus sign.

This yields the following equation, where $\mathbf{B}$ is now interpreted as a vector quantity:

\[
\mathbf{B} = \frac{\mu_0}{4\pi} \frac{1}{||\mathbf{n}||} \left( \frac{\mathbf{n} \times \mathbf{v}}{||\mathbf{n}|| ||\mathbf{v}||} - \frac{\mathbf{n} \times \mathbf{r}}{||\mathbf{n}|| ||\mathbf{r}||} \right)
\]

\[
\mathbf{B} = \frac{\mu_0}{4\pi} \frac{1}{||\mathbf{n}||^2} \left( \frac{\mathbf{n} \times \mathbf{v}}{||\mathbf{v}||} - \frac{\mathbf{n} \times \mathbf{r}}{||\mathbf{r}||} \right). \tag{A.3}
\]

The tricky bit comes next. The vector $\mathbf{n}$ can be neatly expressed in terms of a linear combination of the vectors $\mathbf{r}$ and $\mathbf{m}$ in the following manner

\[
\mathbf{n} = \mathbf{r} + t\mathbf{m}, \tag{A.4}
\]

where $t$ is some real number. Since $\mathbf{n}$ and $\mathbf{m}$ are perpendicular to one another, their
dot product is zero. This gives

\[ \mathbf{n} \cdot \mathbf{m} = 0 \]
\[ = (\mathbf{r} + t\mathbf{m}) \cdot \mathbf{m} \]
\[ = \mathbf{r} \cdot \mathbf{m} + t|\mathbf{m}|^2 \]
\[ \rightarrow t = -\frac{\mathbf{r} \cdot \mathbf{m}}{|\mathbf{m}|^2}. \]

(A.5)

In a similar manner, we can define \( \mathbf{n} \) in terms of \( \mathbf{v} \) and \( \mathbf{m} \), and get another expression for \( t \)

\[ \mathbf{n} = \mathbf{v} - (1 - t)\mathbf{m}, \]
\[ \mathbf{n} \cdot \mathbf{m} = 0 \]
\[ \rightarrow t = 1 - \frac{\mathbf{m} \cdot \mathbf{v}}{|\mathbf{m}|^2}. \]

(A.6)

There is still another way to define \( \mathbf{n} \), only in terms of \( \mathbf{r} \) and \( \mathbf{v} \) alone. Since \( \mathbf{m} = \mathbf{v} - \mathbf{r} \), we can use equation (A.4) and write

\[ \mathbf{n} = \mathbf{r} + t\mathbf{m} \]
\[ = \mathbf{r} + t(\mathbf{v} - \mathbf{r}) \]
\[ = (1 - t)\mathbf{r} + t\mathbf{v}. \]
We are now in position to rewrite the bottom line of equation (A.3):

\[
B = \frac{\mu_0}{4\pi} I \frac{1}{|n|^2} \left( \frac{n \times v}{|v|} - \frac{n \times r}{|r|} \right)
\]

\[
= \frac{\mu_0}{4\pi} I \frac{1}{|n|^2} \left( \frac{(1-t)r \times v + tv \times r}{|v|} - \frac{(1-t)r + tv \times r}{|r|} \right)
\]

\[
= \frac{\mu_0}{4\pi} I \frac{1}{|n|^2} \left( \frac{(1-t)r \times v - tv \times r}{|v|} \right)
\]

\[
= \frac{\mu_0}{4\pi} I \frac{r \times v}{|n|^2} \left( \frac{(1-t)}{|v|} + \frac{t}{|r|} \right),
\]

Using equation (A.6) for the first occurrence of \( t \), and (A.5) for the second, yielding

\[
B = \frac{\mu_0}{4\pi} I \frac{r \times v}{|n|} \left( \frac{1}{|v|} - \frac{m \cdot v}{|m|^2} + \frac{-m \cdot r}{|m|^2} \right)
\]

\[
= \frac{\mu_0}{4\pi} I \frac{r \times v}{|n|^2|m|^2} \left( \frac{m \cdot v}{|v|} - \frac{m \cdot r}{|r|} \right),
\] (A.7)

which looks a bit more reasonable.

The final bit comes from the realization that \( |n|^2|m|^2 \) is the area of the parallelogram described by \( r \times v \) so that we can write

\[
|m|^2|n|^2 = |r \times v|
\]

in the last line of equation (A.7):

\[
B = \frac{\mu_0}{4\pi} I \left( \frac{m \cdot v}{|v|} - \frac{m \cdot r}{|r|} \right) \frac{r \times v}{|r \times v|^2},
\]

which gives both the magnitude and direction of \( B \) in terms of \( m, v \) and \( r \).
A.2 Magnetic Field of a Circular Loop of Wire

We will start of with the repetition of equation 2.31, which is

\[ A_\phi(r, \theta) = \frac{\mu_0 I}{4\pi a} \int \frac{\cos \phi' \sin \theta' \delta(\cos \theta') \delta(r' - a)}{R} \, d\tau. \]  
(A.8)

The interpretation of \( R \) is a tricky one. It represents the distance from the differential volume element to the point \( P \). We can write out the Cartesian coordinates for the vector \( \mathbf{R} \) by noticing that it is constructed from two vectors, \( \mathbf{r} \) and \( \mathbf{r}' \), see Figure A.2

\[ \begin{align*}
\mathbf{r} &= r \sin \theta \hat{i} + 0 \hat{j} + r \cos \theta \hat{k} \\
\mathbf{r}' &= r' \sin \theta' \cos \phi' \hat{i} + r' \sin \theta' \sin \phi' \hat{j} + r' \cos \theta' \hat{k}
\end{align*} \]  
(A.9)

From Figure A.2, we see that \( \mathbf{R} = \mathbf{r} - \mathbf{r}' \). We can write out the Cartesian coordinates for \( \mathbf{r} \) and \( \mathbf{r}' \) explicitly. Remember that \( \mathbf{r} \) is constrained to the \( x \)-axis:
The magnitude of \( \mathbf{R} \) is obtained in the normal manner

\[
R^2 = \mathbf{R} \cdot \mathbf{R} = (\mathbf{r} - \mathbf{r'}) \cdot (\mathbf{r} - \mathbf{r'})
= \mathbf{r} \cdot \mathbf{r} - 2\mathbf{r} \cdot \mathbf{r'} + \mathbf{r'} \cdot \mathbf{r'}
= r^2 + r'^2 - 2\mathbf{r} \cdot \mathbf{r'}.
\]

We can evaluate the dot product directly from (A.9):

\[
\mathbf{r} \cdot \mathbf{r'} = rr'(\sin \theta \sin \theta' \cos \phi') + 0 + rr'(\cos \theta \cos \theta')
= rr'(\sin \theta \sin \theta' \cos \phi' + \cos \theta \cos \theta'),
\]

so that \( R \) becomes

\[
R = \sqrt{r^2 + r'^2 - 2rr'\sin \theta \sin \theta' \cos \phi' + \cos \theta \cos \theta'};
\]

and we place this in the denominator of (A.8) to get

\[
A_\phi(r, \theta) = \frac{\mu_0 I}{4\pi a} \int \frac{\cos \phi' \sin \theta' \delta(\cos \theta') \delta(r' - a)}{\sqrt{r^2 + r'^2 - 2rr'(\sin \theta \sin \theta' \cos \phi' + \cos \theta \cos \theta')}} d\tau.
\]

To evaluate the integral above, we will take advantage of spherical coordinates:

\[
A_\phi(r, \theta) = \frac{\mu_0 I}{4\pi a} \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{\cos \phi' \sin \theta' \delta(\cos \theta') \delta(r' - a)}{\sqrt{r^2 + r'^2 - 2rr'(\sin \theta \sin \theta' \cos \phi' + \cos \theta \cos \theta')}} r'^2 \, dr' \, d\theta' \, d\phi'.
\]

The integrations over the delta functions will fish out the value of \( a \) from \( dr' \) and
\[ \theta' = \pi/2 \] from the integration of \( d\theta' \). This leaves us with

\[
A_\phi(r, \theta) = \frac{\mu_0 I a}{4\pi} \frac{4\pi}{\sqrt{r^2 + a^2 - 2ra \sin \theta \cos \phi'}} \int_0^{2\pi} \frac{\cos \phi' d\phi'}{\sqrt{r^2 + a^2 - 2ra \sin \theta \cos \phi'}}.
\] (A.10)

Following Jackson (1999), this integral can be expressed in terms of the complete elliptic integrals \( K \) and \( E \) with argument \( k \)

\[
A_\phi(r, \theta) = \frac{\mu_0 I a}{4\pi} \frac{4\pi}{\sqrt{a^2 + r^2 + 2ar \sin \theta}} \left( \frac{(2 - k^2)K(k) - 2E(k)}{k^2} \right),
\] (A.11)

where

\[
k^2 = \frac{4ar \sin \theta}{a^2 + r^2 + 2ar \sin \theta}.
\]

Jackson (1999) then gives the components of magnetic induction for spherical coordinates by applying the curl operator to the vector potential expression. In addition, I have applied the curl operator to the vector potential in cylindrical coordinates. The curl operator in cylindrical coordinates yields

\[
B = \nabla \times A = -\frac{\partial}{\partial z} A_\phi \hat{\rho} + 0 \hat{\phi} + \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\phi) \hat{k}.
\] (A.12)

