Derivative Kinematics in Relatively Rotating Coordinate Frames: Investigation on the Razi Acceleration

A thesis submitted in fulfilment of the requirements for the degree of Doctor of Philosophy

by

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Declaration of Authorship

I, Ahmad Salahuddin MOHD HARITHUDDIN, declare that this thesis titled, 'Derivative Kinematics in Relatively Rotating Coordinate Frames: Investigation on the Razi Acceleration' and the work presented in it are my own. I confirm that:

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Abstract

The Razi acceleration has been discovered as a new acceleration term that appears due to relative motion of multiple moving reference frames. When Newton’s laws of motion are transformed between a stationary frame and a rotating frame, additional acceleration terms called inertial accelerations, must be introduced to take into account of the acceleration of the rotating frame itself. Traditionally, there are three types of inertial acceleration that acts on a particle in a rotating frame: the centripetal acceleration, the Coriolis acceleration, and the tangential acceleration. Most of the discussion of kinematics of rotating frames is, however, limited to only two frames and often the attention is restricted to two-dimensional models. Introducing a third relatively-rotating frame, the vector derivative transformation between three frames exhibits a peculiar acceleration term, which is written as a triple product of two angular velocity vectors and a position vector. The acceleration term arises when a displacement vector is twice-differentiated using the angular velocities of the rotating frames and transformed into an inertial reference frame. It appears that the Razi acceleration term is present when one transposes the vector derivatives between multiple rotating frames. This discovery leads to a new perspective of analysing the acceleration acting on rigid bodies in compound rotation, such as a tumbling or nutating motion. Three research questions are formulated in investigating the Razi acceleration: (1) How to improve the theory of vector derivatives to produce a complete equation of motion for a rigid body in multiple rotating coordinate frames; (2) What are the components of acceleration for such rigid body motion, and specifically, is the Razi acceleration a necessary inclusion for a complete rigid body acceleration expression, and; (3) Can the Razi acceleration be observed experimentally. This work aims to answer these questions by revising and extending the classical method of Euler derivative transformation between frames. The modification of the method is done for two purposes: to devise a general derivative kinematics formula for systems with an arbitrary number of rotating frames, and to identify the influence and the characteristics of the Razi acceleration. An experiment is setup using an industrial robotic manipulator to simulate compound rotation motion to investigate the appearance of the Razi acceleration in a mechanical system. It has been demonstrated by both mathematical and experimental methods that the acceleration of a rigid body in complex rotation motion can be resolved into its components to show the appearance of the Razi acceleration.
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Abbreviations

AxOR  Axis Of Rotation
AnOR  Angle Of Rotation
DOF   Degree Of Freedom
IMU   Inertial Measurement Unit
POI   Point Of Interest
Symbols

- **a**: acceleration vector, $ms^{-2}$, $g$
- **a**: tri-axis accelerometer output, $ms^{-2}$, $g$
- **A, B, C**: relatively-rotating coordinate frames
- **B, B_i**: body-fixed coordinate frame
- **d_x, d_y, d_z**: scalar components of vector $d$
- **d**: distance between origins of two coordinate frame, translation vector, $m$
- **D**: translation matrix
- **e**: Euler parameters in quaternion form
- **e_0, e_1, e_2, e_4**: Euler parameters
- **F**: force vector, $N, kgms^{-1}$
- **G**: global-fixed coordinate frame, inertial frame
- **i, j, k**: unit vectors
- **I**: identity matrix
- **j**: jerk vector, $ms^{-3}$
- **n**: number of noninertial coordinate frame
- **o, O, O_i**: point of origin of a coordinate frame
- **p**: momentum vector, $kgms^{-1}$
- **q, p**: quaternion
- **q*, p***: conjugate of quaternion $q$ and $p$
- **r_x, r_y, r_z**: scalar components of vector $r$
- **r**: displacement vector, position vector, radius, $m$
- **R**: rotation matrix
- **T**: homogeneous transformation matrix
- **u**: unit vector of rotational axis
- **u**: unit vector of rotational axis in skew-symmetric form
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<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{v} )</td>
<td>velocity vector</td>
<td>( ms^{-1} )</td>
</tr>
<tr>
<td>( x, y, z )</td>
<td>body coordinate axes</td>
<td></td>
</tr>
<tr>
<td>( x_i, y_i, z_i )</td>
<td>coordinate axes of the ( i )th frame</td>
<td></td>
</tr>
<tr>
<td>( X, Y, Z )</td>
<td>inertial coordinate axes</td>
<td></td>
</tr>
<tr>
<td>( \tilde{\alpha} )</td>
<td>angular acceleration in skew-symmetric form</td>
<td>( rad , s^{-2}, \circ s^{-2} )</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>angular acceleration vector</td>
<td>( rad , s^{-2}, \circ s^{-2} )</td>
</tr>
<tr>
<td>( \epsilon )</td>
<td>arbitrary small number</td>
<td></td>
</tr>
<tr>
<td>( \theta )</td>
<td>rotation angle about axis ( \hat{u} ), yaw angle, spin angle</td>
<td>( rad, \circ )</td>
</tr>
<tr>
<td>( \dot{\theta} )</td>
<td>pitch rate, spin rate</td>
<td>( rad , s^{-1} )</td>
</tr>
<tr>
<td>( (\phi, \theta, \psi) )</td>
<td>three Euler angles</td>
<td>( rad, \circ )</td>
</tr>
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<td>roll angle, precess angle</td>
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<td>( \dot{\phi} )</td>
<td>roll rate, precess rate</td>
<td>( rad , s^{-1} )</td>
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<td>( \chi )</td>
<td>angular jerk vector</td>
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<td>scalar components of vector ( \omega )</td>
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<td>( \tilde{\omega} )</td>
<td>angular velocity in skew-symmetric form</td>
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<td>( \omega )</td>
<td>angular velocity vector</td>
<td>( rad , s^{-1}, \circ s^{-1} )</td>
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<tr>
<td>( \tilde{\omega} )</td>
<td>tri-axis gyroscope output</td>
<td>( rad , s^{-1} )</td>
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To my father and my mother
Chapter 1

Introduction

Background

Classical Newtonian mechanics asserts that a particle either stays at rest or moves with a constant velocity; unless an external force is exerted on the particle giving it acceleration. This law, however, only holds when the reference frame used to describe the motion is inertial. If the particle is observed in a noninertial reference frame, one must take into account the acceleration of that frame itself with respect to an inertial frame in order to describe its motion using the laws of classical mechanics. The relative motion between two frames of reference induces additional acceleration terms in the equation of motion called inertial acceleration\(^1\), or more commonly known by its misnomer “fictitious force”.

The word fictitious here is perhaps used because these acceleration terms do not arise due to physical interactions between bodies, but the relative motion of the bodies with the inertial space. These acceleration terms appear in the equation of motion in order to “compensate” the inadequacy of Newton’s law in non-inertial reference frames. Their effects in dynamics of rotating systems are however real and demonstrable. To avoid misnomers and misconceptions, the phrase *inertial acceleration* will be used in the more technical discussions in this thesis.

Traditionally, there are three types of inertial acceleration that acts on a particle in a rotating frame: the centripetal acceleration, which acts toward the center of curvature, the Coriolis acceleration, due to the velocity of the particle in the rotating frame, and the tangential acceleration, caused by the change of the rate of rotation.

\(^1\)The term is also known as pseudo-force or d’Alembert force.
To visualize the effects of inertial acceleration, consider the merry-go-round example: A person is standing on the edge of a spinning merry-go-round facing out. While the merry-go-round is spinning, he/she experiences an outward force throwing him/her off the edge. This is a reaction corresponding to a centripetal force caused by the rotation. When the person tries to throw a ball straight ahead, the Coriolis acceleration ‘deflects’ the ball path sideways. And if the merry-go-round is sped up or slowed down, a tangential force is felt. Although the preceding example of relative motion is seemingly simple and readily observable—an amusement park is a great place to demonstrate this—the concept of “fictitious” forces is not always understood.

A Historical Perspective

In 1834, a French mechanical engineer by the name of Gaspard-Gustave de Coriolis (1792-1843) wrote down the energy equations of rotating machines in his published work “Sur les equations du mouvement relatif des systemes de corps”. In his derivation, he discovered the force term that he named the “compound centrifugal force”. This is known later as the Coriolis force. Coincidently, Simeon Denis Poisson (1781-1840) was working on calculating the exact trajectory of artillery projectiles and became interested in Coriolis’ works. It is apparent that by this time, the rotation of the Earth was already known to have effects on the motion of objects on its surface. Apart from influencing the dynamics of Coriolis’ waterwheels and frustrating amateur golfers from the other hemisphere, the Coriolis effect has helped geophysicist to deduce many important terrestrial phenomena [2–5]. Coriolis’ legacy does not only owing to Earth spinning motion. Flying insects, like crane flies and moths, have been known to have Coriolis force sensors in the form of halteres and antennae to help their in-flight stability [6, 7]. This concept has been copied in the design of gyroscopes which uses vibrating structures for attitude indication [8].

The Coriolis effect is only one of the many dynamical consequences that arise when the frame of reference used is rotating about a fixed axis. These so-called “fictitious” forces appear as an artifact of the relative rotation between a space-fixed inertial frame and a situation-fixed reference frame. The application is vast, from amusement park rides to orbiting spacecraft, however, the pedagogical discussion on the fictitious force or d’Alembert principle has always been tricky. Even the early, great academics took some time to come up with the correct conceptual definition. The historical exposition on conflicts and confusion of the Coriolis force concept is reported by Persson [9]. The first quantitative analysis of Coriolis force has been published by Leonhard Euler in his work on celestial mechanics in 1749, but it took almost a hundred years before the realization of its quality in physics and engineering by the French mathematicians. Even
the more “popular” fictitious force – the centrifugal force – has undergone a long thought and understanding process. Christiaan Huygens’ “De Vi Centrifuga” (1659) and Isaac Newton’s “Principia” (1687) perceived this force as the result of the curvilinear motion of a physical body. Interestingly, Huygens described this force as centrifugal (center-fleeing) while Newton referred to it as centripetal (center-seeking). It took a century, after the work of d’Alembert and Lagrange in analytical mechanics, before the modern interpretation of “fictitious” force is defined and refined. And it took longer, for Coriolis to be understood [9].

Justification of the Research

That there is a novel finding on the subject of classical mechanics may raise curiosity and skepticism to some. After all, kinematics, as a smaller branch of classical mechanics, has been one of the most applied areas in engineering, with applications including, and not limited to robotics, flight dynamics, dynamics modeling, and simulation. However, the purely application-based papers, which present the majority of the engineering literature on kinematics, and the over-dependence on computing tools and software to solve mechanics problems mean that very little attention is given to the supporting theories and mathematics used.

An interesting remark made by Jorge Angeles in his book “Rational Kinematics” which portrayed the disconnect between the mathematics of rigid body motion in multiple frames and its understanding in engineering practice. He mentioned his surprise that a fundamental question such as the relationship between a vector derivative of an angular velocity and its rotation tensor are “virtually appeared unexplained in the literature” [10].

The need to understand and be proficient in dealing with complex kinematics has also been forewarned by Junkins and Turner [11]:

“It has been our observation that a very large fraction, perhaps the majority, of errors (committed in formulation dynamical equations), are of kinematic origin. When faced with three or more reference frames with general relative translation and rotation, ample room exists for confusion, even in interpretation of linear velocities and accelerations.”

Traditionally, the discussion to the relative motion between frames of reference has been restricted to two frames: fixed and rotating. From this binary frame system, we have been able to identify and determine the inertial accelerations given the angular velocity of
the rotating frame, and the position and velocity of the particle inside that frame. Many complex mechanics problems involve multiple frames of reference relatively-moving with respect to each other. Imagine a typical demonstration gyroscope which has three or more rotating frames spinning about different axes. Can we identify the type and the magnitude of forces acting on the joints and bearings of a gyroscopic structure due to the multi-axis rotations? Compounded rotation motion such as seen in the gyroscope is also apparent in the spinning turbine or rotor rotating structures inside an aerospace vehicle which itself is also making rotational maneuvers. Such elaborate rotation motion is usually reduced to a two-frame system so that a more familiar approach of kinematics derivation can be applied to the analysis. But surely this oversimplification means some subtle accelerations due to small precession and nutation are overlooked.

Advancements in space vehicle technology have seen dynamicists taking a more careful approach in dealing with the gyroscope effects caused by the internal rotating parts in spacecraft. As space flight is not actively and continuously controlled due to limited propellants, the motion of an orbiting satellite is largely affected by the natural dynamics in space. Reaction forces and torques coming from these internal parts, even if they are relatively small in magnitude, can significantly alter the position and attitude of the satellite over time. If previously these small accelerations, forces, and torques are left out to achieve simplicity and computational load reduction, now such effects play a much more significant role in the problem [12].

Can we observe the effect of multi-axis rotation on Earth as we can with the Coriolis effect? The Earth can be assumed to be spinning about a fixed axis, though it has a small precession rate with a precession-to-spin ratio at about 1:26,000. Even when the precession rate is that small, some researchers [13, 14] speculate as to whether the precession affect the motion of the Earth’s liquid core, which in turn influence its magnetic field. Another thoughtful example is a tumbling asteroid – a curious mind cannot help but wonder what it is like for an astronaut to play golf on a huge, wobbling asteroid. How can we predict the golf ball trajectory? What kind of “fictitious” forces will come into play?

**Derivative kinematics and the Razi acceleration**

Derivative kinematics is the study of the mathematics of velocity and acceleration of rigid bodies in relatively-moving frames. It focuses on deriving and relating the time derivatives of vectors and rotation matrices in different coordinate frames and the kinematics term resulting from the derivation. Using the tools of derivative kinematics to solve kinematics in a binary (stationary and rotating) frames system enables one to
find the expressions for the inertial accelerations – centripetal/centrifugal, Coriolis, and tangential – given the information of how the two frames are moving in relative to each other.

Recent work by R. N. Jazar [15, 16] in the field of derivative kinematics has resulted in a discovery of a peculiar component of the total acceleration, which R.N. Jazar named as the *Razi* acceleration. Most of the previous discussions of kinematics of rotating frames are limited to the derivative transformation between two coordinate frames, and often the attention is restricted to two-dimensional models. Even when several coordinate frames are used, the calculation of kinematics is done exclusively between two associated frames [17]. Introducing a third relatively-rotating frame, the transformation of the three-dimensional equation of motion between three frames presents the new Razi acceleration term, written as $(A \omega_B \times B \omega_C) \times r$. The acceleration is described as a term that appears when a displacement vector is successively differentiated from two different coordinate frames, which technique is called *mixed double derivative*. Jazar has shown the derivation of of kinematics between three coordinate frames and the appearance of the Razi acceleration in such system, however, the application of the mixed double derivative and the mechanical explanation is lacking.

This finding raises several new questions. What is the physical description of the Razi acceleration? It is explained that the Razi acceleration is an acceleration akin to the centripetal and Coriolis accelerations, but it appears due to the motion about *two* axes of rotation. If it is similar to the Coriolis acceleration, is there a Razi force? Furthermore, what are the order of magnitude of such acceleration/force compared to centripetal acceleration/force? From the viewpoint of engineering, what type of rotational motion exhibit the Razi acceleration/force and how does the load caused by it can affect dynamical systems?

The work aims to delve deeper into the mathematics of derivative kinematics; in particular, to study the properties and applications of the newly discovered Razi acceleration. With this discovery, combined with a new perspective of viewing kinematics of multiple relatively-rotating frames, there is a need in the literature to revise thoroughly and extend the derivative kinematics method, and clarify the mechanical influence and effect of the Razi acceleration.

### 1.1 Research Objectives

The main objectives of this research are described as follows:
• To produce a unified description of derivative kinematics for systems with arbitrary number of coordinate frames, in view of application to the analysis of complex rigid body motion (compound rotation).

• To investigate the appearance of the Razi acceleration in rigid-body motion kinematics equation.

• To demonstrate the mechanical effect of the Razi acceleration on a rigid body in compound motion.

1.2 Research Questions

To reach the research objectives, several key research questions have been developed as to guide the work that is presented in the thesis.

• How can we improve the theory of vector derivatives to produce the complete expression of the acceleration terms for rigid body in multiple relative motion? Is the Razi term a necessary inclusion for a complete acceleration expression? What is the physical meaning of the Razi acceleration?

• What are the components of acceleration of a rigid body in multiple coordinate frames system? What is the significance of the out-of-plane component which can be only observed using >2 coordinate frames? What type of rigid body motion incurs the maximum/minimum Razi acceleration?

• Can we physically simulate such rigid body motion and describe how the Razi acceleration works? In other words, can the Razi acceleration be observed experimentally?

1.3 Strategies

• Apply rigorous mathematical techniques to show the derivation of the Razi acceleration. This is done by extending the Euler’s derivative transformation procedures to include multiple coordinate frames. Mathematical techniques from vector calculus and Lie algebra are used to prepare the derivative kinematics formula for computer application.

• Experimental investigation using an industrial robotic manipulator to reproduce a compound rotation motion. A 9-DOF inertial measurement unit is used to detect the acceleration and angular velocities of the manipulator. The new derivative
Chapter 1. Introduction

kinematics formula is used to see if it can predict the acceleration of a body in compound motion.

1.4 Thesis Outline

Summaries for each chapter are presented herein. A more detailed description of the objective is presented in the introductory sections of each chapter. The important contributions is listed in the conclusion sections.

The content of this thesis begins with a “Literature Survey” in Chapter 2 which covers the relevant works on the topic of derivative kinematics. It provides a more detailed review of the related works compared to the ones presented in the introductory section. A historical overview of the development of kinematics as a research area in classical mechanics is included with a scope on the derivation and understanding of the inertial acceleration concept. The theory of Euler’s vector derivative transformation formula is given a preliminary discussion here as the concept is central to this thesis. An extensive selection of literature on mathematical methods in kinematics, vector kinematics and related works on engineering research pertaining to the potential application of the Razi acceleration are given.

Chapter 3, ”Foundations of Derivative Kinematics” covers the essential vector kinematics tools and techniques that are used throughout this thesis. The two essential concepts – vectors and coordinate frames – are given an exhaustive treatment here. The first section of this chapter includes a proposal for a standardized vector notation system for the analysis of derivative kinematics in a multiple coordinate frames. The following sections presents an extensive review on the methods of vectors and quaternions, the method of deriving angular velocity and angular acceleration vectors, and the introduction of Euler’s derivative transformation formula. A list of the relationship between vector derivatives in a multi-frame system is also included. A more detailed treatment of the angular acceleration vector to show the acceleration terms that are often overlooked in mechanics analysis due to the two-frame system limitation. This angular acceleration term contributes to the Razi acceleration, which is shown in the next chapter. This chapter should provide the tools for deriving the Razi acceleration which is presented in the following chapter.

Chapter 4, ”On the Razi Acceleration” focuses on the mechanical interpretation of the Razi acceleration. The chapter starts with the application of Euler’s derivative transformation formula for system with more than two coordinate frames and the appearance of inertial acceleration terms. The method of calculating compositions of angular velocity
and angular acceleration vectors are crucial here as it explains the origin of the Razi term. The derivation of the Razi acceleration here is compared to the previous derivation. The physical interpretation of the Razi acceleration is explained here. An analysis on the parameters that influence the magnitude of the Razi acceleration is shown to give a deeper understanding of its characteristics.

Chapter 5, "Experimental Investigation of Razi Acceleration I" presents the first of two chapters dedicated on the experimental work done in this thesis. The experimental setup to detect the Razi acceleration in compound rotation motion is described. A robotic manipulator is used as a physical simulator to produce the rigid body motion. An inertial measurement unit, which combines an accelerometer, a gyroscope, and a magnetometer, is used to measure the acceleration and angular rates of the motion with respect to the inertial (Earth-fixed) coordinate frame. The kinematical equation is derived using the techniques of derivative kinematics presented in Chapter 3 and Chapter 4. Results from the experiment are presented here.

Chapter 6, "Experimental Investigation of Razi Acceleration II" presents a discussion of the experimental study in Chapter 6. The acceleration signal obtained from the sensor is compared with the acceleration terms produced using the derivative kinematics equation. The acquired acceleration data is decomposed into its inertial acceleration components. The predicted Razi acceleration is shown to appear in the out-of-plane component of the rigid body motion. The effect of the Razi acceleration to the overall acceleration of the rigid body is shown based on the experimental results.

Chapter 7, "Conclusion" is the final chapter in which outcomes from each chapter are reported. Research questions are recapitulated and reassessed here to see if the objective of the thesis is reached. This chapter is supplemented with a set of recommendation for future research.
Chapter 2

Literature Survey

This chapter provides a more detailed review of the related works compared to the one presented in the introductory section. An extensive selection of literature describing the historical background, the foundational theory of derivative kinematics, mathematical methods in kinematics, and related works on engineering application side. The review is intended not only to provide the reader with the relevant publications but also to emphasize the importance of the contribution of this work to the engineering research community in general and to support the novelty of the work in this thesis.

Describing motion in Euclidean three space $\mathbb{R}^3$ with six degrees-of-freedom – three for position representation and three for orientation representation – is essential in modeling, simulation and analysis.

The traditional method of using two coordinate frames system – one stationary frame representing the inertial frame, and one moving frame, which is fixed to the body of interest – has contributed significantly to many areas in classical mechanics. The earliest, and in this case the most important application was used by Euler in his study on the motion of rigid bodies. Noninertial referential frame, although does not follow the Newton laws of motion, seems to simplify most, if not every, analyses in classical mechanics.

By using the binary coordinate frames convention, methods for relating physical quantity in between inertial and noninertial reference frames have been developed. With the introduction of vector algebra by Gibbs things been easier for modern engineers to comprehend the complexity of the kinematics and dynamics between moving reference frames.
Chapter 2. Literature Survey

2.1 Definition

The definitions of the most frequently used terms are presented here:

**Vector derivative** is a time-derivative of a vector function that represents a kinematical quantity. For example, the first and second time-derivative of a displacement vector \( \mathbf{r} \) are \( \dot{\mathbf{r}} \) (velocity) and \( \ddot{\mathbf{r}} \) (acceleration).

**Binary reference/coordinate frames system** is a frame system that involves a fixed reference frame and a rotating reference frame.

**Multiple reference/coordinate frames system** is a frame system that involves a fixed reference frame with more than one relatively-rotating reference frames.

**Derivative transformation** is a technique of transforming vector derivatives between difference coordinate frames.

**Inertial acceleration** is an acceleration term that appear when the kinematics of a motion is described in a non-inertial frame. It arises due to the relative acceleration between coordinate frames.

**Compound rotation**, or **nested rotation**, is a term to describe the motion where a rigid frame is rotating inside another rotating frame, hence the term “nested”. Such motion can be described as a rigid body rotating about an axis that is also rotating in space (example shown in Figure 2.1). Common examples of such motion are precession and nutation.

![Figure 2.1: An example of the trajectory of a point on a rigid body undergoing compound rotation motion.](image)
2.2 Theory

Recent work in the derivative kinematics by Jazar [15, 16] has resulted in the formulation of general expression vector derivatives transformation in three coordinate frames. A new term is discovered in the work and is called the Razi acceleration. The Razi acceleration is the double-derivative of a displacement vector of which its first and second derivatives are taken from two different coordinate frames.

To illustrate the discovery of the new acceleration term and to show why it is a novel one, it is useful to review the classical vector differentiation method in a system with two relatively-rotating coordinate frames.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure22.png}
\caption{A two relatively-rotating coordinate frames system.}
\end{figure}

**Theorem 2.1.** Let \( A(OXYZ) \) and \( B(Oxyz) \) be two arbitrary, relatively-rotating coordinate frames sharing a common origin \( O \). as shown in Figure 2.2 And let \( B\square \) be a generic vector expressed in the frame \( B \), \( A\omega_B \) be the angular velocity of the frame \( B \) with respect to frame \( A \). The time derivative of the \( B \)-vector as seen from frame \( A \) is expressed as

\[
\frac{Ad_B}{dt} = \frac{Bd_B}{dt} + A\omega_B \times B\square
\]  

(2.1)

This technique is well known and has been treated and applied extensively throughout the literature and academic textbooks. If such vector is a displacement vector \( \mathbf{r} \), the formula can be used to find the velocity and acceleration transformation between two frames.
Chapter 2. Literature Survey

\[
\frac{A}{dt}B = Bv = B\omega_B \times B = \frac{A}{dt}B = B\omega_B \times B
\] (2.2)

\[
\frac{A}{dt}A \frac{A}{dt}B = A_B = B\omega_B \times B + B\omega_B \times B \frac{A}{dt}B + \frac{A}{dt}A \omega_B \times \frac{A}{dt}B
\] (2.3)

The classical expression of acceleration transformation is useful in dynamics of relative motion as it shows the appearance of inertial accelerations: centrifugal/centripetal acceleration, Coriolis acceleration, and tangential acceleration. These accelerations give rise to what some literature refer to as inertia effects or inertia forces. These additional accelerations appear not because of physical interaction between bodies, but due to the relative, non-uniform motion between the referential coordinate frame.

The acceleration terms appear as the result of differentiating a displacement vector, which is expressed in the B frame, twice from the same relatively-rotating A frame.

The formula is demonstrated as follows: Let A, B, and C be three arbitrary, relatively-rotating coordinate frames sharing an origin as shown in Figure 2.3. Let \( B \square \) be a generic vector expressed in the frame B, \( A\omega_B \) be the angular velocity of the frame B with respect to the frame A, and \( C\omega_B \) be the angular velocity of the frame B with respect to the frame C. The time-derivative of the B-vector as seen from the A frame can be expressed as

\[ A_B = B\omega_B \times B + B\omega_B \times B \frac{A}{dt}B + \frac{A}{dt}A \omega_B \times \frac{A}{dt}B \] (2.3)

Figure 2.3: A three relatively-rotating coordinate frames system.
\[
\frac{Ad_B}{dt} = \frac{C d_B}{dt} + (B \omega_B - B \omega_B) \times B
\]  

(2.4)

The formula presents a more general expression of the classical derivative transformation formula between two relatively-rotating frames in Equation (2.1) where the frame C is assumed to coincide with B (C ≡ B), thus \( B \omega_B = 0 \). One advantage of the general mixed derivative transformation formula is that it leaves the need to compute the derivative of \( \frac{d}{dt} B \) in its local coordinate frame.

### 2.2.1 The Razi Acceleration Term

Consider the three-relatively rotating coordinate frames system. When a displacement vector in \( C \), \( C \mathbf{r} \) is differentiated from \( B \) and \( A \), the resulting acceleration becomes [16]

\[
\begin{align*}
\mathbf{a}_{AB} &= \frac{Ad_B}{dt} C \mathbf{r} \\
&= \frac{Ad_B}{dt} \left( \frac{C d_B}{dt} C \mathbf{r} + \frac{C B}{C} \omega_C \times C \mathbf{r} \right) \\
&= \frac{C d_B}{dt} C \mathbf{r} + \frac{C B}{C} \omega_C \times C \mathbf{r} + \frac{C A}{C} \omega_C \times C \mathbf{r} + \frac{C B}{C} \omega_C \times C \mathbf{r} + \left( \frac{C B}{C} \omega_C \times C \mathbf{r} \right) \times C \mathbf{r} \\
&+ \frac{C B}{C} \omega_C \times \left( \frac{C A}{C} \omega_C \times C \mathbf{v} \right)
\end{align*}
\]  

(2.5)

A new acceleration term \((C \omega_C \times C \omega_C) \times C \mathbf{r}\) appeared in the expression. The expression with such form – a triple product of two angular velocity vectors and a position vector – is called the Razi acceleration \(^1\). Note that the order of product is different from the generic centripetal acceleration term, which is \( \omega \times (\omega \times \mathbf{r}) \). In practical application however, it is quite uncommon to take derivatives of a vector from two different frames. It is unclear as to whether the Razi term has any dynamical meaning; whether it is a kinematic effect or just a product of algebraic manipulation.

To the best of the author’s knowledge, the first time such term is mentioned as an acceleration term is from a paper on geomagnetic study by Malkus [13]. He called the term \((\omega \times \Omega) \times \mathbf{r}\) the Poincaré force. The \( \omega \) represents the Earth’s rotation speed and \( \Omega \) represents its precession speed, while \( \mathbf{r} \) is the Earth’s radius. Malkus did not provide the derivation of the acceleration term, but cited Poincaré’s [18] and Hough’s [19] papers on precessing fluid motion. Both of these papers however did not provide an explicit expression or derivation of the term.

The cross-product of two angular velocities \((\omega_1 \times \omega_2)\) appears in Beatty [20], as an "additional convective" component of angular acceleration vectors in a chained coordinate

\(^1\)A complete derivation of Equation 2.5 can be found in [15] or [16]
frames system. He illustrated this concept of convective rate of change with an example of compound rotation motion by demonstrating the angular acceleration of a motor on a rotating platform.

A tensorial analysis on angular acceleration and this "convective rate of change" can be found in publications by Angeles [10], Géradin and Cardona Alberto [21], and Henderson [22]. An angular acceleration tensor is derived from time-differentiating rotational invariants, which produces two components of an angular acceleration tensor, $\dot{\omega}$ and $\omega^2$. Henderson [22] showed the same explanation using different and further produced the matrix expression of angular acceleration. The tensor expression is also general enough to be used in the case where a rigid body rotates about a non-stationary axis. This is useful for the application of multi-frame kinematics as will be shown in the coming chapters.

The term for convective rate of change of angular velocity, however, disappears in the kinematics equation when the frame system is limited to only one stationary and one rotating (e.g., in Figure 2.2). The reason is, in such binary system, there is only a single angular velocity involved. When the angular velocity is differentiated to get the angular acceleration expression, the term $(A\omega_B \times B\omega_A)$ is zero [23].

### 2.3 Historical Background

Cinématique, or kinematics, is one of the many sub-disciplines of classical mechanics proposed by André-Marie Ampère in 1834 [24] as a method to study motion. While dynamics is devoted to analyzing the forces and torques that causes the motion of physical bodies, kinematics deals purely on the geometrics of motion disregarding the mass of the bodies or the agents that cause the motion. Kinematics studies the trajectories of geometrical objects and its relation to the differential properties of the linear and angular motion, such as velocities, accelerations, and higher derivatives of displacement.

Majority of kinematics discussion in literature deals with planar kinematics, where every motion is constrained to a plane. In planar kinematics, the motion is limited to instantaneous translation and instantaneous rotation only. This was discovered by Johann Bernoulli in his publication “De centro spontaneo rotationis” in 1742 [25] in which he introduced the concept of spontaneous center of rotation. Spherical kinematics is where the motion of bodies are restricted to be about a fixed point. The described motion of bodies is limited to spherical surfaces with respect to each other. For this motion, the concept of instantaneous axis of motion is introduced by D’Alembert in his “Recherches
Chapter 2. Literature Survey

"sur la Précession des Équinoxes et sur la Nutation de l’Axe de la Terre" in 1749 [26]. Spatial kinematics deals with the motion of two Euclidean spaces in relative to each other that can be described as a combination of a translation and a rotation about an axis. The axis is called the instantaneous screw axis, which concept was discovered by Giulio Mozzi in “Discorso matematica sopra il rotamento momentaneo dei corpi” in 1763. Chasles’ work on rigid body has received more attention than Mozzi although his paper was published in 1831. Chasles’s theorem, which states that any rigid body motion can be decomposed into a translation and a rotation along/about a common line, has now become the basis of screw theory. Ball [27] later developed the mathematical framework for screw calculus to be applied on rigid body kinematics and dynamics.

2.3.1 Euler and Coriolis

The building foundation of kinematics is indebted to the work of Euler, almost a century before Ampère’s ‘introduction’ of cinématique. It begins with the work on the general motion of rigid body (tr. Discovery of a new principle in mechanics), and continued with an emphasis on the rotation of solid bodies (tr. “On the motion of a rotating solid round a mobile axis” and tr. “On the movement of rotation of solid bodies around a variable axis”). In the first mentioned paper, after he discovered the concept of instantaneous axis of rotation, he tried to represent the velocities and accelerations according to this axis. It is in this paper where the Coriolis terms is first developed, a century before Coriolis himself rediscovered it. Applying Newton’s law to the body coordinates, Euler wrote, among many others, the term $2dM \frac{dv_x}{dt}$, $2dM \frac{dv_y}{dt}$, $2dM \frac{dv_z}{dt}$ where $M$ is the mass of the body. This is equivalent to today’s vector expression $2m\omega \times v$. He continued this problem in the two later papers, which ultimately gave birth to the equations of motion of a rigid body in space or Euler’s equation [26]. This goes on to show that the mathematical expression for the Coriolis force has been known way before it is actually understood and applied.

Coriolis discovered the force not through coordinate transformation like Euler did, but through using energy equation of which he named as “compound centrifugal force”s [28]. His work started to gain attention when Poisson used Coriolis’ paper to study artillery projectile motion. Persson [9, 28] wrote an excellent series of papers on the history of the conflict and confusion over the Coriolis force. Though its mathematics has been discovered in the 18$t^{th}$ century, there were still confusion among geo-scientists in the early 20$t^{th}$ century on how the Coriolis force actually affect the dynamics of atmospheres and oceans.
Another amusing story about the Coriolis effect [29]: In 1651, Giovanni Battista Riccioli (1598–1671) was trying to prove against the diurnal rotation of the Earth. He argued that if the Earth were rotating, a shot cannonball would show a slight deflection from our point-of-view because the ground beneath it has moved during flight. In other words, his argument is similar to this: If the Earth does rotates, there will be Coriolis force! In actuality, the Earth rotates so slowly such that the Coriolis effect is small. It only becomes apparent in large-scale system, such as wind patterns, ocean currents and large artilleries, and not toilet bowls!

2.4 Mathematical Methods in Kinematics

Describing motion in Euclidean three space \( \mathbb{R}^3 \) with six degrees-of-freedom – three for position representation and three for orientation representation – is essential in modeling, simulation and analysis. Position description is fairly easy and straightforward. The most common types of coordinate system are the rectangular Cartesian coordinate (which is the system that is used throughout this thesis), the cylindrical coordinate system, and the spherical coordinate system. Nonorthogonal coordinate systems are rarely used in rigid body kinematics, however, the continuum mechanics, fluid mechanics, and relativistic mechanics among many others. This thesis uses exclusively the rectangular Cartesian coordinate system due to its familiarity in engineering kinematics and dynamics. Orientation, or attitude, description are not as apparent as they are vast options of parameters that can be used ranging from 3-by-3, 4-by-4 matrices, and three angles with different existing variations. The discussion in the following chapters will only cover the most useful for engineering kinematics study.

