Accurate Induced Drag Prediction for Highly Non-Planar Lifting Systems

A thesis submitted in fulfilment of the requirements for the degree of Doctor of Philosophy

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Declaration

I certify that except where due acknowledgement has been made, the work is that of the author alone; the work has not been submitted previously, in whole or in part, to qualify for any other academic award; the content of the thesis is the result of work which has been carried out since the official commencement date of the approved research program; and, any editorial work, paid or unpaid, carried out by a third party is acknowledged.

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Publications

Portions of the material in this thesis have previously appeared in the following publications:


Abstract

For highly non-planar lifting systems like the box wing, induced drag predictions based on common potential-flow methods can have limited accuracy. This is primarily related to the linear, fixed-wake surrogate models, which neglect the correlation of the effective height-to-span ratio and system angle of attack or insufficiently account for free-wake deformations such as deflection and roll-up effects.

Dependent on the vertical and horizontal wing arrangement of simplified box wing and biplane configurations and the system angle of attack, the present research analyses the unknown impact of wake model effects, investigates the accuracy of potential-flow induced drag predictions against an Euler-flow reference and explores the influence of higher-order wake effects. The computational expense of considered methodologies is assessed to evaluate their applicability within an aerodynamic design and optimization methodology for highly non-planar lifting systems.

Under certain conditions, higher-order wake and wake surrogate effects are confirmed to impact on the induced drag prediction. The body-fixed wake model is found generally inappropriate for induced drag estimation of present lifting systems, whereas the freestream-fixed wake model provides consistent results. Positive-staggered systems at positive angels of attack are found particularly prone to higher-order wake effects, due to the vertical contraction of wake trajectories, which leads to smaller effective height-to-span ratios than compared with negative stagger and thus closer interactions between trailing wakes and lifting surfaces. A relaxed, force-free wake model is found compulsory to enable fast but accurate induced drag predictions when using potential-flow methods for the analysis of highly non-planar lifting systems with significant positive stagger.
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Nomenclature

\( \Delta \bar{u} \) \hspace{1cm} Irreversible axial velocity defect

\( \Delta \nu \) \hspace{1cm} Roll-up angle deviation

\( \Delta H \) \hspace{1cm} Variation of the stagnation enthalpy relative to the freestream state

\( \Delta s \) \hspace{1cm} Variation of the entropy relative to the freestream state

\( \Delta t \) \hspace{1cm} Time-step width

\( \Delta u^* \) \hspace{1cm} Reversible axial velocity defect

\( \dot{q} \) \hspace{1cm} Heat flux

\( \Gamma \) \hspace{1cm} Circulation

\( \gamma \) \hspace{1cm} Vorticity

\( \kappa \) \hspace{1cm} Ratio of specific heats

\( \Lambda = \frac{b^2}{S_{ref}} \) \hspace{1cm} Aspect ratio

\( \lambda = \frac{c_t}{c_r} \) \hspace{1cm} Taper ratio

\( \Lambda_{1,2} = \frac{b_{1,2}^2}{S_{1,2}} \) \hspace{1cm} Single wing aspect ratio

\( \lambda_{1,2} = \frac{c_{1,2}}{c_r, 1,2} \) \hspace{1cm} Single wing taper ratio

\( \Phi \) \hspace{1cm} Velocity potential
\( \phi \)  
Sweep angle

\( \rho \)  
Density

\( \sigma \)  
Interference factor

\( \tau \)  
Viscous stress tensor

\( \Theta \)  
Dihedral angle

\( \vec{V}_i \)  
Induced velocity vector

\( \xi = \frac{x}{c}, \eta = \frac{y}{b}, \zeta = \frac{z}{h} \)  
Relative spatial Cartesian coordinates

\( A, B, C \)  
Circulation constants

\( b \)  
Span

\( c \)  
Chord

\( C_l = \frac{L'}{q_\infty c} \)  
Sectional lift coefficient

\( C_L = \frac{L}{q_\infty S_{ref}} \)  
Lift coefficient

\( C_n = \frac{N'}{q_\infty c} = \frac{2 \Gamma(\eta, \zeta)}{c} \)  
Sectional normal force coefficient

\( C_n, \text{opt} = \frac{2 \Gamma_{\text{opt}}(\eta, \zeta)}{c} \)  
Optimum sectional normal force coefficient

\( C_{Dc} = \frac{D_c}{q_\infty S_{ref}} \)  
Wave drag coefficient

\( C_{Di} = \frac{D_i}{q_\infty S_{ref}} \)  
Induced drag coefficient

\( C_{Dp} = \frac{D_p}{q_\infty S_{ref}} \)  
Parasite or profile drag coefficient

\( c_{ref} = \frac{S_{1,2}}{b} \)  
Reference chord

\( D_i \)  
Induced drag

\( d_{\text{euc}}/b \)  
Relative Euclidean distance

\( D_{Sp} \)  
Spurious entropy drag
$E$  Oswald factor

e  Span efficiency factor

$e_t$  Total energy

$e_{opt} = \frac{C_{D_i, opt}}{C_{D_i, ref}}$  Optimum span efficiency factor

$h$  Height

$h/b$  Height-to-span ratio

$h/b_\infty$  Freestream height-to-span ration

$h_{grid}$  Non-dimensional grid size

$k$  Thermal conductivity

$k_{se}$  Singularity parameter

$L'$  Sectional lift

$L/D$  Lift-to-drag ratio

$M$  Mach number

$N'$  Sectional normal force

$p$  Pressure

$q_\infty = 0.5 \cdot \rho_\infty \cdot V_\infty^2$  Freestream stagnation pressure

$R$  Gas constant

$S_{ref} = S_1 + S_2$  Reference area

$St$  Stagger factor

$St_\infty$  Freestream stagger factor

$T$  Temperature
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CHAPTER 1

Introduction

1.1 Definition of Research Topic

The present research is focused on the accurate and fast induced drag prediction for highly non-planar lifting systems, in particular for box wing and biplane configurations. Dependent on the vertical and horizontal wing arrangement of simplified box wing and biplane configurations and the system angle of attack, this work investigates the accuracy of potential-flow induced drag predictions against an Euler-flow reference and explores the influence of higher-order wake and wake surrogate effects.