The direction of \( \hat{\rho} \) is the planar radial direction away from the origin, and the other direction, \( \hat{k} \), is self-explanatory. We need to derive an expression for \( k \) based on \( \rho \) and \( z \). Notice that

\[
r \sin \theta = \rho,
\]

and

\[
r^2 = \rho^2 + z^2,
\]

which allows us to express \( k \) as

\[
k^2 = \frac{4a\rho}{(a + \rho)^2 + z^2}.
\] (A.13)
Placing (A.13) into (A.11), we get

\[ A_\phi(\rho, z) = \frac{\mu_0}{4\pi} \frac{4Ia}{\sqrt{(a + \rho)^2 + z^2}} \left( \frac{(2 - k^2)K(k) - 2E(k)}{k^2} \right) \]

\[ = \frac{\mu_0}{4\pi} \frac{4Ia}{\sqrt{(a + \rho)^2 + z^2}} \left( \frac{(a + \rho)^2 + z^2}{4a\rho} \right) (2 - k^2)K(k) - 2E(k) \]

\[ = \frac{\mu_0 I}{4\pi \rho} \left( \frac{(a + \rho)^2 + z^2}{4a\rho} \right)^{3/2} \left( 2 - \frac{4a\rho}{(a + \rho)^2 + z^2} \right) K(k) - 2E(k) \].

(A.14)

Thus, we can determine B by placing the last line of (A.14) into the equations of (A.12) and calculating the derivatives. Beginning with the \( \rho \)-direction, we get

\[ B_\rho = -\frac{\partial}{\partial z} A_\phi \]

\[ = -\frac{\partial}{\partial z} \left( \frac{\mu_0 I}{4\pi \rho} \left( \frac{(a + \rho)^2 + z^2}{4a\rho} \right)^{3/2} \left( 2 - \frac{4a\rho}{(a + \rho)^2 + z^2} \right) K(k) - 2E(k) \right) \].

(A.15)

The derivation is tedious: I have completed it using Maple 9.00 (Maplesoft, 2006), and produce the result here:

\[ B_\rho(\rho, z) = \frac{\mu_0 I}{2\pi} \frac{z}{\rho \sqrt{(a + \rho)^2 + z^2}} \left( \frac{a^2 + \rho^2 + z^2}{(a - \rho)^2 + z^2} E(k) - K(k) \right), \]

where

\[ k = 2 \sqrt{\frac{a\rho}{(a + \rho)^2 + z^2}}. \]

I treated the component of B in the z-direction in the same manner. The expression for the \( \phi \)-component of the vector potential is multiplied by \( \rho \) before the
partial derivative is taken.

\[
B_z = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\phi)
\]

\[
= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \mu_0 \frac{I}{4\pi} \rho \left( (a + \rho)^2 + z^2 \right)^{3/2} \left( 2 - \frac{4a\rho}{(a + \rho)^2 + z^2} \right) K(k) - 2E(k) \right)
\]

This derivative was also calculated using Maple 9.00 (Maplesoft, 2006), and the result is

\[
B_z = \frac{\mu_0 I}{2\pi} \frac{1}{\sqrt{(a + \rho)^2 + z^2}} \left( \frac{a^2 - \rho^2 - z^2}{(a - \rho)^2 + z^2} E(k) + K(k) \right),
\]

where

\[
k = 2 \sqrt{\frac{a\rho}{(a + \rho)^2 + z^2}}.
\]

### A.3 Magnetic Field of a Magnetic Dipole

In Chapter 2.6.3, we got as far as determining the \( \phi \)-component of the vector potential. Mathematically,

\[
A = \frac{\mu_0}{4\pi} \frac{mr \sin \theta}{(a^2 + r^2)^{3/2}} \hat{\phi}, \quad (A.16)
\]

and we can calculate \( B \) by applying the curl to \( A \) in the appropriate coordinate frame.
A.3.1 Spherical Coordinates

We can calculate the magnetic induction by applying the curl operator to the vector potential in spherical coordinates:

\[
\mathbf{B} = \nabla \times \mathbf{A} = \begin{cases} 
B_r = \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\phi) \\
B_\theta = -\frac{1}{r} \frac{\partial}{\partial r} (r A_\phi) \\
B_\phi = 0
\end{cases}
\]

We look first at \( B_r \). Putting (A.27) into the above equation gives the following simple differentials:

\[
B_r = \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\phi) \\
= \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\mu_0}{4\pi} \frac{mr \sin \theta}{(a^2 + r^2)^{3/2}} \right) \\
= \frac{\mu_0}{4\pi} \frac{m r}{(a^2 + r^2)^{3/2}} \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin^2 \theta) \\
= \frac{\mu_0}{4\pi} \frac{m}{(a^2 + r^2)^{3/2}} \sin \theta \left(2 \sin \theta \cos \theta\right) \\
= \frac{\mu_0}{2\pi} \frac{m \cos \theta}{(a^2 + r^2)^{3/2}}.
\]

The next computation is the derivative to calculate \( B_\theta \):

\[
B_\theta = -\frac{1}{r} \frac{\partial}{\partial r} (r A_\phi) \\
= -\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\mu_0}{4\pi} \frac{mr \sin \theta}{(a^2 + r^2)^{3/2}} \right) \\
= -\frac{\mu_0}{4\pi} m \sin \theta \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r^2}{(a^2 + r^2)^{3/2}} \right) \\
= -\frac{\mu_0}{4\pi} m \sin \theta \frac{1}{r} \left( r^2 - 2a^2 \frac{2a}{(a^2 + r^2)^{5/2}} \right) \\
= \frac{\mu_0}{4\pi} m \sin \theta \frac{r^2 - 2a^2}{(a^2 + r^2)^{5/2}}.
\]

We now make another use of the approximation that \( r \gg a \). This approximation
A.3. MAGNETIC FIELD OF A MAGNETIC DIPOLE

simplifies both $B_r$ and $B_\theta$ in terms of $r$ and $a$:

\[
\begin{align*}
B_r &\approx \frac{\mu_0 m \cos \theta}{2\pi r^3} \\
B_\theta &\approx \frac{\mu_0 m \sin \theta}{4\pi r^3}
\end{align*}
\]

### A.3.2 Cartesian and Cylindrical Coordinates

We can express the magnetic induction calculated in the previous section in terms of Cartesian coordinates by making use of the following definitions of Cartesian axes:

\[
\begin{align*}
\hat{i} &= \sin \theta \cos \phi \hat{r} + \cos \theta \cos \phi \hat{\theta} - \sin \phi \hat{\phi} \\
\hat{j} &= \sin \theta \sin \phi \hat{r} + \cos \theta \sin \phi \hat{\theta} + \cos \phi \hat{\phi} \\
\hat{k} &= \cos \theta \hat{r} - \sin \theta \hat{\theta}
\end{align*}
\]

Thus, $B_x$ may be expressed in Cartesian coordinates as:

\[
B_x = \mathbf{B} \cdot \hat{i} \\
= \frac{\mu_0}{4\pi} \left( \frac{\pi a^2 I}{r^3} \right) \left( \frac{2 \cos \theta}{r^3} \hat{r} + \frac{\sin \theta}{r^3} \hat{\theta} \right) \cdot \left( \sin \theta \cos \phi \hat{r} + \cos \theta \cos \phi \hat{\theta} - \sin \phi \hat{\phi} \right) \\
= 3 \frac{\mu_0}{4\pi} \left( \frac{\pi a^2 I}{r^3} \right) \cos \theta \sin \theta \cos \phi \\
\]

Similarly, $B_y$ becomes

\[
B_y = \mathbf{B} \cdot \hat{j} \\
= \frac{\mu_0}{4\pi} \left( \frac{\pi a^2 I}{r^3} \right) \left( \frac{2 \cos \theta}{r^3} \hat{r} + \frac{\sin \theta}{r^3} \hat{\theta} \right) \cdot \left( \sin \theta \sin \phi \hat{r} + \cos \theta \sin \phi \hat{\theta} + \cos \phi \hat{\phi} \right) \\
= 3 \frac{\mu_0}{4\pi} \left( \frac{\pi a^2 I}{r^3} \right) \cos \theta \sin \theta \sin \phi \\.\]
Finally, $B_z$ becomes

$$B_z = B \cdot \hat{k}$$

$$= \frac{\mu_0}{4\pi} (\pi a^2 I) \left( \frac{2 \cos \theta}{r^3} \hat{r} + \frac{\sin \theta}{r^3} \hat{\theta} \right) \cdot \left( \cos \theta \hat{r} - \sin \theta \hat{\theta} \right)$$

$$= \frac{\mu_0}{4\pi} (\pi a^2 I) \frac{2 \cos^2 \theta - \sin^2 \theta}{r^3}.$$

Now, we need to define the basis vectors for the cylindrical coordinate system:

$$\begin{align*}
\hat{\rho} &= \cos \phi \hat{i} + \sin \phi \hat{j} \\
\hat{\phi} &= -\sin \phi \hat{i} + \cos \phi \hat{j} \\
\hat{k} &= \hat{k}
\end{align*}$$