Vector methods are almost natural when it comes to analyzing kinematics of a physical system. Any linear quantity, velocity for instance, can be effectively represented by a vector which provides the description for both its magnitude and spatial direction. The vector concept was introduced by J. Willard Gibbs circa late 1800’s (the collected works of J. Willard Gibbs) as a method to express the dynamics of physical quantities, especially for the problems of science and engineering. His work was compiled into a textbook in 1901 by Edwin Bidwell Wilson called Vector Analysis. This textbook, of which has been republished several times until 1943, has is one of the first and most complete book on the then-new mathematics and has introduced the modern standard method of vector analysis [30].

There are alternative methods which were predominantly use before the popularization of the vector methods, and are still appearing in current literature in mechanics. Graphical method is preferred before computer tools become a convention in problem analysis.
Its advantage is that graphics provides clear visuals without delving into heavy computation. Analysis and interpretation of planar kinematics are easily presented on papers. However, even with today’s computing technology, producing graphics are relatively more laborious and often abandoned in favor of numerical analysis.

Dual numbers algebra is introduced by Clifford [31] as a mathematical tool to deal with screws [27]. Its application towards kinematics of rigid bodies was further developed by Denavit [32], Yang [33, 34], Veldkamp [35]. More recent works on kinematics using dual numbers can be found in Brodsky and Shoham [36], and Pennestri and Stefanelli [37]. Screw theory, however, has not received as much attention due to its non-familiarity apart from a small circle in the kinematics community.

The application of matrix as a mathematical object is relatively new. It was not until the work of Grassman, Gibbs, and Heaviside on vector that matrix were used in scientific and mathematical analysis [38]. Its popularity in engineering is helped by its compatibility and applicability within computer programming. The Denavit-Hartenberg matrix [39–41] developed in the 60’s are still being widely used in modern mechanics textbooks, especially robotics [42].

Quaternions have found their niche in aerospace engineering community for representing orientation kinematics in three-dimension [43]. This is due to the problem of gimbal lock (singularity) when Euler angles are used, whereas the quaternion representation avoids this type of singularity [43]. The idea of quaternion was developed by an Irish mathematician/astronomer William Rowan Hamilton in a series of lectures and papers between 1843-1866. This development was further applied in the kinematics of rotation by Cayley [44]. Dual numbers theory as derived by Clifford was also the product of the quaternion development by Hamilton. The vector method was introduced as a more efficient alternative to the cumbersome calculus of quaternions. The departure from the method of quaternions to vectors was also largely helped by Wilson’s excellent book on vector analysis [30]. This is apparent in the modern introductory books and courses on mechanics as vector algebra and calculus are given more emphasis than quaternions.

Other authors propose alternative methods of representing kinematics. For aerospace simulation and modeling, Zipfel [45] put forward the concept of using tensor notation in modeling flight dynamics. He argues that, the tensor form of derivative kinematics is invariant under time-dependent coordinate transformation, based on the covariance principle that states that physical laws are independent of coordinate systems. The tensors are only converted to matrices for computational purposes. Henderson [22] proposes the Einstein summation convention as a method to give the exact components of equations and demonstrate their values, which is especially useful in multi-variable analysis. Tensors have also received attention in robotics [46]. However, the techniques
in tensorial calculus require deep mathematical foundations to fully understand and utilize its advantages.

Screw theory [47, 48] receives more attention in the robotics community [49–51]. Although screw algebra is geometrically convenient to describe both rotation and translation, it is only limited to speed and infinitesimal displacement analysis [52], thus not deemed suitable for acceleration analysis in this work. Featherstone [53] uses elegant, though unconventional, 6-D vector notation to describe velocity, acceleration, and force quantity.

Attitude representation is the parameters that describe the orientation of a rigid body in three-dimensional space. While describing position is fairly easy and straightforward, there are many ways to represent attitude which applicability depends on the situation.

Other than the common attitude representation used in major textbooks on mechanics [11, 54–56], there are three main survey papers on attitude representation used in this work. One is an extensive survey by Shuster [38] on general attitude kinematics representation which covers Euler angles, Euler-Rodrigues parameters, Rodrigues parameters, Cayley-Klein parameters, and modified Rodrigues parameters. The second paper is a review paper by Phillips and Hailey [43] on the attitude representation most used for aircraft kinematics with a focus on the quaternion formulation of kinematics. And the third one is from Diebel [57] which includes the transformation between an attitude representation with its derivatives (angular velocity and angular acceleration). Diebel also introduced another kind of attitude representation called rotation vector which does not have singularities or the quadratic norm constraint of the quaternions.

The various attitude representations existing in the literature in the following chapters on attitude representation, only four most useful are discussed: (1) direction cosine matrix; (2) Euler angles matrix; (3) the axis and angle of rotation; and (4) the quaternion.

### 2.5 Vector Notation System and in Derivative Kinematics

The mathematical framework of this thesis is limited to the consideration of vector and matrix calculus and is limited to the Euclidean $\mathbb{R}^3$ space. In dealing with derivative kinematics in a system with multiple reference frames,

Both Junkins and Turner [11], and Snyder [58] assert that most errors in formulating dynamics equations are due to mistakes in formulating kinematics equations. Difficulty in identifying vectors is especially difficult when one is faced with a number of coordinate frames and multiple different types of vector differentiation between the many coordinate
frames. Shuster [59] argues that this due to, in practice and in the earliest undergraduate courses in dynamics, the vectors and their derivatives are treated as coordinate-free vectors. Kasdin and Paley [60] point out that a careful and rigor notation is essential when one is approaching the subject of dynamics. Since a vector, by itself, only provides the information of its magnitude and direction, the study of derivative kinematics necessitates a complete notation system to associate the column-vector representation to the related coordinate frames environment.

There is no universally accepted system in use [60]. Junkins and Turner [11] noted that most errors in formulating dynamical equations of motion comes from the confusion in distinguishing kinematical vectors. This problem arises when multiple reference coordinate frames are used and the relative motion between the frames are not specified clearly. The most widely-used [11, 23, 60–64] notation system in literature is from a combined system from Likins [65], and Kane et al. [55]. Their notation shows explicitly the measurement frame and the rigid body identifier. Two examples of the notation system, taken from Reference [11] is shown here,

\[ f_{\mathbf{v}^{A/B}} = \text{the velocity} \mathbf{v} \text{ of point} A \text{ with respect to point} B \text{ as seen from} f \]

\[ \omega^{A/B} \text{ or } \omega^{A} = \text{the angular velocity} \mathbf{\omega} \text{ of} B \text{ with respect to} A \]

However, both notations do not tell the representation (expression) frame of the vector. This is only expressed when the vector is expanded to its scalar components representation (Kane et al. [55] pp. 84-86). A full scalar components representation of a vector, although it describes the coordinate frame, can be lengthy and can be burdensome when presenting complex equations of motion.

Hughes [56] acknowledged the problem of notation and introduced the use of a vectrix in dealing with kinematics in multiple reference frames. A vectrix is an inner product of two vectors: a vectrix \( \mathcal{F} \) which represents the basis vectors of a coordinate frame, and a vector \( \mathbf{v} \) which represents the scalar components of the vector itself. For example, a vector \( \mathbf{v} = v_1 \mathbf{a}_1 + v_2 \mathbf{a}_2 + v_3 \mathbf{a}_3 \) expressed in the frame \( A \) can be completely described as

\[ \mathbf{v} = \mathbf{v}^T \mathcal{F}_A = \mathcal{F}_A^T \mathbf{v} \]

Vectrix notation is designed to suit the application of multi-frame kinematics and has been used in the study of aircraft and UAV kinematics [66, 67] and space formation flight [68, 69]. The vectrix system eliminates the source of ambiguities in vector notating, however, the vectrix-and-vector combination expression can become too cumbersome as the kinematic equation gets longer.
This thesis opted for the more conventional vectorial method with a notation system used by Jazar [15, 16] with new improvement (will be introduced in Chapter 3). For linear velocity and acceleration vectors, Jazar’s notation system can explicitly and sufficiently show the related coordinate frames (the frames in which the vector is expressed and differentiated), however, this is not extended to angular quantities, e.g. angular velocity and angular acceleration. Such angular quantities are relative, that is, they are measured with respect to another reference frame. Apart from the basis vectors (reference frame) used to present the scalar quantities of an angular velocity, the vector notation needs to also specify the frame to which it is referred (identifier), and to which it is compared (relative). For example, the vector $\mathbf{C}_A \omega_B$ denotes an angular velocity $\omega$ of the frame $B$ in relative to the frame $A$ with its coordinates expressed in frame $C$. As angular acceleration is a time-derivative of angular velocity, the notation for angular acceleration vector should also specify the frame in which the vector is differentiated. This means, the notation system must explicitly show four frames related to an angular acceleration vector: the frame in which it is expressed, the identifier and the relative frames, and the frame in which it is differentiated. Another improvement made is the removal of some redundancy in the vector notation. For example, an acceleration vector can be represented using Jazar’s system as $\mathbf{C}_B \mathbf{a}_A$, which means an acceleration vector where the first time derivative is taken from frame $A$, and the second derivative is taken from frame $B$, and its basis vectors are of frame $C$. This is simplified to $\mathbf{C}_B \mathbf{a}_A$ due to redundancy, as a vector cannot have its scalar quantities be represented by two sets of basis vectors simultaneously. In other words, the vector $\mathbf{D}_B \mathbf{a}_A$ does not make much mathematical sense as it is expressed in both $C$ and $D$ frames.

### 2.6 Vector Derivative in Multiple Reference Frames

The law of classical mechanics asserts that there exist a reference frame in which the conservation of linear momentum $\mathbf{F} = \frac{d}{dt} \mathbf{m} \mathbf{v}$ applies to every point in that frame. Such a reference frame is called an inertial reference frame. In other words, Newton’s law of motion is only valid when the frame of reference is inertial. However, observers look upon the world mostly from a noninertial perspective; our frame moves relative a fixed point in space. In practice, no frame of reference is truly stationary in space to be considered an inertial frame. Hence, it is necessary to have a method of transforming and relating physical vectors, and their derivatives, between a noninertial frame and an inertial frame.
2.6.1 Frame of Reference

Mainly, there are only two types of frames of reference: inertial and noninertial. An inertial frame is the frame in which the linear and angular momenta are conserved for every body in that frame. The classical law of mechanics $\mathbf{F} = \frac{d\mathbf{p}}{dt}$ holds in an inertial reference frame. Such frame is in a constant state with respect to a fixed point in space. A noninertial frame is a frame of reference which accelerates – whether translationally or rotating – with respect to the inertial frame. This is a frame in which observers may measure and write a description of the equations of motion – the relation between forces and torques acting on a body and the resulting acceleration. To consider motion from a noninertial frame, it is necessary that the calculation of derivatives $\frac{d\mathbf{p}}{dt}$, of which is only valid in the inertial frame, be supplemented with the relative acceleration of the observer’s frame with respect to the inertial frame.

So why opted for a noninertial frame when applying Newton’s laws in an inertial frame is more straightforward? Our natural point-of-view are almost always a noninertial one. For example, a pilot watching another flying object in the sky from his/her cockpit view, a moving frame, and not from a static point on Earth. Even the Earth itself cannot be considered as an inertial frame due to the rotation on its own axis. Similarly the Earth, as a frame of reference, is being accelerated with respect to the Sun. The heliocentric frame of reference cannot be absolutely inertial either, as the whole solar system is rotating about the center of the Milky Way. In application sense, there is no true inertial frame in which the laws of classical mechanics hold, and every observer frame is noninertial.

The definition of a true inertial is indeed problematic. It is sufficient that we assume that our, the observer’s, frame is inertial. This, of course, depends on the application. Geophysicists may use the geocentric frame as the inertial frame, disregarding the frame’s rotation around the Sun, while a helicopter pilot may ignore the effect of the rotation of the Earth and assume the surface of the Earth as the inertial one.

This leads to the consideration of relative motion. Since vectors are used to represent our kinematics, it is necessary to have the method of calculating and relating vector derivatives between different frames of reference.

2.6.2 Vector Derivatives between Frames

The classical theory about vector derivatives transformation between reference frame is surveyed here. The classical vector calculus technique of relating a derivative of a vector
The formula is interpreted as follows: Let $B$ be a reference frame rotating arbitrarily with respect to $A$. Then, for a displacement vector expressed in a reference frame $B$, its velocity from another reference frame $A$ is equal to the vector sum of the velocity in its own reference frame $B$, plus the cross product of the angular velocity of $B$ in $A$ with the position vector expressed in $B$. Using this kinematic law of different frames, one can relate the absolute velocity vector of a particle in one frame to another by the sum of the particle’s absolute velocity vector in the other frame and the cross-product of the frame’s angular velocity vector and the particle’s position vector.

Applying the same formula again to find the second derivative of the vector in $B$ from $A$ yields the classical expression for relative acceleration between two relatively-rotating frames:

$$\frac{A d^2}{dt^2} B \mathbf{r} = \frac{B d^2}{dt^2} B \mathbf{r} + B A \mathbf{\alpha}_B \times B \mathbf{r} + 2 B A \mathbf{\omega}_A \times \frac{B d}{dt} B \mathbf{r} + B A \mathbf{\omega}_A \times (B A \mathbf{\omega}_A \times B \mathbf{r})$$  \hspace{1cm} (2.7)

It is known for quite some time among the engineering community with the earliest accessible literature from the late 1940’s [70, 71]. Despite of its importance in classical mechanics and its ubiquitous application in engineering, there is no universally-accepted name for the Euler derivative transformation formula in (2.6). Several terminology are used: kinematic theorem [55, 63], transport theorem [23, 64], and transport equation [59, 60]. These terms, although terminologically correct, are more prevalent in the subject of fluid mechanics to refer to entirely different physics concepts. The second derivative expression in (2.7) is sometimes called the Coriolis theorem in some literature [45, 72]. Perhaps due to its lack of application, there is no general expression for third derivative of a vector (jerk or jolt) between two reference frame. We use Zipfel’s [45] and Jazar’s [15] terminology for the vector derivative technique in (2.6), Euler derivative transformation, which the author opines as more descriptive and unique to the subject of derivative kinematics. The name also is an homage to Leonard Euler who first used the technique of coordinate transformation in his works on rigid body dynamics [26].

One of the earliest improvement on the Euler derivative transformation formula is Polish mechanical engineering paper by Janik [73]. He extended the formula for the application of a multiple chained reference frames system, that is, a reference frame moving relative to another reference frame which in turn is moving in another reference frame and so on. His formulation relates the velocity and acceleration of a point in any two reference frames in the system, which is useful for calculating forces and torques for serial multibodies such as a robotic manipulator.
Sherby and Chmielewski [17, 74] extended Janik’s formulation to provide general expressions for $n^{th}$-order derivatives and applied the formula for multibody system. The general method is useful for finding jerk, that is the first time-derivative of acceleration, between two moving frames. Higher derivatives than jerk are rarely found in classical mechanics. The second time-derivative of acceleration, snap or jounce, however, can be found in the quantum mechanics literature [75]. Hacisalihoğlu [76] rewrote Sherby and Chmielewski’s vector derivative expression using skew-symmetric matrices, which provide a more compact and elegant representation of the equation of motion.

The topic of vector derivatives seems to receive a wide attention from there on until the late 80’s with works from Kane et al. [55, 62] being the most prominent ones. A classic text on mechanics by Arnold [77] is also often used when topics about inertial acceleration is discussed.

The vector derivative transformation, or derivative kinematics, has drawn serious attention particularly in the mechanism theory and robotics research community [41, 50, 52, 78–84]. Most of these papers are dealing with the study of spatial kinematics using screw algebra convention [50, 52, 80, 82, 83], with a focus on representing higher order screw derivatives like acceleration [50, 52, 82], and jerk [83]. Most of the textbooks on vector derivatives can be also found in the works in mechanism design [85, 86] and modern robotics [42, 51, 87, 88].

In the aerospace community, most of the discussion on derivative kinematics is based on the works by Kane [62] and Kane et al. [55]. The emphasis, however, is not towards the mathematical representation of kinematics but more on the effects of inertial loading on maneuvering aerospace vehicles [12, 89, 90] caused by motion calculation in noninertial frames. Zipfel [45], however, revisit the derivative kinematics problem again by introducing and proposing his tensorial rotational time-derivative technique to be used in computer modeling of aerospace vehicle motion. This work is a continuation from his previous work [89] and has found application in perturbation analysis in flight dynamics [91].

Jazar [15, 16] revised the Euler derivative transformation formula and produced the first general expression for vector derivatives in three coordinate frames, called mixed derivative formula, which was not explored in Sherby and Chmielewski [17, 74]. An offshoot discovery from the new formula is an acceleration term called the Razi acceleration. Presently, the Razi acceleration formula is only a mathematical term that appeared from the mixed derivative formula application on second order motion. However, the properties, characteristics and application of the acceleration term is not thoroughly investigated. The collection of these new discoveries in the study of kinematics in multiple frames is presented under the name of "theory of time derivative" [16].
2.6.3 Inertial Acceleration

The most important result that arises from the vector derivatives in noninertial frames is the inertial acceleration. The name inertial acceleration is always mistakenly referred to as fictitious acceleration. These inertial acceleration terms, which are the acceleration that appeared when a displacement vector is differentiated twice from a relatively-rotating coordinate frame, can be grouped into three [77]

1. $\mathbf{B}_A \omega_A \times (\mathbf{B}_A \omega_A \times \mathbf{B}_r)$ is the centripetal, or centrifugal, acceleration. It is due to the displacement of a body point from the rotation axis of a noninertial frame.

2. $2\mathbf{B}_A \omega_A \times \frac{\partial}{\partial t} \mathbf{B}_r$ is the Coriolis acceleration. It appears due to the velocity of a body point in the rotating frame. Its name is a tribute to its discoverer Gaspard-Gustave Coriolis (1792-1843).

3. $\mathbf{B}_A \alpha_B \times \mathbf{B}_r$ is due to the acceleration of the body rotation rates. It has no historical name. However, the most used term is tangential. Other names used in literature is Euler acceleration and inertial acceleration.

As many kinematics and dynamics analyses in engineering problems require the use of noninertial frames, the role of inertial acceleration (or force, when multiplied by mass) is paramount. Unfortunately, the term “fictitious”, and pedagogical explanation are often confusing and lacking rigor, led some people to believe that it has no “real” effect on dynamical systems. In actuality, inertial acceleration/forces should be treated the same as other types of force as, without its consideration, it is almost impossible to apply Newton’s theories in physical problems. For discussions on the concept of inertial force, readers are encouraged to read References [28, 92]

2.7 Related Engineering Application

Relevant papers are selected here to show the prevalence of the concepts discussed in this thesis – coordinate transformation, inertial acceleration and its influence on engineering structures, compound rotation motion. The aim is two-fold: (1) To show the importance of inertial acceleration consideration in kinematics analysis and how it can affect the dynamics of rotating structures. To give a big picture, we present a number of important papers that have looked in to the problem of inertial loading in rotating structures with specific scope of mechanical and aerospace engineering. (2) And secondly, to suggest several research areas and application in which the discovery the Razi acceleration, or the concept of vector derivatives in general, can contribute.
It can be conjectured that there is no other engineering discipline that deals with more coordinate frames than aerospace engineering, especially in the area of flight mechanics. This is because some type of framework is need to define, observe, and measure the direction and orientation of objects in flight. Among them are heliocentric frame, geocentric inertial frame, Earth-fixed, vehicle-carried frame, wind reference frame, stability axis frame, and flight-path frame. A more complete list of coordinate frames used in flight dynamics can be found in [93]. For a space-flight specific coordinate system, refer to [94]. Inevitably, knowing proper coordinate transformation methods is a must in order to work in multiple reference frame system environment. An review of coordinate transformation techniques using rotation matrices can be found in technical reports [95, 96].

Experimentalist engineers who are working with rotating platform or rotating structures, usually record experimental data using rotating sensors. These acquired data is then transformed to an inertial frame for further analysis. There are also some cases where ground-fixed sensors are used. Afolabi et al. [97] recognized this problem and provided a method of transforming velocity and acceleration data between a rotating and a fixed frame. The intended application was for the modal analysis of vibration systems. Afolabi further elaborate this technique to a more general case in [98]. He approached the problem by manipulating the identities of symplectic matrix used in the generalized vibrating system equation. However, this elegant method is only limited to two-dimensional case where the angular velocity matrix can be symplectic.

The study of load analysis on structures undergoing inertial motion has been explored for quite some time. This indirectly shows that the “fictitious force” has been a concern for engineers and should not be regarded as mere mathematical convenience. The initial studies on rotating systems are more focused on describing the kinematics of various types of rigid body systems and finding the expressions of forces and torques. However, when computational cost became a concern, the aim of system modeling has shifted towards producing a numerically friendlier representation of a system. This sometimes causes the small effects coming from subtle mathematical terms to be ignored completely. Most of the work on inertial forces in rotating systems are directed towards finding the vibration characteristics. Later studies on inertial and gyroscopic motion is more directed towards dynamical analysis of the system, in view of achieving dynamical stability and attitude control, but not on the kinematics of the gyroscopic motion itself.

One of the earliest investigation on the effects of inertial acceleration towards the dynamics of an engineering structure is by Schillians [99]. He demonstrated that neglecting the centrifugal stiffening, which is caused by rotation, yields to incorrect solutions for the describing partial differential equation of a cantilever beam. Houbolt and Brooks [100],
in their investigation of the deformation theory of helicopter rotors and propeller blades, showed that in the presence of centrifugal force increases the first bending frequency of rotating blades. These two early investigations has sparked efforts in developing a more accurate set of equations of motion describing rotating structures to take into account the role of inertial forces.

Carnegie [101], considered the effects of rotary inertia, including the Coriolis acceleration, in his formulation of the equations of motion of pretwisted cantilevered blade. The study is extended in two other papers by Rao and Carnegie [102, 103]. It is found that the Coriolis acceleration gives rise several nonlinear terms in the equations of motion. Several other papers studies the effect of centrifugal and Coriolis acceleration on rotating blades. Subrahmanyam and Kaza [104], Subrahmanyam et al. [105] noted that the effect of Coriolis must be included in analysing the vibration and buckling of rotor blades with geometric nonlinearities. Gans and Anderson [106] incorporated the effect of both Coriolis and centrifugal forces in rotating blades/beams and showed that the Coriolis term has to be included in vibration modeling for a better predictions of vibratory response.

So far, all mentioned literature are based on the case where the angular velocity is constant. The irregularity of angular speed is also a concern among structural engineers as it affects the response of rotating structures as shown by Vyas and Rao [107]. In another paper [108], Vyas and Rao demonstrated that a high tangential acceleration may cause shock to rotor blades hence decreasing its fatigue life. However, at certain frequencies, large tangential acceleration may reduce the blade’s forced vibratory response.

Another important effect due to centrifugal acceleration is centrifugal stiffening. This problem is a concern among researchers in flexible body dynamics [109–111]. Kane et al. [112] pointed that, neglecting the centrifugal stiffening in formulating the dynamics of a rotating cantilever beam results in inaccurate response prediction.

The effects of gyroscopic motion [113–117], or compound/complex rotation [118–121] has been long investigated. As early as 1958, Hirschberg and Mendelson has noticed that when there are rotations axis is itself rotated, then a disk structure under such motion experience forces normal to its surface [113]. The effect of the so called gyroscopic motion on the vibrational characteristics of a disk has been studied as early as [114]. Kessel and Schlack [115–117] has also done an extensive work on “gyroscopically induced” inertia loads, particularly on thin structures such as plates and membranes. They described this type of motion as “rotation about two or more axes simultaneously” Gulyaev et al. [118–121] produced a series of publications on the effects of precession on thin-walled structures.
Inertial loading due to flight maneuvers has gotten the attention of several communities in aerospace engineering. Rotating disks and blades of jet engine rotors are subjected to a gyroscopic moment due to angular velocity when the aircraft moves along a curved path or responds rapidly to disturbances. Barlow [90] applied an acceleration analysis to turbo-fan blades of a gas turbine engines under compound motion. He derived the gyroscopic load as a sum of two centripetal force terms; one from the rotation of the turbo-fan, and another from the banking maneuver of the carrying aircraft. He compared his derivation of acceleration with a numerical solution obtained using finite element displacement analysis. Sakata [122] considered the same problem using finite element analysis and compared his theoretical shaft deflection data with experimental data, showed that formulation need to include gyroscopic effects and the results are verified experimentally. A more recent example of inertial forces due to flight maneuver can be seen in literature on flexible aircraft, for example [123]. In this case, the inertial load has a much more significant effect on the aircraft due to the increased flexibility of the structures.

2.8 Conclusion

The investigation of the Razi acceleration here is likened to the historical story of how the Coriolis force is discovered and understood. It was first discovered as a result of coordinate system manipulation but its physical meaning is hidden in the mathematical abstractions. The concept of the Coriolis effect is easily understood today but it took quite some time for scientists to grasp its definition and apply it to the real world problems. The story line is parallel with the work in this dissertation. The Razi term is discovered as a result of changing vector derivatives from one coordinate frame to another, but its influence, effects, characteristics and overall description in terms of dynamics are yet to be uncovered.
Chapter 3

Foundations of Derivative Kinematics

The subject of derivative kinematics is defined as the mathematics of velocity, acceleration, and higher order derivatives of linear and angular displacements of rigid bodies. The two main building blocks in this subject are (1) vectors, which represent the physical quantities of a rigid body, and (2) coordinate frames, which provide the referential perspective of the rigid body motion, and assign numerical values to the physical vectors. The time-derivative of a kinematical vector is dependent on the coordinate frame in which the vector itself is expressed, and the coordinate frame from which its differential is calculated. Derivative kinematics must also be able to explain, relate, and distinguish between the vectors that are expressed and differentiated in different frames.

This chapter starts with the introduction to the notation method as a system to distinguish between the numerous vector expressions in different frames. The next section provides qualitative definitions of the core concepts in derivative kinematics: a review on vector and quaternion algebra, the concept of frame of reference and coordinate frames, and a review on mostly applied rotation description. The subsequent section discusses the relation of time-derivative of vectors between two relatively moving coordinate frames starting with the establishment of angular velocity as a tool to relate between coordinate frames.

The materials presented in this chapter is intended to give a solid groundwork for the derivation of the Razi acceleration. Some of the introductory materials presented here overlap with many vector and quaternion mechanics literature [15, 42, 54, 55, 60, 64, 77, 124]. This is necessary as to provide continuity in presenting the important concepts in this chapter. The main topics covered in this chapter: (1) A proposal for a standardized vector notation system for the analysis of derivative kinematics in a multiple coordinate
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frames environment; (2) An extensive review on the methods of vectors and quaternions, and the method of deriving angular velocity and angular acceleration from rotation matrix (3) A list of the relationship of vector derivatives in a multi-frame system using the introduced notation system; (4) A more detailed treatment of the angular acceleration vector to show the acceleration terms that are often overlooked in mechanics analysis due to the two-frame system limitation. This angular acceleration term contributes to the Razi acceleration, which will be shown in the next chapter; and (5) the derivation of Euler’s derivative transformation formula.

### 3.1 New Notation System

Analyzing derivative kinematics in multiple reference frames systems necessitates the use of a complete vector notation to avoid ambiguity. A vector per se provides the information of its magnitude and direction, or *quantity*. Whereas the *quality* of a vector – the coordinate frame in which it is expressed or from which it is differentiated – is not obvious without a proper notation system. Mathematical operations on vectors may also be incorrect if incomplete notations are used.

The objective of this proposed notation system is to provide clarity and simplicity. To construct a notation system that can define explicitly the quality of of a kinematic vector, one needs to make a checklist of what can make a vector notation unambiguous and unique. The complete notation should specify

1. **The coordinate frame in which the vector is expressed.**
   A vector is represented in terms of the basis of a coordinate frame. The coordinate frame used to define a vector however is not unique and the vector’s numerical representation might be different from one referential coordinate frame to another.

2. **The coordinate frame from which the time-derivative of the vector is taken.**
   Time-differentiating a vector is... If there are multiple sequential differentiation involving several coordinate frames, the notation should also be able to show the order of the differentiation.

3. **The coordinate frame to which it is related.**
   This is only applied when specifying a relative nature of a kinematic quantity; specifically the angular velocity of frame A with respect to frame B.

Conforming to the convention used in mechanics, the lowercase bold letters are used to indicate a vector. Dot accents on a vector indicates that is a time-derivative of the vector.
• \( \mathbf{r}, \mathbf{v} \) (or \( \dot{\mathbf{r}} \)), \( \mathbf{a} \) (or \( \ddot{\mathbf{r}} \)), and \( \mathbf{j} \) (or \( \dddot{\mathbf{r}} \)) are the position, velocity, acceleration, and jerk vectors.

• \( \omega, \alpha \) (or \( \dot{\omega} \)), and \( \chi \) (or \( \ddot{\omega} \)) are angular velocity, angular acceleration, and angular jerk vectors.

Capital letters \( A, B, C, \) and \( G \) are used to denote a reference frame. In examples where only \( B \) and \( G \) are used, the former indicates a rotating, body coordinate frame and the latter indicates the global-fixed, inertial body frame. When a reference frame is introduced, its origin point and three basis vectors are indicated. For example:

\[
A(\mathbf{oa}_1\mathbf{a}_2\mathbf{a}_3) \quad G(OXYZ) \quad B(\hat{i}\hat{j}\hat{k})
\]

Capital letters \( R \) and \( T \) are reserved for rotation matrix and transformation matrix. The right subscript on a rotation or transformation matrix indicates its departure frame, and the left superscript indicates its destination frame. For example:

\[
B^A R_A = \text{rotation matrix from frame } A \text{ to frame } B
\]
\[
G^B T_B = \text{transformation matrix from body frame } B \text{ to global frame } G
\]

Left superscript is used to denote the coordinate frame in which the vector is expressed. For example, if a vector \( \mathbf{r} \) is expressed in an arbitrary coordinate frame \( A \) which has the basis vector \((\hat{i}, \hat{j}, \hat{k})\), it is written as

\[
A^\mathbf{r} = r_1 \hat{i} + r_2 \hat{j} + r_3 \hat{k}
\]

= vector \( \mathbf{r} \) expressed in frame \( A \)

Left subscript is used to denote the coordinate frame from which the vector is differentiated. For example,

\[
B^A \dot{\mathbf{r}} = \frac{B^d}{dt} A^\mathbf{r}
\]

= vector \( \mathbf{r} \) expressed in frame \( A \) and differentiated from frame \( B \)

\[
B^B \ddot{\mathbf{r}} = \frac{B^d}{dt} \frac{B^d}{dt} A^\mathbf{r}
\]

= vector \( \mathbf{r} \) expressed in frame \( A \) and differentiated twice from frame \( B \)
\[ C^B_A \ddot{\mathbf{r}} = \frac{d}{dt} \frac{d}{dt} C^B_A \mathbf{r} \]

= vector \( \mathbf{r} \) expressed in frame \( A \) and differentiated from frame \( B \) and differentiated again from frame \( C \)

Right subscript is used to denote the coordinate frame to which the vector is referred, and left subscript is the coordinate frame to which it is related. For example, the angular velocity of frame \( A \) with respect to frame \( B \) is written as

\[ \dot{\omega}^B_A = \text{angular velocity vector of } A \text{ with respect to } B \]

If the coordinate frame in which the angular velocity is expressed is specified

\[ \dot{\omega}^{C_B}_A = \text{angular velocity vector of } A \text{ with respect to } B \text{ expressed in } C \]

As it is necessary that one be able to distinguish between the frame in which a derivative is calculated and the coordinates of the frame in which the vector is expressed, this notation provides the characterization naturally. For example, the velocity of a point on a rigid body rotating in the inertial space \( \dot{\mathbf{r}} \) can be calculated using a global inertial frame \( G \) and a local body frame \( B \) and expressed in four ways:

\[ \dot{\mathbf{r}}^G_G, \quad \dot{\mathbf{r}}^G_B, \quad \dot{\mathbf{r}}^B_B, \quad \dot{\mathbf{r}}^G_B \]

where \( \dot{\mathbf{r}}^G_G \) represents the velocity of \( \mathbf{r} \) as seen from \( G \) and expressed in \( G \)-coordinates, \( \dot{\mathbf{r}}^G_B \) represents the velocity of \( \mathbf{r} \) as seen from \( B \) and expressed in \( G \)-coordinates, \( \dot{\mathbf{r}}^B_B \) represents the velocity of \( \mathbf{r} \) as seen from \( B \) and expressed in \( B \)-coordinates, and \( \dot{\mathbf{r}}^G_B \) represents the velocity of \( \mathbf{r} \) as seen from \( G \) and expressed in \( B \)-coordinates. And in general,

\[ \dot{\mathbf{r}}^G_G \neq \dot{\mathbf{r}}^G_B \neq \dot{\mathbf{r}}^B_B \neq \dot{\mathbf{r}}^G_B \]

This notation system presented is sufficient to uniquely identify a vector by associating it with the referential coordinate frames (clarity). It is done by using only subscripted and superscripted variables (simplicity). It offers the completeness of the vectrix notation system by Hughes [56] in which it specifies the associated coordinate frame, and removes some redundancy and ambiguity in Jazar’s notation system [15].
3.2 Vector and Quaternion Algebra

A vector is a physical quantity that can be represented by a directed straight line. Newtonian mechanics primarily focuses on the motion in three-dimensional Euclidean space. Therefore, every vector in this work is assumed to be in $\mathbb{R}^3$ space.

$$\mathbf{r} \in \mathbb{R}^3 \quad (3.1)$$

The length of a vector corresponds to the magnitude of the physical quantity. It is denoted by the Euclidean norm of the vector $\|\mathbf{r}\|$. The direction of a non-zero vector indicates the orientation to which the physical quantity is pointing. The direction of a vector is also known as a unit vector and it can be found by dividing the vector by its magnitude, i.e. $\mathbf{r}/\|\mathbf{r}\|$.