1.2 Research Background

Prior to the discussion of the motivation and the objectives associated with the research presented herein, the scientific background is discussed shortly to establish a basic understanding of the fundamentals aspects and their relative importance. This concerns (i) the impact of induced drag on aircraft performance, (ii) an introduction to highly non-planar lifting systems and (iii) techniques for accurate induced drag predictions. The current chapter further identifies the scientific gaps in existing knowledge and briefly describes the methodological procedures to address the research questions framing this work. A summary of key novel research outcomes and an outline of the thesis is provided.
CHAPTER 1: INTRODUCTION

1.2.1 Induced Drag and Aircraft Performance

The induced drag is an inviscid phenomenon and originates in the opposed spanwise flow patterns due to the pressure imbalance of a finite wing generating lift. It corresponds to the translational and rotational kinetic energy confined in the trailing wake [Schmidt-Göller, 1992].

The impact of induced drag on aircraft performance is profound. For a commercial aircraft, it accounts for approximately 40% of the total drag during cruise flight [Kroo, 2000]. Based on the Brequet range equation [Anderson, 2012], an induced drag reduction by about 1% results in a fuel saving of about 0.4% or an equivalent increase in range. Assessing the potential of induced drag savings beyond simple cruise aerodynamics, other flight conditions have an indirect but decisive influence on aircraft performance. During takeoff, associated with high lift coefficients, induced drag amounts up to 90%. A marginal induced drag reduction can hence translate into considerable savings in fuel and emissions or facilitate a range extension. Considering indirect effects associated with an improved climb performance and a larger maximum takeoff mass, gains can easily multiply as explained by Kroo [2000].

1.2.2 Highly Non-Planar Lifting Systems

Against the background of the objectives set in Flightpath 2050 [Darecki et al., 2011] and in absence of any suitable near-term, non-fossil energy carrier, measures providing lower (induced) drag are of high importance to attain a cutback in fuel consumption and climate reactive emissions.

Different wing concepts for induced drag reduction have been proposed over the last decades. Besides rather conventional approaches like winglets or wing-tip extensions, more radical concepts attain attention. Examples of several non-planar lifting systems and associated theoretical optimum span efficiency factors $e_{opt}$ for equivalent height-to-span ratio $(h/b)$ and lift are given in Figure 1.1. All systems have ideal loadings and show approximately 40% of span efficiency improvement over an equivalently loaded, optimum planar wing. Within linear potential-flow theory, the box wing achieves the highest span
SECTION 1.2: RESEARCH BACKGROUND

Figure 1.1: Non-planar lifting systems with a height-to-span ratio of $(h/b) = 0.20$ and associated optimum span efficiency factors $e_{\text{opt}}$ based on linear potential-flow theory, adapted from Kroo [2000].

Efficiency or lowest induced drag for given height-to-span ratio and lift [Prandtl, 1924; Demasi, 2007; DeYoung, 1980; Frediani and Montanari, 2009; Demasi et al., 2015a,b].

A further sub-classification of non-planar systems can be performed in accordance with Jansen and Perez [2010]. Nevertheless, in the context of the research presented herein, a different approach is proposed. The class of non-planar configurations may include more conventional systems like wings with dihedral or specific wing-tip devices, i.e. winglets. Opposed to that, multi-surface non-planar systems are described by multiple lifting surfaces, having a considerable share on the lift creation. This certainly encompasses configurations, such as box wings, joined-wings, biplanes and with some reservations, canard aircraft and the c-wing concept. In contrast to more common non-planar systems, these, more unconventional configurations are further characterized by comparably large vertical separations of lifting surfaces and may hence be referred to as highly non-planar. Within the scope of this study, the biplane and the box wing configuration are primary examples.

Because of their potential to significantly reduce induced drag, highly non-planar configurations, such as box wings or c-wings, have repeatedly been in the focus of research [Frediani, 2005; Schiktanz, 2011; Seywald et al., 2012; Salam and Bil, 2015; Andrews and Perez, 2015; Gagnon and Zingg, 2015; Skinner and Zare-Behtash, 2016]. The inviscid aerodynamic advantage of these configurations is primarily due to the bound circulation

(a) Winglet, $e_{\text{opt}} = 1.41$.
(b) Biplane, $e_{\text{opt}} = 1.36$.
(c) C-wing, $e_{\text{opt}} = 1.45$.
(d) Box wing, $e_{\text{opt}} = 1.46$. 
being distributed over a larger effective wingspan. This lowers the spanwise loading and reduces the average downwash velocity of the system compared to an optimum planar wing of equivalent span and lift [Kroo, 2000].

1.2.3 Induced Drag Prediction

Computational Methods

Several (computational) flow models are available to predict aerodynamic properties. With respect to their governing equations, a simplified classification and hierarchical arrangement according to their complexity is illustrated in Figure 1.2. Acknowledging that induced drag is an inviscid phenomenon requires to apply inviscid flow models respectively. These models are theoretical and contrast the actual scenario in nature, where a truly inviscid flow does not exist.

The two classes of models, suitable to accomplish induced drag prediction are based on the potential-flow equations and the more elaborate Euler equations. A solution of these equations can be obtained numerically. Thereby, the Euler model constitutes the most complete representation of inviscid flow, but is computationally expensive. With regards to induced drag prediction, it is hence preferably used for validation purpose.

Potential-flow models describe irrotational flows and can be further simplified by linearization, which effectively limits the suitability to incompressible flows. Because of their simplicity compared to the Euler-flow model, computational potential-methods like the vortex-lattice method are frequently used for early design and optimization purpose. Besides a vast availability, this is related to their ability to provide accurate induced drag prediction for a wide range of applications, while requiring a very modest computational effort.

Experimental Methods

A direct measurement of the induced drag $D_i$ is generally not possible [Schmidt-Göller, 1992]. For a single wing, measurements of the total drag contain friction and pressure drag, both associated with viscosity, as well as induced drag parts. The sum of friction
and pressure drag is referred to as profile drag $D_p$. The total drag is then given by Equation 1.1.

$$D = D_p + D_i$$  \hspace{1cm} (1.1)

To estimate the induced drag, profile drag contributions must be separated from the total drag. This however cannot be done with sufficient accuracy to accommodate investigations into higher-order effects [Schmidt-Göller, 1992]. Usually, the profile drag is attained from two-dimensional measurements of airfoil sections. This neglects the impact of the three-dimensional flow on the profile drag and leads to induced drag estimates that contain three-dimensional profile drag contributions [Schmidt-Göller, 1992].