The magnetic field can then be calculated by taking the dot product of $B$, as defined in the Cartesian system, with the basis unit vectors for each component. Thus

$$B_\rho = B \cdot \hat{\rho}$$

$$= \frac{\mu_0}{4\pi} (\pi a^2 I) \frac{1}{r^3} \left( 3 \cos \theta \sin \theta (\cos \phi \hat{i} + \sin \phi \hat{j}) \\
+ (2 \cos^2 \theta - \sin^2 \theta) \hat{k} \right) \cdot \left( \cos \phi \hat{i} + \sin \phi \hat{j} \right)$$

$$= \frac{\mu_0}{4\pi} (\pi a^2 I) \frac{3 \cos \theta \sin \theta}{r^3} (\cos^2 \phi + \sin^2 \phi)$$

$$= \frac{\mu_0}{4\pi} (\pi a^2 I) \frac{3 \cos \theta \sin \theta}{r^3}.$$
The next calculation has the same form

\[ B_\phi = \mathbf{B} \cdot \hat{\phi} \]

\[ = \frac{\mu_0}{4\pi} (\pi a^2 I) \frac{1}{r^3} \left( 3 \cos \theta \sin \theta (\cos \phi \hat{i} + \sin \phi \hat{j}) + (2 \cos^2 \theta - \sin^2 \theta) \hat{k} \right) \cdot \left( -\sin \phi \hat{i} + \cos \phi \hat{j} \right) \]

\[ = \frac{\mu_0}{4\pi} (\pi a^2 I) \frac{3 \cos \theta \sin \theta}{r^3} (-\cos \phi \sin \phi + \sin \phi \cos \phi) \]

\[ = 0, \]

which means, as we expect, that the magnetic induction does not circulate in the \( \phi \)-direction. Finally, we calculate the \( z \)-component of the magnetic field of the dipole.

The calculation is presented here:

\[ B_z = \mathbf{B} \cdot \hat{k} \]

\[ = \frac{\mu_0}{4\pi} (\pi a^2 I) \frac{1}{r^3} \left( 3 \cos \theta \sin \theta (\cos \phi \hat{i} + \sin \phi \hat{j}) + (2 \cos^2 \theta - \sin^2 \theta) \hat{k} \right) \cdot (\hat{k}) \]

\[ = \frac{\mu_0}{4\pi} (\pi a^2 I) \frac{2 \cos^2 \theta - \sin^2 \theta}{r^3} \]

To make sense of all the calculations above, I will make use of Figure A.2, which shows that we can express the sine and cosine of \( \theta \) in the following manner:

\[ \cos \theta = \frac{z}{r} = \frac{z}{\sqrt{\rho^2 + z^2}}, \]

and

\[ \sin \theta = \frac{\rho}{r} = \frac{\rho}{\sqrt{\rho^2 + z^2}}. \]

Using these relations, I will write out the components of \( \mathbf{B} \) in cylindrical coordi-
nates in equations

\[ B_\rho = \frac{\mu_0}{4\pi} \left( \pi a^2 I \right) \frac{1}{(\rho^2 + z^2) (\rho^2 + z^2)^{1/2}} \frac{z\rho}{(\rho^2 + z^2)^{5/2}}, \]

and

\[ B_z = \frac{\mu_0}{4\pi} \left( \pi a^2 I \right) \frac{2\cos^2 \theta - \sin^2 \theta}{(\rho^2 + z^2)^{3/2}} \frac{2z^2 - \rho^2}{(\rho^2 + z^2)^{5/2}}, \]

to get the set

\[
\text{Cylindrical Coordinates} \begin{cases} 
B_\rho = \frac{\mu_0}{4\pi} \left( \pi a^2 I \right) \frac{z\rho}{(\rho^2 + z^2)^{5/2}}, \\
B_z = \frac{\mu_0}{4\pi} \left( \pi a^2 I \right) \frac{2z^2 - \rho^2}{(\rho^2 + z^2)^{5/2}}.
\end{cases}
\tag{A.17}
\]

\section*{A.3.3 Coordinate-Independent}

For this derivation, we need the expression of the vector potential of a magnetic dipole, given in equation (A.16), repeated here:

\[ A = \frac{\mu_0}{4\pi} \frac{mr \sin \theta}{a^2 + r^2)^{3/2}} \hat{\phi}. \]

Using the approximation that \( r \gg a \), the vector potential becomes

\[ A = \frac{\mu_0}{4\pi} \frac{mr \sin \theta}{r^3} \hat{\phi}. \]

It is clear that \( A \) points in the \( \hat{\phi} \)-direction everywhere in space. Using equation (A.1), the vector definition of sine, \( A \) can be expressed in a much more general way...
as

$$A = \frac{\mu_0}{4\pi} m \times \frac{r}{r^3},$$

which can be further expressed as

$$A = \frac{\mu_0}{4\pi} m \times \frac{\hat{r}}{r^2}, \quad \text{(A.18)}$$

where \(\hat{r}\) is the unit vector pointing from the dipole to the point of observation. As usual, the magnetic field is obtained by taking the curl of the vector potential.

In order to derive the coordinate-independent expression for the magnetic field of a dipole, I will begin with the vector identity of the curl of a cross product. This is shown in equation (A.19) below:

$$\nabla \times (a \times b) = a(\nabla \cdot b) - b(\nabla \cdot a) + (b \cdot \nabla)a - (a \cdot \nabla)b. \quad \text{(A.19)}$$

The first step is to put expression (A.18) into equation (A.19).

$$\nabla \times \left( m \times \frac{\hat{r}}{r^2} \right) = m \left( \nabla \cdot \frac{\hat{r}}{r^2} \right) - \frac{\hat{r}}{r^2} (\nabla \cdot m) + \left( \frac{\hat{r}}{r^2} \cdot \nabla \right) m - (m \cdot \nabla)\frac{\hat{r}}{r^2}. \quad \text{(A.20)}$$

Since \(m\) is a fixed vector, the second and third terms drop out of the expression. The first term evaluates to zero, since

$$\nabla \cdot \frac{\hat{r}}{r} = 0.$$

That leaves only the last term in equation (A.20). Let’s look at it more closely. The last term is a spatial derivative type of operator that acts on each component of \(r\).
Written out, the last term of equation (A.20) becomes

\[-(\mathbf{m} \cdot \nabla) \frac{\hat{r}}{r^2} = - \left( m_x \frac{\partial}{\partial x} + m_y \frac{\partial}{\partial y} + m_z \frac{\partial}{\partial z} \right) \frac{\hat{r}}{r^2}.\]

Looking at only the $x$-component, since the others will be the same,

\[B_x = -\frac{\mu_0}{4\pi} \left( m_x \frac{\partial}{\partial x} + m_y \frac{\partial}{\partial y} + m_z \frac{\partial}{\partial z} \right) \frac{x}{(x^2 + y^2 + z^2)^{3/2}}.\]

Computing the partial derivatives on this expression yields

\[B_x = \frac{\mu_0}{4\pi} \left( m_x \left( \frac{3x^2}{(x^2 + y^2 + z^2)^{5/2}} - \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \right) + 3m_y \left( \frac{xy}{(x^2 + y^2 + z^2)^{5/2}} \right) + 3m_z \left( \frac{xz}{(x^2 + y^2 + z^2)^{5/2}} \right) \right).\]

Simplifying and expressing the partial power terms in powers of $r$ gives

\[B_x = \frac{\mu_0}{4\pi} \left( 3x \left( \frac{m_x x + m_y y + m_z z}{r^5} \right) - \frac{m_x}{r^3} \right).\]

The second term is simply the $x$-component of the magnetic dipole vector, while the first term is the $x$-component of the dot product of $\mathbf{m}$ with $\mathbf{r}$ multiplied to $\mathbf{r}$.

Written out in vector format,

\[B_x = \frac{\mu_0}{4\pi} \left( \frac{3(\mathbf{m} \cdot \mathbf{r}) \mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3} \right) \cdot \hat{i}.\]

The complete expression for the magnetic field of a dipole can be recast into the more elegant form

\[B = \frac{\mu_0}{4\pi} \frac{1}{r^3} \left( 3(\mathbf{m} \cdot \hat{r}) \hat{r} - \mathbf{m} \right). \quad (A.21)\]
Appendix B

Mutual and Self Inductance Calculations

B.1 The Mutual Inductance of Two Skew Lines

The calculation of the mutual inductance of two skew lines is a bit tricky—get ready to see some horrible integrals! We will start with a diagram of the geometry of the two wires, Figure B.1. For any two skew lines, it is possible to create a line that is

\[ \text{Figure B.1: Vector representation of skew lines.} \]
perpendicular to both lines. This is represented in the figure above by line $P$. Line $l_1$ is created from the vectors $\mathbf{a}$ and $\mathbf{b}$ such that

$$l_1 = \mathbf{b} - \mathbf{a},$$

while line $l_2$ is formed from $\mathbf{c}$ and $\mathbf{d}$:

$$l_2 = \mathbf{d} - \mathbf{c}.$$ 

Removing the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$, and drawing in new ones that join line 1 to line 2, we get the next vector diagram of interest, Figure B.2.