Any vector $\mathbf{r} \in \mathbb{R}^3$ can be expressed by three noncoplanar basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in \mathbb{R}^3$. These noncoplanar vectors are known as basis vectors. A vector $\mathbf{r}$ is written in terms of its components and basis vectors as

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad (3.2)$$

where $x, y,$ and $z$ are scalars that denote the magnitude of the vector $\mathbf{r}$ in the direction of the basis vectors $\mathbf{i}, \mathbf{j},$ and $\mathbf{k}$ as illustrated in Figure 3.1. The vector $\mathbf{r}$ can also be concisely represented as a $3 \times 1$ matrix

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (3.3)$$

The magnitude, or length, of a vector $\mathbf{r}$ can be found as

$$\|\mathbf{r}\| = \sqrt{x^2 + y^2 + z^2} \quad (3.4)$$

A unit vector is a vector with a length of one. To normalize a vector $\mathbf{r}$ into a unit vector, one needs to divide each component by the vector length.

$$\hat{\mathbf{r}} = \frac{x}{\|\mathbf{r}\|}\mathbf{i} + \frac{y}{\|\mathbf{r}\|}\mathbf{j} + \frac{z}{\|\mathbf{r}\|}\mathbf{k} \quad (3.5)$$
3.2.1 Vector Addition

For all $r_j \in \mathbb{R}^3$ where $j = 1, 2, 3, \ldots$, the vectors can only be added when they are represented in terms of a common axis [126]

$$r_1 = r_1 \hat{u} \quad r_2 = r_2 \hat{u}$$  \hspace{1cm} (3.6)

If the vectors are coaxial, then it has the following properties [124]

1. Commutativity
   $$r_1 + r_2 = r_2 + r_1$$  \hspace{1cm} (3.7)

2. Associativity
   $$(r_1 + r_2) + r_3 = r_1 + (r_2 + r_3)$$  \hspace{1cm} (3.8)

3. Null element
   $$0 + r = r$$  \hspace{1cm} (3.9)

4. Inverse element
   $$r + (-r) = 0$$  \hspace{1cm} (3.10)

Note that the basis vectors act as the common axes. It is shown that two vectors in $\mathbb{R}^3$ can only be added if they are expressed using the same basis vectors. This is one of the
most important concept in derivative kinematics since a same vector \( \mathbf{r} \) can be expressed by many sets of basis vectors. For example, let \( \mathbf{r} \in \mathbb{R}^3 \) and let there be two sets of basis vectors \( B(i, j, k) \) and \( G(\hat{I}, \hat{J}, \hat{K}) \). The expression of the vector can be written two ways

\[
\begin{align*}
B\mathbf{r} &= x\hat{i} + y\hat{j} + z\hat{k} \\
G\mathbf{r} &= X\hat{I} + Y\hat{J} + Z\hat{K}
\end{align*}
\]  

(3.11)

and due to their non-coaxiality, the two vector cannot be added together. To add multiple vectors of different axes (vector bases), all vectors has to be transformed into a common set of axes.

### 3.2.2 Quaternion Addition

A quaternion \( q \) is a sum of a scalar and a vector that has application in mechanics in \( \mathbb{R}^3 \) [15].

\[
q = q_0 + \mathbf{q} = q_0 + q_1\hat{i} + q_2\hat{j} + q_3\hat{k} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}
\]  

(3.12)

where \( q_0 \) is a scalar and \( \mathbf{q} \) is a vector. The addition rule of two quaternions \( q \) and \( p \) are as follows

\[
q + p = q_0 + \mathbf{q} + p_0 + \mathbf{p} = q_0 + p_0 + (q_1 + p_1)\hat{i} + (q_2 + p_2)\hat{j} + (q_3 + p_3)\hat{k}
\]  

(3.13)

Quaternion addition is also both associative and commutative,

\[
q + p = p + q
\]

\[
(q + p) + r = q + (p + r)
\]  

(3.14)

### 3.2.3 Vector Multiplication

Let vectors \( \mathbf{r}_j \in \mathbb{R}^3 \) and scalars \( c \in \mathbb{R} \) where \( j = 1, 2, 3, \ldots \). There are four types vector multiplication [15, 124]:
1. **Scalar multiplication**, which has the following properties

\[
  c(r_1 + r_2) = cr_1 + cr_2 \\
  (c_1 + c_2)r = c_1r + c_2r \\
  (c_1c_2)r = c_1(c_2r) \\
  1r = r \\
  0r = 0 \\
  -1r = -r
\] (3.15)

2. **Dot or inner product.**

\[
  r_1 \cdot r_2 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = x_1x_2 + y_1y_2 + z_1z_2
\] (3.16)

The dot product produces a scalar equals to the product of the lengths of the vectors and the cosine of the angle between them

\[
  r_1 \cdot r_2 = \|r_1\|\|r_2\|\cos(\alpha)
\] (3.17)

For an orthogonal set of basis vectors, the dot product is commutative.

\[
  r_1 \cdot r_2 = r_2 \cdot r_1
\] (3.18)

3. **Cross Product**

\[
  r_1 \times r_2 = \begin{bmatrix} y_1z_2 - y_2z_1 \\ x_2z_1 - x_1z_2 \\ x_1y_2 - x_2y_1 \end{bmatrix}
\] (3.19)

The cross product produces another vector that is perpendicular to the plane of \( r_1 \) and \( r_2 \). The length of the resulting vector is

\[
  \|r_1 \times r_2\| = \|r_1\|\|r_2\|\sin(\alpha)
\] (3.20)

The vector cross product is skew-commutative

\[
  r_1 \times r_2 = -r_2 \times r_1
\] (3.21)

It is also important to note the non-associativity of a vector cross product

\[
  r_1 \times (r_2 \times r_3) \neq (r_1 \times r_2) \times r_3
\] (3.22)
3.2.4 Quaternion Product

A quaternion product is a procedure to multiply two quaternions [15, 43].

\[ qp = q \times p + q \cdot p \] (3.23)

The quaternion product follows the distributive law. If the quaternions are decomposed into their scalar and vector components, the quaternions product can be written as

\[ qp = (q_0 + q)(p_0 + p) \]
\[ = q_0p_0 - q \cdot p + q_0p + p_0q + q \times p \] (3.24)

The quaternion product is not commutative, but associative and distributive over addition

\[ qp \neq pq \]
\[ (pq)r = p(qr) \]
\[ (p + q)r = pr + qr \] (3.25)

A quaternion magnitude, or length, can be found as

\[ \|q\| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} \] (3.26)

A unit quaternion is when the length of the quaternion is equal to 1. A quaternion \( q \) has a conjugate \( q^* \)

\[ q^* = q_0 - q \]
\[ = q_0 - q_1i - q_2j - q_3k \] (3.27)

It can be shown that a product of a quaternion with its conjugate is

\[ qq^* = \|q\|^2 \] (3.28)

A quaternion inverse is obtained by

\[ q^{-1} = \frac{1}{q} = \frac{q^*}{\|q\|^2} \] (3.29)

If \( q \) is a unit quaternion, then its inverse can be simplified to its own conjugate

\[ q^{-1} = q^* \] (3.30)
A quaternion $q$ can also be expressed as a $4 \times 4$ matrix

$$
\begin{bmatrix}
q_0 & -q_1 & -q_2 & -q_3 \\
q_1 & q_0 & -q_3 & q_2 \\
q_2 & q_3 & -q_0 & -q_1 \\
q_3 & -q_2 & q_1 & q_0 \\
\end{bmatrix}
$$

(3.31)

or in a compact form

$$
q = \begin{bmatrix}
q_0 \\
-q^T \\
q \\
q_0 I_3 - \tilde{q}
\end{bmatrix}
$$

(3.32)

where $\tilde{q}$ is a skew-symmetric matrix

$$
\tilde{q} = 
\begin{bmatrix}
0 & -q_3 & q_2 \\
q_3 & 0 & -q_1 \\
-q_2 & q_1 & 0
\end{bmatrix}
$$

(3.33)

It can be shown that the transpose of a matrix quaternion is equal to its inverse, which shows its orthogonality property

$$
q^T = q^{-1}
$$

(3.34)

A quaternion product in matrix form is shown as

$$
qp = 
\begin{bmatrix}
q_0 & -q_1 & -q_2 & -q_3 \\
q_1 & q_0 & -q_3 & q_2 \\
q_2 & q_3 & -q_0 & -q_1 \\
q_3 & -q_2 & q_1 & q_0
\end{bmatrix}
\begin{bmatrix}
p_0 \\
p_1 \\
p_2 \\
p_3
\end{bmatrix}
$$

(3.35)

### 3.3 Frame of Reference

#### 3.3.1 Reference Frame

A reference frame is a model of rigid structure that provides a perspective from which measurements and observations of motion are made. Kinematics quantities such as position, displacement, velocity, and acceleration are defined using the reference frame system.

In accordance to the theories of classical mechanics, a frame can be defined as follows: A frame that consists of a continuous, unbounded set of points in three-dimensional Euclidean space $\mathbb{R}^3$, which contains a subset of at least three noncolinear points, such that the distance between any two points does not change with time. To illustrate, suppose two points are chosen in a frame $G$ and let the distance between the two points
at time $t_1$ be expressed as $r(t_1)$. At some later time $t_2$, every point in $G$ moves to a different position in $\mathbb{R}^3$ and the distance is now expressed as $r(t_2)$. If both the distances are equal for all values of $t_1$ and $t_2$, then the set $G$ is a reference frame. This is illustrated in Figure 3.2.

To enable a frame of reference to describe both position and orientation of a body, it should be constructed by a \textit{triad}, which consists of a reference point $O$ and three lines originating from $O$. By common convention, a triad is described by three orthogonal lines or vectors

$$\hat{u}_1, \hat{u}_2, \hat{u}_3 \quad \text{where} \quad \hat{u}_i^T \hat{u}_j = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} ; \quad i = 1, 2, 3; \quad j = 1, 2, 3 \quad (3.36)$$

A \textit{right-handed triad}, the basis vectors must also satisfy the following equalities:

\begin{align*}
\hat{u}_1 &= \hat{u}_2 \times \hat{u}_3 \\
\hat{u}_2 &= \hat{u}_3 \times \hat{u}_1 \\
\hat{u}_3 &= \hat{u}_1 \times \hat{u}_3
\end{align*} \quad (3.37)

Using the definition, the basis vectors of a right-handed triad must satisfy three properties [64]

1. The basis vectors are unit vectors
2. The basis vectors are mutually orthogonal
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Figure 3.3: A frame can describe position and orientation in space as illustrated by two orthogonal triads $A(a_1a_2a_3)$ and $B(b_1b_2b_3)$.

3. The basis vectors satisfy the right-hand rule

A triad must have a system of ordered marks at fixed intervals in order to be used as a reference in motion measurement. When the three basis lines have scales, such a triad is called a coordinate frame.

3.3.2 Inertiality of a Reference Frame

Essentially, there are two types of reference frame: (1) a fixed, or inertial frame, and (2) a moving, or noninertial frame. If the set of points in a frame is absolutely fixed in space, or at most moves with respect to the absolutely fixed points in space with a constant velocity, then the reference frame is called an inertial reference frame. If the set of points in a frame accelerates with respect to the inertially-fixed points, the frame is called a noninertial frame. In other words, a noninertial frame may translate with nonconstant velocity and/or rotate in space.

Newton’s laws of motion are only valid if the frame of reference in which the motion of a particle is described is inertial.

\[ \mathbf{F} = m\mathbf{a} \]

Otherwise, the acceleration of the frame itself must be included in the equation of motion of the particle.

\[ \mathbf{F} = m(\mathbf{a} + \mathbf{a}_{\text{frame}}) \]
Generally, the letter $G$ is used to denote a global, inertially-fixed reference frame and the letter $B$ is used to denote a body-fixed, noninertial reference frame.

### 3.3.3 Orthogonal Coordinate Frames

The orthogonality of a coordinate frame greatly simplifies the mathematical tasks of describing the motion in it. Orthogonality condition is used as the principal rule to express any vector in an orthogonal coordinate frame. Therefore, the orthogonality condition has to be established first.

Consider a coordinate frame $G$ with unit vectors $\hat{u}_1$, $\hat{u}_2$, and $\hat{u}_3$ and an origin point $O$. For $G$ to be orthogonal, the unit vectors must be perpendicular to each other. Therefore, condition in Equation (3.37) must hold

\[
\hat{u}_1 \cdot \hat{u}_2 = 0 \\
\hat{u}_2 \cdot \hat{u}_3 = 0 \\
\hat{u}_3 \cdot \hat{u}_1 = 0
\]

(3.38)

It can be shown that every vector $\mathbf{r}$ can be described as a composition of the unit vectors

\[
\mathbf{r} = (\mathbf{r} \cdot \hat{u}_1)\hat{u}_1 + (\mathbf{r} \cdot \hat{u}_2)\hat{u}_2 + (\mathbf{r} \cdot \hat{u}_3)\hat{u}_3
\]

(3.39)

**Proof.** Assume that the coordinate frame $(O\hat{u}_1\hat{u}_2\hat{u}_3)$ is orthogonal, and a vector $\mathbf{r}$ is expressed by its components as

\[
\mathbf{r} = r_1\hat{u}_1 + r_2\hat{u}_2 + r_3\hat{u}_3
\]

(3.40)

Since, it is assumed that the coordinate frame is orthogonal, the inner products of $\mathbf{r}$ with the unit vectors are

\[
\mathbf{r} \cdot \hat{u}_1 = (r_1\hat{u}_1 + r_2\hat{u}_2 + r_3\hat{u}_3) \cdot \hat{u}_1 = r_1 \\
\mathbf{r} \cdot \hat{u}_2 = (r_1\hat{u}_1 + r_2\hat{u}_2 + r_3\hat{u}_3) \cdot \hat{u}_2 = r_2 \\
\mathbf{r} \cdot \hat{u}_3 = (r_1\hat{u}_1 + r_2\hat{u}_2 + r_3\hat{u}_3) \cdot \hat{u}_3 = r_3
\]

(3.41)

Substituting the components of $\mathbf{r}$ back into (3.40) yields

\[
\mathbf{r} = (\mathbf{r} \cdot \hat{u}_1)\hat{u}_1 + (\mathbf{r} \cdot \hat{u}_2)\hat{u}_2 + (\mathbf{r} \cdot \hat{u}_3)\hat{u}_3
\]

(3.42)
3.4 General Transformation Matrix

Any kinematic information represented by a vector expressed in one orthogonal coordinate frame can be transformed to another orthogonal coordinate frame. In derivative kinematics, *transformation matrix* is used to change the coordinates of a vector, or to change the point-of-view of a kinematic vector between two arbitrary reference frames. The coordinate orthogonality condition with basis vectors \( \hat{i} \), \( \hat{j} \), and \( \hat{k} \)

\[
r = (r \cdot \hat{i})\hat{i} + (r \cdot \hat{j})\hat{j} + (r \cdot \hat{k})\hat{k}
\]  

(3.43)

is used to develop the coordinate transformation method.

3.4.1 Rotation Matrix

Consider a general rotation of frame \( B(\text{Oxyz}) \) with respect to \( G(\text{OXYZ}) \) about the origin point. Let \( G{r} \), called "\( r \) in \( G \)", represents the vector \( r \) expressed by the basis vectors of \( G \), and \( B{r} \) is expressed by the basis vectors of \( B \). There is always a *rotation matrix* \( R \) to map the components of a vector \( B{r} \) to \( G{r} \), and vice versa

\[
G{r} = G{R_R^{B}}B{r}
B{r} = B{R_R^{G}}G{r}
\]  

(3.44)
Proof. Let the vector \( B_r \) be decomposed into the basis vectors of \( B \), following the orthogonality condition in Equation (3.43)

\[
B_r = (r \cdot \hat{i})\hat{i} + (r \cdot \hat{j})\hat{j} + (r \cdot \hat{k})\hat{k}
\]  
(3.45)

To express the vector \( r \) using \( G \)-coordinates, i.e. to transform the expression of \( r \) from \( B \) to \( G \), the basis vectors of \( G \) need to be expressed in terms of the basis vectors of \( B \),

\[
\hat{I} = (\hat{I} \cdot \hat{i})\hat{i} + (\hat{I} \cdot \hat{j})\hat{j} + (\hat{I} \cdot \hat{k})\hat{k}
\]
\[
\hat{J} = (\hat{J} \cdot \hat{i})\hat{i} + (\hat{J} \cdot \hat{j})\hat{j} + (\hat{J} \cdot \hat{k})\hat{k}
\]
\[
\hat{K} = (\hat{K} \cdot \hat{i})\hat{i} + (\hat{K} \cdot \hat{j})\hat{j} + (\hat{K} \cdot \hat{k})\hat{k}
\]  
(3.46)

Rearranging in matrix form, one obtains the coordinate transformation matrix from \( B \) to \( G \)

\[
\begin{bmatrix}
\hat{I} \\
\hat{J} \\
\hat{K}
\end{bmatrix} =
\begin{bmatrix}
\hat{i} \\
\hat{j} \\
\hat{k}
\end{bmatrix}
\]

\[
= G R_B
\begin{bmatrix}
\hat{i} \\
\hat{j} \\
\hat{k}
\end{bmatrix}
\]  
(3.47)

Thus, to change the basis vector of \( B_r \) from \( B \) to \( G \)

\[
G r = G R_B B_r
\]  
(3.48)

Similarly, the transformation matrix from \( G \) to \( B \)

\[
\begin{bmatrix}
\hat{i} \\
\hat{j} \\
\hat{k}
\end{bmatrix} =
\begin{bmatrix}
\hat{i} \cdot \hat{I} & \hat{i} \cdot \hat{J} & \hat{i} \cdot \hat{K} \\
\hat{j} \cdot \hat{I} & \hat{j} \cdot \hat{J} & \hat{j} \cdot \hat{K} \\
\hat{k} \cdot \hat{I} & \hat{k} \cdot \hat{J} & \hat{k} \cdot \hat{K}
\end{bmatrix}
\begin{bmatrix}
\hat{I} \\
\hat{J} \\
\hat{K}
\end{bmatrix}
\]

\[
= B R_G
\begin{bmatrix}
\hat{i} \\
\hat{j} \\
\hat{k}
\end{bmatrix}
\]  
(3.49)

and to change the basis vector of \( G r \) from \( G \) to \( B \)

\[
B r = B R_G G r
\]  
(3.50)
A series of $n$ sequential rotations can be represented as a single rotation matrix.

\[
^G R_B = R_n, R_{n-1} \ldots R_2 R_1 \tag{3.51}
\]

The order of sequence is important as matrix multiplication is noncommutative. However, sequential rotations are associative. For example,

\[
R_3 R_2 R_1 = (R_3 R_2) R_1 = R_3 (R_2 R_1) \tag{3.52}
\]

The transpose of a sequential rotation can be represented as

\[
{^G R_B}^T = (R_n, R_{n-1} \ldots R_2 R_1)^T
\]

\[
= R_1^T R_2^T \ldots R_{n-1}^T R_n^T \tag{3.53}
\]

Every 3-by-3 rotation matrices that maps an orthonormal basis of $\mathbb{R}^3$ to another orthonormal basis, is called an orthogonal matrix. Thus it has the following properties:

1. $\det R = \pm 1$

An orthogonal matrix that has determinant $+1$ is called proper and is used to describe the orientation kinematics of a rigid body. An orthogonal matrix with determinant $-1$ is a reflection matrix which reflect a vector about an axis that goes through the origin of an orthogonal frame. This is called improper rotation, which is generally a rotation followed by an inversion. An improper rotation is not considered due to it not being a rigid body transformation.

2. $R^{-1} = R^T$

The inverse of a rotation matrix is equal to its transpose. This means, for the case of two orthogonal coordinate frames $G$ and $B$ involved, then

\[
^G R_B = ^B {R_G}^T = ^B R_G^{-1}
\]

\[
^B R_G = ^G {R_B}^T = ^G R_B^{-1} \tag{3.54}
\]

### 3.4.2 Group Property

A set $S$, with a binary operation $\otimes$ between its elements, is a called group $(S, \otimes)$ if it satisfies the following conditions called the group axiom

- **Closure**: If $s_1, s_2 \in S$, then $s_1 \otimes s_2 \in S$.
- **Associativity**: If $s_1, s_2, s_3 \in S$, then $(s_1 \otimes s_2) \otimes s_3 = s_1 \otimes (s_2 \otimes s_3)$
• **Identity** There exist \( s_0 \in S \) such that for every \( s \in S \), \( s_0 \otimes s = s \otimes s_0 = s \) holds.

• **Invertibility** For each \( s \in S \), there exists an element \( s^{-1} \in S \) such that \( s^{-1} \otimes s = s \otimes s^{-1} = s_0 \)

The set of three-dimensional coordinate transformation matrices \( R \) with the operation of matrix multiplication \( \otimes \) qualifies as a group \( S \)

\[
S = R \in \mathcal{R} : RR^T = R^T R = I, \|R\| = 1
\]

The elements of the set satisfy closure as the multiplication of two rotations results in another rotation; associativity as matrix multiplication has the same property; invertibility as every rotation \( R \) has its unique inverse \( R^{-1} \); and the identity matrix \( I \) is, by definition, a rotation.

The three-dimensional orthogonal group which consists of \( 3 \times 3 \) orthogonal matrices is a three-dimensional Lie group. Every orthogonal rotation matrix that has determinant \(+1\)\(^1\) and belongs to the Lie subgroup called the special orthogonal group \( SO(3) \).

### 3.4.3 Homogeneous Transformation Matrix

![](image)

**Figure 3.5:** Frame \( B \) with its origin separated with distance \( d \) from the origin of frame \( G \)

\(^1\)An orthogonal matrix with determinant \(-1\) is a reflection matrix which reflects a vector about an axis that goes through the origin of an orthogonal frame that goes through the origin of an orthogonal frame.
If the origins of the frame $B(oxyz)$ and $G(OXYZ)$ are separated by a distance $d$, as shown in Figure 3.5, then to transform a vector in the $G$ coordinate frame to the $B$ coordinate frame, one may write,

$$G_r = G_{RB} B_r + G_d$$  \hspace{1cm} (3.56)

The rotation matrix $G_{RB}$ and translation vector $G_d$ can be combined to a 4-by-4 transformation matrix called the homogeneous transformation matrix $G_{TB}$ [42, 52, 88]

$$G_r = G_{TB} B_r$$  \hspace{1cm} (3.57)

where

$$G_r = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}, \quad B_r = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}, \quad G_d = \begin{bmatrix} dX \\ dY \\ dZ \end{bmatrix}$$  \hspace{1cm} (3.58)

The 3-by-3 upper left submatrix denotes the 3-by-3 rotation matrix, and the 3-by-1 upper right submatrix denotes the distance between the origin points of the two coordinate frames. The “1” appended on the fourth row of the $r$ vector is the scale factor. The scale factor can be used to represent the magnitude multiplied to the vector itself. If the length of the vector is preserved, the scale factor should be a unity. The lower left 1-by-3 zero matrix denotes the perspective transformation.

The homogeneous transformation matrix can be decomposed into a pure 4-by-4 rotation matrix $G_{RB}$ and pure 4-by-4 translation matrix $G_{DB}$ as follows

$$G_{TB} = G_{DB} G_{RB}$$

$$= \begin{bmatrix} I_{3\times3} & G_d \\ 0_{1\times3} & 1 \end{bmatrix} \begin{bmatrix} G_{RB} & 0_{3\times1} \\ 0_{1\times3} & 1 \end{bmatrix}$$  \hspace{1cm} (3.59)

The inverse of a homogeneous transformation matrix is defined as

$$G_{TB}^{-1} = \begin{bmatrix} G_{RB}^T & -G_{RB}^T G_d \\ 0_{1\times3} & 1 \end{bmatrix}$$  \hspace{1cm} (3.60)

**Proof.** If a matrix can be decomposed into arbitrary-sized submatrices $A$, $B$, $C$, and $D$, where $A$ and $B$ are square matrices, and $D$ and $A - BD^{-1}C$ are non singular, one can
use the blockwise inversion formula [127] to find its inverse

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} \end{bmatrix}$$

(3.61)

Letting

$$A = G_R B \quad B = G_d \quad C = 0_{1 \times 3} \quad D = 1$$

(3.62)

gives

$$\begin{bmatrix} G_R B & G_d \\ 0_{1 \times 3} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} G_R^T & -G_R^T G_d \\ 0_{1 \times 3} & 1 \end{bmatrix}$$

(3.63)

Thus, a homogeneous transformation matrix from $G$ to $B$, $^B T_G$ can be written as the inverse of $^G T_B$

$$^B T_G = ^G T_B^{-1}$$

(3.64)

Since $T$ is not an orthogonal matrix, its inverse is not equal to its transpose

$$^G T_B^{-1} \neq ^G T_B^T$$

(3.65)

The reverse motion of $^G T_B$ is

$$^G T_{-B} = \begin{bmatrix} G_R^T & -G_R^T G_d \\ 0_{1 \times 3} & 1 \end{bmatrix}$$

(3.66)

### 3.5 Representing Rotation

There are a number of methods to describe the orientation of a vector and transforming it from one coordinate frame to another. Four of the mostly used methods of representing the attitude kinematics in engineering are presented here. In the examples, different representations are shown for a rotation between two distinct coordinate frame $G(\hat{I}, \hat{J}, \hat{K})$ and $B(\hat{i}, \hat{j}, \hat{k})$. For each attitude representation, its 3-by-3 rotation matrix form and its inverse are given.

#### 3.5.1 Direction Cosine

The rotation matrix between $G$ and $B$ can be found by calculating the angle differences between the orthogonal unit vectors of both departure and destination coordinate frames. As shown in Equation (3.45)-(3.46), the unit vectors of one frame can be described in
terms of the unit vectors of the other. Following the definition of vector dot product, the rotation matrix from $B$ to $G$ is shown to be

$$
\begin{bmatrix}
\hat{i} \cdot \hat{i} & \hat{j} \cdot \hat{i} & \hat{k} \cdot \hat{i} \\
\hat{i} \cdot \hat{j} & \hat{j} \cdot \hat{j} & \hat{k} \cdot \hat{j} \\
\hat{i} \cdot \hat{k} & \hat{j} \cdot \hat{k} & \hat{k} \cdot \hat{k}
\end{bmatrix} =
\begin{bmatrix}
\cos(\hat{i}, \hat{i}) & \cos(\hat{i}, \hat{j}) & \cos(\hat{i}, \hat{k}) \\
\cos(\hat{j}, \hat{i}) & \cos(\hat{j}, \hat{j}) & \cos(\hat{j}, \hat{k}) \\
\cos(\hat{k}, \hat{i}) & \cos(\hat{k}, \hat{j}) & \cos(\hat{k}, \hat{k})
\end{bmatrix}
$$

(3.67)

Similarly, the rotation matrix from $G$ to $B$

$$
\begin{bmatrix}
\hat{i} \cdot \hat{i} & \hat{i} \cdot \hat{j} & \hat{i} \cdot \hat{k} \\
\hat{j} \cdot \hat{i} & \hat{j} \cdot \hat{j} & \hat{j} \cdot \hat{k} \\
\hat{k} \cdot \hat{i} & \hat{k} \cdot \hat{j} & \hat{k} \cdot \hat{k}
\end{bmatrix} =
\begin{bmatrix}
\cos(\hat{i}, \hat{i}) & \cos(\hat{i}, \hat{j}) & \cos(\hat{i}, \hat{k}) \\
\cos(\hat{j}, \hat{i}) & \cos(\hat{j}, \hat{j}) & \cos(\hat{j}, \hat{k}) \\
\cos(\hat{k}, \hat{i}) & \cos(\hat{k}, \hat{j}) & \cos(\hat{k}, \hat{k})
\end{bmatrix}
$$

(3.68)

Direction cosine matrix is an orthogonal rotation matrix, thus, its inverse are equal to its transpose.

Direction cosines uses nine components to be able to describe orientation and avoid singularity. Due to its non-singularity, it is often used in aeronautics community to avoid gimbal lock. However, since the description involves so many numbers, it is computationally time-consuming and numerical errors may accumulate over time [15, 43].

### 3.5.2 Euler Angles

Rotation by Euler angles is performed by three sequential basic rotations about the body-referenced local axes. Using the notation $x, y, z$ to name the three local axes, there 27 possible combinations of a sequence of three axes but only 12 satisfy Euler’s theorem constraint that no two sequential rotations can be the same.

- **Euler angles** (symmetric Euler angles sets)
  
  1. $R_{z,\psi}R_{x,\theta}R_{z,\phi}$
  2. $R_{x,\psi}R_{y,\theta}R_{x,\phi}$
  3. $R_{y,\psi}R_{z,\theta}R_{y,\phi}$
  4. $R_{y,\psi}R_{x,\theta}R_{y,\phi}$
5. $R_{z,\psi}R_{y,\theta}R_{z,\phi}$
6. $R_{x,\psi}R_{z,\theta}R_{x,\phi}$

- Tait-Bryant or Cardan angles or yaw-pitch-roll (asymmetric Euler angles sets)

1. $R_{z,\psi}R_{y,\theta}R_{x,\phi}$
2. $R_{y,\psi}R_{z,\theta}R_{x,\phi}$
3. $R_{z,\psi}R_{x,\theta}R_{y,\phi}$
4. $R_{x,\psi}R_{z,\theta}R_{y,\phi}$
5. $R_{x,\psi}R_{y,\theta}R_{z,\phi}$
6. $R_{y,\psi}R_{x,\theta}R_{z,\phi}$

Any of these Euler angle sequences can be used to describe attitude. The most common Euler sequence is 3-1-3, which means the first rotation is about the $z$-axis, and the second rotation is about the $x$-axis, and the final rotation is about the $z$-axis. The angles $\phi$, $\theta$, and $\psi$ are known as precession, nutation, and spin respectively. These terms are primarily used in general rigid body dynamics [77], and orbital mechanics [128]. The rotation matrix built by the 3-1-3 Euler sequence can be found as

$$
B_{RG} = R_{z,\psi}R_{x,\theta}R_{z,\phi} = \begin{bmatrix}
c_{\theta}c_{\psi} - c_{\theta}s_{\phi}s_{\psi} & c_{\psi}s_{\phi} + c_{\theta}c_{\phi}s_{\psi} & s_{\theta}s_{\psi} \\
-s_{\theta}c_{\phi}s_{\psi} - c_{\theta}c_{\psi}s_{\phi} & c_{\theta}s_{\phi} + c_{\theta}c_{\phi}s_{\psi} & s_{\theta}c_{\psi} \\
-s_{\theta}s_{\phi} & c_{\phi}s_{\theta} & c_{\theta}
\end{bmatrix} \quad (3.69)
$$

where $c_{\theta} = \cos \theta$, $s_{\phi} = \sin \phi$, et cetera.

The other most commonly used local rotations sequence is 1-2-3, known as Tait-Bryant angles or Cardan angles. Its application can be found in aeronautics community [43], in which the angles $\phi$, $\theta$, and $\psi$ are known as roll, pitch, and yaw\(^2\). The rotation matrix built by the 1-2-3 Euler sequence can be found as

$$
B_{RG} = R_{z,\psi}R_{y,\theta}R_{x,\phi} = \begin{bmatrix}
c_{\theta}c_{\psi} & c_{\psi}s_{\phi} + s_{\theta}c_{\phi}s_{\psi} & s_{\theta}s_{\phi} - c_{\phi}s_{\theta}c_{\psi} \\
-s_{\theta}c_{\phi}s_{\psi} & c_{\theta}s_{\phi} - s_{\theta}c_{\psi}s_{\phi} & c_{\theta}s_{\phi} + c_{\theta}c_{\phi}s_{\psi} \\
-s_{\theta}s_{\phi} & c_{\phi}s_{\theta} & c_{\theta}
\end{bmatrix} \quad (3.70)
$$

The full matrix representation of all 12 Euler sequence matrix is given in [15]. The advantages of Euler angles are that they are integrable, intuitive (illustration by graphics are presented in [129, 130]), and use only three numbers. However, the Euler\(^2\) also known as banking, elevation, and azimuth/heading respectively
sequence matrix is not orthogonal. This is due to the Euler coordinate frame $O_\phi \theta \psi$ is not an orthogonal frame. Hence, an inverse of a rotation matrix described an Euler angle sequence is not equal to its transpose.

### 3.5.3 Euler Axis-angle

According to Euler’s rotation theorem, any general rotation about a point is equivalent to a rotation about some line passing through the point [131]. Therefore, a rotation matrix $R$ can be completely defined by a fixed rotation axis $\hat{u}$, called eigenaxis or Euler axis, and a rotation $\theta$ about this axis. This gives rise to four components to describe the orientation: the rotation angle $\theta$, and the three scalar components of the unit vector that describes the Euler axis $u_1, u_2,$ and $u_3$.