From a very stringent perspective, an experimental investigation, even when based on instantaneous, contactless velocity measurements in the wake (i.e. Particle Image Velocimetry (PIV)), is generally prevented by the viscosity inherent to real flow. Because tangential velocities around the side-edge of a wingtip can become very large, a viscosity induced flow separation occurs. This causes initial roll-up of the wake and shifts the tip-vortex inwards [Smith, 1995], ultimately altering the trailing wake shed compared to an inviscid computational solution. Acknowledging that it is the wake shape and its vorticity which define the induced drag [Smith, 1995], it is apparent that the induced drag in a real, viscous flow and in a theoretical, inviscid flow-simulation is fundamentally different. An

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**Figure 1.2:** Hierarchy of (computational) flow models.
experimental investigation is thus not feasible in principle and can therefore not function as validation baseline.

Adopting a less rigorous position, it can be argued that for well-designed wing-tip shapes the impact of the side-edge is limited [Smith, 1995]. This may allow one to experimentally predict the induced drag with an accuracy, which is sufficient for engineering estimates of wing performance [Smith, 1995] and may serve to detect trends. However, for the present study an experimental determination is not considered expedient to facilitate an accurate and detailed study of induced drag characteristics of highly non-planar lifting systems, especially as higher-order wake effects are to be included [Schmidt-Göller, 1992].

1.3 Motivation and Objectives

1.3.1 Problem Statement

Numerical accuracy and computational efficiency are fundamental but often conflicting objectives with regards to the suitability of a computational design method. Within the early phase of aircraft design, a high level of accuracy is desired to reduce uncertainties in the performance projection [Gage, 1995]. This is important, because initial (induced) drag estimation dictates the selection of a specific concept in the early design phase and affects the projected configuration dimensions and costs [Cummings et al., 2015]. Commonly, about 70% of the total avoidable life-cycle costs of an aircraft are associated with the early product definition phase [Thokala et al., 2012]. Related to the strong impact of induced drag on aircraft performance, marginal incorrect predictions can cause distorted performance estimates and non-optimal designs, requiring cost-intensive revisions in later development stages.

This is especially valid for unconventional aircraft configurations, deviating from the classical wing-tailplane arrangement, where only a limited amount of empirical values is available to substantiate the validity of numerical estimates. It is hence desired to integrate a higher computational fidelity early in the design to increase the specific design knowledge as indicated in Figure 1.3.

However, to complicate matters, within early design or as a part of a multidisciplinary
design and optimization (MDO), an extremely large, usually unknown number of computational investigations have to be performed to fully explore the parameter space. Despite advances in CPU performance, employing a high computational fidelity early in the design or optimization process, erodes the computational efficiency. A direct application of Euler-flow methodology and traditional gradient-based optimization techniques entails a high computational burden [Leifsson et al., 2014], often unacceptable in early design, even if adjoint sensitivity information are available [Jameson, 1988]. Although surrogate or metamodels, i.e. based on quadratic interpolation or Kriging techniques can be established to accelerate the prediction [Paiva et al., 2009], potential-flow methods perform significantly faster. These have proven to provide accurate induced drag estimates for a wide range of applications [Cummings et al., 2015] and are hence predestined for design purpose.

Despite highly non-planar concepts offering large induced drag savings over equivalent planar systems, an accurate prediction remains a prerequisite to an overall efficient design. Their total aerodynamic performance must be evaluated as a trade-off between induced drag and contributions from other drag sources. In example for the box wing, viscous drag penalties [Jansen et al., 2010], associated with the increased wetted area [Kroo, 2000], diminish the advantage of induced drag savings considerably. To enable meaningful
CHAPTER 1: INTRODUCTION

performance projections, an accurate induced drag prediction is required.

For these types of configurations however, potential-based methodologies can become inaccurate under certain conditions. Excluding compressibility effects, this is primarily related to the linear, fixed-wake model approaches not necessarily providing appropriate surrogates of the actual trailing vortical flow phenomena. In particular the body-fixed wake placement, which neglects the correlation of the system angle of attack and the aerodynamic or effective height-to-span ratio, a key design parameter for highly non-planar concepts, can potentially affect the estimation. Although the error introduced by its application may still be acceptable for single planar wings at moderate angles of attack, this wake placement is generally incorrect. In contrast to a freestream-fixed model, the wake is aligned with the trailing edge bisector or the mean camber surface. Hence, the wake is not drag-free and supports longitudinal forces, that distort an accurate induced drag estimation based on farfield velocities [Kroo and Smith, 1990]. This is especially true when the wake is non-planar [Kroo and Smith, 1990] as apparent herein. Wake deflection is considered to have a first-order impact on the induced drag [Kroo and Smith, 1990].

The linear, fixed-wake model approaches systematically neglect higher-order wake effects due to the roll-up and deflection of the physical wake. Their impact is considered small for single planar wings, but can become significant in the analysis of configurations such as joined-wings and canard aircraft [Kroo and Smith, 1990]. The influence of these effects is dependent on the system angle of attack and the geometrical arrangement and design of the lifting surfaces. This particularly refers to the height-to-span ratio and the stagger factor. With regards to the height-to-span ratio, the sensitivity to higher-order wake effects is certainly more pronounced for near-planar multi-surface configurations and progressively reduces with larger vertical separations. Nevertheless, the system angle of attack and the longitudinal arrangement modify (i.e. reduce) the effective vertical gap and may thus change relative sensitivities. As inferred from the wake substitution concept [Smith, 1995], especially the influence of the longitudinal arrangement is decisive for highly non-planar configurations like the box wing. Stagger is favorable for practical applications to attain high aerodynamic efficiency under static longitudinal stability and trim constraints [Andrews and Perez, 2015] and easily results in significant wake roll-up
and deflection between the trailing edges of two staggered lifting surfaces.

### 1.3.2 Significance and Originality

No publication in open literature is found to sufficiently address the impact of wake surrogate models on the induced drag estimation for highly non-planar lifting concepts. The mechanisms of how higher-order wake effects affect the induced drag (prediction) for these systems have not yet been explored explicitly. The extent of these effects has not been quantified for configurations like the biplane or the box wing and is currently unknown.

Only incidentally, Bramesfeld and Malik [2015] evaluated the effect of different wake models for a micro aerial vehicle (MAV) based on a biplane concept. Overall, only a minor impact of wake models on the performance projection is found. Investigating the potential of wing-tip extensions and winglets on tiltrotor wings, Cole et al. [2013] found that wake relaxation has an effect on the induced drag, especially in case of a winglet, whereas the impact on the lift distribution is negligible. The effect of wake modeling with regards to a down and upward-oriented winglet was investigated i.e. by Leyser [1996]. While equivalent induced drag savings are associated with the freestream-fixed wake, using a relaxed-wake reveals an upward-oriented winglet to be superior to a downward-oriented. With regards to its flutter characteristics, the wake model is found of importance for joined-wing concepts, as discussed by Cavallaro et al. [2015]. The effect of wake roll-up on the induced drag of two wings in formation flight was stressed by Bramesfeld and Maughmer [2008a], but revealed only small performance differences between wake surrogate models.