Now, I will write some distances and factors that are relevant to the figures above. The distance $\lambda$ that I defined in equation (B.1) is the projection of $\mathbf{c} - \mathbf{a}$ onto the line $P$, defined by the cross product of $l_1$ and $l_2$. The distance $\xi$ is the distance from line $P$ to the start of line $l_1$, while the distance $\upsilon$ is the distance along $l_2$ from line $P$ to the start of $l_2$. I have marked these distances in Figure B.2. I have also included the calculation of $\lambda$, $\xi$ and $\upsilon$ below.

$$\xi = \frac{|l_1| |(\mathbf{c} - \mathbf{a}) \times l_2| \cdot (l_2 \times l_1)|}{|l_1 \times l_2|^2},$$

$$\upsilon = \frac{|l_2| |(\mathbf{c} - \mathbf{a}) \times l_1| \cdot (l_2 \times l_1)|}{|l_1 \times l_2|^2}$$

and

$$\lambda = \frac{|(\mathbf{c} - \mathbf{a}) \cdot (l_1 \times l_2)|}{|l_1 \times l_2|}. \quad (B.1)$$

So now we turn to actually calculating the mutual inductance. For this derivation, I will use the Neumann integration formula (equation (2.44)), repeated again here:

$$M_{12} = \frac{\mu_0}{4\pi} \int \int \frac{dl_1 \cdot dl_2}{R}. \quad (B.2)$$
The two vectors, \( \mathbf{l}_1 \) and \( \mathbf{l}_2 \) make an angle \( \epsilon \) with each other such that the integral in (B.2) simplifies to

\[
M = \frac{\mu_0}{4\pi} \frac{\mathbf{l}_1 \cdot \mathbf{l}_2}{|\mathbf{l}_1| |\mathbf{l}_2|} \int \int \frac{ds\, dt}{R},
\]

where \( ds \) and \( dt \) are integrations along the lines \( \mathbf{l}_1 \) and \( \mathbf{l}_2 \), respectively, over the lengths of line 1 and line 2. The distance \( R \) is calculated using the parametric lengths \( s \) and \( t \), as well as the length \( \lambda \),

\[
R = \sqrt{\lambda^2 + s^2 + t^2 - 2st \cos \epsilon}.
\]

\[\text{Figure B.2: Lines } \mathbf{l}_1 \text{ and } \mathbf{l}_2, \text{ constructed from vectors } \mathbf{a}, \mathbf{b}, \mathbf{c} \text{ and } \mathbf{d}.\]

Campbell (1915) recommends using a mathematical trick to make the integrals more tractable. He suggests writing the integral over \( 1/R \) as the following

\[
M = \frac{\mu_0}{4\pi} \frac{\mathbf{l}_1 \cdot \mathbf{l}_2}{|\mathbf{l}_1| |\mathbf{l}_2|} \int \int \frac{ds\, dt}{R} \\
M = \frac{\mu_0}{4\pi} \frac{\mathbf{l}_1 \cdot \mathbf{l}_2}{|\mathbf{l}_1| |\mathbf{l}_2|} \int \int \left( \frac{\partial}{\partial s} \frac{s}{R} + \frac{\partial}{\partial t} \frac{t}{R} - \frac{\lambda^2}{R^3} \right) ds\, dt.
\]
This makes the integration possible: it is still laborious. Before I evaluate the
integrals, we must discuss the limits on the integrations. For the integral along
line \( l_1 \), the limit of integration is from \( \xi \) to \( \xi + |l_1| \). Along the \( s \) direction (that is, along \( l_2 \)), the limit of integration is from \( \upsilon \) to \( \upsilon + |l_2| \).

We are now in a position to integrate the separate terms. I will start with the
very first term in the integral:

\[
I_1 = \int_{\xi}^{\xi + |l_1|} \int_{\upsilon}^{\upsilon + |l_2|} \frac{\partial s}{\partial s} R \, ds \, dt
\]

\[
I_1 = \int_{\xi}^{\xi + |l_1|} \int_{\upsilon}^{\upsilon + |l_2|} \frac{d(s)}{ds} \frac{1}{R} \, ds \, dt
\]

\[
I_1 = \int_{\xi}^{\xi + |l_1|} \left( \frac{s}{R} \right) \left[ \int_{\upsilon}^{\upsilon + |l_2|} dt \right]
\]

\[
I_1 = \int_{\xi}^{\xi + |l_1|} \left( \frac{s}{\sqrt{\lambda^2 + s^2 + t^2 - 2st \cos \epsilon}} \right) \left[ \int_{\upsilon}^{\upsilon + |l_2|} dt \right]
\]

\[
I_1 = s \log \left( \sqrt{\lambda^2 + s^2 + t^2 - 2st \cos \epsilon + t - s \cos \epsilon} \right) \bigg|_{s=\upsilon}^{s=\upsilon + |l_2|} \bigg|_{t=\xi}^{t=\xi + |l_1|}
\]

(B.4)

This will give us 4 terms from the first part of equation (B.3). Writing them all out:

\[
I_1 = (v + |l_2|) \log \left( \sqrt{\lambda^2 + (v + |l_2|)^2 + (\xi + |l_1|)^2 - 2(v + |l_2|)(\xi + |l_1|) \cos \epsilon} \right)
\]

\[
+ (\xi + |l_1|) - (v + |l_2|) \cos \epsilon
\]

\[
- (v + |l_2|) \log \left( \sqrt{\lambda^2 + (v + |l_2|)^2 + \xi^2 - 2(v + |l_2|) \xi \cos \epsilon} \right)
\]

\[
+ \xi - (v + |l_2|) \cos \epsilon
\]

\[
- v \log \left( \sqrt{\lambda^2 + v^2 + (\xi + |l_1|)^2 - 2v(\xi + |l_1|) \cos \epsilon} + (\xi + |l_1|) - v \cos \epsilon \right)
\]

\[
+ v \log \left( \sqrt{\lambda^2 + v^2 + \xi^2 - 2v \xi \cos \epsilon + \xi - v \cos \epsilon} \right)
\]

(B.5)
Let us look at the first logarithmic term. The term under the radical is the length of the vector from the end of \( \mathbf{l}_1 \) to the end of \( \mathbf{l}_2 \), in other words, it is the length of the vector \( \mathbf{d} - \mathbf{b} \). The term \((v + |\mathbf{l}_2|)\cos \epsilon\) is the projection of the entire length of line 2, from the intersection of line \( P \) to the tip of \( \mathbf{l}_2 \), onto the line of \( \mathbf{l}_1 \). This means that the value

\[
\xi + |\mathbf{l}_1| - (v + |\mathbf{l}_2|)\cos \epsilon = -\frac{(\mathbf{d} - \mathbf{b}) \cdot \mathbf{l}_1}{|\mathbf{l}_1|},
\]

or that the term not under the radical is the negative of the projection of vector \( \mathbf{d} - \mathbf{b} \) onto the vector \( \mathbf{l}_1 \). The second logarithmic has a similar interpretation. The term under the radical is the magnitude of the vector \( \mathbf{d} - \mathbf{a} \), while the terms outside represent the negative projection of vector \( \mathbf{d} - \mathbf{a} \) onto \( \mathbf{l}_1 \). Gathering the first two terms, we get

\[
(v + |\mathbf{l}_2|) \left( \log \left| \frac{\mathbf{d} - \mathbf{b} - (\mathbf{d} - \mathbf{b}) \cdot \mathbf{l}_1}{|\mathbf{l}_1|} \right| - \log \left| \frac{\mathbf{d} - \mathbf{a} - (\mathbf{d} - \mathbf{a}) \cdot \mathbf{l}_1}{|\mathbf{l}_1|} \right| \right)
= (v + |\mathbf{l}_2|) \left( \log \left| \frac{\mathbf{d} - \mathbf{b} - (\mathbf{b} - \mathbf{d}) \cdot \mathbf{l}_1}{|\mathbf{b} - \mathbf{a}|} \right| + \log \left| \frac{\mathbf{d} - \mathbf{a} - (\mathbf{a} - \mathbf{d}) \cdot \mathbf{l}_1}{|\mathbf{b} - \mathbf{a}|} \right| \right)
= 2(v + |\mathbf{l}_2|) \tanh^{-1} \left( \frac{|\mathbf{l}_1|}{|\mathbf{d} - \mathbf{b}| + |\mathbf{b} - \mathbf{a}|} \right). \quad (B.6)
\]

The last result is from a nice little identity that I discovered some time ago. Imagine a triangle made of three vectors, \( \mathbf{a} \), \( \mathbf{b} \), \( \mathbf{c} \), such that \( \mathbf{c} = \mathbf{b} - \mathbf{a} \). A diagram is shown in Figure B.3.