To demonstrate the axis-angle transformation matrix, consider the body frame $B$ rotates $\theta$ about a fixed line $\hat{u}$ that goes through $O$ in the global frame $G$ where

$$\hat{u} = u_1 \hat{I} + u_2 \hat{J} + u_3 \hat{K}$$  \hspace{1cm} (3.71)

Then the transformation matrix $^G R_B$ that changes the coordinates of a vector expressed in $B$ to be expressed in $G$ by the Euler’s formula [132]

$$^G \mathbf{r} = ^G R_B \mathbf{r}$$  \hspace{1cm} (3.72)
Chapter 3. Foundations of Derivative Kinematics

is \[133, 134\]

\[ G^B_R = R_{\theta, \hat{u}} = I \cos \theta + \hat{u} \sin \theta + \hat{u}^2 (1 - \cos \theta) \] (3.73)

where \( I \) is a \( 3 \times 3 \) identity matrix and \( \hat{u} \) is defined as a skew-symmetric matrix

\[ \hat{u} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \] (3.74)

The \( 3 \times 3 \) matrix representation of \( G^B_R \) is

\[ G^B_R = \begin{bmatrix} u_1^2(1 - c_\theta) + c_\theta & u_1u_2(1 - c_\theta) - u_3s_\theta & u_1u_3(1 - c_\theta) + u_2s_\theta \\ u_1u_2(1 - c_\theta) + u_3s_\theta & u_2^2(1 - c_\theta) + c_\theta & u_2u_3(1 - c_\theta) - u_1s_\theta \\ u_1u_3(1 - c_\theta) - u_2s_\theta & u_2u_3(1 - c_\theta) + u_1s_\theta & u_3^2(1 - c_\theta) + c_\theta \end{bmatrix} \] (3.75)

To inverse the rotation \( GR_B \), one can simply change the sign of the rotation angle to negative, giving

\[ G^{-1}_R B = \begin{bmatrix} u_1^2(1 - c_\theta) + c_\theta & u_1u_2(1 - c_\theta) + u_3s_\theta & u_1u_3(1 - c_\theta) - u_2s_\theta \\ u_1u_2(1 - c_\theta) - u_3s_\theta & u_2^2(1 - c_\theta) + c_\theta & u_2u_3(1 - c_\theta) + u_1s_\theta \\ u_1u_3(1 - c_\theta) + u_2s_\theta & u_2u_3(1 - c_\theta) - u_1s_\theta & u_3^2(1 - c_\theta) + c_\theta \end{bmatrix} \] (3.76)

To find the angle of rotation \( \theta \) and the rotation axis \( \hat{u} \) given a transformation matrix \( G^R_B \), one may use the following

\[ \theta = \arccos \left( \frac{1}{2} (\text{tr}(G^R_B) - 1) \right) \] (3.77)

\[ \hat{u} = \frac{1}{2 \sin \theta} (G^R_B - G^R_B^T) \] (3.78)

### 3.5.4 Quaternion

Since any general rotation can be represented the angle \( \theta \) and axis \( \hat{u} \), a unit quaternion can be used to represent rotation matrix \( R \). The angle and axis of rotation can be
represented as the Euler parameters $e_0$ and $\mathbf{e}$, where

$$

e_0 = \cos \frac{\theta}{2}
$$

$$
\mathbf{e} = e_1 \hat{i} + e_2 \hat{j} + e_3 \hat{k}
$$

This formulation is known as the Euler-Rodrigues formula \[133, 135\] A rotation expressed in quaternion form is

$$
\mathbf{e} (\theta, \hat{u}) = e_0 + \mathbf{e} = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \hat{u}
$$

where its length is

$$
\| \mathbf{e} (\theta, \hat{u}) \| = 1
$$

To obtain the angle $\theta$ and axis of rotation $\hat{u}$ from a rotation quaternion, one may use

$$
\theta = 2 \arctan \left( \frac{\| \mathbf{e} \|}{e_0} \right)
$$

$$
\hat{u} = \frac{\mathbf{e}}{\| \mathbf{e} \|}
$$

Using the quaternion method, the Euler parameters, which are used to represent the angle $\theta$ and axis of rotation $\hat{u}$, are used as the quaternion elements.

Assuming the Euler parameters $\theta$ and $\hat{u}$ to rotate the orientation of $B$ to $G$ are known, the rotation of $G^r_r$ to $G^t_t$ can be found using the rotation quaternion $\mathbf{e}$ as

$$
G^g_g = \mathbf{e} (\theta, \hat{u}) B^r_r \mathbf{e}^* (\theta, \hat{u})
$$

If $n$ successive rotation quaternions are applied, the sequence can be simplified by the product of all rotation quaternions

$$
\mathbf{e} (\theta, \hat{u}) = e_n (\theta_n, \hat{u}_n) e_{n-1} (\theta_{n-1}, \hat{u}_{n-1}) \cdots e_2 (\theta_2, \hat{u}_2) e_1 (\theta_1, \hat{u}_1)
$$

To represent the rotation matrix $G^R_B$ as a $3 \times 3$ matrix using the elements of a quaternion, one can use the modified Rodrigues’ rotation formula for quaternions

$$
G^R_B = (q_0^2 - \mathbf{q}^2) \mathbf{I} + 2 \mathbf{q} \mathbf{q}^T + 2 q_0 \tilde{\mathbf{q}}
$$
where \( \tilde{q} \) is a skew-symmetric matrix defined in Equation (3.33). Using the Euler parameters as the quaternion elements, Equation (3.84) can be rewritten as

\[
G_B = ((e_0^2 - e^2) + 2ee^T + 2e_0e) B_r
\]

\[
= \begin{bmatrix}
2e_0^2 + 2e_1^2 - 1 & 2(e_1e_2 - e_0e_3) & 2(e_0e_2 + e_1e_3) \\
2(e_0e_3 + e_1e_2) & 2e_0^2 + 2e_2^2 - 1 & 2(e_2e_3 - e_0e_1) \\
2(e_1e_3 - e_0e_2) & 2(e_0e_1 + e_2e_3) & 2e_0^2 + 2e_3^2 - 1
\end{bmatrix} B_r
\]

\[ (3.86) \]

\[
= G_B B_r
\]

The rotation matrix \( G_B \) has the inverse \( G_B^{-1} = B_G \) as

\[
G_B^{-1} = \begin{bmatrix}
2e_0^2 + 2e_1^2 - 1 & 2(e_1e_2 + e_0e_3) & 2(e_0e_2 + e_1e_3) \\
2(-e_0e_3 + e_1e_2) & 2e_0^2 + 2e_2^2 - 1 & 2(e_2e_3 + e_0e_1) \\
2(e_1e_3 - e_0e_2) & 2(-e_0e_1 + e_2e_3) & 2e_0^2 + 2e_3^2 - 1
\end{bmatrix} B_r
\]

\[ (3.87) \]
### 3-by-3 Rotation Matrix

<table>
<thead>
<tr>
<th>No. of parameters</th>
<th>9 parameters 3 are independent</th>
<th>4 parameters</th>
</tr>
</thead>
</table>

### Rotation of \( \mathbf{r} \) from \( B \) to \( G \)

\[
G \mathbf{r} = G R_B B \mathbf{r}
\]

### Inverse Rotation

\[
\begin{bmatrix}
0 \\
B \mathbf{r}
\end{bmatrix} = G R_B^{-1} \mathbf{r} = G R_B^T B \mathbf{r}
\]

\[
B \mathbf{r} = q^* G \mathbf{r} q
\]

### Composition of Rotation

\[
q = q_n, q_{n-1} \ldots q_2 q_1
\]

### Angular Velocity

\[
G \dot{\omega}_B = G \dot{R}_B G R_B^T
\]

\[
G \omega_B = 2qq^*
\]

### Angular Acceleration

\[
G \ddot{\omega}_B = G \ddot{R}_B G R_B^T + G \dot{R}_B \dot{R}_B^T
\]

\[
G \dot{\omega}_B = 2(\ddot{q}q^* + \dot{q}q^*)
\]

### Conversion between Rotation Matrix and Unit Quaternion

\[q_0 = \frac{1}{2} \sqrt{tr(R)} + 1\]

\[q_1 = (R_{23} - R_{32})/4q_0\]

\[q_2 = (R_{31} - R_{13})/4q_0\]

\[q_3 = (R_{12} - R_{21})/4q_0\]

\[
G R_B = (q_0^2 - q_i^2) \mathbf{I} + 2qq^T + 2q_iq_i^*
\]

### Table 3.1: Comparison between rotation matrix and quaternion representation of rotation.

This is the most used mapping out four different ones [136]. The other three mappings of rotation matrix to quaternion can be found in Reference [38].
3.6 Derivative Kinematics

3.6.1 Angular Velocity

*Angular velocity* is defined as the rate of change of orientation of a rigid frame with respect to a reference frame. It can be derived from the time derivative of rotation parameters and be expressed as a vectorial quantity using the three orthonormal basis vectors of a frame. To reduce rotations to vectors, one needs to start with the assumption of small rotation.

\[
\epsilon_r = r(t) - r(t_0)
\]

Consider a vector \( r \) with fixed magnitude in a reference frame shown in Figure 3.7. The vector is rotated about the origin point of the frame at time \( t_0 \). At time \( t \) after a small angle rotation performed by a rotation matrix \( R \), the vector \( r(t) \) can be expressed as

\[
r(t) = Rr(t_0)
\]

(3.88)

The difference between the \( r(t) \) and \( r(t_0) \) is denoted by

\[
\epsilon_r = r(t) - r(t_0)
\]

(3.89)

Using finite difference to approximate the time derivative, one can write

\[
r'(t) = \lim_{\Delta t \to 0} \frac{r(t) - r(t_0)}{\Delta t}
\]

(3.90)

Substituting Equation (3.89) in this equation

\[
r'(t) = \lim_{\Delta t \to 0} \frac{\epsilon_r}{\Delta t} = \frac{d(\epsilon r)}{dt}
\]

(3.91)

Substituting Equation (3.88) in the equation

\[
r'(t) = \frac{d(\epsilon R)}{dt}r(t_0)
\]

(3.92)
Figure 3.8: Frame $B$ is rotating relative to a global-fixed frame $G$ with an angular velocity $\omega$.

Representing the constant $\mathbf{r}(t_0)$ using the transposed rotation matrix $R^T$

$$\mathbf{r}'(t) = \frac{d(\epsilon R)}{dt} R^T \mathbf{r}(t)$$  \hspace{1cm} (3.93)

This final equation is shown in the next section as the representation of the classical vector product of the form

$$\frac{d}{dt} \mathbf{r} = \mathbf{\omega} \times \mathbf{r}$$  \hspace{1cm} (3.94)

Within the classical mechanics point-of-view, angular velocity is relative, i.e. one cannot express rotation without referring it to another reference frame in space. The definition of angular velocity here is derived from a case of rotating a rigid frame relative to a stationary frame.

Consider a globally-fixed coordinate frame $G(OXYZ)$ and a rotating coordinate frame $B(Oxyz)$ sharing a common origin point, as shown in Figure 3.8. To transform the coordinates of a position vector $\mathbf{r}$ from $B$ to $G$ or vice versa, one may use the time-varying rotation transformation matrix $R$

$$G \mathbf{r}(t) = ^G R_B(t) ^B \mathbf{r}$$

$$^B \mathbf{r} = ^G R_B(t)^T ^G \mathbf{r}(t)$$  \hspace{1cm} (3.95)
Assuming the position vector $r$ is constant in $B$, the expression for the velocity from the point-of-view of the globally-fixed $G$ using the time derivative of rotation matrix in Equation (3.93), one may write

$$G\mathbf{v}(t) = G\dot{R}_B(t) B = G\dot{R}_B(t) G R_B^T(t) G \mathbf{r} = \hat{G} \omega_B \times G \mathbf{r}(t)$$

(3.96)

where $\hat{G} \omega_B$ is defined as the angular velocity of $B$ relative to $G$, expressed in $G$-coordinates. Angular velocity can be defined as the rotation rate $\dot{\theta}$ about an instantaneous axis of rotation $\hat{u}$.

$$\omega = \dot{\theta} \hat{u}$$

(3.97)

Transforming the coordinates of velocity $G\mathbf{v}$ in $G$ to the coordinates of $B$, one can apply the rotation transformation matrix $G R_B^T$

$$B\mathbf{v}(t) = G R_B^T(t) G \mathbf{v} = G R_B^T(t) G \dot{R}_B(t) G \mathbf{r} = G R_B^T(t) G \dot{R}_B(t) B \mathbf{r} = B \omega_B \times B \mathbf{r}(t)$$

(3.98)

From Equation (4.10) and Equation (3.98), we can define angular velocity in terms of the rate of rotation matrix as

$$\hat{B} \omega_B = G R_B^T G \dot{R}_B \quad \hat{G} \omega_B = G R_B^T G \dot{R}_B$$

(3.99)

The angular velocity expressions can be transformed to each other’s coordinates using the rotation matrix $G R_B$

$$\hat{G} \omega_B = G R_B^T B \omega_B G R_B^T \quad \hat{B} \omega_B = G R_B^T G \omega_B G R_B$$

(3.100)

Since angular velocity is relative, it can also be expressed with respect to a rotating frame. For example, let the position vector $r$ be constant in the globally-fixed $G$, the expression for the velocity from the point-of-view of rotating $B$ is

$$B\mathbf{v}(t) = G \dot{R}_B^T(t) G \mathbf{r} = G \dot{R}_B^T(t) G R_B(t) B \mathbf{r} = B \omega_G \times B \mathbf{r}(t)$$

(3.101)

where $B \omega_G$ is defined as the angular velocity of $G$ relative to $B$, expressed in $B$-coordinates. Important to note that even though the frame $G$ is stationary in space, but from the perspective of an observer fixed in the rotating frame $B$, frame $G$ is rotating. The angular velocity of $G$ with respect to $B$ $B \omega_G$ is found to be the negative of $G \omega_B$

$$B \omega_G = -G \omega_B$$

(3.102)
provided both angular velocity vectors are expressed in the same coordinate frame.

The angular velocity vector can also be defined in terms of time-derivative of basis vectors in relation to a reference frame. For example, the angular velocity of a rotating frame \( B(\hat{i}, \hat{j}, \hat{k}) \) in reference frame \( G(\hat{I}, \hat{J}, \hat{K}) \) expressed in the \( B \)-coordinates is

\[
\begin{align*}
\frac{d}{dt} \omega_B &= \left( \frac{\hat{k} \cdot \omega_{G}}{dt} \right) \hat{i} + \left( \frac{\hat{i} \cdot \omega_{G}}{dt} \right) \hat{j} + \left( \frac{\hat{j} \cdot \omega_{G}}{dt} \right) \hat{k} \\
&= \left( \frac{\hat{k} \cdot \omega_{G}}{dt} \right) \hat{i} + \left( \frac{\hat{i} \cdot \omega_{G}}{dt} \right) \hat{j} + \left( \frac{\hat{j} \cdot \omega_{G}}{dt} \right) \hat{k}
\end{align*}
\tag{3.103}
\]

**Proof:** Recall the expression of \( \frac{d}{dt} \omega_B \) in terms of rotation matrices in (3.99)

\[
\frac{d}{dt} \omega_B = \frac{d}{dt} \begin{bmatrix} I \hat{I} \hat{I} \hat{J} \hat{J} \hat{K} \hat{K} \hat{K} \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} I \hat{I} \hat{I} \hat{J} \hat{J} \hat{K} \hat{K} \hat{K} \end{bmatrix}
\tag{3.104}
\]

Expanding the rotation matrix as direction cosines in Equation (3.67), one obtains

\[
\begin{align*}
\frac{d}{dt} \omega_B &= \left( \frac{\hat{k} \cdot \omega_{G}}{dt} \right) \hat{i} + \left( \frac{\hat{i} \cdot \omega_{G}}{dt} \right) \hat{j} + \left( \frac{\hat{j} \cdot \omega_{G}}{dt} \right) \hat{k}
\end{align*}
\tag{3.105}
\]

Employing the unit vector relationships in (3.36) and

\[
e_i \cdot d e_j = \begin{cases} -e_j \cdot d e_i & \text{for } i \neq j \\ 0 & \text{for } i = j \end{cases} ; \quad i = 1, 2, 3; \quad j = 1, 2, 3
\tag{3.106}
\]

one obtains,

\[
\frac{d}{dt} \omega_B = \left[ \begin{array}{ccc} 0 & \frac{d}{dt} \hat{i} & \frac{d}{dt} \hat{j} \\ \frac{d}{dt} \hat{j} & 0 & \frac{d}{dt} \hat{k} \\ \frac{d}{dt} \hat{k} & \frac{d}{dt} \hat{j} & 0 \end{array} \right]
\tag{3.107}
\]

Comparing the matrix product and a vector cross product with an arbitrary vector \( [\hat{x} \hat{y} \hat{z}]^T \)

\[
\begin{align*}
\mathbf{\omega} &\mathbf{\times} \hat{\mathbf{x}} = \begin{bmatrix} \mathbf{\omega}_1 \\ \mathbf{\omega}_2 \\ \mathbf{\omega}_3 \end{bmatrix} \times \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \begin{bmatrix} \mathbf{\omega}_1 \\ \mathbf{\omega}_2 \\ \mathbf{\omega}_3 \end{bmatrix} \\
&= \begin{bmatrix} i \cdot d \hat{j} \cdot \hat{y} + i \cdot d \hat{k} \cdot \hat{z} \\ j \cdot d \hat{i} \cdot \hat{x} + j \cdot d \hat{k} \cdot \hat{z} \\ k \cdot d \hat{i} \cdot \hat{x} + k \cdot d \hat{j} \cdot \hat{y} \end{bmatrix}
\end{align*}
\tag{3.108}
\]
which shows that the angular velocity vector is

\[
\begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3 \\
\end{bmatrix} = \begin{bmatrix}
\hat{k} \cdot \frac{G dj}{dt} \\
\hat{i} \cdot \frac{G dk}{dt} \\
\hat{j} \cdot \frac{G di}{dt} \\
\end{bmatrix} = \left( \hat{k} \cdot \frac{G dj}{dt} \right) \hat{i} + \left( \hat{i} \cdot \frac{G dk}{dt} \right) \hat{j} + \left( \hat{j} \cdot \frac{G di}{dt} \right) \hat{k} \\
\text{(3.109)}
\]

An angular velocity \( \omega \) has been so far written in a \( 3 \times 1 \) vector form. It can also be represented as a \( 3 \times 3 \) skew-symmetric matrix. A skew-symmetric matrix is a square matrix of which its transpose is equal to its negative. A \( 3 \times 3 \) skew-symmetric matrix has the form

\[
\tilde{x} = \begin{bmatrix}
0 & -x_3 & x_2 \\
x_3 & 0 & -x_1 \\
-x_2 & x_1 & 0 \\
\end{bmatrix} \\
\text{(3.110)}
\]

An angular velocity matrix can be shown to be a skew-symmetric matrix by using the orthogonality property of rotation matrix \( R \).

**Proof.** Recall the orthogonality property of a rotation matrix \( R \)

\[
RR^T = I \\
\text{(3.111)}
\]

Differentiating both sides with respect to time leads to

\[
\dot{RR}^T + R\dot{R}^T = 0 \\
\text{(3.112)}
\]

From the reverse transpose formula

\[
\dot{RR}^T + (\dot{RR})^T = 0 \\
\text{(3.113)}
\]

Following Equation (3.99)

\[
\omega + \omega^T = 0 \\
\text{(3.114)}
\]

which shows that \( \omega \) has the property of a skew-symmetric matrix. Hence, an angular velocity can also be represented as

\[
\tilde{\omega} = \begin{bmatrix}
0 & -\omega_3 & \omega_2 \\
\omega_3 & 0 & -\omega_1 \\
-\omega_2 & \omega_1 & 0 \\
\end{bmatrix} \\
\text{(3.115)}
\]

\(^3\text{For matrices } A \text{ and } B, (AB)^T = B^T A^T\)
where $\omega_1, \omega_2$, and $\omega_3$ are its components in the first, second, and third axis respectively.

The cross product of an angular velocity vector and another vector can be expressed more succinctly as

$$\mathbf{\omega} \times \mathbf{r} = \bar{\omega} \mathbf{r}$$

(3.116)

This technique is well known in mathematics literature [137, 138]. A skew-symmetric matrix multiplication is preferred over vector cross products is because it requires fewer arithmetic operations, thus reducing round-off errors [139].

<table>
<thead>
<tr>
<th>Angular velocity $\mathbf{\omega}$ relationship between two frames $G$ and $B$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>In terms of rotation matrix</strong> $\mathbf{G} \mathbf{R}_B$</td>
</tr>
<tr>
<td>$\mathbf{G} \mathbf{\omega}_B = \mathbf{G} \mathbf{R}_B \mathbf{G}_B^T$</td>
</tr>
<tr>
<td>$\mathbf{B} \mathbf{\omega}_B = \mathbf{G}_B^T \mathbf{G} \mathbf{R}_B$</td>
</tr>
<tr>
<td><strong>In terms of $\dot{u}$ and $\theta$</strong></td>
</tr>
<tr>
<td>$\mathbf{\omega} = \dot{\theta} \mathbf{u}$</td>
</tr>
<tr>
<td><strong>Relation between two frames $B &amp; G$</strong></td>
</tr>
<tr>
<td>$\mathbf{G} \mathbf{\omega}_B = -\mathbf{B}_G \mathbf{\omega}_G$</td>
</tr>
<tr>
<td>$\mathbf{B} \mathbf{\omega}_B = -\mathbf{B}_G \mathbf{\omega}_G$</td>
</tr>
<tr>
<td><strong>Changing expression frame</strong></td>
</tr>
<tr>
<td>$\mathbf{G} \mathbf{\omega}_B = \mathbf{G} \mathbf{R}_B \mathbf{B} \mathbf{\omega}_B \mathbf{G} \mathbf{R}_B^T$</td>
</tr>
<tr>
<td>$\mathbf{B} \mathbf{\omega}_B = \mathbf{G} \mathbf{R}_B^T \mathbf{G} \mathbf{\omega}_B \mathbf{G} \mathbf{R}_B$</td>
</tr>
</tbody>
</table>

In matrix form

$$\bar{\mathbf{\omega}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\dot{\theta}_3 & \dot{\theta}_2 \\ \dot{\theta}_3 & 0 & -\dot{\theta}_1 \\ -\dot{\theta}_2 & \dot{\theta}_1 & 0 \end{bmatrix}$$

Table 3.2: Summary of the relationships of angular velocity vector in a system with two coordinate frames – one fixed and one rotating frame. The frames $G$ and $B$ are arbitrary

### 3.6.2 Angular Acceleration

Recalling the two coordinate frames system of $G$ and $B$ in Section 3.6.1, the relation between a vector $\mathbf{r}$ expressed in $G$ and the same vector expressed in $B$ can be related
with a rotation matrix $^GR_B$

$$^G\mathbf{r} = ^GR_B^B\mathbf{r} \quad (3.117)$$

Its first derivative with respect to the $G$-frame is

$$^G\dot{\mathbf{r}} = ^GR_B^B\dot{\mathbf{r}} = ^GR_B^G\dot{R}_B^T \dot{\mathbf{r}} = \dot{\mathbf{r}} = ^G\omega_B \times ^G\mathbf{r} \quad (3.118)$$

Differentiating again from $G$-frame gives

$$^G\ddot{\mathbf{r}} = \frac{d}{dt}(^GR_B^B\dot{\mathbf{r}}) = \frac{d}{dt}(^GR_B^G\dot{R}_B^T \dot{\mathbf{r}})$$

$$= \dot{\mathbf{r}} = ^G\omega_B \times ^G\dot{\mathbf{r}} + ^G\dot{\omega}_B \times (^G\omega_B \times ^G\mathbf{r})$$

$$= ^G\mathbf{S}_B^B\mathbf{r} \quad (3.119)$$

where the angular acceleration matrix is represented by

$$^G\dot{\tilde{\omega}}_B = ^G\ddot{R}_B^G\dot{R}_B^T + ^G\dot{R}_B^G\ddot{R}_B^T$$

$$= \ddot{\theta} \hat{u} + \dot{\theta} \ddot{\hat{u}} \quad (3.120)$$

which is a skew-symmetric matrix. Thus, its components can be represented as a 3-by-1 vector.

If the instantaneous axis $\hat{u}$ does not change with respect to the differential frame $G$, then $\dot{\hat{u}} = 0$

$$^G\dot{\tilde{\omega}}_B = \ddot{\theta} \hat{u} \quad (3.121)$$

In other words, when the angular acceleration vector is parallel with the angular velocity vector, the expression is reduced to Equation (3.121). The cross product of the term in Equation (3.121) with a radius vector gives the tangential acceleration.

The centripetal acceleration term in Equation (3.119) comes from

$$^G\omega_B \times (^G\omega_B \times ^G\mathbf{r}) = ^G\omega_B^2 \dot{\mathbf{r}}$$

$$= ^GR_B^G\dot{R}_B^T \left(^GR_B^G\dot{R}_B^T \dot{\mathbf{r}} \right)$$

$$= ^GR_B^G\dot{R}_B^T ^GR_B^G\dot{G}^G\mathbf{r} \quad (3.122)$$

which gives a symmetric matrix.

The matrix $^G\mathbf{S}_B$ is the acceleration transformation matrix, which rotates the acceleration of $^B\mathbf{r}$ to $G$, and gives both angular acceleration and centripetal acceleration terms. It
can be represented as

\[ G_{SB} = G_{\tilde{R}B}^{G} R_{B}^{T} \]

\[ = \left( \frac{G_{d} dG_{\tilde{R}B}^{G} R_{B}^{T}}{dt} \right) - G_{\tilde{R}B}^{G} R_{B}^{T} \]

\[ = G_{G\tilde{\alpha}B} + G_{\omega B}^{2} \]

\[ = \dot{\theta} \ddot{u} + \dddot{\theta} \dddot{u} + \dot{\theta} \dddot{u} \]

\[ = \dddot{\theta} \hat{u} + \dot{\theta} \dddot{u} + \dot{\theta} \dddot{u} \]

\[ = \begin{bmatrix}
0 & -\left( \dot{\theta} u_{3} + \dot{\theta} u_{3} \right) & \left( \dot{\theta} u_{2} + \dot{\theta} u_{2} \right) \\
\left( \dot{\theta} u_{3} + \dot{\theta} u_{3} \right) & 0 & -\left( \dot{\theta} u_{1} + \dot{\theta} u_{1} \right) \\
\left( \dot{\theta} u_{2} + \dot{\theta} u_{2} \right) & \left( \dot{\theta} u_{1} + \dot{\theta} u_{1} \right) & 0
\end{bmatrix} \]

\[ = \dot{\theta} \ddot{u} + \dot{\theta} \dddot{u} \]

\[ = \dddot{\theta} \hat{u} + \dot{\theta} \dddot{u} + \dot{\theta} \dddot{u} \]

There are some inconsistencies and ambiguities between the definition of angular acceleration matrix and the acceleration transformation matrix in the literature. Angeles [10] developed the acceleration transformation \( G_{SB} \) by means of angular velocity differentiation. However, he named it as angular acceleration tensor. Geradin and Cordona [21], named the term given by \( G_{SB} \) as spatial angular acceleration, and later but separates the tensor by naming \( \frac{G_{d} dG_{\tilde{R}B}^{G} R_{B}^{T}}{dt} \) as the angular acceleration and \( -G_{\tilde{R}B}^{G} R_{B}^{T} \) as the centripetal acceleration. In both references, the definition of the actual angular acceleration matrix in Equation (3.120) is not explicitly given. Jazar [15] recognizes \( G_{SB} \) and \( G_{G\tilde{\alpha}B} \) as two different matrices; the former is a rotational acceleration transformation matrix, and the latter is the angular acceleration matrix.

The previous formulation is correct, but it does not show explicitly the actual components of an angular acceleration, which is \( G_{G\tilde{\alpha}B} \)

\[ G_{G\tilde{\alpha}B} = G_{\dot{R}B}^{G} R_{B}^{T} + G_{\ddot{R}B}^{G} R_{B}^{T} \]

\[ = \dot{\theta} \ddot{u} + \dot{\theta} \dddot{u} \]

\[ = \begin{bmatrix}
(\dot{\theta} u_{3} + \dot{\theta} u_{3}) & 0 & -\left( \dot{\theta} u_{1} + \dot{\theta} u_{1} \right) \\
0 & (\dot{\theta} u_{2} + \dot{\theta} u_{2}) & \left( \dot{\theta} u_{1} + \dot{\theta} u_{1} \right) \\
-\left( \dot{\theta} u_{2} + \dot{\theta} u_{2} \right) & 0 & (\dot{\theta} u_{1} + \dot{\theta} u_{1})
\end{bmatrix} \]

Comparing it with \( G_{SB} \), or spatial angular acceleration [21], or rotational acceleration matrix [15]

\[ G_{SB} = \begin{bmatrix}
-\dot{\theta}^2 & -\left( \dot{\theta} u_{3} + \dot{\theta} u_{3} \right) + u_{1} u_{2} \dot{\theta} & \left( \dot{\theta} u_{2} + \dot{\theta} u_{2} \right) + u_{1} u_{3} \dot{\theta}^2 \\
(\dot{\theta} u_{3} + \dot{\theta} u_{3}) + u_{1} u_{2} \dot{\theta}^2 & -\dot{\theta}^2 & -\left( \dot{\theta} u_{1} + \dot{\theta} u_{1} \right) + u_{2} u_{3} \dot{\theta}^2 \\
\left( \dot{\theta} u_{2} + \dot{\theta} u_{2} \right) + \dot{\theta}^2 & \left( \dot{\theta} u_{1} + \dot{\theta} u_{1} \right) + u_{2} u_{3} \dot{\theta}^2 & -\dot{\theta}^2
\end{bmatrix} \]
Due to the ambiguity in the definition and distinction between $\ddot{\theta} \hat{u} + \dot{\theta} \ddot{\hat{u}}$, it is proposed that both terms are given different notation:

$$\frac{\partial}{\partial t} \mathbf{G}_{\dot{\omega}B} = \frac{\partial}{\partial t} \mathbf{G}_{\dot{\alpha}B} + \frac{\partial}{\partial t} \mathbf{G}_{\dot{\omega}G} \times \frac{\partial}{\partial t} \mathbf{G}_{\omega B}$$

$$\frac{\partial}{\partial t} \mathbf{G}_{\dot{\alpha}B} = \ddot{\theta} \hat{u}$$

$$\frac{\partial}{\partial t} \mathbf{G}_{\dot{\omega}G} \times \frac{\partial}{\partial t} \mathbf{G}_{\omega B} = \dot{\theta} \ddot{\hat{u}}$$

and in matrix form, generally

$$\mathbf{G}_{\dot{\alpha}B} = \begin{bmatrix} 0 & -\dot{\theta} & \dot{\hat{u}}_2 \\ \dot{\hat{u}}_3 & 0 & -\dot{\theta} \\ \dot{\hat{u}}_2 & \dot{\theta} & 0 \end{bmatrix}$$

$$\mathbf{G_{\dot{\omega}B} \times B\dot{\omega}G} = \begin{bmatrix} 0 & -\dot{\theta} & \dot{\hat{u}}_2 \\ \dot{\hat{u}}_3 & 0 & -\dot{\theta} \\ -\dot{\hat{u}}_2 & \dot{\theta} & 0 \end{bmatrix}$$

In this specification, the *angular acceleration matrix* $\frac{\partial}{\partial t} \mathbf{G}_{\dot{\alpha}B}$ is exclusively defined as equal to Equation (3.121); whereas the vector cross product is defined as the rate of change of $\mathbf{G}_{\dot{\omega}B}$ due to the change of axis of rotation $\hat{u}$ in space\(^4\). Notice that the cross product term is zero because $\mathbf{G}_{\dot{\omega}G} = -\mathbf{G}_{\omega B}$. This shows that, for two coordinate frames system only, this relation holds

$$\frac{\partial}{\partial t} \mathbf{G}_{\dot{\omega}B} = \frac{\partial}{\partial t} \mathbf{G}_{\dot{\alpha}B} = \ddot{\theta} \hat{u}$$

The cross-product term that produces to $\dot{\theta} \ddot{\hat{u}}$ is normally discarded, or not given appropriate attention, because for the conventional two coordinate frames system with fixed rotation axis, the term is always canceled out. However, in multi-frames system with moving axes of rotation, this term is in general not zero.

No literature has discussed the mechanical interpretation of this acceleration term separately. The cross-product term, for a system with more than two coordinate frames, will be shown in the next chapter as the contributor to the Razi acceleration term.

### 3.7 Euler Derivative Transformation Formula

#### 3.7.1 Particle Motion in Rotating Frame

With the thorough definition of angular velocity, one can now write a general formula for derivative transformation between coordinate frames. Let $A(OXYZ)$ and $B(Oxyz)$

---

\(^4\)The proof that the rate of change of basis vectors (which represent the unit axis of rotation) produces a cross product between two vectors is shown in Equation (3.104)– (3.109)
Angular acceleration $\alpha$ relationship between two frames $G$ and $B$

In terms of rotation matrix $^{G}R_{B}$

$$
^{G}\alpha_{B} = ^{G}R_{B}^{G}R_{B}^{T}B
$$

$$
^{B}\alpha_{B} = ^{G}R_{B}^{T}B\dot{R}_{B}
$$

In terms of $\dot{u}$ and $\theta$

$$
\alpha = \ddot{\theta}\dot{u}
$$

Relation between two frames $B \& G$

$$
^{G}\alpha_{B} = ^{-G}\dot{B}\alpha_{G}
$$

$$
^{B}\alpha_{B} = ^{-B}\dot{B}\alpha_{G}
$$

Changing expression frame

$$
^{G}\alpha_{B} = ^{G}R_{B}\dot{B}\alpha_{B}^{G}R_{B}^{T}B
$$

$$
^{B}\alpha_{B} = ^{G}R_{B}^{T}\dot{G}\alpha_{B}^{G}R_{B}
$$

In matrix form

$$
\dot{\omega} = 
\begin{bmatrix}
0 & -(\dot{\theta}u_{3} + \dot{\theta}\dot{u}_{3}) & (\dot{\theta}u_{2} + \dot{\theta}\dot{u}_{2}) \\
(\dot{\theta}u_{3} + \dot{\theta}\dot{u}_{3}) & 0 & -(\dot{\theta}u_{1} + \dot{\theta}\dot{u}_{1}) \\
-(\dot{\theta}u_{2} + \dot{\theta}\dot{u}_{2}) & (\dot{\theta}u_{1} + \dot{\theta}\dot{u}_{1}) & 0 \\
\end{bmatrix}
$$

| Table 3.3: Summary of the relationships of angular acceleration vector in a system with two coordinate frames – one fixed and one rotating frame. The frames $G$ and $B$ are arbitrary |

be two arbitrary frames in $\mathbb{R}^{3}$ which motion is related by the angular velocity $\omega$, for every vector $\Box$ in $B$, its time derivative with respect to $A$ is given by

$$
\frac{Ad}{dt}B\Box = \frac{Bd}{dt}B\Box + ^{B}A\omega_{B} \times B\Box
$$

(3.129)

Using the new notation system, the formula is expressed as

$$
^{B}A\dot{\Box} = ^{B}B\dot{\Box} + ^{A}B\omega_{B} \times B\Box
$$

(3.130)
Proof: Let $B \mathbf{r}$ be a vector expressed in $B$-coordinates $i, j, k$. Its derivative in relation to frame $A$ can be written as

$$\frac{A}{dt} B \mathbf{r} = \frac{A}{dt} (x \hat{i} + y \hat{j} + z \hat{k}) = \dot{x} \hat{i} + \dot{y} \hat{j} + \dot{z} \hat{k} + x \frac{A}{dt} (\hat{i}) + y \frac{A}{dt} (\hat{j}) + z \frac{A}{dt} (\hat{k}) = \dot{x} \hat{i} + \dot{y} \hat{j} + \dot{z} \hat{k} + x (\omega^B_A \times \hat{i}) + y (\omega^B_A \times \hat{j}) + z (\omega^B_A \times \hat{k}) = \frac{A}{dt} B \mathbf{r} + \omega^B_A \times B \mathbf{r} \quad (3.131)$$

Another proof: The time-derivative of transformation matrix is used to show the formula for the differentiation between two frames. By definition, $A B$-expressed vector $\mathbf{r}$ differentiated from $A$ is equal to $A$-expressed vector $\mathbf{r}$ differentiated from $A$ that is rotated to $B$ by $A R_B$

$$\frac{A}{dt} B \mathbf{r} = \frac{A}{dt} (A R_B \mathbf{r}) \quad (3.132)$$

Using the result in Equation (4.10) and assuming the vector $B \mathbf{r}$ is non-constant in $B$, Equation (3.132) becomes

$$\frac{A}{dt} B \mathbf{r} = \frac{A}{dt} (A R_B A R_B ^T A \mathbf{r}) = \frac{A}{dt} (A \omega_B \times A \mathbf{r} + A R_B B \mathbf{r}) = \frac{A}{dt} A \omega_B \times B \mathbf{r} + B \mathbf{r} \quad (3.133)$$

which is similar Equation (4.20)

Although the frames $A$ and $B$ can be any two arbitrary coordinate frame, it is important to note the order of differentiation, as

$$\frac{A}{dt} B \mathbf{r} \neq \frac{A}{dt} A \mathbf{r} \quad (3.134)$$

This formula is called the Euler derivative transformation formula and it is the most fundamental result of kinematics in multiple coordinate frames. The formula relates the first time-derivative of a vector, which is expressed in a coordinate frame, with respect to another coordinate frame by means of angular velocity. To demonstrate its application, consider the point $P$ in Figure???. Its position vector in the rotating $B$ frame is expressed as $B \mathbf{r}$. The first time-derivative of $B \mathbf{r}$ (velocity) with respect to frame $A$ is

$$\frac{A}{dt} B \mathbf{r} = \frac{A}{dt} B \mathbf{r} + \omega^B_A \times B \mathbf{r} \quad (3.135)$$
In words, Equation (3.135) means that the velocity of point $P$ in from the point-of-view of frame $A$ is equal to the sum of the velocity of $P$ in $B$ and the cross product of the angular velocity of $B$ with respect to $A$ and the position vector in $B$. Every vector is expressed using the $B$-coordinates. Another natural application of the derivative transformation formula is the introduction of angular acceleration. It is defined as the rate of change of angular velocity.