Limited to planar systems, the impact of the wake model approach and contributions by higher-order wake effects were studied in the past. Smith and Kroo [1993] compared freestream-fixed and relaxed-wake model estimates for a crescent and elliptical wing, but noticed only minor efficiency differences in between due to near-planar wakes. A comprehensive study was presented by Smith [1995] involving a planar split-tip wing. The system induces wake related higher-order effect promoted by a high angle of attack and a special wing-tip shape, causing major efficiency differences between freestream-fixed and relaxed-wake model. The impact of wake shapes on high-lift system aerodynamic predic-
tions was considered by Bissonnette and Bramesfeld [2016]. In terms of lift and induced drag, improved consistency is evident for relaxed-wake based estimates compared to an experimental baseline. In conjunction with horizontal-axis wind turbines, Basom and Maughmer [2011] conclude that other wake models, i.e. the freestream-fixed surrogate, generally compare poorly to the relaxed-wake results.

The dependency of higher-order wake effects on the height-to-span ratio, the stagger factor and the system angle of attack has not yet been determined for highly non-planar systems like the box wing or biplane. The vast majority of existing research employs linear potential-flow methodologies and generally neglects these contributions. Investigations previously conducted are therefore limited to linear analyses of the impact of both key design parameters and the system angle of attack.

In this context, the influence of the height-to-span ratio on the optimum efficiency of an unstaggered biplane and box wing configuration was first described by Munk [1923b] and Prandtl [1924]. Later, Pistolesi [1932], DeYoung [1980], Rizzo [2007], Demasi [2006] and Frediani and Montanari [2009] derived comparable expressions for these or various other (highly) non-planar systems. By means of a variational approach and using a lifting-line model, Demasi [2006] demonstrated that the optimum circulation distribution for a biplane is commonly not elliptical. A related study, involving box wing and biplane configurations is presented in Demasi [2007], which is further complemented by an analysis on the condition of minimum induced drag and further refinements to theory by Demasi et al. [2014, 2015a,b, 2016].

Andrews and Perez [2015] studied the influence of the height-to-span ratio and the stagger factor on the efficiency of a box wing. It is confirmed that besides the height-to-span ratio, especially the longitudinal separation is decisive, which agrees well with Salam and Bil [2012a,b], concluding that the box wing is most effective for low horizontal separations. Mamla and Galinski [2009] for a box wing, as well as Selberg [1983] for a staggered biplane and Selberg and Cronin [1986] for a joined-wing configuration showed, that positive stagger, according to its definition presented herein, is less efficient than equivalent negative for any positive angle of attack. This is in-line with Munk [1923b] and qualitatively corresponds to experimental measurements performed by Norton [1921]
for a staggered biplane concept, indicating that the maximum aerodynamic efficiency increases with negative stagger. A similar result is obtained by Moschetta and Thipyopas [2007], Kang et al. [2009a,b], Maqsood and Go [2013] and Bramesfeld and Malik [2015]. Nevertheless, a sufficiently self-contained discussion of physical reasons and mechanisms enabling these performance gains is not conveyed. Only Munk [1923b] argues, that the aerodynamic effective vertical gap is increased for negative-staggered arrangements at any positive angle of attack, ultimately causing smaller induced drag.

Despite this finding is of high practical interest, it is based on the assumption of equivalent geometric height-to-span ratios, which results in different effective vertical gaps due to non-zero system angle of attack. To facilitate an investigation into the principal impact of gap and stagger variations and to avoid overlayed effects induced by the system angle of attack, lifting systems with similar aerodynamic rather than geometric design parameters are additionally concerned herein.

For highly non-planar concepts, the accuracy associated with potential-flow induced drag predictions is presently unknown. No firmly and systematic evaluation of the accuracy compared to an Euler-flow model, including the impact of the employed wake model, geometric arrangement and the system angle of attack has yet been conducted.

Potentially capable to resolve higher-order wake effects, Hicken and Zingg [2008] performed induced drag minimization for several non-planar concepts by means of an Euler-flow model. For a simplified, unstaggered box wing configuration with a height-to-span ratio of \((h/b) = 0.10\), they found their non-linear optimization to agree well with linear predictions. This is confirmed in principle by Gagnon and Zingg [2015] for an equivalent system and height-to-span ratios between \((h/b) = 0.10\) and \((h/b) = 0.30\) using high-fidelity inviscid aerodynamic optimization. While this is a reasonable result based on the wake substitution concept [Smith, 1995], it is not overly surprising as such kind of unstaggered planforms likely do not exhibit any influence by higher-order wake effects, in particular due to wake deflection.

The computational expense of wake relaxation has not yet been evaluated for highly non-planar lifting systems yet. The application of the relaxed-wake model is supposed to improve the accuracy of induced drag predictions compared to a freestream-fixed wake.
model approach, but requires a higher computational overhead as the wake shape needs to be computed iteratively. To draw conclusions regarding the applicability of the relaxed-wake model within a design methodology suited for early design stages or a MDO effort, the associated computational expense compared to the common freestream-fixed wake model and Euler-flow method needs to be evaluated.

Using a space marching technique to produce the relaxed-wake shape, Smith and Kroo [1997] comment, that approximately 12 to 15 iterations are required to attain Trefftz plane drag convergence. This corresponds at least to an equivalent overhead in computational time compared to the freestream-fixed surrogate. Based on a time-stepping approach, the computational expense associated with the wake relaxation is evaluated by Bissonnette and Bramesfeld [2016] for high-lift systems. The relaxed-wake model is found to increase the computational effort by about 35% compared to an iteratively shed wake aligned with the freestream-fixed direction. To improve the computational efficiency, Basom [2010] studied the potential of a variable timestep size and a wake extrapolation technique for wind turbine applications. Computational time reductions in the range of 75% compared to the relaxed-wake case were attained.

1.3.3 Research Questions

Derived from the problem definition and the identified gaps in existing knowledge detailed in the preceding sections, the research presented herein is framed by the following questions:

RQ-1. What impact has the employed wake model approach on accurate potential-flow induced drag prediction for highly non-planar concepts?

a) How and to what extent is the induced drag (prediction) for highly non-planar concepts affected by higher-order wake effects?

b) How do higher-order wake effects depend on the geometrical arrangement of the system and the system angle of attack?