The derivation is as follows:

\[
\frac{\mathbf{b} \cdot \mathbf{c} + |\mathbf{b}| |\mathbf{c}|}{\mathbf{a} \cdot \mathbf{c} + |\mathbf{a}| |\mathbf{c}|} = \frac{\mathbf{b} \cdot (\mathbf{b} - \mathbf{a}) + |\mathbf{b}| |\mathbf{c}|}{\mathbf{a} \cdot (\mathbf{b} - \mathbf{a}) + |\mathbf{a}| |\mathbf{c}|}
= \frac{|\mathbf{b}|^2 - \mathbf{a} \cdot \mathbf{b} + |\mathbf{b}| |\mathbf{c}|}{-|\mathbf{a}|^2 + \mathbf{a} \cdot \mathbf{b} + |\mathbf{a}| |\mathbf{c}|}, \quad \text{but} \quad \mathbf{a} \cdot \mathbf{b} = \frac{|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{c}|^2}{2}
= \frac{2|\mathbf{b}|^2 - |\mathbf{a}|^2 - |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2|\mathbf{b}| |\mathbf{c}|}{-2|\mathbf{a}|^2 + |\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{c}|^2 + 2|\mathbf{a}| |\mathbf{c}|}.
\]

The last line was achieved by use of the cosine rule on the top and the bottom.
Figure B.3: Vectors $a$, $b$, and $c$, where $c = b - a$.

Gathering terms and factoring, we arrive at

\[
\frac{b \cdot c + |b||c|}{a \cdot c + |a||c|} = \frac{|b|^2 - |a|^2 + |c|^2 + 2|b||c|}{-|a|^2 + |b|^2 - |c|^2 + 2|a||c|} = \frac{|a|^2 + |b|^2 + |c|^2}{|a|^2 + |b|^2 - |c|^2}, \quad \frac{-|a|^2 + |b|^2 + |c|^2}{-|a|^2 + |b|^2 + |c|^2}
\]

\[
= \frac{|a|^2 + |b|^2 + |c|^2}{|a|^2 + |b|^2 - |c|^2}
\]

\[
= 1 + \frac{|c|^2}{|a|^2 + |b|^2}
\]

\[
= 1 - \frac{|c|^2}{|a|^2 + |b|^2}
\]

We use this result to prove that

\[
\log \left( \frac{b \cdot c + |b||c|}{a \cdot c + |a||c|} \right) = \log \left( \frac{1 + \frac{|c|^2}{|a|^2 + |b|^2}}{1 - \frac{|c|^2}{|a|^2 + |b|^2}} \right) = 2 \tanh^{-1} \left( \frac{|c|^2}{|a|^2 + |b|^2} \right),
\]

using the relationship between the inverse hyperbolic tangent and log functions.
Going back to equation (B.5) and collecting the last two terms, repeated here for clarity,

\[-v \log \left( \sqrt{\lambda^2 + v^2 + (\xi + |l_1|)^2} - 2v(\xi + |l_1|) \cos \epsilon + (\xi + |l_1|) - v \cos \epsilon \right) \]

\[+ v \log \left( \sqrt{\lambda^2 + v^2 + \xi^2 - 2v \xi \cos \epsilon + \xi - v \cos \epsilon} \right), \quad (B.7)\]

we apply the same reasoning as we did for equation (B.6). The term under the radical is the length of the vector from the end of \(l_1\) to the beginning of \(l_2\), i.e. it is the length of \(c - b\). The last term in the first logarithm is the negative of the projection of \(c - b\) onto \(l_1\). For the second logarithm in (B.7), the radical represents the length of the vector \(c - a\), while the last two terms have the geometrical interpretation of the negative projection of \(c - a\) onto \(l_1\). Thus, we arrive at

\[-v \log \left( \sqrt{\lambda^2 + v^2 + (\xi + |l_1|)^2} - 2v(\xi + |l_1|) \cos \epsilon + (\xi + |l_1|) - v \cos \epsilon \right) \]

\[+ v \log \left( \sqrt{\lambda^2 + v^2 + \xi^2 - 2v \xi \cos \epsilon + \xi - v \cos \epsilon} \right) \]

\[= -v \left( \log \left( \frac{|c - b||l_1| - (c - b) \cdot l_1}{|c - a||l_1| - (c - a) \cdot l_1} \right) - \log \left( |c - a||l_1| - (c - a) \cdot l_1 \right) \right) \]

\[= -v \left( \log \left( \frac{|c - b||l_1| + (b - c) \cdot l_1}{|c - a||l_1| + (a - c) \cdot l_1} \right) \right) \]

\[= -2v \tanh^{-1} \left( \frac{|l_1|}{|a - c| + |c - b|} \right). \]

Thus, the expression (B.4) integrates to

\[I_1 = 2(v + |l_2|) \tanh^{-1} \left( \frac{|l_1|}{|d - b| + |d - a|} \right) - 2v \tanh^{-1} \left( \frac{|l_1|}{|a - c| + |c - b|} \right). \quad (B.8)\]

We now turn to the second integration of equation (B.3):

\[I_2 = \int_{\xi}^{\xi + |l_1|} \int_{v}^{v + |l_2|} \frac{\partial}{\partial t} R \, ds \, dt.\]
Moving along, the evaluation of the integrals will yield four logarithmic terms in exact accordance with the form of the last line of (B.4). Written out, term $I_2$ is

$$I_2 = t \log \left( \frac{\lambda^2 + s^2 + t^2 - 2st \cos \epsilon + s - t \cos \epsilon}{\xi + |\mathbf{l}_1|} \right)_{s=v}^{t=\xi+|\mathbf{l}_1|}$$

$$= (\xi + |\mathbf{l}_1|) \log \left( \frac{\lambda^2 + (v + |\mathbf{l}_2|)^2 + (\xi + |\mathbf{l}_1|)^2 - 2(v + |\mathbf{l}_2|)(\xi + |\mathbf{l}_1|) \cos \epsilon}{(v + |\mathbf{l}_2|) - (\xi + |\mathbf{l}_1|) \cos \epsilon} \right)$$

$$- \xi \log \left( \frac{\lambda^2 + (v + |\mathbf{l}_2|)^2 + \xi^2 - 2\xi(v + |\mathbf{l}_2|) \cos \epsilon + (v + |\mathbf{l}_2|) - \xi \cos \epsilon}{\xi + |\mathbf{l}_1|} \right) + \xi \log \left( \frac{\lambda^2 + v^2 + \xi^2 - 2\xi \cos \epsilon + v - \xi \cos \epsilon}{\xi + |\mathbf{l}_1|} \right).$$

(B.9)

With the same procedure as before, we look to the first logarithm of equation (B.9). The radical represents the length of the vector $\mathbf{d} - \mathbf{b}$. The other term is now the negative projection of $\mathbf{d} - \mathbf{b}$ onto the vector $\mathbf{l}_2$ (for, note the interchange of $s$ and $t$ in the first line of (B.9)). The second logarithm has a close interpretation for the radical, the length of $\mathbf{c} - \mathbf{b}$, while outside the radical, the other terms are the projection of $\mathbf{c} - \mathbf{b}$ onto $\mathbf{l}_2$. The third and fourth logarithms follow in the same manner: length of $\mathbf{d} - \mathbf{a}$, projection of $\mathbf{d} - \mathbf{a}$ onto $\mathbf{l}_2$ and length of $\mathbf{c} - \mathbf{a}$, projection of $\mathbf{c} - \mathbf{a}$ onto $\mathbf{l}_2$. Gathering terms, we get

$$I_2 = (\xi + |\mathbf{l}_1|) \left( \log |\mathbf{d} - \mathbf{b}| |\mathbf{l}_2| - (\mathbf{d} - \mathbf{b}) \cdot \mathbf{l}_2 \right) - \xi \log \left| \frac{\mathbf{c} - \mathbf{b}}{|\mathbf{c} - \mathbf{b}|} \right| - (\mathbf{c} - \mathbf{b}) \cdot \mathbf{l}_2 \right)$$

$$- \xi \left( \log |\mathbf{d} - \mathbf{a}| |\mathbf{l}_2| - (\mathbf{d} - \mathbf{a}) \cdot \mathbf{l}_2 \right) - \log \left| \frac{\mathbf{c} - \mathbf{a}}{|\mathbf{c} - \mathbf{a}|} \right| (\mathbf{c} - \mathbf{a}) \cdot \mathbf{l}_2 \right),$$

which can be expressed as

$$I_2 = (\xi + |\mathbf{l}_1|) \log \left( \frac{|\mathbf{d} - \mathbf{b}| |\mathbf{l}_2| - (\mathbf{d} - \mathbf{b}) \cdot \mathbf{l}_2}{|\mathbf{c} - \mathbf{b}| |\mathbf{l}_2| - (\mathbf{c} - \mathbf{b}) \cdot \mathbf{l}_2} \right) - \xi \log \left( \frac{|\mathbf{d} - \mathbf{a}| |\mathbf{l}_2| - (\mathbf{d} - \mathbf{a}) \cdot \mathbf{l}_2}{|\mathbf{c} - \mathbf{a}| |\mathbf{l}_2| - (\mathbf{c} - \mathbf{a}) \cdot \mathbf{l}_2} \right),$$

292
and further reduced to

\[ I_2 = 2(\xi + |l_1|) \tanh^{-1}\left( \frac{|l_2|}{|d - b| + |c - b|} \right) - 2\xi \tanh^{-1}\left( \frac{|l_2|}{|d - a| + |c - a|} \right). \]

(B.10)

Both of the first two terms have now been directly integrated and transformed to hyperbolic tangent forms that take vector magnitudes as their input.