Despite of its importance in classical mechanics and its ubiquitous application in engineering, there is no universally-accepted name for the formula in Equation (4.14). Kane et al. [55] refers to it as the first\textsuperscript{5} kinematic theorem, which nomenclature is also used by Tenenbaum [63]\textsuperscript{6}. However, the term is also used in kinematic synthesis, usually in bar-linkages systems, to refer to a totally different set of theorems. Rao [64] and Baruh [23] calls it the transport theorem, which might be confused with the transport theorem in fluid mechanics. Kasdin and Paley [60] suggest the term transport equation, although this term is also used in fluid mechanics. Zipfel [45] uses Euler transformation and Coriolis transformation interchangeably depending on the application. This thesis uses the terminology by Jazar [15], the Euler derivative transformation formula, which the author opines as more descriptive and explanatory.
3.7.2 Particle Motion in Translating and Rotating Frame

Consider the case where frame $A(OXYZ)$ and frame $B(oxyz)$ are not sharing a common origin. Let the distance between the origin of frame $A$ and the origin of frame $B$ be $d$ and the moving frame $B$ is rotating with angular velocity $\omega_B$. For every vector $\square$ in $B$, its time derivative with respect to $A$ is given by

$$\frac{A d B}{dt} \square = \frac{B d}{dt} d + \frac{B d}{dt} \square + \frac{B \omega_B}{A} \times \frac{B}{B} \square$$ (3.136)

3.7.3 Conclusion

This chapter is intended to provide the groundwork for working with vector derivatives in the following chapters of this thesis. A proposal for a new and improved notation system for the purpose of kinematics analysis in multiple coordinate frame environment is presented. The earlier part of this chapter is an overview of the mathematics of coordinate frames and vectors. After this is established, the derivation of rotation matrix and its derivatives (angular velocity vector and angular acceleration vector) is shown. The four most widely-used rotation (attitude) representations – direction cosines, Euler angles, Axis-angle representation, and quaternions – are given for the sake of completeness of the

---

5 the second theorem refers to the vector summation of velocities between two coordinate frames

6 Tenenbaum calls it the zeroth theorem, with the other two theorems referring to the calculation of velocity and acceleration in two coordinate frames
attitude kinematics discussion. In the angular velocity section, the relationship between angular velocities in a multiple coordinate frames system is given. This is important in establishing the vector derivatives technique in the next chapter.

There are three important contributions in this chapter. Firstly, the development of the technique of differentiating vectors in different coordinate frames. A renewed and improved notation system is used, which provides explicit information on the kinematical vectors involved. Descriptions of rotation matrix, angular velocity, and angular acceleration are given a more detailed look, in particular, the methods of transforming the expressions of angular velocity and angular acceleration between frames. This is done in view of applying these vectors in a multiple co-rotating frames environment in the next chapter.

Secondly, the derivation of angular acceleration vector is given a more detailed description to show the often overlooked components of the vector. It is proposed that the first time-derivative of angular velocity (angular acceleration) is written in its complete form as in Equation 3.126 as to not neglecting the acceleration caused by the motion of the rotation axis. In the next chapter, it will be shown that the Razi term is derived from the component $\dot{\theta}\dot{u}$. This kinematics treatment on the angular acceleration components are previously non-existent in the literature.

Lastly, the general formulas in Equation 4.20 and Equation 3.136 are given using the renewed notation system to provide explicit information about the expression frame and the derivative frame. In the following chapter, the application of the Euler derivative transformation formula will be extended to a system with more than two coordinate frames to derive the Razi acceleration term.
Chapter 4

On the Razi Acceleration

The time derivative of a vector depends on the coordinate frame in which it is differentiated. In a multiple coordinate frames environment, it is important to know from which coordinate the derivative vectors, e.g. velocity and acceleration, are calculated and to which coordinate frame it is related. In this section, we reproduce the updated derivative transformation formula in view of extending its application for a reference frame system involving more than two coordinate frames. Further, we show the formulation of the composition of multiple angular velocity and angular acceleration vectors. This formula, combined with the derivative kinematics method, will be used to show how the Razi acceleration take place in the expression of inertial acceleration. The kinematical characteristics of the Razi acceleration is described with an illustrative example and by comparing its magnitude with the centripetal acceleration.

The purpose of this chapter is primarily to explain the physical meaning of the Razi acceleration. Previously, in [16], the Razi acceleration term appears when a position vector is differentiated from two different coordinate frames. Such method is called the mixed double-derivative method [15]. The method, however, is only applicable in very specific and limited circumstances. Here, we show that the Razi acceleration can also appear when a position vector is successively differentiated from one coordinate frame. The term is shown to originate from the differentiation of angular velocity vectors. The result shown here is consistent with the tensorial analysis of angular acceleration found in the literature. From this angular acceleration vector analysis, the Razi acceleration can be shown to be an inertial acceleration that results from the rotation about a rotating frame.
4.1 Derivative Transformation Formula in Two Coordinate Frames

An expression for a vector derivatives is dependent on the coordinate frame in which it is expressed, and the coordinate frames in which it is differentiated. The content in this section provides the definitions and proofs of different types of vector derivatives. Here, the notation system that is introduced in the previous chapter is used to distinguish between different types of vector derivatives. Only derivatives up to the second-order are considered here. The number of coordinate frames are limited to two here as to provide a first step in understanding and developing vector derivatives techniques for multiple-frame system in the later sections.

This section presents the Euler derivative transformation formula and provides the definitions of the second derivative terms, i.e. the centripetal, Coriolis, and tangential accelerations.

4.1.1 Time Derivative of the Position Vector

Any vector \( \mathbf{r} \) in a coordinate frame can be represented by its components using the basis vectors of the frame. Let \( B(O\hat{i}\hat{j}\hat{k}) \) be an arbitrary coordinate frame in which \( \mathbf{r}(t) \) is expressed. The vector is written using the basis vectors of \( B \) as

\[
B\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}
\]  

(4.1)

For \( \mathbf{r}(t) \), a vector function of time, its components \( x(t), y(t), \) and \( z(t) \) are scalar functions of time. The time-derivative of the vector \( \mathbf{r}(t) \) is

\[
\frac{B\mathbf{d}}{dt}B\mathbf{r} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k} + x\hat{i} + y\hat{j} + z\hat{k}
\]  

(4.2)

Since the basis vector is constant with respect to its own frame, the time-derivative of a vector expressed in \( B \) differentiated from its own frame \( B \) is

\[
\frac{B\mathbf{r}}{B} = \frac{B\mathbf{d}}{dt}B\mathbf{r} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}
\]  

(4.3)

This is called a local or simple derivative since the vector derivative is taken from the coordinate frame in which it is also expressed. The result is a simple derivative of its scalar components.

Now consider global-fixed coordinate frame \( G(O\hat{I}\hat{J}\hat{K}) \) and a rotating body-fixed coordinate frame \( B(O\hat{i}\hat{j}\hat{k}) \) as shown in Figure 4.1. Due to the relative motion of the coordinate
frames, the basis vectors of one frame are not constant in relation to the other frame. The time-derivative of a vector expressed in $B$ differentiated from frame $G$ is expressed as

$$
\frac{d}{dt} B \mathbf{r} = G d \frac{d}{dt} G \mathbf{r} = \hat{x} + \hat{y} \hat{y} + \hat{z} \hat{z} + x \frac{d}{dt} \hat{x} + y \frac{d}{dt} \hat{y} + z \frac{d}{dt} \hat{z} 
$$

(4.4)

Similarly, the time-derivative of a vector expressed in $G$ differentiated from frame $B$ is expressed as

$$
\frac{d}{dt} G \mathbf{r} = B d \frac{d}{dt} B \mathbf{r} = \hat{x} \hat{x} + \hat{y} \hat{y} + \hat{z} \hat{z} + \hat{x} \frac{d}{dt} \hat{x} + \hat{y} \frac{d}{dt} \hat{y} + \hat{z} \frac{d}{dt} \hat{z} 
$$

(4.5)

An angular velocity vector can be defined in terms of a time-derivative of basis vectors coordinate frame in relation to another coordinate frame. The angular velocity of a rotating frame $B$ in coordinate frame $G$, $G\omega_B$, expressed using the $B$-coordinates, is written as

$$
\frac{d}{dt} G \mathbf{r} = \hat{k} \cdot B d \frac{d}{dt} \hat{J} + \hat{i} \cdot B d \frac{d}{dt} \hat{K} + \hat{J} \cdot B d \frac{d}{dt} \hat{I} 
$$

(4.6)

And the angular velocity of a rotating frame $G$ in coordinate frame $B$, $B\omega_G$, expressed using the $G$-coordinates, is written as

$$
\frac{d}{dt} G \mathbf{r} = \hat{k} \cdot G d \frac{d}{dt} \hat{J} + \hat{i} \cdot G d \frac{d}{dt} \hat{K} + \hat{J} \cdot G d \frac{d}{dt} \hat{I} 
$$

(4.7)

Equation (4.4) and Equation (4.5) can be represented as

$$
\frac{d}{dt} B \mathbf{r} = \frac{d}{dt} G \mathbf{r} = \frac{d}{dt} G \mathbf{r} + G \omega_B \times B \mathbf{r} 
$$

$$
\frac{d}{dt} G \mathbf{r} = \frac{d}{dt} G \mathbf{r} = \frac{d}{dt} G \mathbf{r} + B \omega_G \times G \mathbf{r} 
$$

(4.8)
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Proof. This is a proof for Equation (4.8) using angular velocity as defined by the time-derivative of rotation matrix. A vector expressed in two arbitrary coordinate frames $G$ and $B$.

$$G \mathbf{r} = G R_B B \mathbf{r}$$  \hspace{1cm} (4.9)

Finding the derivative of the $G$-vector from $G$, we get

$$\frac{d}{dt} G \mathbf{r} = \frac{d}{dt} G R_B B \mathbf{r} = \frac{d}{dt} G \dot{R}_B G R_B T G \mathbf{r} = \frac{d}{dt} G \omega_B \times G \mathbf{r}$$  \hspace{1cm} (4.10)

Here, it is assumed that the vector $B \mathbf{r}$ is constant in $B$. If it is not constant, we have to include the simple derivative of the vector in $B$ and rotate it to $G$

$$\frac{d}{dt} G \mathbf{r} = \frac{d}{dt} G \dot{R}_B B \mathbf{r} + B \frac{d}{dt} B \mathbf{r} = \frac{d}{dt} G \omega_B \times G \mathbf{r} + B \dot{r}$$  \hspace{1cm} (4.11)

The angular velocities between two arbitrary frames are equal and oppositely directed given that both are expressed in the same frame

$$G \omega_B = -B \omega_G$$  \hspace{1cm} (4.12)

Using this angular velocity property, rearranging Equation (4.11) gives

$$\frac{d}{dt} B \mathbf{r} = \frac{d}{dt} G \mathbf{r} + B \omega_G \times G \mathbf{r}$$  \hspace{1cm} (4.13)

Using this results, one can present a general method of calculating and transforming the derivative of a vector expressed in one frame to another: Let $A$ and $B$ be two arbitrary frames which motion is related by the angular velocity $\omega$, for every vector $\Box$ in $B$, its time derivative with respect to $A$ is given by

$$\frac{d}{dt} A \Box = \frac{d}{dt} B \Box + B \omega_B \times B \Box$$  \hspace{1cm} (4.14)

This formula is called the Euler derivative transformation formula. It relates the first time-derivative of a vector with respect to another coordinate frame by means of angular velocity.

Despite of its importance in classical mechanics and its ubiquitous application in engineering, there is no universally-accepted name for the Euler derivative transformation formula in Equation (4.14). Several terminology are used: kinematic theorem [55, 63], transport theorem [23, 64], and transport equation [59, 60]. These terms, although terminologically correct, are more prevalent in the subject of fluid mechanics to refer to
entirely different concepts. We use the name *Euler derivative transformation formula*, which the author opines as more descriptive and unique to the subject of derivative kinematics. The name also is an homage to Leonard Euler who first used the technique of coordinate transformation in his works on rigid body dynamics [26].

### 4.1.2 Second Derivative

One of the most important applications resulting from the vector derivative transformation method is the development of inertial accelerations. Inertial accelerations, or sometimes called d’Alembert accelerations or, misnomerly, “fictitious” accelerations, are the acceleration terms that appears when the double-derivative of a position vector is transformed from a rotating frame to an inertial frame or vice versa.

To demonstrate this, let \( G \) be an inertial coordinate frame and \( B \) a non-inertial coordinate frame rotating with respect to \( G \), as shown in Figure 4.1, with an angular velocity \( G \omega_B \). Using the Euler derivative transformation formula, the first derivative of a vector \( B \mathbf{r} \) in \( B \) as seen from \( G \) is

\[
\frac{G}{d} d\frac{B}{dt} \mathbf{r} = \frac{B}{d} d\frac{B}{dt} \mathbf{r} + \frac{B}{G} \omega_B \times \frac{B}{d} \mathbf{r} \quad (4.15)
\]

Using the formula again to the vector \( \frac{B}{d} \mathbf{r} \) in order to get the second derivative of the vector \( B \mathbf{r} \) from \( G \) yields

\[
\frac{G}{d} d\frac{G}{d} d\frac{B}{dt} \mathbf{r} = \frac{G}{d} d\left( \frac{B}{d} d\frac{B}{dt} \mathbf{r} + \frac{B}{G} \omega_B \times \frac{B}{d} \mathbf{r} \right)
\]

\[
= \frac{B}{d} d\left( \frac{B}{d} d\frac{B}{dt} \mathbf{r} + \frac{B}{G} \omega_B \times \frac{B}{d} \mathbf{r} \right) + \frac{B}{G} \omega_B \times \left( \frac{B}{d} d\frac{B}{dt} \mathbf{r} + \frac{B}{G} \omega_B \times \frac{B}{d} \mathbf{r} \right) \quad (4.16)
\]

\[
= \frac{B}{d} d^2 \mathbf{r} + \frac{B}{G} \alpha_B \times \frac{B}{d} \mathbf{r} + 2\frac{B}{G} \omega_B \times \frac{B}{d} d\frac{B}{dt} \mathbf{r} + \frac{B}{G} \omega_B \times \frac{B}{G} \omega_B \times \frac{B}{d} \mathbf{r}
\]

\[
= \frac{B}{d} d\mathbf{a} + \frac{B}{G} \alpha_B \times \frac{B}{d} \mathbf{r} + 2\frac{B}{G} \omega_B \times \frac{B}{d} \mathbf{v} + \frac{B}{G} \omega_B \times \frac{B}{G} \omega_B \times \frac{B}{d} \mathbf{r}
\]

Equation (4.16) is the classical expression for the acceleration of a point in a rotating frame as seen from an inertial frame. The term \( \frac{B}{d} d\mathbf{a} \) is the local acceleration of a point in \( B \) with respect to the frame itself regardless of the rotation of \( B \) in \( G \). The second term \( \frac{B}{G} \alpha_B \times \frac{B}{d} \mathbf{r} \) is called the tangential acceleration as its direction is always tangential to the rotation path. Tangential acceleration term is represented by a vector product of an angular acceleration and a position vector. The third term \( 2\frac{B}{G} \omega_B \times \frac{B}{d} \mathbf{v} \) is an acceleration term traditionally named after Coriolis. The Coriolis acceleration term appears from the two different sources: half of the term comes from a local differentiation of \( \frac{B}{G} \omega_B \times \frac{B}{d} \mathbf{r} \),
and the other half is from the product of angular velocity and the local linear velocity vector. The Coriolis acceleration in the classical second derivative between two frames is written as twice the vector product of an angular velocity and a linear velocity. The final term in the expression $\frac{\partial}{\partial t} \omega_B \times (\frac{\partial}{\partial t} \omega_B \times \frac{\partial}{\partial t} r)$ is called the centripetal acceleration as its direction is towards the center of rotation. The centripetal acceleration has the form of a vector product of an angular velocity with another vector product of an angular velocity and a position vector.

4.2 Derivative Transformation Formula in Three Coordinate Frames

The Euler derivative transformation formula is now extended to include a third coordinate frame. Let $A$, $B$, and $C$ be three arbitrary relatively-rotating coordinate frames sharing a same origin as shown in Figure 4.2.

\[ \frac{\partial}{\partial t} \frac{\partial}{\partial t} r_B = \frac{\partial}{\partial t} v_B = \frac{\partial}{\partial t} \frac{\partial}{\partial t} \omega_B \times \frac{\partial}{\partial t} r_B \] (4.17)

And the first time derivative of the same $B$-vector in relative to the $C$-frame is written as

\[ \frac{\partial}{\partial t} \frac{\partial}{\partial t} r_C = \frac{\partial}{\partial t} v_C = \frac{\partial}{\partial t} \frac{\partial}{\partial t} \omega_B \times \frac{\partial}{\partial t} r_C \] (4.18)
Combining Equations (4.17) and Equations (4.18) yields

\[ \dot{\mathbf{r}}_A = \dot{\mathbf{r}}_C + (\dot{\mathbf{r}}_A - \dot{\mathbf{r}}_C) \times \mathbf{r}_B \]  
(4.19)

The resulting equation relates the A-derivative of a B-vector and the C-derivative of the same B-vector by means of the angular velocity differences between the three coordinate frames. This result also shows that if the relative velocities and angular velocities with respect to an auxiliary coordinate frames are known, the simple derivative in local frame can be neglected.

The extended Euler derivative transformation in three coordinate frames can be generalized as

\[ \dot{\mathbf{r}}_A = \dot{\mathbf{r}}_C + (\dot{\mathbf{r}}_A - \dot{\mathbf{r}}_C) \times \mathbf{r}_B \]  
(4.20)

This formula is called the mixed derivative transformation formula [16]. It can be used to relate vector derivatives in three frames by means of angular velocities difference between the three coordinate frames.

### 4.2.1 Razi Acceleration as Mixed Acceleration Term

We reproduce here the Razi acceleration term as derived by differentiating a position vector from two different coordinate frames. This is the Razi acceleration in its “original” form as explained in [15, 16]. It is mentioned in Chapter 2 that the Razi acceleration as derived here has little meaning kinematically. The reason is that we rarely find a problem where a vector needs to be sequentially differentiated in two different frames. Hence, it is quite problematic to give a physical interpretation of this Razi acceleration.

To illustrate this, consider three relatively-rotating coordinate frames \( A, B, \) and \( C \), where the frame \( A \) is considered as a global-fixed frame, while the frame \( B \) is rotating in \( A \), and a body frame \( C \) is rotating in \( B \). This motion is called compound rotation or nested rotations, since the rotation of \( C \) is inside the frame \( B \), which in turn is also rotating inside another frame, \( A \). Assume a position vector \( \mathbf{r}_C \) is expressed in its local frame \( C \). Taking the first derivative of \( \mathbf{r}_C \) from the \( B \)-frame using the vector derivative transformation formula yields

\[ \frac{\dot{\mathbf{r}}_C}{\dot{t}} = \frac{\mathbf{C}}{\dot{t}} \mathbf{C} \mathbf{r} + \frac{\mathbf{C}}{\dot{t}} \mathbf{C} \omega \times \mathbf{r}_C \]  
(4.21)
Taking the derivative of $\frac{B_d C}{dt}$ from the $A$-frame now yields

$$\frac{C}{AB} a = \frac{A d B}{dt} \frac{d C}{dt} r$$

$$= \frac{A d}{dt} \left( \frac{C d}{dt} C r + \frac{C}{B} \omega_C \times C r \right)$$

$$= \frac{C d}{dt} \left( \frac{C d}{dt} C r + \frac{C}{B} \omega_C \times C r \right) + \frac{C}{A} \omega_C \times \left( \frac{C d}{dt} C r + \frac{C}{B} \omega_C \times C r \right)$$

$$= \frac{C d}{dt} \left( \frac{C d}{dt} C r + \frac{C}{B} \omega_C \times C r \right) + \frac{C}{A} \omega_C \times \left( \frac{C d}{dt} C r + \frac{C}{B} \omega_C \times C r \right)$$

$$= \frac{C d}{dt} \left( \frac{C d}{dt} C r + \frac{C}{B} \omega_C \times C r \right) + \frac{C}{A} \omega_C \times \left( \frac{C d}{dt} C r + \frac{C}{B} \omega_C \times C r \right)$$

From the expression, the mixed acceleration expression contains the local acceleration term $C^a_C a$; two acceleration terms that have the Coriolis form of a product of an angular velocity and a linear velocity, which are $C^a_A \omega C \times C v$ and $C^a_B \omega C \times C v$; the tangential acceleration term $C^a_B \alpha C \times C r$; and a mixed centripetal acceleration term $C^a_B \omega C \times (C^a_A \omega C \times C r)$.

The term $(C^a_A \omega C \times C^a_B \omega C) \times C r$ is called the Razi acceleration. Note the difference between the Razi acceleration and the mixed centripetal acceleration is the order of cross product. This indicates that when a coordinate body frame is rotating relative to other two rotating coordinate frame, the mixed double derivative of $C r$ contains these inertial acceleration terms: mixed Coriolis acceleration, tangential acceleration, mixed centripetal acceleration, and the Razi acceleration. The mixed acceleration $\frac{C}{AB} a = \frac{A d C}{dt} B v$ can be used to complement the acceleration of a point in $C$ from $B$ using the mixed derivative transformation formula in Equation (4.20)

$$\frac{B d}{dt} C v = \frac{A d}{dt} B v + \left( \frac{B}{A} \omega C - \frac{C}{A} \omega C \right) \times B v$$

However, the meaning of the mixed acceleration is not clear. In practice, the double derivative of a position vector is usually directly transformed from the local frame to the inertial frame without undergoing a mid-frame differentiation. The reason is that a double-differentiation of a point position in a rotating frame from the inertial frame gives the additional acceleration terms, which contribute to inertial effect such as centrifugal and Coriolis forces. In contrast, the mixed acceleration calculates velocity (the first time-derivative) in one frame, and acceleration in another. Consequently, it is problematic to define the terms that constitute the mixed acceleration expression and to find practical applications for it. This remains an open problem.
4.3 Razi Acceleration in Derivative Kinematics of Multiple Coordinate Frames System

In this section, we will show that the Razi acceleration can appear in the same way the Coriolis and centripetal accelerations are derived, provided there are more than one rotating coordinate frame involved in the referential frame system. To do this, we need to revise the relation of series of angular velocity and angular acceleration vectors in multiple coordinate frames system.

The following analysis present a method computing vector derivatives in a system with an arbitrary number of relatively-rotating coordinate frames. In order to arrive at the general derivative kinematics formula for \( n \) coordinate frames, the composition rule for multiple angular velocity and angular acceleration vectors must be established first.

4.3.1 Successive Rotations

It is well known that a composition of \( n \) rotation matrices \( R_1, R_2, \ldots, R_n \) can be represented by a single rotation matrix. For example, if there are \( n \) coordinate frame, and a sequence of rotations is performed from the first frame to the \( n^{th} \), the rotation matrix can be written as

\[
^nR_1 = ^nR_{(n-1)}^{(n-1)}R_{(n-2)}^{(n-2)} \cdots R_2^2R_1
\]

4.3.2 Angular Velocity Transformation

To extend the Euler derivative transformation to account for arbitrary number of moving coordinate frames, we derive the general formula for the composition of relative angular velocity vectors. This is useful for, but not limited to, a system with a chain of coordinate frames where the motion of one frame is subject to another. One example of such system is depicted in Figure 4.3.

Consider now \( n \) angular velocities representing a set of \( n \) non-inertial, rotating coordinate frames. The composition of the relative angular velocities satisfy the relation

\[
^0\omega_n = ^0\omega_1 + ^1\omega_2 + ^2\omega_3 + \cdots + ^{n-1}\omega_n
\]

\[
= \sum_{i=1}^{n} ^i\omega_i
\]

if and only if they all are expressed in the same coordinate frame, which in this case it is the frame 0. The angular velocity \( ^i\omega_i \) represents a simple angular velocity of frame
i to its direct preceding frame \(i-1\), expressed in its local frame \(i\). Using the rule, the composition of the angular velocity \(0\omega_n\) can be represented in any coordinate frame \(f\) in the set.

\[
0\omega_n^f = fR_00\omega_n = \sum_{i=1}^{n} f_0^{-1}\omega_i 
\]  \hspace{1cm} (4.26)

Therefore, any equation of angular velocities that has no indication of the coordinate frame in which they are expressed is incomplete and generally incorrect.

\[
0\omega_n \neq 0\omega_1 + 1\omega_2 + 2\omega_3 + \cdots + n-1\omega_n 
\]  \hspace{1cm} (4.27)

Equation (4.26) represents the additional rule of the composition of multiple angular velocity vectors.
4.3.3 Angular Acceleration Transformation

Angular acceleration is the first time-derivative of an angular velocity vector. For an angular velocity vector of $B$ with respect to $G$ for example

$$B_G \omega_B = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k} \quad (4.28)$$

the angular acceleration is defined as

$$B_G \alpha_B = \frac{B_d}{dt} B_G \omega_B = \dot{\omega}_x \hat{i} + \dot{\omega}_y \hat{j} + \dot{\omega}_z \hat{k} \quad (4.29)$$

As pointed out in Section 3.6.2, this is only true if the axis of rotation between $B$ and $G$ is constant, hence making the rate of change of the basis vector $\dot{\mathbf{u}}$ equals to zero.

For general case where there are an arbitrary number of coordinate frames, with each frame’s rotation is represented by a simple angular velocity with respect to its preceding frame.

Suppose there are $n$ angular velocity vectors representing a set of $n$ coordinate frames expressed in an arbitrary frame $f$ in the set

$$f_0 \omega_n = f_0 \omega_1 + f_1 \omega_2 + f_2 \omega_3 + \cdots + f_{n-1} \omega_n \quad (4.30)$$

To differentiate the composition of angular velocities, which are expressed in $f$, from an arbitrary frame $g$ in the set, one has to apply the Euler derivative transformation formula in Equation (4.14)

$$f_{g,0} \alpha_n = \frac{g_d}{dt} \left( \sum_{i=1}^{n} f_{i-1} \omega_i \right) = \frac{g_d}{dt} f_0 \omega_n$$

$$= \frac{g_d}{dt} \left( f_0 \omega_1 + f_1 \omega_2 + \cdots + f_{n-1} \omega_n \right)$$

$$= \left( \frac{f}{g} \alpha_1 + f_0 \omega_1 \times f_0 \omega_1 \right) + \left( \frac{f}{g} \alpha_2 + f_1 \omega_2 \times f_1 \omega_2 \right) + \cdots$$

$$\cdots + \left( \frac{f}{g} \alpha_{n-1} + f_{n-1} \omega_n \times f_{n-1} \omega_n \right)$$

$$= \sum_{i=1}^{n} f_{i-1} \alpha_i + f_0 \omega_i \times f_{i-1} \omega_i \quad (4.31)$$

The formula provides a method to express the first derivative of a composition of angular velocity vectors, which completely describes the coordinate frame in which the vectors are expressed, the coordinate frame from which it is differentiated, and the coordinate frame to which the angular velocities and angular accelerations are compared.
Proof. To prove Equation (4.31), we start with a composition of \( n \) angular velocity vectors which are all expressed in an arbitrary frame \( g \).

\[
\dot{g}\omega_n = \sum_{i=1}^{n} \dot{g}\omega_i
\]

\[
= \dot{g}\omega_1 + \dot{g}\omega_2 + \cdots + \dot{g}\omega_n
\]

\[
= \dot{g}R_1\omega_1 + \dot{g}R_2\omega_2 + \cdots + \dot{g}R_n\omega_n
\]

We use the technique of differentiating from another frame in Equation (4.10). Differentiating the angular velocities from an arbitrary frame \( g \) gives

\[
\frac{gd}{dt}\sum_{i=1}^{n} \dot{g}\omega_i = \frac{gd}{dt}\left(\dot{g}R_1\omega_1 + \dot{g}R_2\omega_2 + \cdots + \dot{g}R_n\omega_n\right)
\]

\[
= (\dot{g}R_1\alpha_1 + \dot{g}R_2\alpha_2 + \cdots + \dot{g}R_n\alpha_n)
\]

\[
= \frac{gd}{dt}\left(\dot{g}R_1\omega_1 + \dot{g}R_2\omega_2 + \cdots + \dot{g}R_n\omega_n\right)
\]

\[
= \frac{g}{d}\left(\sum_{i=1}^{n} \dot{g}\alpha_i + \dot{g}\omega_i \times \dot{g}\omega_i\right)
\]

The whole expression can be transformed to any arbitrary frame \( f \) by applying the rotation matrix from \( g \) to \( f \), \( fR_g \)

\[
\frac{gd}{dt}\left(\sum_{i=1}^{n} \dot{g}\omega_i\right) = \left(\frac{fR_g}{d} \sum_{i=1}^{n} \dot{g}\omega_i\right)
\]

\[
= \frac{g}{d}\left(\sum_{i=1}^{n} \dot{g}\omega_i\right)
\]

\[
= \sum_{i=1}^{n} \dot{g}\omega_i + \dot{g}\omega_i \times \dot{g}\omega_i
\]

The formula (4.31) presents a more general and complete expression of angular acceleration multiple coordinate frames. The result is consistent with References [20, 62, 63] and also with the tensorial definition of angular acceleration in [10, 22].

We give an example to show how to apply Equation (4.31) into a system with multiple coordinate frames.
Example 4.1. Let the number of rotating coordinate frames be \( n = 3 \). The angular velocity of frame 3 with respect to the inertial frame 0 is written as a composition of three angular velocities, expressed in frame 3.

\[
\mathbf{0}_0 \mathbf{w}_3 = \mathbf{0}_0 \mathbf{w}_1 + \mathbf{0}_0 \mathbf{w}_2 + \mathbf{0}_0 \mathbf{w}_3 \tag{4.33}
\]

Applying differentiation from frame 0, using the formula in Equation (4.31), one obtains

\[
\frac{d}{dt} \mathbf{0}_0 \mathbf{w}_3 = \left( \mathbf{0}_0 \alpha_1 + \mathbf{0}_0 \mathbf{w}_1 \times \mathbf{0}_0 \mathbf{w}_1 \right) + \left( \mathbf{0}_0 \alpha_2 + \mathbf{0}_0 \mathbf{w}_2 \times \mathbf{0}_0 \mathbf{w}_2 \right) + \left( \mathbf{0}_0 \alpha_3 + \mathbf{0}_0 \mathbf{w}_3 \times \mathbf{0}_0 \mathbf{w}_3 \right) \tag{4.34}
\]

Looking at the third term in the right-hand side, \( \mathbf{0}_0 \alpha_3 \) is the rate of magnitudinal change of \( \mathbf{0}_0 \mathbf{w}_3 \) in frame 3, and \( \mathbf{0}_0 \mathbf{w}_3 \times \mathbf{0}_0 \mathbf{w}_3 \) is the convective rate of change of \( \mathbf{0}_0 \mathbf{w}_3 \) due to its angular velocity of frame 3 with respect to the frame it is differentiated, \( \mathbf{0}_0 \mathbf{w}_3 \). The second right-hand side term can be interpreted similarly. The total angular acceleration of frame 1 with respect to frame 0 is only \( \mathbf{0}_0 \alpha_1 \) as the direction of the angular velocity of frame 1 with respect to frame 0 \( \mathbf{0}_0 \mathbf{w}_1 \) does not change in frame from which it is differentiated as \( \mathbf{0}_0 \mathbf{w}_1 \times \mathbf{0}_0 \mathbf{w}_1 = 0 \).