RQ-2. With what accuracy can potential-based methodologies predict the induced drag of highly non-planar concepts within the subsonic flow regime?
compared to an Euler-flow reference?

a) How does this depend on the employed wake model approach, the geometrical arrangement of the system and the system angle of attack?

RQ-3. Which computational expense is related to the relaxed-wake compared to a freestream-fixed wake approach and an Euler-flow reference and how does it apply within a design method for highly non-planar concepts?

1.3.4 Scientific Contribution

The work described herein makes novel contributions in the field of induced-drag prediction for highly non-planar lifting systems. The primary outcomes are briefly summarized in the following:

- With regards to highly non-planar lifting systems and in particular for biplane and box wing configurations, the present research for the first time consistently identifies the impact of trailing wake modeling on accurate potential-flow induced drag prediction. Besides quantifying the extent of higher-order wake effects in correlation to the geometrical arrangement and the system angle of attack, the causing physical mechanisms are explored to gain deeper insight into higher-order wake and wake surrogate effects. In contrast to investigations conducted previously, it is explicitly differentiated between wake-related effects predominately promoted by the system angle of attack and those induced by the geometrical arrangement. Their specific contribution can thus be decomposed and assessed more in detail. This new knowledge may subsequently be utilized to develop design guidelines that take advantage of wake effects for these particular systems and enable performance achievements beyond linear theory.

- In this context, the accuracy associated with potential-flow induced drag predictions is systematically evaluated in comparison to a high-fidelity estimation based on an Euler-flow methodology. The ability to attain accurate induced drag projections is investigated in dependency on the wake model, the geometrical arrangement of the
system and the system angle of attack. This specifically aims to verify the accuracy of the freestream and the relaxed-wake model subjected to certain geometrical properties or flow conditions and to identify currently unknown limitations occurring in the scope of their application. The uncertainty associated with accurate potential-flow induced drag prediction, in particular with respect to planform arrangement and angle of attack sensitivities, is thus clearly reduced. The detailing of the comprehensive Euler-based reference baseline established for this purpose depicts a further novel aspect.

- Likely alleviating accuracy deficits of linear potential-flow induced drag projections based on freestream-fixed wake, the present research effort is further complemented by assessing the computational expense of wake relaxation against the common freestream-fixed surrogate and the Euler-flow reference to evaluate its applicability within an aerodynamic design and optimization methodology for highly non-planar lifting systems. With respect to a fast but accurate induced drag prediction, this knowledge permits to preselect an appropriate computational methodology in dependency of the given flow problem.

- Further important contribution is made by the modification and validation established in the induced drag calculation methodology within the utilized higher-order potential-flow model. In contrast to the current implementation, a collocation point projection in compliance with the stagger theorem is suggested. Actually, this enables accurate induced drag predictions based on the higher-order potential-flow model in the first place.

1.4 Methodological Procedure

Several computational potential-flow methodologies have been originally considered in the scope of this research. These are in particular a general vortex-lattice method (AVL) [Drela and Youngren, 2013], a multiple-lifting line formulation (LiftingLine) [Horstmann, 1987], a higher-order potential-flow technique (FreeWake) [Bramesfeld, 2006] and a higher-
order panel method (*PanAir*) [Magnus and Epton, 1990]. To effectively reduce the computational and modeling effort, but also motivated by preliminary study results [Schirra et al., 2014a,b,c], the extent of detailed potential-flow investigations is essentially limited to the higher-order potential-flow method by Bramesfeld [2006], incorporating a linear freestream-fixed and a relaxed-wake model to compute the physical, force-free wake shape. Identification and quantification of higher-order wake effects is thus enabled by comparison between estimates originating from the freestream-fixed and the relaxed-wake model. Nevertheless, comparison is made towards other potential-flow methodologies where appropriate, especially to demonstrate issues related to the application of a body-fixed wake model.

A commercial CFD-code (*STAR-CCM+*) [CD-Adapco, 2013] is employed to establish a reference baseline by means of the Euler-flow equations and to assess the accuracy associated with potential-flow methodologies with regards to their wake model. Higher-order wake effects, systematically neglected by linear-potential methods are inherently included. To prevent issues in the induced drag prediction, originating in the application of surface pressure integration techniques, a farfield approach [Destarac and van der Vooren, 2004] is utilized. To establish a consolidated reference baseline for the Euler-flow model, particularly in the context of the farfield induced drag prediction, and to further assess the accuracy associated with the higher-order potential-flow model, an investigation of well-documented planar lifting systems is conducted.

With regards to their general wing arrangement, the analysis of highly non-planar lifting systems involves a simplified box wing and a geometrically equivalent biplane concept. Both configuration types depict important special cases within theory related to highly non-planar systems [Munk, 1923b; Prandtl, 1924], are of recent scientific [Demasi et al., 2015b,a] and practical interest (compare Section 2.3.2) and, caused by their inherent aerodynamic characteristics, permit mutual comparison.

To explore the influence of the angle of attack and in parts of the stagger factor on the extent of higher-order wake effects (RQ-1) and to further assess its impact on the accuracy with regards to the Euler-flow reference (RQ-2), a positive and a negative-staggered system with a fixed geometric height-to-span ratio is derived for both lifting
CHAPTER 1: INTRODUCTION

concepts. The isolated dependency of higher-order wake effects on the stagger factor and height-to-span ratio (RQ-1), as well as the accuracy related to the applied wake model approach (RQ-2) is concerned by means of a dedicated parametric investigation. This contrasts the procedure involving effects induced by the angle of attack, which is referred to as the classical analysis case, but also other recent research effort.

Insight into the reasons and mechanism causing higher-order wake effects is conveyed by analyses of trailing flowfield quantities like the wake trace and the vorticity distribution (RQ-1). In addition, the computational expense of employed methodologies (RQ-3), only considering the effective CPU time, is evaluated by comparison of most critical test cases involved.