So much for the first two terms of equation (B.3). The last term is an integral of \(1/R^3\):

\[ I_3 = -\int_{\xi}^{\xi+|l_1|} \int_{\nu}^{\nu+|l_2|} \frac{\lambda^2}{R^3} ds dt. \]

In order to evaluate this integral, Campbell (1915) recommends taking one of the terms of \(\lambda\) out of the integral, and multiplying through by \(\sin \epsilon\):

\[ I_3 = -\frac{\lambda}{\sin \epsilon} \int_{\xi}^{\xi+|l_1|} \int_{\nu}^{\nu+|l_2|} \frac{\lambda \sin \epsilon}{R^3} ds dt. \]

The first integral computes readily enough to give

\[ I_3 = -\frac{\lambda}{\sin \epsilon} \int_{\xi}^{\xi+|l_1|} \left( \frac{\lambda \sin \epsilon (s - t \cos \epsilon)}{(\lambda^2 + t^2 - t^2 \cos^2 \epsilon)R} \right) ds \bigg|_{s=\nu+|l_2|} \bigg|_{s=\nu} \]

\[ = -\frac{\lambda}{\sin \epsilon} \int_{\xi}^{\xi+|l_1|} \frac{\lambda \sin \epsilon (s - t \cos \epsilon)}{(\lambda^2 + t^2 \sin^2 \epsilon)R} ds \bigg|_{s=\nu+|l_2|} \bigg|_{s=\nu} \]  

(B.11)

In order to evaluate the final integral, I will use an exchange of variables and exploit the following integral form

\[ \int \frac{1}{1 + x^2} dx = \tan^{-1}(x), \]

(B.12)

and prove that the integrand in the bottom line of (B.11) can be put into that form.
To begin, let’s define the form of $x$

$$x = \frac{\lambda^2 \cos \epsilon + st \sin^2 \epsilon}{\lambda R \sin \epsilon}. $$

Taking the differential of $x$ with respect to $t$ yields

$$dx = \frac{\lambda^2 t \cos \epsilon - s^3 + s^2 t \cos \epsilon - s \lambda^2 + s^3 \cos^2 \epsilon - s^2 t \cos^3 \epsilon}{\lambda R^3 \sin \epsilon} dt. $$

The top term can be factored so we get

$$dx = \frac{(-1)(s - t \cos \epsilon)(\lambda^2 + s^2 \sin^2 \epsilon)}{\lambda R^3 \sin \epsilon} dt \quad (B.13)$$

Taking the integrand of equation (B.12) and putting in the new definition of $x$, we get

$$\frac{1}{1 + x^2} = \frac{\lambda^2 R^2 \sin^2 \epsilon}{\lambda^2 R^2 \sin^2 \epsilon + (\lambda^2 \cos \epsilon + st \sin^2 \epsilon)^2}. $$

This term can be simplified remarkably by expanding $R$, and using the cosine identity

$$\cos^2 \epsilon + \sin^2 \epsilon = 1$$

$$\frac{1}{1 + x^2} = \frac{\lambda^2 R^2 \sin^2 \epsilon}{(\lambda^2 + t^2 \sin^2 \epsilon)(\lambda^2 + s^2 \sin^2 \epsilon)}. \quad (B.14)$$

Multiplying together equations (B.13) and (B.14) yields

$$\frac{1}{1 + x^2} dx = \frac{\lambda^2 R^2 \sin^2 \epsilon}{(\lambda^2 + t^2 \sin^2 \epsilon)(\lambda^2 + s^2 \sin^2 \epsilon)} \frac{(-1)(s - t \cos \epsilon)(\lambda^2 + s^2 \sin^2 \epsilon)}{\lambda R^3 \sin \epsilon} dt$$

which simplifies to

$$\frac{1}{1 + x^2} dx = -\frac{\lambda \sin \epsilon(s - t \cos \epsilon)}{(\lambda^2 + t^2 \sin^2 \epsilon)R} dt,$$

which is precisely the integrand of equation (B.11).
The last term of integration is now as straightforward as the other two were. The term $I_3$ becomes

$$I_3 = -\frac{\lambda}{\sin \epsilon} \tan^{-1}\left(\frac{\lambda^2 \cos \epsilon + st \sin^2 \epsilon}{R \lambda \sin \epsilon}\right)_{s=v+|l_2|}^{s=\xi+|l_1|} \left|t=\xi+|l_1|\right|$$

$$I_3 = -\frac{\lambda |l_1||l_2|}{|l_1 \times l_2|} \left(\tan^{-1}\left(\frac{\lambda^2 \cos \epsilon + (v + |l_2|)(\xi + |l_1|) \sin^2 \epsilon}{|d - b| \lambda \sin \epsilon}\right)
- \tan^{-1}\left(\frac{\lambda^2 \cos \epsilon + (v + |l_2|) \xi \sin^2 \epsilon}{|d - a| \lambda \sin \epsilon}\right)
- \tan^{-1}\left(\frac{\lambda^2 \cos \epsilon + v(\xi + |l_1|) \sin^2 \epsilon}{|c - b| \lambda \sin \epsilon}\right)
+ \tan^{-1}\left(\frac{\lambda^2 \cos \epsilon + v \xi \sin^2 \epsilon}{|c - a| \lambda \sin \epsilon}\right)\right)$$

(B.15)

When we gather together equations (B.8), (B.10), and (B.15), the mutual inductance of two skew lines becomes

$$M = \frac{\mu_0}{2\pi} \frac{l_1 \cdot l_2}{|l_1||l_2|} \left((v + |l_2|) \tan^{-1}\left(\frac{|l_1|}{|d - b| + |d - a|}\right)
- 2v \tan^{-1}\left(\frac{|l_1|}{|c - a| + |c - b|}\right) + (\xi + |l_1|) \tan^{-1}\left(\frac{|l_2|}{|d - b| + |c - b|}\right)
- 2\xi \tan^{-1}\left(\frac{|l_2|}{|d - a| + |c - a|}\right)\right)
- \frac{\lambda |l_1||l_2|}{|l_1 \times l_2|} \left(\tan^{-1}\left(\frac{\lambda^2 \cos \epsilon + (v + |l_2|)(\xi + |l_1|) \sin^2 \epsilon}{|d - b| \lambda \sin \epsilon}\right)
- \tan^{-1}\left(\frac{\lambda^2 \cos \epsilon + (v + |l_2|) \xi \sin^2 \epsilon}{|d - a| \lambda \sin \epsilon}\right)
- \tan^{-1}\left(\frac{\lambda^2 \cos \epsilon + v(\xi + |l_1|) \sin^2 \epsilon}{|c - b| \lambda \sin \epsilon}\right)
+ \tan^{-1}\left(\frac{\lambda^2 \cos \epsilon + v \xi \sin^2 \epsilon}{|c - a| \lambda \sin \epsilon}\right)\right)$$

(B.16)

which is exactly what we set out to prove. This derivation is consistent with the notation of both Campbell (1915) and Grover (1946), and simplifies to their results for cases such as parallel wires of equal length.
B.2 Mutual Inductance Between Two Dipoles

The mutual inductance calculation that I wish to present in this section is the calculation of the mutual inductance between the image source and the receiver of Section 3.2. The image and the receiver are separated by distance $R$, and the dipole vectors point in opposite directions. The diagram for the setup is Figure 3.1, repeated here in Figure B.4. The mutual inductance for two dipoles has already been calculated in equation (2.48), and I will also repeat that here:

$$M = \frac{\mu_0}{4\pi} \frac{1}{r^3} (3(a_1 \cdot \hat{r}) \hat{r} - a_1) \cdot a_2.$$

In this configuration, the distance $r$ is given by

$$r = R = \sqrt{(4h^2 + d^2)}.$$

I will assume the dipole of the receiver is pointing in the same direction as the Tx source. The vector $\hat{r}$ points from the Tx image to the receiver. Thus, the mutual

![Diagram](image.png)

**Figure B.4:** Copy of Figure 3.1, repeated here for convenience.
inductance between the image and the receiver becomes

\[ M_{IR} = \frac{\mu_0}{4\pi} \frac{1}{\sqrt{(4h^2 + d^2)^3}} \left( -3 \left( a_1^2 \frac{2h}{4h^2 + d^2} + a_2^2 \right) a_2^2 \right) \]

\[ = -\frac{\mu_0}{4\pi} \frac{a_1^2 a_2^2}{\sqrt{(4h^2 + d^2)^5}} \left( 12h^2 - (4h^2 + d^2) \right) \]

\[ = -\frac{\mu_0}{4\pi} \frac{a_1^2 a_2^2 (8h^2 - d^2)}{\sqrt{(4h^2 + d^2)^5}} \]

The mutual inductance of the transmitter to the receiver is that of a horizontal coplanar dipole system of separation \( d \), equation (2.49):

\[ M_{TR} = \frac{\mu_0}{4\pi} \frac{a_1^2 a_2^2}{d^3} \]

The geometric factor at the inductive limit is the ratio of the two

\[ G = 10^6 \frac{M_{IR}}{M_{TR}} = \frac{10^6 d^3(8h^2 - d^2)}{\sqrt{(4h^2 + d^2)^5}} \]

where the factor 10^6 is put in to normalise the geometric factor to parts per million.
Appendix C

The RESOLVE and DIGHEM$^VRES$ AEM Systems

C.1 The RESOLVE System

The RESOLVE system is a helicopter borne frequency domain system owned and operated by Fugro Airborne Surveys (2005c). It has six transmitter-receiver pairs, each operating at a different base frequency. Five of the pairs are aligned in the HCP orientation, and one pair is aligned in the VCP orientation. Figure C.1 shows three of the transmitters.