Therefore, any equation of angular acceleration that has no indication of in which coordinate frame they are expressed, or without the convective rate of change term

\[
\sum_{i=1}^{n} \mathbf{f}_i \mathbf{w}_i \times \mathbf{i}_i \mathbf{w}_i \text{ is incomplete and generally incorrect.}
\]

\[
\frac{d}{dt} \left( \sum_{i=1}^{n} \mathbf{i}_i \mathbf{w}_i \right) \neq \mathbf{0}_0 \alpha_1 + \mathbf{0}_2 \alpha_2 + \cdots + \mathbf{n}_{n-1} \alpha_n \tag{4.35}
\]

In general, it should be noted that the angular velocity \( \mathbf{i}_i \mathbf{w}_i \) represents a simple angular velocity of frame \( i \) to its direct preceding frame \( i - 1 \), expressed in its local frame \( i \). Thus, the differentiation of \( \mathbf{i}_i \mathbf{w}_i \) in its local frame yields the rate of change of the angular velocity vector in its frame \( i \), only. This follows the definition of an angular acceleration vector shown in Equation (3.121). This also means that \( \mathbf{i}_i \mathbf{w}_i \) and \( \mathbf{i}_i \alpha_i \) share the same axis of rotation \( \mathbf{u}_i \).

The convective rate of change in Equation (4.31), \( \mathbf{f}_i \mathbf{w}_i \times \mathbf{i}_i \mathbf{w}_i \), describes the rate of change caused by the motion of \( \mathbf{u}_i \) with respect to \( i - 1 \). Note also that the convective rate of change does not appear in a two reference frames system due to a cross product two equal vectors. However, this is not the case in a multi-frame system. Hence, it is necessary to distinguish which term contributes to the tangential acceleration \( \mathbf{i}_i \alpha_i \) and the Razi acceleration \( \mathbf{f}_i \mathbf{w}_i \times \mathbf{i}_i \mathbf{w}_i \).
4.3.4 Jacobi-Lie Bracket Expression

The cross product of two angular velocity vectors can be easily found with the 3-by-1 vector representation. However, when the angular velocities are represented as 3-by-3 matrix, an alternative operation to the vector cross-product is needed. The Lie algebra of the rotation group $SO(3)$ consists of skew-symmetric 3x3 matrices (Section 3.4.2). Recalling the skew symmetric representation of angular velocity in Equation (3.115), let two angular velocity matrices be

$$\tilde{\omega}_A = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \quad \tilde{\omega}_B = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

(4.36)

To perform the equivalent cross-product operation, we use the matrix commutator or the Lie bracket $[\cdot, \cdot]$. For any two $n \times n$ matrices, $A$ and $B$, the Jacobi-Lie bracket is defined as

$$[A, B] = AB - BA$$

(4.37)

Thus, the equivalent cross-product operation of two vectors $\omega_A \times \omega_B$ is

$$[\tilde{\omega}_A, \tilde{\omega}_B] = \tilde{\omega}_A \tilde{\omega}_B - \tilde{\omega}_B \tilde{\omega}_A$$

(4.38)

The Jacobi-Lie bracket has the following properties

1. $[A, B] = -[B, A]$

2. It satisfies the Jacobi identity $[A, [B, C]] = [[A, B], C] + [B, [A, C]]$

This is equivalent to the properties of the cross product

1. $a \times b = -b \times a$

2. It satisfies the Jacobi identity $a \times (b \times c) = (a \times b) \times c + b \times (a \times c)$

A matrix Lie algebra is closed under bracket. This means the skew-symmetry is preserved under matrix commutator

$$[\tilde{\omega}_A, \tilde{\omega}_B] = \begin{bmatrix} 0 & \omega_z \omega_y - \omega_y \omega_z & \omega_x \omega_y - \omega_y \omega_x \\ \omega_y \omega_z - \omega_z \omega_y & 0 & \omega_x \omega_z - \omega_z \omega_x \\ \omega_y \omega_x - \omega_x \omega_y & \omega_z \omega_x - \omega_x \omega_z & 0 \end{bmatrix}$$

(4.39)
Equation (4.31) can therefore be alternatively represented in matrix form as

\[
\frac{g}{d} \left( \sum_{i=1}^{n} i^{-1} \dot{\omega}_i \right) = \sum_{i=1}^{n} i^{-1} \ddot{\alpha}_i + \left[ i, i^{-1} \ddot{\omega}_i \right]
\] (4.40)

The Jacobi-Lie bracket \([\dot{\omega}_A, \dot{\omega}_B]\) can be defined as the derivative of \(\dot{\omega}_B\) in the direction of a vector field \(\dot{\omega}_A\) ([140], pp. 20). This is consistent with the definition of \(\omega_A \times \omega_B\) as the rate of convective, or directional change of angular velocity \(\omega_B\) due to the rotation by \(\omega_A\).

If one has an option of using both matrix and vector, a cross product operation between matrix \(\dot{\omega}\) and vector \(r\) can be written as a simple multiplication

\[
\dot{\omega} r = \omega \times r
\] (4.41)

### 4.4 Razi Acceleration as an Inertial Acceleration

We examine the system of three enclosed coordinate frames - a body-fixed coordinate frame \((C)\) is rotating in another coordinate frame \((B)\), which in turn is rotating in a globally-fixed, inertial frame \((A)\), with all coordinate frames sharing a common origin. The position vector of a body point \(C r\) is expressed in the body frame \(C\) and its double derivative, hence the acceleration of the body point, is taken from the inertial frame \(A\). Any resulting acceleration term, apart from the local absolute acceleration \(C C a\), that appears from the transformation of the vector derivatives is defined as an inertial acceleration. The derivative transformation between two coordinate frames produces inertial accelerations such as centripetal, Coriolis, and tangential acceleration as shown in Equation (4.16). We show that enclosed rotations involving more than two coordinate frames produces more inertial accelerations terms, in which one of them is the Razi acceleration.

#### 4.4.1 Acceleration Transformation in Three Coordinate Frames

The derivation of the Razi acceleration here uses the Euler derivative transformation formula (4.14) in conjunction with the kinematic chain rules in Equation (4.26) and Equation (4.31). It differs from the mixed acceleration method in such way that the both derivatives of a body point in a local coordinate frame is taken from the inertial frame, the same way the Coriolis, centripetal, and tangential accelerations are derived. To demonstrate, consider a two coordinate frames system with \(G\) as the global-fixed inertial
frame and $B$ as the rotating body-fixed frame. Since the relative rotation between the two frames can be represented by one angular velocity $G\omega_B$, we can use Equation (4.14) twice to find the double derivative of a body-fixed position vector $B\mathbf{r}$ from the inertial frame $G$. Looking at the resulting expression of the acceleration in two frames, shown in Equation (4.16), one can separate the acceleration terms as follows

\[ \begin{align*}
    B_B^B a & \equiv \text{local acceleration in } B \\
    B_B^C \alpha_B \times B\mathbf{r} & \equiv \text{tangential acceleration} \\
    2B_B^C \omega_B \times B\mathbf{v} & \equiv \text{Coriolis acceleration} \\
    B_B^C \omega_B \times (B_B^C \omega_B \times B\mathbf{r}) & \equiv \text{centripetal acceleration}
\end{align*} \]

(4.42)

Now, consider a slightly complex case of enclosed rotations. Consider three relatively rotating frames $A$, $B$, and $C$ where the $A$-frame is treated as the inertial frame, the $B$-frame is rotating in $A$, and the $C$-frame, which the body-fixed frame, is rotating in $B$. The motion of a rigid body which is spinning in one or more rotating coordinate frame is called compound rotation or "nested" rotations. Nested rotation is usually seen in the gyroscopic motion, or the precession and nutation of a rigid body. To express the velocity of a point in $C$ as seen from the inertial frame, the Euler derivative transformation formula in (4.14) is invoked to transform the velocity from the local frame $C$ to the inertial frame $A$

\[ \frac{A}{dt}C\mathbf{r} = C\mathbf{\dot{r}} = C\mathbf{\dot{r}} + C_B^A \omega_B \times C\mathbf{r} \]  

(4.43)

Since, the frame $C$ is in $B$, and the frame $B$ is in $A$, the angular velocity $C_B^A \omega_C$ can be expanded using the angular velocity addition rule from Equation (4.26).

\[ C_B^A \omega_C = C_B^A \omega_B + C_A^B \omega_C \]  

(4.44)

The expression of the acceleration of a point in $C$ from the inertial frame can be expressed as

\[ \frac{A}{dt} \frac{A}{dt}C\mathbf{r} = C\mathbf{a} + \frac{A}{dt}C\mathbf{\alpha_C} \times C\mathbf{r} + 2\frac{A}{dt}C\mathbf{\omega_C} \times C\mathbf{v} + C_A^B \omega_C \times (C_A^B \omega_C \times C\mathbf{r}) \]  

(4.45)

The angular velocity derivative $\frac{A}{dt}C\mathbf{\omega_C}$ can be expanded using the rule of addition for angular acceleration in (4.31) with $n = 2$

\[ \frac{A}{dt}C\mathbf{\omega_C} = \frac{A}{dt} (C_B^A \omega_B + C_A^B \omega_C) \]  

\[ = C_B^A \alpha_B + C_B^A \alpha_B \times C_A^B \omega_B + C_A^B \omega_C + C_A^B \omega_C \times C_A^B \omega_C \]  

(4.46)
The expanded expression of the acceleration of a point in \( C \) from the inertial frame now becomes

\[
\frac{A}{dt} \frac{dA}{dt} C \mathbf{r} = C_a + \alpha_B \times C \mathbf{r} + (\omega_C \times B \omega_C) \times C \mathbf{r} + C_B \alpha_C \times C \mathbf{r} + 2A \omega_C \times C \mathbf{v} + C_A \omega_B \times (A \omega_C \times C \mathbf{r}),
\]

(4.47)

One can now list the acceleration terms expressed in local frame \( C \) and include the Razi term

\[
C_a \equiv \text{local acceleration in } C, \\
\alpha \times C \mathbf{r} \equiv \text{tangential acceleration}, \\
2A \omega_C \times C \mathbf{v} \equiv \text{Coriolis acceleration}, \\
(C_A \omega_C \times C \mathbf{r}) \times C \mathbf{r} \equiv \text{centripetal acceleration}, \\
(C_A \omega_C \times B \omega_C) \times C \mathbf{r} \equiv \text{Razi acceleration}.
\]

(4.48)

The Razi acceleration \((C_A \omega_C \times C \omega_B) \times C \mathbf{r}\) comes from differentiating the composition of angular velocity vectors from a different frame as shown in Equation (4.46). Next, we show that for \( n > 2 \) enclosed rotating frames, Razi acceleration terms will appear in the equation of motion.

It is important to note that the Razi acceleration can only appear in compound rotation motion. A rigid body in a rotation about a fixed axis does not experience an inertial force caused by the Razi acceleration. This is because a rotation about a fixed axis can always be simplified to a two coordinate frames system. In such case the Razi acceleration vanishes as there is only one angular velocity, making \( A \omega_B \times A \omega_B = 0 \).

### 4.4.2 Acceleration Transformation in Spherical Motion with Multiple Coordinate Frames

Given a set of \( n \) non-inertial rotating coordinate frames in enclosed rotations with \( i-1 \omega_i \) representing the angular velocity of \( i \)-th coordinate frame with respect to the preceding coordinate frame, we present the general expression for the acceleration (the double derivative of a vector) of the \( n \)-th coordinate frame as seen from the inertial coordinate frame 0.

\[
\frac{0}{dt}^2 n \mathbf{r} = 0a + n \mathbf{a} + \sum_{i=1}^{n} n_{i-1} \alpha_i \times n \mathbf{r} + \sum_{i=1}^{n} (n_{i} \omega_i \times n_{i-1} \omega_i) \times n \mathbf{r}
\]

\[
\sum_{i=1}^{n} n_{i} \omega_i \times n \mathbf{v} + \sum_{i=1}^{n} n_{i} \omega_i \times \left( \sum_{i=1}^{n} n_{i} \omega_i \times n \mathbf{r} \right)
\]

(4.49)
<table>
<thead>
<tr>
<th>No. of Coordinate Frames</th>
<th>Acceleration terms</th>
<th>Enclosed system</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inertial frame (n = 0)</td>
<td>$A_a$</td>
<td></td>
</tr>
<tr>
<td>Inertial frame + 1 rotating frame (n = 1)</td>
<td>$B_a$ + $\frac{B}{A}\omega_B \times (\frac{B}{A}\omega_B \times B r)$ + $2\frac{B}{A}\omega_B \times B v$ + $\frac{B}{A}\alpha_B \times B r$</td>
<td></td>
</tr>
<tr>
<td>Inertial frame + 2 rotating frames (n = 2)</td>
<td>$C_a$ + $\frac{C}{A}\omega_C \times (\frac{C}{A}\omega_C \times C r)$ + $2\frac{C}{A}\omega_C \times C v$ + $\frac{C}{A}\alpha_C \times C r$ + $(\frac{C}{B}\omega_B \times \frac{C}{B}\omega_C) \times C r$</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.1: Inertial acceleration terms for describing rigid body in multiple frames system
Chapter 4. On the Razi Acceleration

<table>
<thead>
<tr>
<th>No. of Coordinate Frames</th>
<th>Acceleration terms</th>
<th>Enclosed system</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inertial frame + 3 rotating frames $(n = 3)$</td>
<td>$\dot{D}{a}$ $+$ $\dot{A}^D{\omega_D} \times \left( \dot{A}^D{\omega_D} \times D{r} \right)$ $+$ $2 \dot{A}^D{\omega_D} \times D{v}$ $+$ $\dot{A}^D{\alpha_D} \times D{r}$ $+$ $\left( \dot{A}^D{\omega_C} \times \dot{B}^D{\omega_C} \right) \times C{r}$ $+$ $\left( \dot{A}^D{\omega_D} \times \dot{D}^A{\omega_D} \right) \times C{r}$</td>
<td></td>
</tr>
<tr>
<td>Inertial frame + 4 rotating frames $(n = 4)$</td>
<td>$\dot{E}{a}$ $+$ $\dot{A}^E{\omega_E} \times \left( \dot{A}^E{\omega_E} \times E{r} \right)$ $+$ $2 \dot{A}^E{\omega_E} \times E{v}$ $+$ $\dot{A}^E{\alpha_E} \times E{r}$ $+$ $\left( \dot{A}^E{\omega_C} \times \dot{B}^E{\omega_C} \right) \times E{r}$ $+$ $\left( \dot{A}^E{\omega_D} \times \dot{D}^E{\omega_D} \right) \times E{r}$ $+$ $\left( \dot{A}^E{\omega_E} \times \dot{E}^A{\omega_E} \right) \times E{r}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.1: (continued)

where

$$n_{nn}{a} \equiv \text{local acceleration}$$

$$\sum_{i=1}^{n} \dot{n}_{d}{\alpha_i} \times \dot{n}{r} \equiv \text{tangential acceleration}$$

$$2 \sum_{i=1}^{n} \dot{B}{\omega_i} \times \dot{n}{v} \equiv \text{Coriolis acceleration}$$

$$(4.50)$$

$$\sum_{i=1}^{n} \dot{n}{\omega_i} \times \left( \sum_{i=1}^{n} \dot{B}{\omega_i} \times \dot{n}{r} \right) \equiv \text{centripetal acceleration}$$

$$\sum_{i=1}^{n} \left( \dot{n}{\omega_i} \times \dot{n}_{i-1}{\omega_i} \right) \times \dot{n}{r} \equiv \text{Razi acceleration}$$

The general expressions for acceleration when a position vector in a local frame is differentiated twice from the inertial frame are shown in Table 4.1. Examples are given for up to a five coordinate frames system. The frame $A$ is dedicated to the inertial frame.
and the successive relatively-rotating frame are named $B$, $C$, and so on, with the last letter indicating the local coordinate frame. For example, in a system of 3 coordinate frames, $A$ is the inertial frame, $B$ is the intermediate frame, and $C$ is the local frame. All vectors are expressed in the local coordinate frame as indicated by the left superscript on each vector. Addition of angular acceleration vectors with the cross-product terms is simplified as $\sum_{i=1}^{n} n_i \alpha_i$.

The resulting acceleration expression in Table 4.1 shows that if a rigid body moves linearly with respect to an inertial space, its motion can be sufficiently described by a linear acceleration vector $^{A}\mathbf{a}$. When a rotating reference frame is introduced and the motion is described in this frame, we can see the additional acceleration terms – centripetal, Coriolis, and tangential accelerations – which act as “supplementary” accelerations to describe the motion in a rotating reference frame. When another rotating frame is introduced in the existing rotating frame making a three coordinate frames system, the expression adds another acceleration term: the Razi acceleration. As more coordinate frames are introduced in the enclosed rotating frames system, the acceleration expression becomes more complex to accommodate the relative motion between the coordinate frames. In general, Equation (4.49) includes all of the relative accelerations acting on a rigid body in compound motion which are otherwise unobservable with the classical derivative kinematics method and not intuitive to the analyst.

### 4.5 Mechanical Interpretation of the Razi Acceleration

Classical dynamics asserts that there exists a rigid frame of reference in which Newton’s second law of motion is valid. Assuming the mass is constant at all times, the law of motion is written as

$$\mathbf{F} = m\mathbf{a} \tag{4.51}$$

where $m$ indicates a mass scalar and $\mathbf{a}$ represents the acceleration vector. The reference frame in which this relationship is valid is called an inertial frame. An inertial frame is fixed in space, or at most, moves in relation to fixed points in space with a constant velocity. On the other hand, a non-inertial reference frame can be defined as a frame which accelerates with respect to the inertially-fixed points. The calculation of motion observed from a non-inertial reference frame does not follow Newton’s second law due to the relative acceleration between the observer’s frame and the inertial frame.

Starting from an inertial frame, the methods of derivative kinematics is used to calculate and transform the vector derivatives to the non-inertial observer. Such transformation produces additional acceleration terms to the Newton’s law in Equation (4.51). For the
classical case of one stationary inertial frame and a body frame which rotates about a fixed axis, these additional acceleration terms are generally divided into tangential acceleration, Coriolis acceleration, and centripetal acceleration, as shown in the expression in Equation (4.16).

\[ F = m (a + a_{tangential} + a_{Coriolis} + a_{centripetal}) \]  

Multiplying the acceleration terms with mass gives the expression of the forces that is acting on a point mass in the rotating coordinate frame.

In the case of multiple enclosed coordinate frames, the classical method of transforming derivative between two frames is limiting. This method is sufficient for any motion consideration in two-dimensional space, due to the assumption that the direction of a rigid body’s angular velocity is fixed with respect to the inertial frame at all time. However, a general formula for a three-dimensional space is needed to account for the subtle acceleration due to a non-constant angular velocity.

The derivative transformation method can be extended to include the effects of moving angular velocities. To do this, the complex rotation caused by the non-constant angular velocity vector is decomposed into several simpler coordinate frame representation such that each coordinate frame then will have a fixed-axis angular velocity vector. Such technique results on multiple coordinate frames in enclosed rotations. To analyze such system, we incorporate the extended derivative transformation formula and the kinematic chain rule, which method reveals new results and application.

For the case of three coordinate frames system (two enclosed rotating frames in an inertial frame), we have seen that the acceleration expression includes a new effect called which is the Razi acceleration. We also have shown the general expression for all four inertial acceleration – tangential, Coriolis, centripetal, and Razi – for an arbitrary number of enclosed rotating frames.

\[ F = m (a + a_{tangential} + a_{Coriolis} + a_{centripetal} + a_{Razi}) \]  

Extending the derivative transformation formula and incorporating the kinematic chain rule reveal several new result and application. The Razi acceleration is given a focus due to its appearance as one of the additional acceleration terms in the system of more than two coordinate frames.

We take the simplest case of the Razi acceleration where there are three enclosed coordinate frames – \( A, B, \) and \( C \) – sharing an origin with the body coordinate frame \( C \) being the “innermost” frame. Such system can be better illustrated by a flat disk in
compound rotation motion (nested rotations), as shown in Figure 4.4. We fix the \( C \) coordinate frame in the disk, which is spinning on its own axis with an angular speed \( \dot{\theta} \) and tilted on a turning shaft by \( \psi \) degrees. We fix the \( B \) coordinate frame on the shaft, which is turning with respect to the global-fixed frame with an angular speed \( \dot{\phi} \). The two angular velocities can be written as \( B\omega_C = \hat{\theta}i \) and \( A\omega_B = \hat{\phi}j \).

The total acceleration acting on a point \( r \) on the disc, as seen from the inertial frame, can be expressed by Equation (4.47). Decomposing the expression into multiple terms, the Razi acceleration, if expressed in the local body coordinates of \( C \), is written as

\[
a_{\text{Razi}} = (\alpha_A \times \omega_B C) \times C r \tag{4.54}
\]

The cross product of two angular velocities \( \omega_A C \times \omega_B C \) is the convective rate of change of \( B\omega_C \) due to the angular velocity of \( C \) in \( A \), \( A\omega_B \). Therefore, the Razi acceleration can be described as an inertial effect caused by the change of the local angular velocity vector direction. This is separable from the tangential direction which is due to the change of an angular velocity magnitude, hence producing acceleration in the direction which is tangent to the rotation curve.

**Example 4.2.** To examine the direction of action of the Razi acceleration, we look at the visualization of vectors in Figure 4.5. Assuming the position vector of the point of interest \( r \) is constant in \( C \). Since the angular velocity of \( C \) in \( A \) can be decomposed using the addition rule, i.e. \( \omega_C A = \omega_B C + \omega_B C \), the Razi acceleration is written as

\[
a_{\text{Razi}} = (\alpha_B C \times \omega_B C) \times C r \tag{4.55}
\]

The total acceleration is, thus, can be expressed as

\[
A \frac{d}{dt} A r = (\alpha_B C \times \omega_B C) \times C r + \alpha_C C \times (\alpha_C C \times C r) \tag{4.56}
\]

In Figure 4.5, the total acceleration vector of a body point in \( C \) is decomposed into the centripetal acceleration vector \( \alpha_B C \times (\alpha_C C \times C r) \) and the Razi acceleration vector \( (\alpha_B C \times \omega_B C) \times C r \). Using the disc to represent the plane of rotation of the \( C \)-frame, we can see the resultant vector of the Razi acceleration is always in the out-of-plane direction. This can be tested by finding the direction of the cross products in the Razi term using the right-hand rule.
(A) Body coordinate frame $C$ spinning in shaft coordinate frame $B$. The $B$-frame, in turn, is rotating in inertial frame $A$.

(b) $C$-frame rotating in $B$-frame with $\dot{\theta}_i = \omega_{C_B}$.  

(c) $B$-frame rotating in $A$-frame with $\dot{\phi}_j = \omega_{A_B}$.

**Figure 4.4:** Describing compound rotation motion using three coordinate frames system
\[ a_{\text{centripetal}} = A\omega_C \times (A\omega_C \times r) \]

\[ a_{\text{Razi}} = \left( A\omega_B \times B\omega_C \right) \times r \]

**Figure 4.5:** The resultant Razi acceleration vector (red arrow) are always normal to the disk. The total centripetal acceleration (blue arrow) is always directed towards the instantaneous axis of rotation.
4.6 Comparisons of Razi Acceleration and Centripetal Acceleration

In general, the magnitude of the Razi acceleration on a disk with two simultaneous rotations depends on a number of parameters: the magnitudes of both spin and precession rate, the angle between the spin axis and precession axis, the magnitude of the radius vector, and the angle between the radius vector and the product of the two angular velocities. Here, the Razi acceleration is plotted against the nutation angle to see the effect of increasing the tilt angle, or the nutation angle, on its magnitude.

Figure 4.6 to 4.11 shows the magnitudes of Razi and centripetal acceleration experienced by the body point \( C_r \) in the example (Figure 4.4 and Figure 4.5). The plots show the effect of \( \psi \) to the magnitude of Razi acceleration as a function of nutation angle for different combinations of angular velocities. We compare its magnitude with the magnitude of the combined centripetal acceleration.

\[
\begin{align*}
a_{\text{Razi}} &= (C_A \omega_B \times C_B \omega_C) \times C_r \\
a_{\text{centripetal}} &= C_A \omega_B \times (C_B \omega_B \times C_r) + C_B \omega_C \times (C_B \omega_C \times C_r)
\end{align*}
\] (4.57)

For all cases, it can be said that in general the Razi acceleration is always smaller than the centripetal acceleration. As the ratio \( \dot{\phi}/\dot{\theta} \) becomes smaller, the difference between the magnitudes of centripetal and Razi increases for all nutation angle. The magnitude of the Razi acceleration increases as nutation increases from 0° to 90° and starts decreasing back to 0° as the nutation angle approaches 180°. This means the Razi acceleration is maximized when the disk’s spin axis (\( x_C \)-axis) is perpendicular to the precession axis (\( y_B \)-axis). At zero nutation angle, i.e. when the spin axis and the precession axis is parallel to each other, the Razi acceleration is zero. Figure 4.11 shows that if either angular velocity is zero, the Razi acceleration vanishes.
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Figure 4.6: $\dot{\phi} = 5\, \text{rad/s}; \, \dot{\theta} = 1\, \text{rad/s}$

Figure 4.7: $\dot{\phi} = 4\, \text{rad/s}; \, \dot{\theta} = 2\, \text{rad/s}$
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**Figure 4.8:** $\dot{\phi} = 3 \text{rad/s}; \ \dot{\theta} = 3 \text{rad/s}$

**Figure 4.9:** $\dot{\phi} = 2 \text{rad/s}; \ \dot{\theta} = 4 \text{rad/s}$
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**Figure 4.10:** \( \dot{\phi} = 1 \text{ rad/s}; \dot{\theta} = 5 \text{ rad/s} \)

**Figure 4.11:** \( \dot{\phi} = 5 \text{ rad/s}; \dot{\theta} = 0 \text{ rad/s} \)
4.7 Conclusion

By revising and extending the classical Euler vector derivative transformation formula, we update the interpretation of the Razi acceleration and reveal its application in rigid body motion. The Razi acceleration is revealed as one of the inertial effects, apart from the classical Coriolis, centripetal, and tangential accelerations, affecting rigid bodies in compound rotation motion. It is shown to appear as a result of the convective rate of change of a body-frame’s angular velocity in enclosed rotation. Razi’s mathematical expression, \( \sum_{i=1}^{n} (\hat{n}_i \times \hat{n}_{i-1} \omega_i) \times ^n \mathbf{r} \), can only be discovered by using multiple (more than one rotating) coordinate frames where the angular velocity of each frame is multidirectional. A practical example is illustrated to show the typical direction of action of Razi acceleration. Its magnitude is compared to the more widely-acknowledged centripetal acceleration to show that the Razi acceleration can have a significant impact to the structures of a body in compound rotation.

Three important and novel contributions from this chapter: Firstly, the derivation of the Razi acceleration as an inertial acceleration, i.e. a supplementary acceleration due to nonuniform relative motion of three or more reference frames. In addition, this result is complemented by the revised expression for the composition of multiple angular acceleration vectors. To revise, the composition of relative angular velocities for \( n \) non-inertial rotating coordinate frames, expressed in arbitrary frame \( f \) is

\[
\dot{f}_0 \omega_n = \dot{f} R_0 \dot{0}_0 \omega_n \\
= \dot{f} R_0 (\dot{0}_0 \omega_1 + \dot{1}_1 \omega_2 + \dot{2}_2 \omega_3 + \cdots + \dot{n-1}_n \omega_n) \\
= \sum_{i=1}^{n} \dot{f}_i \omega_i
\]

and the composition of relative angular accelerations for \( n \) non-inertial rotating coordinate frames, expressed in arbitrary frame \( f \) is

\[
\frac{g d}{d t} \dot{f}_0 \omega_n = \frac{g d}{d t} (\dot{0}_0 \omega_1 + \dot{1}_1 \omega_2 + \cdots + \dot{n-1}_n \omega_n) \\
= \sum_{i=1}^{n} \dot{f}_i \alpha_i + \dot{f}_i \omega_i \times \dot{f}_i \omega_i \quad (4.58)
\]

In this expression, it is shown that when a series of angular velocities are differentiated, additional terms \( \sum_{i=1}^{n} (\hat{n}_i \times \hat{n}_{i-1} \omega_i) \) must be included to take into account the convective rate of change of the relative angular velocities as seen from the observer’s frame (coordinate frame of choice).
Secondly, the nature and characteristics of the Razi acceleration is explained, specifically, the type of rotational motion in which the Razi acceleration comes into play and the direction of action of the acceleration. We distinguish the difference between tangential acceleration, which is contributed by the rate of magnitudinal change of rotation about a stationary axis, and the Razi acceleration, which is induced by the rate of directional change of rotation axis. In most cases, it is shown that the effect of the Razi acceleration is less than the effect of centripetal acceleration. However, it is important to consider the inertial force that is caused by the Razi acceleration in cases where engineering structures are subject to compound rotation motion.

Lastly, a unified description of derivative kinematics for systems with arbitrary number of relatively-rotating coordinate frames is presented. Shown in this chapter is a multi-frame system with all origin sharing the same point. Examples are given for referential systems with up to five coordinate frames. The general expression for centripetal, Coriolis, tangential, and Razi acceleration terms are also given for referential system with arbitrary number of coordinate frames.
Chapter 5

Experimental Investigation of Razi Acceleration I

This chapter is the first part of two chapters that present the study of the Razi acceleration in compound rotation motion by experiment. While the previous chapters discuss extensively the appearance of Razi acceleration from a mathematical perspective, this chapter is focused on presenting the experimental work in investigating of the effect of the Razi acceleration in a rigid body motion.

The purpose of this study is to support the theory of Razi acceleration by showing that the effect of the Razi acceleration is observable and measurable in a physical setting. To achieve this purpose, three strategies are devised:

1. Devise an experimental setup using multi-axis manipulator to create a physical simulation of a rigid body undergoing compound (nested) rotation motion.

2. Demonstrate how the multi-frame vector derivative transformation formula can be used to calculate the acceleration acting on a rigid body in motion and show the components of the acceleration.

3. Validate the results of the derivative kinematics technique with the acceleration data obtained from the experiment.

This chapter presents an overview of the experiment setup, an explanation of the coordinate frame system used, a velocity and acceleration analysis, and the acceleration and angular rate data obtained from the experiment. The results are discussed in the second part of this study, which is presented in the following chapter.
5.1 Experimental Investigation

To support the discovery of the Razi acceleration, we plan an investigation of Razi acceleration by experiment. We physically simulate the motion a rigid body undergoing a “nested” rotation motion in a laboratory environment and record the acceleration acting on the rigid body. The nested rotation can reproduced by using a multi-joint robotic manipulator. A sensor is attached at a point on the manipulator to record its acceleration – including the static acceleration (gravity) and the dynamic accelerations – and the angular rates. To validate the derivative transformation formula method and to show the appearance of the Razi acceleration, we compare the acceleration signals with the formulaic calculation (using the angular rates of the motion) to show that the Razi term describes the acceleration in the out-of-plane direction.

5.1.1 IRB 1400 Robotic Arm

At RMIT University, an industrial-grade six-axis robotic arm IRB 1400 is used as a motion simulator to replicate the simultaneous rotations about multiple axes. The motion of its end effector and individual external axes are programmable using its pre-equipped operating system BaseWare OS.

Apart from its motion control capabilities, the other reason a robotic manipulator is used is to ensure both the accuracy and the repeatability of motion. The simulator has to be able to follow a prescribed path with minimum vibration and positional error. Acceleration due to small vibration may produce unwanted noise in the acceleration readings. Since the motion and measurements will be repeated, precision is an important aspect to ensure each test run can be reproduced with the same point-to-point displacements, and velocities.

The IRB-1400 robotic manipulator consists of a floor-mounted base and six arms, connected by active revolute joints called axis (shown in Figure 5.1). Each axis provides one rotary degree-of-freedom. The sixth arm is the end-effector to which tools are attached. In this setup, the sensor is attached (gripped) by the end-effector. The type of motion and range of motion for each axis is shown in Table 5.1. The manipulator’s repeatability, accuracy and resolution of motion is presented in Table 5.2.

5.1.2 Inertial Measurement Unit

An inertial measurement unit (IMU) is attached to the end-effector of the robotic arm. The unit consists of a three-axis accelerometer, a three-axis gyroscope, and a three-axis
Figure 5.1: IRB 1400 with six movable axes. The type of motion of each axis is shown in Table 5.1

<table>
<thead>
<tr>
<th>Axis</th>
<th>Motion classification</th>
<th>Range of rotation</th>
<th>Max. speed</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Rotation</td>
<td>$-170^\circ$ to $+170^\circ$</td>
<td>$120^\circ/s$</td>
</tr>
<tr>
<td>2</td>
<td>Arm motion</td>
<td>$-100^\circ$ to $+20^\circ$</td>
<td>$120^\circ/s$</td>
</tr>
<tr>
<td>3</td>
<td>Arm motion</td>
<td>$-65^\circ$ to $+70^\circ$</td>
<td>$120^\circ/s$</td>
</tr>
<tr>
<td>4</td>
<td>Wrist motion</td>
<td>$-150^\circ$ to $+150^\circ$</td>
<td>$280^\circ/s$</td>
</tr>
<tr>
<td>5</td>
<td>Bend motion</td>
<td>$-115^\circ$ to $+115^\circ$</td>
<td>$280^\circ/s$</td>
</tr>
<tr>
<td>6</td>
<td>Turn motion</td>
<td>$-300^\circ$ to $+300^\circ$</td>
<td>$280^\circ/s$</td>
</tr>
</tbody>
</table>

Table 5.1: IRB 1400 range of motion [1].

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Unidirectional pose repeatability</td>
<td>0.05 mm</td>
</tr>
<tr>
<td>Linear path accuracy</td>
<td>0.45-1.0 mm</td>
</tr>
<tr>
<td>Linear path repeatability</td>
<td>0.14-0.25 mm</td>
</tr>
<tr>
<td>Resolution</td>
<td>$\sim 0.01$ deg</td>
</tr>
</tbody>
</table>

Table 5.2: IRB 1400 repeatability, accuracy and resolution [1].
magnetometer with a Bluetooth data-transfer capability. The accelerometer records the dynamic and static (gravity) acceleration in three orthogonal directions relative to the unit itself. The gyroscope is a separate sensor that measures the angular rates experienced by the unit under rotation.