1.5 Thesis Outline

The thesis is composed of nine chapters. Chapter 2 discusses general aspects of induced drag, its impact on aircraft performance and provides an overview of different concepts for its reduction. This comprises an overview of existing research related to highly non-planar systems and a brief summary of theory relevant to the present work. The governing equations of inviscid flow are described in Chapter 3, including a review of associated numerical methodologies. The implemented computational approaches and techniques enabling an accurate prediction of the induced drag are discussed in detail. To establish a confident validation baseline, Chapter 4 stresses the investigation of planar reference systems by means of implemented computational methodologies and assess their ability to provide accurate induced drag predictions. A review of relevant existing research on planar reference systems is given. The preliminary investigation in Chapter 5 concerns the induced drag estimation of a simplified box wing concept. Issues occurring in its prediction for the higher-order potential-flow method are analyzed by means of an investigation of trailing flowfield properties. Chapter 6 discusses a methodological simplification in the implemented induced drag estimation technique within the higher-order potential-flow method and presents a modification to enable accurate induced drag prediction for highly non-planar lifting systems. Potential-flow induced drag estimates for the box wing concept
found in Chapter 5 are recomputed in Chapter 7, utilizing the modified technique. The impact and reasons of higher-order wake effects is subsequently explored in dependency of the geometrical arrangement and the system angle of attack. The accuracy of potential-flow induced drag prediction is assessed against the Euler-flow reference. An equivalent biplane concept is concerned in Chapter 8 to substantiate findings and conclusions. Introduced by its finite wing span, the influence of relaxed-wake parameters on the solution is resolved in detail. The computational expense associated with employed methodologies is determined in Chapter 7 for the box wing and Chapter 8 for the biplane configuration, providing conclusions of their suitably within early design or MDO purpose. Chapter 9 summaries the present work and its scientific outcomes with regards to research questions formulated.
The current chapter aims to provide the fundamentals of induced drag and related aspects. To emphasize on the importance of accurate induced drag prediction, an assessment of its share and influence on the performance of commercial aircraft is performed. The physical reasons for the creation of induced drag will be presented, including an introduction into finite wing analysis and other theories relevant to the present research. A concise review of existing concepts and related research that lead to a reduction in induced drag is provided.

2.1 Drag Breakdown for Commercial Aircraft

To access the potential of induced drag reduction for commercial aircraft, the contribution by the different sources of drag need to be quantified. A breakdown based on a fluid mechanic perspective facilitates a separation into physical components [Cummings et al., 2015]. These are in particular:

- Friction drag
- Pressure drag
- Induced drag
- Wave drag
SECTION 2.1: DRAG BREAKDOWN FOR COMMERCIAL AIRCRAFT

Employing a coefficient based notation, the total aircraft drag can be approximately described by Equation 2.1:

\[ C_D = C_{Dp}(Re, c_l^2) + \frac{C_L^2}{e \cdot \Lambda \cdot \pi} + C_{Dc} \]  \hspace{1cm} (2.1)

The first term is referred to as parasite drag coefficient [Bertin and Cummings, 2009] and typically varies with the Reynolds number and the squared sectional lift coefficient. Its physical origin is related to skin friction and pressure drag [Kroo, 2000]. The creation of friction drag is caused by surfaces shear stresses. The magnitude depends on whether the boundary layer is of a laminar or a turbulent type. This is related to the Reynolds and Mach number, as well as to the surface roughness [Anderson, 2001]. Pressure drag correlates to incomplete pressure recovery at the trailing edge, caused by boundary layer separation [Anderson, 1999]. For attached flows, the amount of pressure drag remains small for subsonic flight [Anderson, 2012], but can increase significantly within the transonic flight regime. Additional pressure drag stems from the mutual aerodynamic interaction of individuals components of aircraft and is termed interference drag.

The second term corresponds to the induced drag coefficient \( C_{Di} \), which varies quadratically with the lift coefficient \( C_L \) and is dependent on the aspect ratio \( \Lambda \) and the span efficiency factor \( e \). It is an inviscid phenomenon, caused by the pressure imbalance and the merging of opposite-oriented spanwise flows, shedding vorticity from the trailing edge of a finite wing [Maughmer, 2003]. The third component is referred to as wave drag coefficient \( C_{Dc} \) and accounts for compressibility effects. Even though the decomposition based on Equation 2.1 is very common, it may not reflect all details involved in the creation of drag, as i.e. lift-dependent viscous effects are not included [Kroo, 2000].

To approximate the contribution by the parasite and induced drag component, it is expedient to neglect effects associated with flow compressibility. Equation 2.1 then simplifies to:

\[ C_D = C_{Dp} + \frac{C_L^2}{e \cdot \Lambda \cdot \pi} \]  \hspace{1cm} (2.2)
Stationary cruise flight may be performed under conditions leading to maximum lift-to-drag ratio:

\[
\left( \frac{C_L}{C_D} \right)_{\text{max}}
\]  

(2.3)

The flying velocity associated with this condition is termed minimum drag speed and facilitates maximum endurance. Substituting \( C_D \) in Equation 2.3 with the approximation of Equation 2.2 yields the following expression:

\[
\frac{C_L}{C_D} = \frac{C_L}{C_{Dp} + \frac{C^2_L}{\varepsilon \Lambda \pi}}
\]  

(2.4)

Differentiating for \( \left( \frac{C_L}{C_D} \right)_{\text{max}} \) with respect to \( C_L \) leads to Equation 2.5, which implies an equivalent drag share between the parasite and induced drag component.

\[
C_{Dp} = \frac{C^2_L}{\varepsilon \Lambda \pi}
\]  

(2.5)

Due to range and productivity considerations, commercial aircraft are usually operated at higher cruise flight velocities, comprising a smaller lift-to-drag ratio of about 90 percent of the maximum value according to Roskam [1985]. Induced drag may hence constitute only about 40% of the total drag for typical commercial aircraft at cruise conditions [Kroo, 2000]. A similar result is evident from Thomas [1985].

For commercial aircraft, flight conditions providing maximum range are associated with higher cruise flight velocities and are hence identified as a point of interest. This demands a lift-to-drag ratio of:

\[
\left( \frac{\sqrt{C_L}}{C_D} \right)_{\text{max}}
\]  

(2.6)

Replacing \( C_D \) in Equation 2.6 with the simplified expression for the total drag of Equation 2.2 and differentiating for \( \left( \frac{\sqrt{C_L}}{C_D} \right)_{\text{max}} \) results in a reduction of the induced drag share to approximately 30% of the total drag according Equation 2.7. Consequently,
the induced drag fraction can be assumed to account for about 30\% to 50\% of the total drag as explained by Schmidt-Göller [1992].

\[ C_{D_p} = 3 \cdot \frac{C_L^2}{e \cdot A \cdot \pi} \]  

(2.7)

Based on the Brequet range equation an induced drag reduction by about 1\% hence results in a fuel saving of about 0.3\% to 0.5\% or an equivalent increase in range. Taking into account that fuel continues to be the largest single cost item for the global airline industry, with a share of about 33\% of the total operating cost in the year 2008 [IATA, 2010], it becomes apparent that even minor induced drag savings are profitable. In addition, fuel savings directly translate into reduced emissions of environment affecting pollutants, i.e. with regards to carbon dioxides, and aid to achieve the objectives set in *Flightpath 2050* [Darecki et al., 2011].