All pairs are housed in a cylindrical casing that also contains a GPS antenna and receiver, a laser altimeter, a magnetometer, bucking, or calibration coils, and digital processing hardware. The RESOLVE system is suspended about 30 m from the bottom of a helicopter, and flown at survey altitudes of 30 m. A photograph of the RESOLVE system is shown in Figure C.2 and the laser altimeter is pointed out in Figure C.3. The total length of the towed bird is about 10.2 m from nose to tail. The GPS antenna is typically mounted about 0.9 m behind the nose, on the upper surface of the bird shell. The laser altimeter is housed inside the bird near the nose in the underside of the chassis. It is difficult to see in Figure C.2, but the distance
of $\sim$2.6 m from the nose of the bird is marked in the photograph. There is a drag skirt mounted near the rear of the bird that serves to minimise yaw when the bird is towed on survey. The RESOLVE system is slung beneath the helicopter with an inextensible rope. About 3 m above the bird, the rope forms a Y and attaches to the bird at the fore and aft of the chassis. This is shown in Figure C.3.

The red dots drawn on the fore and aft of the bird chassis in Figure C.2 mark the points that I tried to pick out in the manual picking of the helicopter and bird mentioned in Section 3.4. The GPS antenna position on the top of the bird was calculated using the distance 0.9 m aft and $\sim$0.3 m above of the front picking point.

![Figure C.1: Photograph of the inside of the nose of the RESOLVE HEM system. Two of the horizontal coplanar (HCP) transmitters are visible, while the 3300 Hz vertical coaxial (VCA) transmitter is shown lining the inner diameter of the housing.](image)

For all of the surveys discussed in this thesis, the RESOLVE system was towed beneath AS350 helicopters. A photograph of the AS350 helicopter used in the Chowilla plains is shown in Figure C.4. The overall length of the helicopter fuselage is 10.93 m. The red dots in the figure are the nose and tail points that I have tried to pick in the video footage mentioned in Chapter 3. The distance between the
C.1. THE RESOLVE SYSTEM

Figure C.2: Photograph of the RESOLVE HEM system on the ground in Renmark, South Australia.

Figure C.3: Photograph of RESOLVE during lift-off. Cable length is approximately 30.5 m. The Y where the cable splits and joins the fore and aft of the bird is clear in this photo.
two points is 10.2 m. The GPS antenna is on the top of the tail at the rear of the helicopter. From the rear picking point, the GPS antenna is 0.7 m further back and about 1.9 m up. From the GPS antenna of the helicopter, the hitching point is forward 7.7 m and 2.7 m below. These distances were used to draw the white dotted line in Figure 3.10, and to calculate the helicopter and towed bird separations in Figure 3.8.

Each frequency in the RESOLVE electromagnetic system samples data at a rate

### Table C.1: Nominal RESOLVE operating frequencies.

<table>
<thead>
<tr>
<th>Dipole configuration</th>
<th>Frequency (Hz)</th>
<th>Coil separation (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>400</td>
<td>7.9</td>
</tr>
<tr>
<td></td>
<td>1 800</td>
<td>7.9</td>
</tr>
<tr>
<td>HCP</td>
<td>8 200</td>
<td>7.9</td>
</tr>
<tr>
<td></td>
<td>40 000</td>
<td>7.9</td>
</tr>
<tr>
<td></td>
<td>133 000</td>
<td>7.9</td>
</tr>
<tr>
<td>VCA</td>
<td>3 300</td>
<td>9.0</td>
</tr>
</tbody>
</table>
C.2. THE DIGHEM\textsuperscript{VRES} SYSTEM

of 10 Hz, and the system is flown at a speed of about 40 m/s, making the data acquisition rate about 1 sample every 4 m. The operating frequencies of the RESOLVE system are set using tuned circuitry, so they can be subject to some change. Typically, survey reports give the nominal frequencies, shown in Table C.1.

C.2 The DIGHEM\textsuperscript{VRES} System

The DIGHEM\textsuperscript{VRES} system is very similar to the RESOLVE system mentioned above. It was the HEM system originally operated by Geoterrex (Fountain, 1998; Fraser, 1978) before they were taken over by Fugro Airborne Surveys in January 2000 (Fugro Airborne Surveys, 2005\textsuperscript{b}). The DIGHEM\textsuperscript{VRES} system studied in this thesis is the DIGHEM\textsuperscript{VRES}, which operates five horizontal coplanar dipoles in a bird housing like that used for the RESOLVE system. The nominal frequency of each transmitter-receiver coil is listed in Table C.2.

\begin{table}[h]
\centering
\begin{tabular}{lll}
\hline
Dipole configuration & Frequency (Hz) & Coil separation (m) \\
\hline
 & 380 & 7.9 \\
 & 1500 & 7.9 \\
HCP & 6200 & 7.9 \\
 & 25000 & 7.9 \\
 & 102000 & 7.9 \\
\hline
\end{tabular}
\caption{Nominal DIGHEM\textsuperscript{VRES} operating frequencies.}
\end{table}
Appendix D

Video Library

In this Appendix, I have attached a collection of digital videos that I have constructed from video-recorded and sequentially photographed footage. Obviously, these videos are only available in the thesis in an electronic format. Each video carries the unfortunate text reference of Figure, this is a failing of the word processing package used. For the reviewers, I am including a CD that contains all of the videos. In the paper version, the text in the frame where the video should be located is the file name of the actual video, and can be accessed from the CD.

D.1 Chowilla Floodplain Videos: RESOLVE

The videos from the Chowilla Floodplain survey are presented in this section. I have ensured that the helicopter is centered horizontally and occupies the top of the video screen. In each video, the nose and tail points of the towed bird and helicopter are marked with red dots. The calculated position of the GPS antennas are marked with yellow dots.
**Figure D.1:** Video footage, line 10510 of the RESOLVE Chowilla plains survey, 5 July 2005.

**Figure D.2:** Video footage, line 10520 of the RESOLVE Chowilla plains survey, 5 July 2005.
Figure D.3: Video footage, line 10581 of the RESOLVE Chowilla plains survey, 6 July 2005.

Figure D.4: Video footage, line 10591 of the RESOLVE Chowilla plains survey, 6 July 2005.
D.2 Sunraysia Video: RESOLVE

The photo-recording of the RESOLVE HEM system as it was towed on line 21020 is shown in this section. I would like to acknowledge the support of Nick Ebner, Luke Garde and Tegan Kocijan for the actual recordings; and the Ebner family for being hospitable hosts when we were staying in Mildura.

![Normal Fast Play/Pause Stop](FigureD5SunraysiaLine21020.avi)

*Figure D.5: Video footage, line 21020 of the RESOLVE Sunraysia survey, 23 September 2006.*

D.3 Model RESOLVE Videos

The RESOLVE toy model that I built to observe bird oscillations was filmed with a digital video camera. The first video below is the grayscale video footage (Figure D.6). The frame rate is 25 frames per second.

After intensity scanning and hole finding, I ended up with a logical map of each frame that allowed me to calculate the centre of each LED. From this, I made an $x, y$ map of the positions of the LEDs. The logical map is shown in Figure D.7.
D.3. MODEL RESOLVE VIDEOS

Figure D.6: Video footage of the RESOLVE model built to explore bird swing.

Figure D.7: Binary movie of the RESOLVE model built to explore bird swing.
D.4 SkyTEM Video

Multiple photographs were taken from two different locations (one slightly off the flight lines, and one directly in line with the flight direction) by Nick Ebner and Dr. Tim Munday as part of the ground loop calibration tests of the SkyTEM system outside Berri, South Australia. Using the digital photographs headers and employing my patience, I have organised the photographs into two simultaneous videos. The top panel of Figure D.8 shows the SkyTEM from the off-line position, whilst the bottom panel shows it from the on-line position. These videos are extremely useful for seeing the pitch and rolling of SkyTEM as it is being flown. It’s a good thing they have pitch and roll sensors!
Normal Fast Play/Pause Stop

**Figure D.8:** Video made from multiple photographs meshed together to form two movies of the SkyTEM system as it is being flown over the ground loop outside Berri, South Australia. The top panel is footage from the off-line position, while the bottom panel is footage from the on-line position.
Appendix E

TDEM Windowing Schematics

E.1 HoistEM System

The HoistEM system, first mentioned in Chapter 5, and described by Boyd (2004), has a complicated receiver sampling structure. One window of the receiver is $25.325 \ \mu s$ wide, and they come in groups of 4 with 1 dead-time window of $11.4 \ \mu s$. The contractors suggest that the voltage measurement in the receiver begins with an $11.4 \ \mu s$ dead-time and then samples. In addition to this, the measurement after the transmitter pulse begins $14 \ \mu s$ after the transmitter has fully shut off. The total time for a half-cycle pulse from the transmitter is reported to be $5.04 \ ms$. The shut-off begins at $5 \ ms$ and takes $40 \ \mu s$. Thus, the first measurement window is at $5.0654 \ ms$.