The inertial sensor is called the x-IMU from x-io Technologies Limited (http://www.x-io.co.uk/). The sensor consists of a tri-axis accelerometer, a tri-axis gyroscope, and a tri-axis magnetometer. It has the capabilities to track translational and rotational motions, and orientation with respect to the direction of gravity. The magnetometer feature is particularly useful for this experiment as it provides the Earth-based reference frame which is used as the inertial coordinate frame in this experiment. Raw sensor data and instantaneous orientation values from the sensor are transmitted wirelessly via Class 1 Bluetooth with an effective transmitting/receiving range of 100 m. Sampling rate is set at 256 Hz for all experiments.

### 5.1.3 Experimental Setup

The point-of-interest (POI) is located at the end effector of the robotic manipulator. The sensor (IMU) is attached at the POI to record the acceleration and angular rate data. The sensor is made secure at the gripper at the end-effector. A small piece of foam was initially placed between the gripper and the sensor to absorb the small vibration due to the robot motion. Preliminary tests showed that the the foam did not add any significant reduction in vibration. The sensor’s x-axis is aligned parallel to Arm 1 and its z-axis is aligned parallel to Arm 2, as shown in Figure 5.4- 5.6.
To simulate the motion, only two out of six joint-axes are rotated: Axis 1 and Axis 4. The distance between the origin of Axis 1 and the POI is denoted by $r_1$. The distance of between the origin of Axis 4 and the POI is $r_2$, fixed at 90° with respect to Arm 1.
Figure 5.4: IRB 1400 Manipulator: The rotating axes are setup perpendicular to each other with three coordinate frames assigned to each rotating joint. The horizontal arm between the purple and blue coordinate frames is called Arm 1. The arm between the blue and red coordinate frames is called Arm 2.

Figure 5.5: IRB 1400 Manipulator: The sensor is placed at the end of Arm 2. The red coordinate frame indicates the axes of the sensor. Currently, its z-axis points to the ground.
Figure 5.6: IRB 1400 Manipulator: The side view shows the placement of the sensor at the end of the effector (red coordinate frame). The blue coordinate frame follows the orientation of the sensor coordinate frame at all times during simulation.
5.2 Kinematic Modeling

A multiple coordinate frames system is setup for the analysis of velocity and acceleration. Four coordinate frames are used – one inertial frame and three rotating coordinate frames. With the coordinate frames setup, the velocity and acceleration equations can be presented using the derivative kinematics presented in the preceding chapters. The acceleration equation presented here is decomposed into its inertial acceleration terms, including the Razi acceleration term. The acceleration data calculated from the acceleration equations will be compared with the acceleration data obtained from the experiment in the following chapter.

5.2.1 Coordinate Frame Assignment

The global-fixed coordinate frame $G(OXYZ)$ defines the inertial system with its $Z$–axis defines the zenith axis (normal to the Earth’s surface). The base coordinate frame $B_1(Ox_1y_1z_1)$ is fixed to the origin of Axis 4 with its $z_1$-axis aligned to the $Z$-axis, and its $x_1$-axis aligned and fixed to Arm 1. The body-fixed coordinate frame $B_2(O_2x_2y_2z_2)$ is attached to the end of Arm 1, with its $x_1$-axis parallel to Arm 1. Its $z_2$-axis coincides with Arm 2, making the $y_2z_2$-plane always perpendicular to Arm 1. The IMU is attached on the POI located on the $z_2$-axis with a distance $r = 0.2660m$ from the origin $O_2$. The acceleration and angular rate data is given in coordinate frame $B_3(O_3x_3y_3z_3)$, which is attached on the POI. Its $z_3$-axis always pointing towards $O_2$, and its $x_3$-axis parallel with Arm 14 ($x_1$-axis) and pointing away from $O_2$. Its $y_3$-axis completes the orthogonal triad. This coordinate frame system is summarized in Figure 5.7-5.9.
Figure 5.7: (Isometric view) The $B_1$-frame (purple) is attached at the origin of Arm 1, the $B_2$-frame (blue) is attached at the origin of Arm 2, and the $B_3$-frame (red) is attached at point-of-interest (where an IMU is attached). The inertial frame is not shown.
Figure 5.8: (Side view) The distance between the frame $B_1$ and $B_2$ is indicated by the length of Arm 1 $r_1 = 870\, \text{mm}$. And the distance between the frame $B_2$ and $B_3$ (IMU) is indicated by the length of Arm 2 $r_2 = 266\, \text{mm}$.
5.2.2 Vectors and Rotation Matrices

The simultaneous nested rotation is described as follows: Assuming the rotation rates of the arms are constant. The $B_2$-frame is rotating about the $x_2$-axis (parallel with the $x_1$-axis) with an angular velocity $\dot{\theta}$. The $B_1$-frame is rotating about the $z_1$-axis (parallel with the global $Z$-axis) with an angular velocity $\dot{\phi}$.

\[
\begin{align*}
{^2}_1\omega_2 &= \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix} \\
{^1}_G\omega_1 &= \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix}
\end{align*}
\]

(5.1)

With this configuration, the angle between the two angular velocity vectors, the nutation angle $\psi$, is constant at $90^\circ$. The distance from the $z_1$-axis of the $B_1$-frame to the POI is expressed in the $B_1$-frame as

\[
{^1}_r = \begin{bmatrix} 0.870 \\ 0 \\ 0 \end{bmatrix}
\]

(5.2)

The position of the POI in the $B_2$-frame is

\[
{^2}_r = \begin{bmatrix} 0 \\ 0 \\ 0.266 \end{bmatrix}
\]

(5.3)
Table 5.3: Angular velocities between the three coordinate frames – $G(OXYZ)$, $B_1(Ox_1y_1z_1)$, and $B_2(Ox_2y_2z_2)$ – and the radial distances of the point of interest (POI) from their respective origin points.

<table>
<thead>
<tr>
<th>Frame</th>
<th>Angular velocity $\omega$ (deg/s)</th>
<th>Radius $r$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G(OXYZ)$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$B_1(Ox_1y_1z_1)$</td>
<td>$\dot{\phi} \hat{k}_1$</td>
<td>0.870 $\hat{i}_1$</td>
</tr>
<tr>
<td>$B_2(Ox_2y_2z_2)$</td>
<td>$\dot{\theta} \hat{i}_2$</td>
<td>0.226 $\hat{k}_2$</td>
</tr>
</tbody>
</table>

The transformation between the frames are

$$2R_1 = R_{x_2,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$1R_G = R_{z_1,\phi} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(5.4)

where $2R_1$ represents the rotation from $B_1$ to $B_2$ and $1R_G$ represents the rotation from the inertial frame $G$ to $B_1$.

The summary for the kinematics characteristics of each frame is presented in Table 5.3.

Figure 5.10 gives a frame-by-frame motion capture of the nested rotation motion.

### 5.2.3 Velocity and Acceleration Analysis

In this configuration, the total number of coordinate frames used, excluding the inertial frame $G$, are three: the base coordinate frame $B_1$, the effector coordinate frame $B_2$, and the sensor coordinate frame $B_3$. However, since the sensor coordinate frame $B_3$ is not moving in $B_2$, the number of rotating noninertial coordinate frame $n$ can be reduced to 2.

To find the composition of angular velocities of the body-frame $B_2$ due to the two rotations, the angular velocity $\dot{\theta} \hat{i}_2$ needs to be rotated to the $B_2$-frame so that the addition operation between two angular velocities can be applied (Section 4.3.2). Thus, the total angular velocity can be expressed in the body-frame $B_2$ as

$$\frac{2}{\frac{2}{\dot{\phi}} \dot{\phi} \frac{2}{2} \omega_2 = 2R_1 \frac{1}{\frac{1}{\dot{\phi}}} \omega_1 + 2 \frac{1}{\frac{1}{\dot{\phi}}} \omega_2$$

(5.5)
Chapter 5. Experimental Investigation of Razi Acceleration

The velocity experienced by the point at $^2r$ due to the compound rotation about $z_1$-axis ($^1G_ω_1$) and $x_2$-axis ($^1G_ω_1$) is

$$\frac{^2Gv}{dt} = ^2Gω_1 \times (^2R_1^1r + ^2r)$$

$$= ^2Gω_1 \times ^2R_1^1r + ^2ω_2 \times ^2r$$

(5.6)

The acceleration is obtained by differentiating $^2Gv$ with respect to the $G$-frame again. Using the composition of angular acceleration in Equation (4.31), the first derivative of $^2Gω_2$ from the $G$-frame is

$$\frac{^Gd}{dt}^2Gω_2 = ^2Gα_1 + ^2Gω_2 \times ^2ω_2$$

$$= ^2Gω_1 \times ^2ω_2$$

(5.7)

as both angular velocities are constant in magnitude, both $^2Gα_1$ and $^1Gα_2$ are zero. Thus, the double derivative of $^2r$ can be written and expanded into its inertial acceleration components as

$$^GG^a = ^2Gω_1 \times (^2Gω_1 \times ^2R_1^1r) + ^2Gω_1 \times (^2ω_2 \times ^2r)$$

$$+ ^2ω_2 \times (^2ω_2 \times ^2r) + (^2Gω_1 \times ^2ω_2) \times ^2r$$

(5.8)

The first term in Equation (5.8), $^2Gω_1 \times (^2Gω_1 \times ^2R_1^1r)$, is the centripetal acceleration due to the rotation of Axis 1, and the third term, $^2ω_2 \times (^2ω_2 \times ^2r)$, is the centripetal acceleration due to the rotation of Axis 4. The second term, $^2Gω_1 \times (^2ω_2 \times ^2r)$ is the mixed centripetal acceleration. And the last term $(^2Gω_1 \times ^2ω_2) \times ^2r$ is the Razi acceleration.

The IMU records the acceleration and angular velocity in the three axes of the $B_2$-frame. We denote the recorded acceleration as $\vec{a}$ and the angular velocity as $\vec{ω}$

$$\vec{a} = \begin{bmatrix} \ddot{a}_x \\ \ddot{a}_y \\ \ddot{a}_z \end{bmatrix}, \quad \vec{ω} = \begin{bmatrix} \dot{ω}_x \\ \dot{ω}_y \\ \dot{ω}_z \end{bmatrix}$$

(5.9)

The Razi acceleration, as one of the components of inertial acceleration induced by rotation motion, should also be recorded by the IMU. From previous discussions in Chapter 3 and 4, the direction of the Razi acceleration in nested rotation motion is normal to the “innermost” plane of rotation. With the assigned coordinate frame system, the Razi acceleration in Equation (5.8) can be shown to appear in the sensor’s $x$-axis, which corresponds to the $x_3$-axis.
Figure 5.10: The nested rotation motion captured frame-by-frame (1-15). Axis 1 (the mounting base) is rotated from left to right, and simultaneously, Axis 4 is rotated in anti-clockwise direction.
5.3 Results

Three different rotation maneuvers are done: A single rotation about Axis 1, a single rotation about Axis 4, and a nested rotations motion with Axis 1 and Axis 4 rotating simultaneously. The two single rotation tests results are grouped under the name Test 1; Test 1(a) is for rotation about Axis 1 and Test 1(b) is for rotating about Axis 4. The nested rotations tests are grouped under Test 2. For the nested rotations testing, three different angular speed combinations are used. The results are presented into three sections: Test 2(a), Test 2(b), and Test 2(c). For each nested rotation experiment, the motion is repeated 60-80 times to get reliable and significant results.

Note: The acceleration and angular rate data are collected in the $B_3$-frame. Since the directions of the frames $B_2$ and $B_3$ are parallel at all times, both the acceleration and angular rates in $B_3$ can be expressed in $B_2$ without applying any rotation procedure.

$$3\vec{a} = 2\vec{a} \quad 3\vec{\omega} = 2\vec{\omega}$$

5.3.1 Test 1(a): Single Rotation about Axis 1

Axis 1 is rotated from its initial position at 0 degrees to 164.6 degrees. The angles of Joint 3, Joint 4 and Joint 5 are fixed at 0 degree, -145 degrees and 90 degrees, respectively. Arm 1 is parallel to the ground and Arm 2 is perpendicular to Arm 1.

The acceleration data for this test are shown in Figure 5.11 to 5.12. The angular rates data are shown in Figure 5.13 to 5.14. The approximated value is summarized in Table 6.1. The rotation rates for each axis, which are obtained from the angular rate signals, are given in Table 5.5. These constant rotation rates will be used to calculate the theoretical acceleration by exerted by this rotation maneuver.
Figure 5.11: Acceleration signal for a single rotation about an axis in the y-z plane. A centrifugal acceleration reading is shown in the x-axis (red). The big "spikes" in the signal indicate jerk (the first time-derivative of acceleration), which comes from the sudden start-stop motion of the manipulator.

Figure 5.12: Acceleration signals (truncated at between 30.0 and 33.5 seconds) are separated in three directions: x-, y-, and z-axis. Neglecting the surge in acceleration due to jerk motion at the beginning and the end of the maneuver, an acceleration with a magnitude of approximately 3.86 m/s² is detected in the x-axis.
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Figure 5.13: Gyroscope signal for a single rotation about Axis 1. Maneuver started at around 31 seconds and lasted about 1.5 seconds. Angular velocity reading in y-axis (green) and z-axis (blue) shows that Axis 1 lies in the y-z plane of the body-frame \((B_2)\).

Figure 5.14: Gyroscope signals (truncated at between 30.0 and 33.5 seconds) are separated in three directions: x-, y-, and z-axis. Constant angular speeds of approximately 
-68.31 °/s and -98.64 ° are recorded in y- and z-axis respectively.

To get the rotation rate \(\dot{\phi}\) about Axis 1, the gyroscope reading, which is taken in the \(B_2\)-frame, needs to be rotated to the \(B_1\)-frame. The rotation matrix \(^1R_2\) is given (by the IMU output) as

\[
^1R_2 = \begin{bmatrix}
  0.4100 & 0.5229 & -0.7473 \\
  0.8107 & -0.5844 & 0.0358 \\
 -0.4180 & -0.6205 & -0.6635 
\end{bmatrix}
\]
Angular Velocity (°/s)  Acceleration (m/s²)

<table>
<thead>
<tr>
<th>Axis</th>
<th>$\vec{ω}_1$</th>
<th>$\vec{a}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>1.327</td>
<td>-3.855</td>
</tr>
<tr>
<td>$y$</td>
<td>-68.31</td>
<td>0</td>
</tr>
<tr>
<td>$z$</td>
<td>-98.64</td>
<td>0.4429</td>
</tr>
</tbody>
</table>

Table 5.4: Summary of the angular rate and acceleration data obtained from the sensor for the Single Rotation (Axis 1) test. The signal is approximated using averaging method to remove noise and get a constant reading.

<table>
<thead>
<tr>
<th>Robot Axis</th>
<th>Coordinate Frame Axis</th>
<th>Estimated Angular Velocity</th>
<th>Initial Position</th>
<th>Final Position</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$z_1$-axis ($B_1$)</td>
<td>$\dot{φ} = 120.0^\circ/s$</td>
<td>0°</td>
<td>164.6°</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>0°/s</td>
<td>0°</td>
<td>0°</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>0°/s</td>
<td>0°</td>
<td>0°</td>
</tr>
<tr>
<td>4</td>
<td>$x_2$-axis ($B_2$)</td>
<td>$\dot{θ} = 0^\circ/s$</td>
<td>-145°</td>
<td>-145°</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>0°/s</td>
<td>90°</td>
<td>90°</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>0°/s</td>
<td>0°</td>
<td>0°</td>
</tr>
</tbody>
</table>

Table 5.5: Angular rates in terms of $\dot{θ}$ and $\dot{φ}$, and the range of angular motion of each axis for Single Rotation (Axis 1) test.
5.3.2 Test 1(b): Single Rotation about Axis 4

Axis 4 is rotated from its initial position at -145 degrees to 145 degrees. The total rotation is from its initial to its final position is 290 degrees. The angular speed is recorded at approximately 280 degrees/second about the local x2-axis. The angles of Joint 1 and Joint 3 are fixed at 0 degree, and Joint 5 is 90 degrees, respectively. Arm 1 is always parallel to the ground and the rotation axis is parallel to Arm 1.

The acceleration data for this test are shown in Figure 5.15 to 5.16. The angular rates data are shown in Figure 5.17 to 5.18. The approximated value is summarized in Table 6.2. The rotation rates for each axis, which are obtained from the angular rate signals, are given in Table 5.7.

The rotation rate \( \dot{\theta} \) about Axis 4 can be directly obtained from the gyroscope reading in the x-axis. We assume the y- and z-components of the angular rate are negligible as they are small as compared to the x-component. These small readings are due to the imperfect orientation of the sensor on the end-effector. Therefore, we take the total magnitude of the angular velocity and assume it is directed perfectly along the x-axis (shown in Table 5.7).

<table>
<thead>
<tr>
<th>Axis</th>
<th>( \dot{\omega}_1 ) (°/s)</th>
<th>( \ddot{a} ) (m/s²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>280.3</td>
<td>0.3323</td>
</tr>
<tr>
<td>y</td>
<td>8.247</td>
<td>0.6253</td>
</tr>
<tr>
<td>z</td>
<td>5.316</td>
<td>6.253</td>
</tr>
</tbody>
</table>

Table 5.6: Summary of the angular rate and acceleration data obtained from the sensor for the Single Rotation (Axis 4) test. The signal is approximated using averaging method to remove noise and get a constant reading.
Chapter 5. Experimental Investigation of Razi Acceleration

Figure 5.15: Acceleration signal for a single rotation about the x-axis. A dominant acceleration reading is shown in the z-axis (blue).

Figure 5.16: Acceleration signals (truncated at between 24.0 and 27.5 seconds) are separated in three directions: x-, y-, and z-axis. Neglecting the surge in acceleration due to jerk motion at the beginning and the end of the maneuver, an acceleration with a magnitude of approximately 6.25 m/s^2 is detected in the z-axis.
Figure 5.17: Gyroscope signal for a single rotation about Axis 4. Maneuver started at around 25.2 second and lasted about 1 second. An angular speed of 280.3 °/s is recorded in x-axis. Small readings in y- and z-axis can be attributed to mounting orientation error.

Figure 5.18: Gyroscope signals (truncated at between 24.0 and 27.5 seconds) are separated in three directions: x-, y-, and z-axis. The angular speed about the x-axis is approximated at 280.3 °/s. Small angular speeds in the other axes are due to inexact mounting orientation.
<table>
<thead>
<tr>
<th>Robot Axis</th>
<th>Coordinate Frame Axis</th>
<th>Estimated Angular Velocity</th>
<th>Initial Position</th>
<th>Final Position</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$z_1$-axis ($B_1$)</td>
<td>$\dot{\phi} = 0^\circ/s$</td>
<td>$0^\circ$</td>
<td>$0^\circ$</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>$0^\circ/s$</td>
<td>$0^\circ$</td>
<td>$0^\circ$</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>$0^\circ/s$</td>
<td>$0^\circ$</td>
<td>$0^\circ$</td>
</tr>
<tr>
<td>4</td>
<td>$x_2$-axis ($B_2$)</td>
<td>$\dot{\theta} = 280.4^\circ/s$</td>
<td>$145^\circ$</td>
<td>$-145^\circ$</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>$0^\circ/s$</td>
<td>$90^\circ$</td>
<td>$90^\circ$</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>$0^\circ/s$</td>
<td>$0^\circ$</td>
<td>$0^\circ$</td>
</tr>
</tbody>
</table>

Table 5.7: Angular rates in terms of $\dot{\theta}$ and $\dot{\phi}$, and the range of angular motion of each axis for Single Rotation (Axis 4) test.
5.3.3 Nested Rotation Motion: Overview

For the nested rotation maneuver, Axis 1 and Axis 4 are rotated simultaneously. Axis 1 is rotated from 0 to 164.6 degrees and Axis 4 is rotated from -145 to 145 degrees. Joint 3 and Joint 5 are fixed at 0 degree and 90 degrees, respectively. The recorded angular speed for Axis 1 is 203.2 degrees/second and the angular speed for Axis 4 is at 124.5 degrees/second. Arm 1 is kept parallel to the ground and the angle between Arm 1 and Arm 2, nutation angle, is at 90 degrees.

Three different tests with varying angular velocities are conducted.

1. **Test 2(a):** Both axes rotate in full speed and undergo the full range of motion.

2. **Test 2(b):** Axis 1 rotates in full speed and Axis 4 rotates in a reduced speed. To achieve this, the range of motion of Axis 4 is reduced to -145 to 0 degrees, while the range of motion of Axis 1 is maintained at 0 to 164.6 degrees.

3. **Test 2(c):** Axis 4 rotates in full speed and Axis 1 rotates in a reduced speed. To achieve this, the range of motion of Axis 1 is reduced to 0 to 90 degrees, while the range of motion of Axis 4 is maintained at -145 to 145 degrees.

The rotation motion is repeated back and forth to get more readings. For example, for Test 2(a), Axis 1 is rotated from 0 to 164.6 degrees, and back to 0 degree again, with Axis 4 doing the same within its range. This can be seen in the gyroscope signal plots.

The gyroscope only provides the angular rates according to the sensor’s coordinate frame. To get the angular speeds of Axis 1 ($\dot{\phi}^k_1$) and Axis 4 ($\dot{\theta}^k_2$), a coordinate transformation procedure has to be applied. The angular velocity of Axis 4, $\dot{\theta}^k_2$, can be straightforwardly obtained from the $x$-component of the gyroscope signal. This means that the components of $\dot{\phi}^G_1$ lies in the $y$- and $z$-components. Therefore, we can find the magnitude of $\dot{\phi}$ by $\sqrt{\omega_y^2 + \omega_z^2}$.

5.3.4 Test 2(a): Nested Rotation Motion ($\dot{\phi}^G_1 = 128^\circ/s$ $\dot{\theta}^G_2 = 210^\circ/s$)  

The nested rotation is simulated by rotating Axis 1 at full speed and Axis 4 at full speed simultaneously. Figure 5.19 shows significant changes in acceleration in the $x$- and $z$-components every time the axes are rotated. Figure 5.20 shows the angular rates for all three axes of the sensor, whereas Figure 5.21 shows the angular rate of the $B_2$-frame with respect to the $B_1$-frame (top), and the angular rate of the $B_1$-frame with respect to the initial frame $G$ (bottom). The approximated value is summarized in Table 5.8. The acceleration and gyroscope reading for one cycle of motion is shown in Figure 5.22. The
acceleration signals will be compared with the computed acceleration using the angular velocity readings and the derivative kinematics formula in Section 6.1.2 Figure 6.1.

**Figure 5.19:** Acceleration signals showing five repeated motion in a period between 34 and 49 seconds. The x- and z-components are showing significant changes every time the sensor is rotated. The y-component is mainly showing jerk signals and small vibration.

**Figure 5.20:** Gyroscope signals showing the angular rates for the motion of the same period. Rotation is repeated back and forth, hence giving positive and negative values for the x- and z-components.
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Figure 5.21: Gyroscope signals showing the angular rates $\dot{\bar{\omega}}_2$ (from gyroscope’s $x$-component $\bar{\omega}_x$) and $\dot{\bar{\omega}}_1$. The latter is obtained from $\sqrt{\omega_y^2 + \omega_z^2}$, which is the magnitude of angular velocity in the $yz$-plane.

<table>
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<tr>
<th>Robot Axis</th>
<th>Coordinate Frame Axis</th>
<th>Estimated Angular Velocity</th>
<th>Initial Position</th>
<th>Final Position</th>
</tr>
</thead>
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<td>$0^\circ$</td>
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</table>

Table 5.8: Angular rates in terms of $\dot{\theta}$ and $\dot{\phi}$, and the range of angular motion of each axis for the first nested rotation test.
Figure 5.22: Acceleration signal in all three axes (left, red) as the result of angular velocities \( \omega_2 \) and \( \omega_1 \) (right, blue). The rotation range is from \( 0^\circ \) to \( 164.6^\circ \) for Axis 1, and \( -145^\circ \) to \( 145^\circ \) for Axis 4.
5.3.5 Test 2(b): Nested Rotation Motion ($\omega_1 = 95^\circ/s$, $\omega_2 = 280^\circ/s$)

This nested rotation is simulated by rotating Axis 1 at a higher speed and Axis 4 at a lower speed. Figure 5.23 shows significant changes in acceleration in the $x$- and $z$-components every time the axes are rotated. Figure 5.24 shows the angular rates for all three axes of the sensor, whereas Figure 5.25 shows the angular rate of the $B_2$-frame with respect to the $B_1$-frame (top), and the angular rate of the $B_1$-frame with respect to the initial frame $G$ (bottom). The approximated value is summarized in Table 5.9. The acceleration and gyroscope reading for one cycle of motion is shown in Figure 5.26. The acceleration signals will be compared with the computed acceleration using the angular velocity readings and the derivative kinematics formula in Section 6.1.2 Figure 6.2.

![Figure 5.23: Acceleration signals showing five repeated motion in a period between 251 and 264 seconds. The $x$- and $z$-components are showing significant changes every time the sensor is rotated. The $y$-component is mainly showing jerk signals and small vibration.](image-url)
1.1 seconds

Figure 5.24: Gyroscope signals showing the angular rates for the motion of the same period.

Figure 5.25: Gyroscope signals showing the angular rates $\frac{2}{G} \dot{\omega}_2$ (from gyroscope’s $x$-component $\dot{\omega}_x$) and $\frac{1}{G} \dot{\omega}_1$. 
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<th>Coordinate Frame Axis</th>
<th>Estimated Angular Velocity</th>
<th>Initial Position</th>
<th>Final Position</th>
</tr>
</thead>
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</tr>
<tr>
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<td>0°</td>
<td>0°</td>
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<tr>
<td>3</td>
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<td>0°</td>
<td>0°</td>
</tr>
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Table 5.9: Angular rates in terms of $\dot{\theta}$ and $\dot{\phi}$, and the range of angular motion of each axis for the second nested rotation test.
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Figure 5.26: Acceleration signal in all three axes (left, red) as the result of angular velocities $\omega_2$ and $\omega_1$ (right, blue). The rotation range is from $0^\circ$ to $164.6^\circ$ for Axis 1 and $-145^\circ$ to $0^\circ$ for Axis 4.
5.3.6 Test 2(c): Nested Rotation Motion ($\omega_1 = 124^\circ/s$, $\omega_2 = 100^\circ/s$)

This nested rotation is simulated by rotating Axis 1 at a lower speed and Axis 4 at a full speed. Figure 5.27 shows significant changes in acceleration in the $x$- and $z$-components every time the axes are rotated. Figure 5.28 shows the angular rates for all three axes of the sensor, whereas Figure 5.29 shows the angular rate of the $B_2$-frame with respect to the $B_1$-frame (top), and the angular rate of the $B_1$-frame with respect to the initial frame $G$ (bottom). The approximated value is summarized in Table 5.10. The acceleration and gyroscope reading for one cycle of motion is shown in Figure 5.30. The acceleration signals will be compared with the computed acceleration using the angular velocity readings and the derivative kinematics formula in Section 6.1.2 Figure 6.3.

Figure 5.27: Acceleration signals showing five repeated motion in a period between 36 and 51 seconds.
Chapter 5. Experimental Investigation of Razi Acceleration

1.1 seconds

Figure 5.28: Gyroscope signals showing the angular rates for the motion of the same period.

Figure 5.29: Gyroscope signals showing the angular rates $\dot{\omega}_2$ (from gyroscope’s $x$-component $\dot{\omega}_x$) and $\dot{\omega}_1$.
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<th>Robot Axis</th>
<th>Coordinate Frame Axis</th>
<th>Estimated Angular Velocity</th>
<th>Initial Position</th>
<th>Final Position</th>
</tr>
</thead>
<tbody>
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<td>$90^\circ$</td>
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<tr>
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<td>$0^\circ$</td>
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Table 5.10: Angular rates in terms of $\dot{\theta}$ and $\dot{\phi}$, and the range of angular motion of each arm/axis for the third nested rotation test.
Figure 5.30: Acceleration signal in all three axes (left, red) as the result of angular velocities $\omega_2$ and $\omega_1$ (right, blue). The rotation range is from $0^\circ$ to $90^\circ$ for Axis 1, and $-145^\circ$ to $145^\circ$ for Axis 4.
5.4 Conclusion

Three important materials are presented in this chapter: (1) The experimental setup with the description of the simulator (IRB 1400 robotic manipulator) and the inertial measurement unit (x-IMU); (2) the kinematic modeling for the setup, including the multiple coordinate frames system used, and the analysis of velocity and acceleration. The acceleration equation, which is written as a function of displacement and angular velocity vectors, will be used to compare with the experimental results in the following chapter; and (3) the presentation of the experimental data obtained from the experiment.

The Result section is organized this way:

- **Chapter 6.4.1** provides the results for Test 1(a): Single Rotation about Axis 1.
  1. Acceleration: Figure 5.11 and Figure 5.12 (closeup view)
  2. Angular velocity: Figure 5.13 and Figure 5.14 (closeup view)
  3. Approximated constant acceleration and angular rate values: Table 6.1
  4. Angular rates for each Axis: Table 5.5

- **Chapter 6.4.2** provides the results for Test 1(b): Single Rotation about Axis 4.
  1. Acceleration: Figure 5.15 and Figure 5.16 (closeup view)
  2. Angular velocity: Figure 5.17 and Figure 5.18 (closeup view)
  3. Approximated constant acceleration and angular rate values: Table 6.2
  4. Angular rates for each Axis: Table 5.7

- **Chapter 6.4.3** provides the results for Test 2: Nested Rotation Motion.
  **Test 2(a)**
  1. Acceleration: Figure 5.19
  2. Angular velocity in sensor’s frame: Figure 5.20
  3. Angular rates for Axis 1 and Axis 4: Figure 5.21 and Table 5.8
  4. Characteristic acceleration and angular velocity response plot: Figure 5.22
  **Test 2(b)**
  1. Acceleration: Figure 5.23
  2. Angular velocity in sensor’s frame: Figure 5.24
  3. Angular rates for Axis 1 and Axis 4: Figure 5.25 and Table 5.9
  4. Characteristic acceleration and angular velocity response plot: Figure 5.26
Test 2(c)

1. Acceleration: Figure 5.27
2. Angular velocity in sensor’s frame: Figure 5.28
3. Angular rates for Axis 1 and Axis 4: Figure 5.29 and Table 5.10
4. Characteristic acceleration and angular velocity response plot: Figure 5.30
Chapter 6

Experimental Investigation of Razi Acceleration II: Discussion

This chapter is the second part of two chapters on the experimental study of the Razi acceleration. As the preceding chapter presents the experimental setup, kinematic modeling, and the experiment results, this chapter provides an in-depth analysis by comparing the observed sensor output signals with the theoretical value obtained from the acceleration expression in Equation (5.8).

6.1 Comparison of Experimental and Theoretical Data

The equations for inertial acceleration are a function of angular velocity and position vectors. To calculate the acceleration, the angular rates data from the gyroscope are used. The position vectors of $^1r$ and $^2r$ are measured to be 0.870 m and 0.266 m, respectively. Note: Symbols with a bar accent indicate values obtained from experiment.

6.1.1 Test 1: Single Rotation About an Axis

For a single rotation about Axis 1, the acceleration of the POI is written as

$$\overset{\text{G}}{a} = \overset{\text{G}}{\omega}_1 \times (\overset{\text{G}}{\omega}_1 \times ^1r) \quad (6.1)$$

which indicates a centripetal acceleration. Since $\overset{\text{G}}{\omega}_1$ is directed towards $z_1$-axis, and $^1r$ is located along the $x_1$-axis, the resulting centripetal acceleration should also be along the $x_1$-axis. This is apparent in the acceleration plot in Figure 5.12. The result summarized in Table 6.1.
Table 6.1: Test 1(a): Comparison of acceleration values for a single rotation motion about Axis 1.

For the second single rotation test, the acceleration exerted by rotation about Axis 4 is expressed as
\[
G^2 a = \hat{2}^2 \omega_2 \times (\hat{1}^1 \omega_1 \times \hat{2}^2 r)
\] (6.2)

The centripetal acceleration is expected to act in the direction of $z_2$-axis. This is also apparent in the acceleration plot in Figure 5.16. The result summarized in Table 6.2.

Table 6.2: Test 1(b): Comparison of acceleration values for a single rotation motion about Axis 4.

In both experiments, the small acceleration that appear in the other axes can be attributed to the small orientation error in placing the sensor. Even with this small error, the calculated acceleration predicts the accelerometer output very well.

Although these two early testing seems pedestrian and straightforward, it is essential in demonstrating the basic mechanics of this experimental study by providing simple, easily understood examples. It also shows that inertial acceleration can be detected and calculated.

### 6.1.2 Test 2: Nested Rotation Motion

Equation 5.8 is treated as acceleration as a function of angular velocities and the displacement vectors.
\[
a = f(\hat{1}^1 \omega_1, \hat{2}^2 \omega_2, \hat{1}^1 r, \hat{2}^2 r)
\] (6.3)
The values of the displacement vectors are given in Table 5.3 and the angular rates data from the gyroscope are taken as the input for the angular velocity vectors. This is done due to the inconsistence of the angular rate in $x$, $y$, and $z$-component, unlike the first two testings which are done with a constant angular velocity about one axis. To do this, a coordinate transformation procedure is applied to the gyroscope readings $\tilde{\omega}$ to get both $\frac{1}{G}\tilde{\omega}_1$ and $\frac{1}{L}\tilde{\omega}_2$. The results of this transformation are shown in Figure 5.21, Figure 5.25, and Figure 5.29 in the previous chapter. These two vectors then are used as inputs to the acceleration function Equation (6.3).

Both experimental (red lines) and theoretical values (black lines) for acceleration are plotted and compared in Figure 6.1 for Test 2(a), Figure 6.2 for Test 2(b), and Figure 6.3 for Test 2(c). Samples of fifteen repeated maneuvers are taken from three different time-frames. The reason this is done is to show the typical acceleration response shapes obtained from the maneuvers.