The interpretation of the previous analysis has to be refined when taking other flight conditions like takeoff or initial climb phase into account. During takeoff induced drag accounts for up to 90\% of the total drag. Although takeoff constitutes only a small fraction of the entire flight envelope, drag savings in this flight regime facilitate an improved climb performance, a larger maximum takeoff mass and hence an increase in range several times than that based on cruise aerodynamics [Kroo, 2000].

### 2.2 Induced Drag

#### 2.2.1 Reference Frame Convention

The following reference frames are utilized throughout the present work unless otherwise explicitly defined, i.e. with regards to the higher-order potential-flow method and the description of the distributed vorticity element. As shown in Figure 2.1, the body-fixed reference frame is attached to the lifting surface. The \( x_a \)-axis of the aerodynamic system is aligned with the freestream velocity vector, whereas the \( y_a \) and \( z_a \)-axis are defined perpendicular to it.
2.2.2 Physical Origin

Induced drag is an inviscid phenomenon. Its existence is directly related to the generation of lift by a finite wing, which is caused by the pressure difference between the upper and lower surface. In contrast to an airfoil, a finite wing is a three-dimensional body and consequently the flow is three-dimensional as well. Resulting from the pressure imbalance, which tends to equalize at the tips, a spanwise crossflow component is introduced on the wing.

![Figure 2.1: Body-fixed and aerodynamic reference frame.](image)

**Figure 2.1:** Body-fixed and aerodynamic reference frame.

![Figure 2.2: Trailing wake of a Boeing 747-400 during cruise flight visualized by contrails.](image)

**Figure 2.2:** Trailing wake of a Boeing 747-400 during cruise flight visualized by contrails.¹

¹http://www.airliners.net/photo/Virgin-Atlantic-Airways/Boeing-747-4Q8/1122883/L/
Due to the low pressure region on the upper surface, the crossflow is directed inboards on the upper and outboards on the lower surfaces of the wing, which causes the streamlines to bend accordingly [Anderson, 2001]. Additionally, the imbalance of the crossflow velocities will produce a sheet of vorticity, which is shed from the trailing edge and rolls-up into two large vortices just inboard of the wing-tips [Bertin and Cummings, 2009]. This phenomenon is well illustrated in Figure 2.2.

The trailing wake induces a downward directed velocity component termed downwash \( w \). This tilts the local lift vector, defined perpendicular to the local freestream velocity vector \( (z_a\text{-direction}) \) and produces a parallel component in the \( x_a\text{-direction} \), the induced drag [Anderson, 2001]. Alternatively, the induced drag can be considered as the translational and rotational kinetic energy confined in the trailing wake, which must be provided by the aircraft [Schmidt-Göller, 1992]. This was first observed by Lanchester [1907].

2.2.3 The Span Efficiency Factor

To generally ensure comparability among various lifting systems and predictions from different methodological sources, it is expedient to introduce the span efficiency factor \( e \). Often misinterpreted as the Oswald factor \( E \), the span efficiency of a lifting system is defined as the ratio of the induced drag of an elliptically-loaded planar monoplane to the induced drag of the system under investigation, each with the same total lift and projected span. It corresponds to the reciprocal value of the relative induced drag of the lifting system and is defined according Equation 2.8.

\[
e = \frac{D_{i,\text{ell}}}{D_i} = \frac{(L/q_\infty)^2}{\pi \cdot b^2 \cdot (D_i/q_\infty)} \tag{2.8}
\]

The present notation is especially favorable for highly non-planar lifting systems. This is because it avoids the definition of the system aspect ratio, which is not uniquely defined for multiple-surface lifting systems such as box wings and biplanes.\(^2\) An equivalent definition of the span efficiency, more common for monoplanes is given by Equation 4.3.

\(^2\)Cone [1962] introduced the effective aspect ratio, however this definition has not been widely adapted throughout existing literature.
In contrast to the span efficiency factor, the Oswald factor contains additional losses due to the fuselage and lift-dependent viscous effects [Cummings et al., 2015; Smith, 1995].

2.2.4 Finite Wing Analysis

The first mathematical model to predict aerodynamic properties of a finite wing, in particular the lift and the induced drag, was developed by Prandtl [1923] and is referred to as the lifting-line concept. A detailed discourse of the concept and related theories is contained in Katz and Plotkin [1991], Anderson [2001] and Bertin and Cummings [2009]. It is based on the assumption of potential-flow and the work of Lanchester [1907], who recognized that the bound circulation, generating a lifting force when superimposed with a transverse flow (Kutta-Joukowsky theorem), cannot terminate at the tips, but requires a trailing vortex system to satisfy the Helmholtz theorem\(^3\).

![Lifting-line concept](image)

**Figure 2.3:** Schematic illustration of the lifting-line concept.

Within lifting-line theory, the finite wing is represented by a concentrated bound lifting vortex, varying its strength in the spanwise direction, which results in a continuous vorticity sheet aligned with the freestream direction, as illustrated in Figure 2.3. The bound circulation is commonly placed at the quarter-chord line of the wing. This location\(^3\) requires constant strength along its length and must extend to the boundaries of the fluid (i.e. \(±∞\)) or form a closed path [Anderson, 2001].
SECTION 2.2: INDUCED DRAG

represents the aerodynamic centers of airfoil sections based on thin airfoil theory [Anderson, 2001]. For a given spanwise circulation $\Gamma(y_a)$, the local vortex sheet vorticity is equivalent to the local variation in bound circulation $\gamma_{\text{wake}} = (d\Gamma/dy_a)$. The aerodynamic forces are obtained by the Kutta-Joukowsky theorem. The lift is given as:

$$L = \rho_\infty \cdot V_\infty \cdot \frac{b}{2} \int_{-b/2}^{b/2} \Gamma(y_a) \, dy_a$$  \hspace{1cm} (2.9)

In the classical lifting-line theory, induced lift contributions are neglected [Schmidt-Göller, 1992]. Thus, the computation of induced velocities in the streamwise direction is not required. The induced drag is attained from:

$$D_i = \rho_\infty \cdot \frac{b}{2} \int_{-b/2}^{b/2} \Gamma(y_a) \cdot w_i(y_a) \, dy_a$$  \hspace{1cm} (2.10)

The induced downwash velocities from each infinitesimal trailing vortex at the bound lifting-line can be computed according to the law of Biot-Savart and require an integration over the vortex sheet.

$$w_i(y_{a,0}) = -\frac{1}{4} \cdot \pi \cdot \frac{b}{2} \int_{-b/2}^{b/2} \left( \frac{d\Gamma}{dy_a} \cdot \frac{1}{y_a - y_{a,0}} \right) \, dy_a$$  \hspace{1cm} (2.11)

The method is limited to straight and unswept wings with large aspect ratio, but has been extensively developed from its original formulation to extend its applicability and to improve its accuracy [Multhopp, 1938; Weissinger, 1947]. It further provides the basis for several computational methodologies, i.e. the vortex-lattice method. Details of these, more advanced techniques are discussed in Section 3.3.1.