For the moment, assume that the first window starts at a reference time of $0 \ s$. The first window measures for $25.325 \ \mu s$; the second immediately following. The first four channels are equally spaced so that the start and end times are simply

$$\text{Channels 1–4} \left\{ \begin{array}{c} \text{start time} = (i - 1) \cdot (25.325 \ \mu s), \\ \text{end time} = i \cdot (25.325 \ \mu s), \end{array} \right\} \text{ where } i = [1, 4].$$

Then there is the dead time of $11.4 \ \mu s$, during which no measurements are taken.
APPENDIX E. TDEM WINDOWING SCHEMATICS

The 5th channel starts at \((4 \cdot 25.325 + 11.4) \, \mu s\) and ends at \((5 \cdot 25.325 + 11.4) \, \mu s\); and the 6th to 8th channels follow similarly:

Channels 5–8
\[
\begin{align*}
\text{start time} &= (i + 4) \cdot (25.325 \, \mu s) + 11.4 \, \mu s, \\
\text{end time} &= (i + 5) \cdot (25.325 \, \mu s) + 11.4 \, \mu s,
\end{align*}
\]
where \(i = [5, 8]\).

Channels 9–12 are exactly the same:

Channels 9–12
\[
\begin{align*}
\text{start time} &= (i + 8) \cdot (25.325 \, \mu s) + 2 \cdot 11.4 \, \mu s, \\
\text{end time} &= (i + 9) \cdot (25.325 \, \mu s) + 2 \cdot 11.4 \, \mu s,
\end{align*}
\]
where \(i = [9, 12]\).

Channel 13 is composed of 4 complete receiver windows, after the dead time:

Channel 13
\[
\begin{align*}
\text{start time} &= (13) \cdot (25.325 \, \mu s) + 3 \cdot 11.4 \, \mu s, \\
\text{end time} &= (17) \cdot (25.325 \, \mu s) + 3 \cdot 11.4 \, \mu s.
\end{align*}
\]

The remaining channels are built up from linear accumulations of 4 receiver windows and dead times. Channel 14 uses 4 receiver windows, so it appears exactly as 13 but advanced in time. After the dead time, channel 15 accumulates 2 sets of 4 windows, but also one dead time; channel 16: 3 sets of 4 windows and 2 dead, and so on, except for channel 27, which takes in two extra windows. The schematic is shown in Table E.1

After channel 13, the timing becomes more complicated. I found it far simpler to write an \texttt{m}-file using MATLAB (The MathWorks, 2007). The file, named \texttt{HoistEMStackingSchematic.m}, calculates the start time, end time, channel midpoint and total receiving time of the receiver channel. The output of the \texttt{m}-file is given in Table 5.4. The code is repeated below:
Table E.1: HoistEM window stacking scheme. “Live” windows (25.325 μs) are marked with a 1 and the channel number (in bold), “dead” windows are marked with a 0 and a - (where applicable).

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3 4</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>5 6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7 8</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>9 10</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>11 12</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>13 14</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>15 16</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>17 18</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>19 20</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>21 22</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>23 24</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>25 26</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

315
%% HoistEM Sampling and Stacking
clear all;
clc;
%
startDelay = 5000 + 40 + 14; % Waveform peak + shut-off +
% clearing time
deadTime = 11.4; % microseconds
measureTime = 25.325; %microSeconds

% There are 27 channels in all, out of 121 windows of
% 25.325 microseconds. The first 12 channels are 1 window
% each, but there is some dead-time there as well.
% The next 2 are 4channels + dead time wide. From there on
% its (4+dead)*n-13.

%% Window Open Times
% Channels 1:12
multiplePatch = repmat((0:2), 4, 1);
multiplePatch = multiplePatch(:)*4;
windowSubs = repmat([1; 2; 3; 4], 3, 1) + multiplePatch;
% Channel 13
temp = 13*ones(4, 1);
windowSubs = [windowSubs; temp];
% Channels 14:26
for i = 1:13
    temp = repmat(i+13, 4*(i), 1);
    windowSubs = [windowSubs; temp];
end
% Channel 27
temp = repmat(27, 17*4, 1);
windowSubs = [windowSubs; temp];

hoistemWindowing = 25.325*ones(length(windowSubs), 1);
channelOnWidth = accumarray(windowSubs, ...
    hoistemWindowing);

%% Start and End Times
receiverTiming = [deadTime; measureTime*ones(4,1)];
receiverTiming = repmat(receiverTiming, 200, 1);
receiverTiming = cumsum(receiverTiming);
% Channels 1:12
multiplePatch = repmat((0:2), 4, 1);
multiplePatch = multiplePatch(:)*5;
temp = repmat((1:4)', 3, 1);
temp = temp + multiplePatch;
startTimes = receiverTiming(temp);
endTimes = receiverTiming(temp+1);

% Channel 13: has 4 live windows
startTimes = [startTimes; receiverTiming(temp(end)+2)];
endTimes = [endTimes; receiverTiming(temp(end)+2+4)];

% Channels 14:26: each accumulates 1 more group of 4
% than the last.
qq = cumsum(1:13);
for i = 1:13
    eTemp(i, 1) = 21 + 4*qq(i) + qq(i)-1;
end
sTemp = [21; eTemp(1:end-1)+1];
startTimes = [startTimes; receiverTiming(sTemp)];
endTimes = [endTimes; receiverTiming(eTemp)];

% Channel 27: has 17*4 live windows, 16 dead windows
channel27Start = eTemp(end) + 1;
startTimes = [startTimes; ...
    receiverTiming(channel27Start)];
endTimes = [endTimes; ...
    receiverTiming(channel27Start + 17*4 + 16)];

%% Total Start Times, End Time, Channel Centre and 'On Time'
num2str([(1:27)' [startTimes endTimes ...
    (endTimes+startTimes)/2] + startDelay channelOnWidth], ...
    ['%' 5d &' repmat('% 12.3f &',1,3) '% 12.3f \\
    
    %]

%%
HoistEMWindowSampling = ...
    [startTimes endTimes (endTimes+startTimes)/2] + ...
    startDelay;
save HoistEMWindowSampling HoistEMWindowSampling;
Appendix F

Ground Loop Calibration Recommendations

F.1 Test Area

The test area for the ground loop calibration should be:

- Resistive as possible.
- Relatively flat over, say, a 300×300 m area.
- Provided with easy access.

F.2 Wire Loop

The wire loop:

- Must be accurately surveyed \((x, y, z)\) with at least the positions of the corners known,
- Should be about double the size of the nominal flight height,
- Should be multi-turn and insulated,
- Must have either the resistance \(R\) or the self inductance \(L\) known, preferably both,
- Should have a time constant \(\tau = \frac{L}{R}\) that is a reasonable fraction of the system half-period but effectively decayed before the end of the off-time.

I strongly recommend using a 1 \(\Omega\) damping resistor to close the ground loop if you are going to monitor the current in the ground loop. I have found that the systems tested produced current on the order of 100 mA in a 100×100 m 3-turn loop made from housing wire \((\tau \approx 0.5 \text{ ms})\). It is good to have response in all delay times, but this requires more or heavier wire.


F.3 Current Measurement

Communication with the AEM system contractor must be established so that the calibration crew can be on site, after equipment set-up and testing, when the flyover is to take place.

GPS synchronised time marks are very useful to determine time-lags between ground and airborne systems, even though this can be estimated in the fitting process. My tests used a 24-bit/96 kHz inexpensive audio sound acquisition device that plugged into a laptop via USB and wrote directly to the hard drive. One channel was used to measure the ground loop response, while the other recorded GPS timing signals (pulse per second), NMEA, and a reference signal (a sine wave of known $V_{RMS}$). Any system of good accuracy and high speed is sufficient, of course.

F.4 Photography & Videos

Photography and videos provide an excellent way to estimate the pitch and roll of the AEM system during flyover. At RMIT, we have been very fortunate in getting third-year and honours students to accompany us on field trips to join the calibration crew and photograph/video the system.

F.5 Processing

When the ground data in resistive terrain is combined with: a) the raw streamed AEM data and b) the processed AEM data, we can recover:

1. The transmitter current waveform (by deconvolution of the observed current in the ground loop).

2. A check on the system geometry and altimeter, provided the loop coordinates and AEM flight path are accurately recovered, through fitting the measured ground loop current. This part depends only on the transmitter (and tilts).

3. A further check on geometry by comparing the measured ground loop current to the received response (amplitude check).

4. The effects of stacking, windowing, filtering and other processing on the resolution of the received AEM signal.

You need to obtain as much data as possible from the contractors:

- Flight path of aircraft or bird (preferably bird).
- Altitude of aircraft or bird (preferably bird).
- Measured response at receiver.
- Peak current or monitored peak current.
- Waveform of transmitter (you can get this independently from ground loop data).
- Receiver windowing scheme.
- Geometry of transmitter (shape of loop, number of turns).
- Geometry of receiver (shape of loop, number of turns, and position and orientation relative to transmitter).
References


REFERENCES


REFERENCES


327