As the gravity acceleration (static acceleration) has been filtered out from the accelerometer readings, the zero flat line in between the curves indicates that the both axes has stopped moving. Only the dynamic acceleration, i.e. acceleration caused by motion, is recorded. The accelerometer signals show jerk peaks at the start and the end of every maneuver. The reason this is not predicted by the formula is because the gyroscope did not detect any drastic change of angular acceleration during the maneuver. It is understandable that the peaks appear due to the sudden start-and-stop motion of the manipulator. Furthermore, the jerk analysis is not our concern in this study. The $y$-component for all test does not show any significant reading but small vibration from the interaction of the robotic manipulator arm and the attached sensor. For all experiments, the calculated values show a very good agreement with the accelerometer output. This proves that the general expression for acceleration in Equation (5.8) can predict the acceleration acting on a rigid body in compound rotation motion.
Chapter 6. *Experimental Investigation of Razi Acceleration II: Discussion*

Figure 6.1: Test 2(a): Nested Rotation Motion ($\frac{1}{1} \omega_1 = 128^\circ/s$; $\frac{1}{2} \omega_2 = 210^\circ/s$). Both experimental (red) and theoretical (black) values are plotted.
Chapter 6. *Experimental Investigation of Razi Acceleration II: Discussion*

(A) Five repeated nested rotation motion from $t = 61\ s$ to $t = 74\ s$.

(B) Five repeated nested rotation motion from $t = 100\ s$ to $t = 113\ s$.

(C) Five repeated nested rotation motion from $t = 251\ s$ to $t = 254\ s$.

**Figure 6.2:** Test 2(b): Nested Rotation Motion ($\omega_1 = 95^\circ/s$ ? $\omega_2 = 280^\circ/s$). Both experimental (red) and theoretical (black) values are plotted.
(A) Five repeated nested rotation motion from \( t = 36 \) s to \( t = 51 \) s.

(b) Five repeated nested rotation motion from \( t = 74 \) s to \( t = 89 \) s.

(c) Five repeated nested rotation motion from \( t = 186 \) s to \( t = 201 \) s.

Figure 6.3: Test 2(c): Nested Rotation Motion \((\theta_1 \omega_1 = 124^\circ/s \quad \theta_2 \omega_2 = 100^\circ/s)\). Both experimental (red) and theoretical (black) values are plotted.
6.2 The Razi Acceleration Appearance in Experimental Data

It was predicted that the Razi acceleration will appear in the out-of-plane direction of the “innermost” rotation. Therefore, in this case, the Razi acceleration is expected to show in the $x$-component of the accelerometer.

To check the influence of the Razi acceleration, the term $(\frac{2}{3} \omega_1 \times \frac{2}{3} \omega_2) \times r$ is intentionally left out in Equation (5.8). For all three tests involving nested rotation motion, these two equations are compared side-by-side. Only the $x$-component of acceleration are shown.

**Figure 6.4:** Test 2(a): $\dot{\phi} = 128^\circ/s$ and $\dot{\theta} = 210^\circ/s$

**Figure 6.5:** Test 2(b): $\dot{\phi} = 95^\circ/s$ and $\dot{\theta} = 280^\circ/s$
Figures 6.4 to 6.6 show the theoretical predictions (black line) of the total acceleration acting on a body in nested rotations for Test 2(a), Test 2(b), and Test 2(c), respectively. The plots on the left compare the acceleration calculation without incorporating the Razi acceleration, and the plots on the right side shows how close the predictions are when the Razi acceleration is taken into account. It is apparent that the omission of the Razi term here has shown a significant error in predicting the acceleration caused by multiple rotations. Although the shape of the acceleration response is maintained, there is about $2 \text{m/s}^2$ difference in predicting the maximum magnitude in Test 2(a). When the precession rate $\dot{\phi}$ is reduced 25.8% and the body spin rate is increased 33.3%, the difference in the maximum acceleration is still around $1.5 \text{m/s}^2$. This error can be considered huge since the maximum acceleration in the $x$-direction is around $8 \text{m/s}^2$ and $6 \text{m/s}^2$ for Test 2(a) and Test 2(b) respectively (roughly 25% difference).

It is stressed here that the numbers and percentages of error given are not typical for all cases. The magnitude and sensitivity of the Razi acceleration, as with centripetal and Coriolis acceleration, depends heavily on the parameters involved, such as the two angular velocities, the angle between the angular velocity vectors, and the radii of rotation.

With the limitation of the IRB 1400 manipulator, the rotations produced by the axes are not full, continuous rotations. For Test 2(a), Axis 1 is rotated from 0 to 164.4 degrees, and Axis 4 is rotated, in total, 290 degrees. For Test 2(b), the rotation distance of Axis 4 is 145 degrees. Even with this limitation, the setup still managed to get the instance where the out-of-plane acceleration is maximum. However, for Test 2(c), Axis 1 is only rotated up to 90 degrees, and consequently, it is unsure that the maximum
$x$-acceleration is reached within the range of rotation. This is noticeable in the shape of the acceleration response from Test 2(c). In general, it can be observed that the Razi acceleration term improves the accuracy of acceleration prediction, but it is uncertain whether the error difference in maximum acceleration is still around the 25% mark as noted in the other two cases.
6.2.1 Inertial Acceleration Terms

Equation (5.8) shows three different types of inertial acceleration:

1. centripetal acceleration, which are expressed $\vec{a} = \hat{\vec{c}} \times (\hat{\vec{c}} \times \vec{r})$ (about Axis 1) and $\vec{a}_2 = \hat{\vec{c}} \times (\hat{\vec{c}} \times \vec{r})$ (about Axis 4).

2. mixed centripetal acceleration, $\vec{a}_m = \hat{\vec{c}} \times (\hat{\vec{c}} \times \vec{r})$

3. Razi acceleration, $(\hat{\vec{c}} \times \hat{\vec{c}}) \times \vec{r}$

In the sensor’s frame, its $x$-axis is directed normal (out-of-plane) to Arm 2’s rotation plane. Its $y$- and $z$-axes lie in the rotation plane, with $\vec{r}$ lies in the $z$-axis. The centripetal acceleration corresponding to Axis 1 is directed in the $x_1$-axis, which is coincident with the sensor’s $x$-axis. The second centripetal term is directed towards the sensor’s $z$-axis. Both the mixed centripetal acceleration and the Razi acceleration are in the direction of the sensor’s $x$-axis.

To see the effects of these inertial accelerations, one can observe the $x$- and $z$-components of the accelerometer. It is fairly straightforward for the vector $\vec{a}_m = \hat{\vec{c}} \times (\hat{\vec{c}} \times \vec{r})$ because it is the only acceleration acting along the $z$-axis. It is a bit tricky for the components of the $x$-acceleration as it is influenced by the centripetal $\vec{a}_1 = \hat{\vec{c}} \times (\hat{\vec{c}} \times \vec{r})$, the mixed centripetal $\vec{a}_m = \hat{\vec{c}} \times (\hat{\vec{c}} \times \vec{r})$, and the Razi $(\hat{\vec{c}} \times \hat{\vec{c}}) \times \vec{r}$.

To show the contribution of these terms we plot these terms individually in Figure 6.7 to 6.12. The red lines corresponds to the centripetal acceleration, the blue lines refers to the mixed centripetal acceleration, and the green line refers to the Razi acceleration.

The centripetal acceleration is observed in the $x$-acceleration and $z$-acceleration. The red line in the $x$-acceleration refers to the centripetal term $\vec{a}_1 = \hat{\vec{c}} \times (\hat{\vec{c}} \times \vec{r})$ (from $\dot{\phi}$, the rotation of Axis 1), whereas the red line in the $z$-acceleration refers to the centripetal term $\vec{a}_2 = \hat{\vec{c}} \times (\hat{\vec{c}} \times \vec{r})$ (from $\dot{\theta}$, rotation of Axis 4). The other acceleration terms appear strictly in the $x$-acceleration only. Tangential acceleration, if it existed, would appear in the $y$-direction. However, the accelerometer indicates that there is no tangentially-acting acceleration for all three nested rotation motion testing. This is because the angular velocities $\dot{\phi}$ and $\dot{\theta}$ are constant in their respective frames. In other words, the terms that contribute to $\alpha \times \vec{r}$, $\ddot{\phi}$ and $\ddot{\theta}$, are zero.

Time-histories of the three inertial acceleration terms are shown in Figure 6.7, Figure 6.9 and Figure 6.11. Figure 6.8, Figure 6.10 and Figure 6.12 show the ‘representative’ response shapes of the acceleration signals.
Figure 6.7: Test 2(a): \((1\omega_1 = 128^\circ/s, 2\omega_2 = 210^\circ/s)\). Time histories for centripetal (red), mixed centripetal (blue), and Razi (green) accelerations in three axes.

Figure 6.8: Inertial acceleration appearance in the three axes. Centripetal (red), mixed centripetal (blue), and Razi (green).
Figure 6.9: Test 2(b): $\dot{\phi} = 95^\circ/s$ and $\dot{\theta} = 280^\circ/s$. Time histories for centripetal (red), mixed centripetal (blue), and Razi (green) accelerations in three axes.

Figure 6.10: Inertial acceleration appearance in the three axes. Centripetal (red), mixed centripetal (blue), and Razi (green).
Figure 6.11: Test 2(c): $\dot{\phi} = 124^\circ/s$ and $\dot{\theta} = 100^\circ/s$. Time histories for centripetal (red), mixed centripetal (blue), and Razi (green) accelerations in three axes.

Figure 6.12: Inertial acceleration appearance in the three axes. Centripetal (red), mixed centripetal (blue), and Razi (green).
6.2.2 The Razi Acceleration Appearance in the Out-of-plane Direction

The acceleration component in the out-of-plane (body rotation plane) direction is of interest here. Comparing the centripetal and mixed centripetal acceleration with the Razi acceleration, it is observed that the Razi acceleration has a small but significant effect. This is shown by the green line in Figures 6.13 to 6.15\(^1\). The red line represents the other acceleration terms in the \(x\)-direction, and the blue line represents the total \(x\)-acceleration. For Test 2(a), the peak of the Razi acceleration is recorded at 2.16 \(m/s^2\) (\(\approx 0.22 \text{g}\)), compared to the total magnitude in the \(x\)-direction 8.69 \(m/s^2\) (\(\approx 0.89 \text{g}\)). For Test 2(b), the Razi peak is recorded at 2.15 \(m/s^2\) (\(\approx 0.22 \text{g}\)), compared to the total magnitude in the \(x\)-direction 6.84 \(m/s^2\) (\(\approx 0.70 \text{g}\)). It is uncertain whether the maximum Razi acceleration shown in Figure 6.15 in Test 2(c) is the actual maximum obtainable Razi value from the nested rotation configuration. However, out of the maximum \(x\)-acceleration value of 6.05 \(m/s^2\) (\(\approx 0.62 \text{g}\)), 1.02 \(m/s^2\) (\(\approx 0.10 \text{g}\)) is contributed by the Razi acceleration.

From these nested rotation configuration, the total acceleration in the \(x\)-direction has about 15–30% Razi acceleration. Again, this numbers are specific for the nested rotation simulated in this experiment only and not all-conclusive for all dynamics system with compound rotation motion. It is to show that, for certain configuration, the Razi acceleration effect can be very significant. The Earth for instance, if we take into account the precession rate of the Earth, the spin rate, and the radius of the earth, the effect of Razi acceleration in the equatorial is in the order of magnitude of -10. This is obviously too small to affect the motion on Earth.

It appears also that the Razi acceleration changes its direction depending on the body’s (sensor’s) angular position. The range of rotation about Axis 1 is denoted by \(\phi\) and for Axis 4 it is denoted by \(\theta\). For Test 2(a), the peak happened when Axis 4 is at \(\theta = 0^\circ\). This happens when the sensor’s \(z\)-axis points down towards Earth’s surface, as illustrated in Panel #7 in Figure 6.16. The maximum Razi acceleration in Test 2(b) also occurred at the instance where the sensor’s \(z\)-axis points downward. This is because it is the point where the vectors \((\vec{\omega}_1 \times \vec{\omega}_2)\) and \(\vec{r}\) are perpendicular to each other. Hence it can be shown that the "next" point where the Razi acceleration is maximized is when \(\theta = 180^\circ\), when the sensor’s \(z\)-axis is pointing upward. This confirms that, in Test 2(c) Figure 6.15, the point where the maneuver ends at \(\theta = 0^\circ\) is where the maximum Razi acceleration occurs.

\(^1\)Note that the all acceleration in Figures 6.13 to 6.15 are multiplied by -1.
Figure 6.13: Test 2(a): Components of the out-of-plane acceleration. Shown here are the total acceleration (blue), the centrifugal and mixed centrifugal (red), and the Razi acceleration (green).
Figure 6.14: Test 2(b): Components of the out-of-plane acceleration. Shown here are the total acceleration (blue), the centrifugal and mixed centrifugal (red), and the Razi acceleration (green).
Figure 6.15: Test 2(c): Components of the out-of-plane acceleration. Shown here are the total acceleration (blue), the centrifugal and mixed centrifugal (red), and the Razi acceleration (green).
Figure 6.16
6.3 Alternative Method of Finding the Razi Effect

The coordinate frame technique used in for this problem can be described as follows: For every axis that rotates, attach a noninertial coordinate frame on it plus an inertial reference frame \((n + 1)\) frames. Consequently, each coordinate frame, bar the inertial frame, will only have one constant axis about which it rotates. One big advantage of this is that the kinematic treatment of angular velocity and angular acceleration is much easier as they are constant in their own frame. Any changes in direction due to the rotation of the other frames is covered by the term \(\sum_{i=1}^{n} (\vec{\omega}_i \times \vec{r}_i)\).

An example of the coordinate frame assignment technique is a merry-go-round, which has a single axis about which it rotates. Therefore, the number of coordinate frames that is used to describe the motion of a point on it is two. Another example is the rotating blades of a gas turbine on an aircraft that is making a turning maneuver, the number of coordinate frames used is three: an inertial frame, one for the axis of rotation of the turbine blades, and another is the axis about which the aircraft is making the maneuver.

For this experiment/simulation setup, two out of six axes of the IRB 14000 manipulator are rotated. Therefore, with two rotating coordinate frames, \(n + 1 = 3\). They are \(G\), \(B_1\), and \(B_2\). From this system, we get the acceleration expression in Equation (5.8). It is quite a simple expression because the angular velocities are constant in magnitude and the point-of-interest is not moving in \(B_2\), therefore eliminating the tangential and Coriolis terms.

Rimrott [141] (pp. 153–154) approached a similar problem using multiple coordinate frames system. For this similar problem, he coined a term called “reversed Coriolis”, which has a same effect as the Razi acceleration ([141], pp. 185). He used that term, perhaps because he found the acceleration terms that acts in the out-of-plane direction come from the Coriolis term. To illustrate this, consider our experimental setup of nested rotation motion. We fixed a rotating coordinate frame on Axis 1 and called it \(B_1\). First, we ignore the rotation of the POI in Axis 4 but instead, let its displacement be defined as \(\vec{r}^*\), and velocity as \(\vec{v}^*\). Now, the velocity of the POI in frame \(B_1\) is written as

\[
\frac{1}{\vec{G}} \vec{v} = \vec{v}^* + \frac{1}{\vec{G}} \vec{\omega}_1 \times \vec{r}^* \quad (6.4)
\]

Applying another differentiation from \(B_1\) and canceling out \(\frac{1}{\vec{G}} \dot{\vec{\omega}}_1\),

\[
\frac{1}{\vec{G}} \vec{a} = \frac{1}{\vec{G}} \dot{\vec{v}}^* + 2\frac{1}{\vec{G}} \vec{\omega}_1 \times \vec{v}^* + \frac{1}{\vec{G}} \vec{\omega}_1 \times \left(\frac{1}{\vec{G}} \vec{\omega}_1 \times \vec{r}^*\right) \quad (6.5)
\]

Initially there was another dedicated coordinate frame for the sensor \(B_3\). However, since \(B_2\) is always coincident with sensor, \(B_3\) is considered redundant.
The Coriolis term here is \(2\frac{1}{\text{G}}\omega_1 \times \mathbf{v}^*\). Next we substitute \(\mathbf{v}^* = \frac{2}{\text{1}}\omega_2 \times \mathbf{2r}\), where \(\mathbf{2r}\) is constant. This leads to,

\[
\frac{1}{\text{G}}\mathbf{a} = \frac{2}{\text{1}}\omega_2 \times (\frac{2}{\text{1}}\omega_2 \times \mathbf{2r}) + 2 (\frac{1}{\text{G}}\omega_1 \times (\frac{2}{\text{1}}\omega_2 \times \mathbf{2r})) + \frac{1}{\text{G}}\omega_1 \times (\frac{1}{\text{G}}\omega_1 \times \mathbf{r}^*) 
\] (6.6)

This expression, when plotted, gives the same result as Equation (5.8).

This shows that there is no specific way to design a coordinate frame system. In this experimental study, two noninertial rotating frames are used for each rotating Axis. Initially, we had one more coordinate frame dedicated to the sensor frame \(\mathbf{B_3}\), however, it was realized that everything can be transformed into \(\mathbf{B_2}\) with just a translation \(\mathbf{r}\) (the length of Arm 2) and without any rotation procedure. Using this procedure, one can see and understand the derivation the Razi acceleration term and its definition according to the angular acceleration theory presented in Chapter 3. In Rimrott’s method, for this particular case, the Razi acceleration is hidden in an acceleration term that resembles a Coriolis acceleration term. Most importantly, we can now see the Razi effect, which is described by Hirschberg and Mendelson [113] as the normal-to-disk force as a result of “axis of rotation itself is being rotated”.

### 6.4 Conclusion

This experimental study has shown that the Razi acceleration can be detected in the acceleration signals from a body going through compound/nested rotation motion. The magnitude and direction of the Razi acceleration can be found using an acceleration equation developed using multi-frame derivative kinematics technique. This is shown by plotting the equation and comparing it with the acceleration signals, which have been presented to show a good agreement between the theoretical and experimental data (Figures 6.1 to 6.3). The effect of ignoring Razi acceleration in calculating the inertial force acting on a rigid body is shown in Figures 6.4 to 6.6. The Razi effect contributes to a small percentage of the total inertial acceleration acting on a body in rotation motion. However, in general, the magnitude is very much dependent on the parameters involved. This study has also succeeded in decomposing the total acceleration such that the inertial terms such as centripetal and Razi accelerations can be identified and distinguished. This is shown in Figures 6.8 to 6.11 with different colors representing different types of inertial accelerations. Interestingly, unlike the nature of centripetal acceleration which is constant as long as the displacement vector is constant, the Razi acceleration shows a sinusoidal pattern throughout the rotation motion. This means for a single point on rigid body, the Razi acceleration action on it is periodic and bi-directional. The maximum value happens when the vector \((\frac{2}{\text{1}}\omega_1 \times \frac{2}{\text{1}}\omega_2)\) is perpendicular
to $2\mathbf{r}$. Understandably, the Razi acceleration becomes zero when the directions of these two vectors are parallel to each other.
Chapter 7

Summary

The discovery of the Razi acceleration term becomes the starting point of the work in this thesis. This research attempts to analyze and explain the Razi acceleration from the perspective of kinematics. This is done by approaching the problem from a vector mathematics point-of-view by using the revised and extended Euler derivative transformation method. The underlying formulas and rules in multi-frame derivative kinematics are emphasized as to explain the influence and effect of the Razi acceleration on dynamical systems. An experimental investigation is performed to translate the mathematics abstraction into an observable and measurable physical effects.

Razi acceleration is not a new mechanics effect but a term that exists in a general non-inertial motion equation that has been so far undervalued and ignored completely. Like the centripetal/centrifugal, Coriolis, and tangential acceleration, this effect is an apparent force acting on an object as it is viewed in a rotating frame of reference. While the rest of the inertial accelerations can be observed in a binary coordinate frame system, the Razi acceleration is non-measurable in planar kinematics. To get the explicit expression of a force acting in the out-of-plane direction, the coordinate frames system has to involve more than two frames. The reason is the effect appears in “nested rotations” or “compound rotation”, i.e. a rotation motion that occurs in another rotating frame. Such motion can be seen in a typical demonstration gyroscope which has several frames rotating inside each other. Here, auxiliary coordinate frames are used to decompose the complex motion of such rigid body into several, more manageable coordinate systems. And when the equation of motion of such rigid body is transformed to the “outside coordinate frame”, the Razi term can be distinguished and measured.

The Razi acceleration can be summarized here as an inertial acceleration that is caused by the rotational motion of the rotation axes themselves. A three relatively-rotating
coordinate frames system is used as the prime example in demonstrating the Razi acceleration, although it can appear for any relatively-rotating system with more than one rotating frames. It is shown from the experimental studies that the amount of acceleration magnitude attributed to the Razi term is relatively small if compared with centripetal acceleration. However, the effect is significant and quantifiable, and we have seen the overall influence of the Razi acceleration on a rigid body in compound motion.

It is common to theoretically simplify a situation to provide convenience for describing physical phenomena; however, we are also aware of the danger of analytical underestimation. It is very much possible that the future engineering missions require more sophisticated analyses. Modern structures are increasingly becoming more intricate. As we are venturing into unacquainted dynamic environments, we are very dependable on the analytics to make better scientific decisions. It is important that the forces be determined accurately as pessimism in mechanics analysis results in over-engineering and design over-complexity, whereas optimistic evaluation could lead to wrong assumptions hence inviting unprecedented failures and disasters.

The novel contributions provided by this thesis are summarized below chapter by chapter. And then, we revisit the research questions devised in the Introduction section and correlate them with the results presented in the chapters. We supplement this chapter with a set of recommendations for future work in topics related with derivative kinematics and the Razi acceleration.

7.1 Recapitulation and Conclusions

As this dissertation is demarcated by sections and chapters presenting different results, it may be convenient to the reader to have the inferences from each chapter briefly recapitulated here.

Chapter 3 Foundations of Derivative Kinematics

This chapter is intended to provide the groundwork for working with vector derivatives throughout this thesis. Most of the materials in this chapter functions as an introductory materials to prepare the reader with the complex mathematics, symbols and notation, and concepts in the succeeding chapters.

A proposal for a new and improved vector notation system for the purpose of kinematics analysis in multiple coordinate frame environment is presented to provide explicit information about the expression frame and the derivative frames. An extensive review
on the methods of vectors and quaternion are given with a focus on expressing rotation matrix, angular velocity and angular acceleration. In the angular velocity section, the relationship between angular velocities in a multiple coordinate frames system is given. Next, the derivation and proof of the Euler derivative transformation formula is given.

The original contribution in this chapter is the derivation of angular acceleration vector terms for a multiple coordinate frames system. It is proposed that the first time-derivative of angular velocity (angular acceleration) is written in its complete form as in Equation 3.126 as to not neglecting the acceleration caused by the motion of the rotation axis. In the next chapter, it is shown that the Razi term is derived from the component $\dot{\theta}\ddot{u}$ which contributes to the $\sum_{i=1}^{n} (\hat{u}_i \times \hat{\omega}_i)$ part of the Razi acceleration term. This subtle acceleration effect is often overlooked in mechanics analysis due to the two-frame system limitation where the rotation axis is considered fixed at all times. The result is consistent with Angeles [10] and Henderson [22]; however, the separation of the components along with their kinematics and dynamics interpretation on the angular acceleration components are previously non-existent in the literature.

Chapter 4 On the Razi Acceleration

There are three novel contributions from this chapter: Firstly, the demonstration of the Razi acceleration as a type of inertial acceleration. In addition, this result is complemented by the revised expression of the composition of multiple angular acceleration vectors in Chapter 3. The composition of relative angular velocities for $n$ non-inertial rotating coordinate frames expressed in arbitrary frame $f$ is

$$\omega_n^f = \sum_{i=1}^{n} \omega_i^f$$

and the composition of relative angular accelerations for $n$ non-inertial rotating coordinate frames expressed in arbitrary frame $f$ is

$$\frac{d}{dt} \omega_n^f = \sum_{i=1}^{n} (\alpha_i + g \omega_i^f \times \omega_i^f)$$

In this expression, it is shown that when a series of angular velocities are differentiated, additional terms $\sum_{i=1}^{n} (\hat{u}_i \times \hat{\omega}_i)$ must be included to take into account the convective rate of change of the relative angular velocities as seen from the observer’s frame (coordinate frame of choice).

Secondly, the characteristics of the Razi acceleration is explained. It is the acceleration that acts normal to the plane of rotation of the body frame (out-of-plane acceleration).
We distinguish the difference between the tangential acceleration and the Razi. The former is contributed by the rate of magnitudinal change of rotation about a stationary axis, whereas the latter is induced by the rate of directional change of rotation axes.

Lastly, a unified description of derivative kinematics for systems with arbitrary number of relatively-rotating coordinate frames is presented. Shown in this chapter is a multi-frame system with all origin sharing the same point. Examples are given for referential systems with up to five coordinate frames. The general expression for centripetal, Coriolis, tangential, and Razi acceleration terms are also given for referential system with arbitrary number of coordinate frames.

**Chapter 5 and 6 Experimental Investigation of Razi Acceleration**

The experiment starts with two preliminary tests involving rotation about a single axis. The purpose is to demonstrate the basic mechanics of this experimental study by providing simple, easily understood examples. It also shows that the centripetal acceleration exerted by this motion can be detected and calculated. No Razi or out-of-plane components of acceleration are detected in a rotation about a fixed axis.

The nested rotation motion about two axes is simulated using three different angular velocity combinations. The angle between the two axes is set at 90 degrees to maximize the effect of Razi acceleration. For all three configurations, the Razi acceleration is detected in the out-of-plane components. The Razi effect contributes to a relatively small but significant percentage of the total inertial acceleration acting on a body in rotation motion. This study has also succeeded in decomposing the total acceleration such that the inertial terms such as centripetal and Razi accelerations can be identified and distinguished, such that their individual values can be measured in each direction. The Razi acceleration is found to have a sinusoidal-like appearance depending on the angular position of the point of interest in the rotation plane. The result also shows that the magnitude of the Razi acceleration fluctuates along the normal-to-the-plane direction. This is an important finding in that we have discovered that the Razi force is an oscillatory force. This characteristic is a stark difference from centripetal acceleration which is constant at all times given the POI’s radius does not change.

**7.2 Research Outcomes**

All research questions have been exhaustively answered. The outcomes of this research questions are summarized here to show the correlation between the research questions and the outcomes from this work.
To recap, the research questions were:

- **How can we improve the theory of vector derivatives to produce the complete expression of the acceleration terms for rigid body in multiple relative motion? Is the Razi term a necessary inclusion for a complete acceleration expression? What is the physical meaning of the Razi acceleration?**

- **What are the components of acceleration of a rigid body in multiple coordinate frames system? What is the significance of the out-of-plane component which can be only observed using >2 coordinate frames? What type of rigid body motion incurs the maximum/minimum Razi acceleration?**

- **Can we physically simulate such rigid body motion and describe how the Razi acceleration works? In other words, can the Razi acceleration be observed experimentally?**

The Euler derivative transformation formula has been revised and extended for the application of systems with more than two relatively-rotating coordinate frames. This is done with the improvement of the expression of the composition of angular velocity and angular acceleration vectors in Chapter 3 and 4. Using this rule, the angular velocities and angular accelerations of an arbitrary number of coordinate frames can be expressed in any single reference frame of choice (the observer’s frame). The essential consideration is the expression of the angular acceleration, in particular, is different from the traditional definition of angular acceleration in a binary frame system. The expanded angular acceleration expression in Equation (7.2) shows the term that makes up the Razi acceleration. Thus, one can write a complete velocity and acceleration expression of a rigid body in multiple relative motion, and separate the inertial acceleration terms individually.

The general formula for derivative transformation in multi-frame systems has been given in Chapter 4. The value of the Razi term depends on the structure of the coordinate frame system and the values of angular velocities and accelerations. For a simple example of three coordinate frames system with two angular velocities $\omega_A$ and $\omega_B$, the important parameters are the angle between these two vectors, the angle between the POI’s position vector and the product of the two angular velocities, and the magnitudes of these three vectors. The Razi expressions for four and five coordinate frame systems are given explicitly in Chapter 4 for a more complex frame system.

As shown by the alternative method that a different coordinate frames structuring strategy may produce a slightly different expression of acceleration. Chapter 6.3 shows a different approach in finding the acceleration terms. In this approach, the Razi term is
found by substituting the velocity of the body frame into the “reversed Coriolis” term. The approach is different, but upon comparing the result of this derivative with our derivation, it shows that this method can also predict the out-of-plane acceleration for three coordinate frames system with nutation angle of 90 degrees. However, it is unknown whether this technique can be generalized to systems with an arbitrary number of coordinate frame and varying nutation angles.

Chapter 4 explains the Razi acceleration as one of the inertial acceleration terms along the centripetal, Coriolis, and tangential acceleration. It is an acceleration that is exhibited by the relative rotation between three coordinate frames or more. It is evident in the experimental study that has been done that the Razi acceleration appears when an axis of rotation is itself rotating in the inertial space. Experiments in Chapter 5 and 6 shows that the Razi acceleration does not appear when the axis of rotation is fixed as in the cases of Test 1(a) and Test 1(b) in Section 6.1.1.

By doing a vectorial analysis of the Razi term $\sum_{i=1}^{n} (\beta_{i}\omega_{i} \times \gamma_{i}\omega_{i}) \times \eta_{r}$, and verifying it based on the experimental investigation using multi-axis robotic manipulator, it is shown that the Razi acceleration acts in the out-of-plane direction. There are other terms that contribute to the out-of-plane component of acceleration. In the nested rotation configuration, where the angle between two axes of rotation is perpendicular, the centripetal and the mixed centripetal terms also appear in this direction. It is fairly clear from the order of cross-product of the Razi term that its magnitude depends on the angles between the three involved vectors.

While the centripetal term is constant during the full motion, the mixed centripetal and the Razi accelerations are shown to be oscillatory. This is an important aspect to consider in designing rotating structures. The treatment of load and stress coming from an oscillatory source is less straightforward than dealing with a constant source like a centrifugal force. This can be significantly more complicated if the angular velocity of each rotating axis is not constant. This is where the general derivative transformation formula and the study of the Razi acceleration can play a role in determining the pattern of inertial force exerted on the structures of a rotating system.

In the experiment, a nested rotation type of motion is simulated using a multi-axis robotic manipulator to produce multiple rotations in different directions. This study is a necessary part in our search for the physical interpretation of the Razi acceleration. It serves as a purpose to translate the mathematics into an observable and quantifiable physical effects. Experiments have shown that the Razi acceleration is detectable and measurable by the use of acceleration and angular rate sensors. Although the effect from the Razi acceleration has been known for a while, however, here we can isolate its
mathematical expression, calculate its value, and observe its behaviors and patterns in dynamical systems.

7.3 Recommendations for Future Work

The objective of this dissertation has been completed. With the current work, we have discovered more new aspects that can be considered in the further studies in this exciting field. We take the liberty to present several recommendations for future work that can be extended from this thesis.

Small-sized Multi-axis Rate Table/Motion Simulator

The IRB 1400 robotic manipulator is used in the experimental study as a motion simulator to reproduce nested rotation motion. It has several limitations: non-continuous 360 degrees rotation, offset axis of rotation, and limited workable rotation axis. We propose that a similar experiment is performed using the motion simulator that is often used for gyro system testing in military and industry laboratories called rate table. Other manufacturers use the term multi-axis motion simulator. The system provides a perfect example of a nested rotation motion with unlimited and continuous rotational motions, a common rotating origin, and multiple controllable axes, with a big rotating platform for users to place their units under test (UUTs).

Unfortunately, the rate-table system, especially with 2 or more rotating frames, is very expensive as the targeted markets are military laboratories and big aerospace industries. It is doubtful that a university-level research institutes can obtain a funding generous enough to have it in their labs. One of the main reasons the multi-axis rate table system is so expensive is its size. The rate table system is designed to handle heavy UUTs, and to rotate such load, the system need to be able to produce large torque. This is usually done when sensor units are tested in big bulk. However, for academic research investigation purpose, this is rarely necessary. Furthermore, with the development of MEMS in inertial sensing technology, most accelerometer or gyroscopes now are less than 100 g in mass. There is right now a gap in this area – most manufacturers are not interested in producing a small-sized rate tables due to its ineffectuality for military applications and big industries, whereas research groups in academia may not need an oversized and multi-million dollars system to do academic research. We propose the idea to develop and build a micro-sized rate table which quality is comparable with its industrial-grade counterparts but not required to handle heavy loads (>500 g). So far, we are only aware of two undergraduate final year projects –from School of Aerospace, Mechanical and
Manufacturing Engineering, RMIT University and University of Central Florida – that have been involved in developing micro-sized multi-axis rate table. However, currently, there is no found literature on both projects.

Complex Razi Acceleration

This work provides an in-depth analysis of the Razi acceleration in three coordinate frames system. The general term for Razi for $n$ of coordinate frames is given in Chapter 4, which shows that with more coordinate frames involved, the Razi acceleration becomes more complex. With a capable multi-axis rate table, experimental investigation can be done on the Razi acceleration in a more complex rotation motion system. It is mentioned in Chapter 4 that the mixed double derivative method is still an open problem in terms of the practicality of the formula. With a multi-axis rate table and multiple sensors attached to different frames, the mixed double derivative may find some applications amidst the complexity in transforming kinematics equations from one frame to another.

An Extensive Review of Attitude Representation and Attitude Derivatives

Attitude representation is one of the broadest and most difficult disciplines in attitude dynamics and control. Currently, there is no survey or review paper that covers all types of attitude representation. This thesis has provided the method of obtaining attitude derivatives (such as angular velocity and angular acceleration) from a 3-by-3 rotation matrix. However, only a handful of attitude representations are covered: direction cosines, Euler angles, quaternion representation of Euler parameters. Attempts have been made to collect all the applicable and useful results in attitude representations. The most cited survey papers are from Shuster [38] which was published in 1993. In addition to the typical attitude representations used in aerospace engineering problems, it also covers the lesser known attitude representations such as the Cayley-Klein parameters, The Rodrigues parameters, and the Modified Rodrigues parameters. An unpublished, but available online paper from Diebel [57] is another excellent account on attitude representations and attitude derivatives. The article provides, in our opinion, the most comprehensive and useful reference for quaternion representation of attitude and its derivatives. Kuiper [136] is another highly-cited, very accessible book on general quaternions and rotation sequence. However, apart from the exposition of attitude representation by Diebel, there is a need for more references or tutorials on how to transform other forms of attitude representation into its derivatives, and relate them between different coordinate frames.
Bibliography