Even more sophisticated, computational methods to predict the aerodynamic properties of a finite wing became practical with the advent of efficient computer systems, commonly providing numerical solutions of the Euler or RANS equations. With regards to the purpose of the present study, a methodology based on the Euler equations, depicting the most comprehensive inviscid flow model, is expedient. An introduction into this topic
is given in Section 3.2. Related issues evident in Euler-based induced drag prediction as well as measures implemented to avoid these effects will be discussed discussed.

2.3 Concepts for Induced Drag Reduction

2.3.1 Increase in Wing Span

As indicated by 2.1, an induced drag reduction can simply be attained by a higher aspect ratio at fixed wing area or more directly by an increase in span. However, this approach is often prevented by practical considerations and is also constrained by airport regulations. The increased structural weight alleviates potential performance gains and requires a compromise between aerodynamic and structural efficiency. An induced drag reduction by means of a span increase beyond existing limits may hence not be practical, but must be carefully reviewed when new technologies change the relative sensitivities of induced drag and structural weight to the wing span [Kroo, 2000]. This implies, that induced drag minimization is certainly a multidisciplinary topic.

Figure 2.4: Strut-braced wing concept for the subsonic ultra green aircraft research SUGAR Freeze.\(^4\)

To avoid structural weight penalties while exploiting the benefits of a high aspect ratio wing, strut or truss-braced wing concepts have been proposed recently [Gern et al., 2001a,b, 2005; Bradley and Droney, 2012].

\(^4\)http://www.boeing.com/aboutus/environment/environment_report_14/2.3_future_flight.html
2.3.2 Planform Design

Induced drag reduction by means of planform design aims to take positive influence on the span efficiency factor. This can be accomplished i.e. by adapting a specific lift distribution, by promoting favorable wing wake interactions or by a reallocation of the shed vorticity over a larger effective span.

Planar Lifting Systems

Within lifting-line theory, an elliptical circulation distribution, producing constant downwash, leads to minimum induced drag for a planar wing of given span and lift. Expressing this in terms of the span efficiency factor yields the theoretical optimum of $e = 1.00$, which provides the reference baseline for comparisons among different wing concepts. For a system without geometric nor aerodynamic twist, minimum induced drag is obtained with an elliptical distribution of the chord length along its span. However, employing more sophisticated numerical models it has been demonstrated that this is only (approximately) correct in the case of a straight and unswept trailing edge [Mortara and Maughner, 1993; Lam and Maull, 1993; Smith and Kroo, 1993; DeHaan, 1990]. This is in contrast to the classical elliptical planform and an unswept quarter-chord line considered by Prandtl [1923].

Figure 2.5: Raked wing-tip on a Boeing 767-400.\(^5\)

\(^5\)http://www.aerospaceweb.org/aircraft/jetliner/b767/pics03.shtml
Originating from the work of van Dam [1987], the shape and in particular the sweep of planar wing-tips was studied extensively to reduce the induced drag [Smith and Kroo, 1993; DeHaan, 1990], which results from wake deformation effects and changes in spanwise load distribution [Vigen et al., 1989]. This effort has successfully developed into real application, i.e. for the Boeing 767-400, the Boeing 777-300ER or the Dornier 228, and is commonly referred to as a sheared or raked wing-tip, as shown in Figure 2.5.

### Wing-Tip Devices

A wide range of different wing-tip devices has been explored [Berens, 2008]. Among these the wing-grid [La Roche and Palffy, 1996], tip-sails [Spillman and Allen, 1978], the vortex-diffuser [Hackett, 1980] or the spiroid-winglet [Gratzer, 1992]. Most common are winglets, which are largely based on the work of Whitcomb [1976]. They extend the wing in the vertical direction and shift the shed vorticity away from the wing plane [Kroo, 2000], reducing the average downwash.

![Figure 2.6: Split-tip winglet proposed for the Boeing 737-MAX series.](http://www.boeing.com/commercial/737max/)

Interestingly, an equivalent winglet performs quite similarly to a box wing. The reason for this is the concentration of vorticity near the wing tip. It is thus the vertical extent...
SECTION 2.3: CONCEPTS FOR INDUCED DRAG REDUCTION

or the so-called spanwise camber which is the critical parameter [Lowson, 1990]. Based on aero-structural optimization, winglets may be preferred over span extensions, especially if the weight or span is constrained [Jansen et al., 2010]. However, the interdisciplinary complexity to attain an optimal design is involved and depends on the employed methodology and the application scenario, preventing a general conclusion [Kroo, 2000].

An advanced and recent example of a split-tip winglet proposed for the Boeing 737-MAX series is given in Figure 2.6. The concept is inspired by the outer primaries of birds, diffusing the single tip-vortex into a number of smaller vortices to gain additional induced drag savings.

Highly Non-Planar Lifting Systems

The class of highly non-planar concepts encompasses a variety of different configurations such as the box wing, the biplane, the joined-wing and the c-wing concept. Their physical principle of induced drag reduction is basically related to a redistribution of the shed vorticity over an increased effective span, ultimately reducing the average downwash velocity of the system compared to a planar wing of equivalent span and lift [Kroo, 2000].

The box wing is a closed lifting system, which is composed of two horizontal wings that are joined at their tips by a vertical wing. The horizontal elements are separated vertically, may additionally be staggered in the longitudinal direction and can incorporate sweep. Within linear potential-theory, the box wing achieves the highest span efficiency or lowest induced drag compared to an optimum monoplane system of equivalent lift and span. This was shown by Prandtl [1924] based on lifting-line theory, or more recently by DeYoung [1980], Frediani and Montanari [2009] and Demasi [2007] employing lifting-line theory and the small-perturbation acceleration potential, providing minimum induced drag theorems for joined-wings, closed systems and generic biplanes [Demasi et al., 2015a,b].

The fact that the box wing is closed does physically allow it to attain non-zero circulation at the wing-tips. This prevents the occurrence of high circulation gradients associated with strong tip-vortices for wings of finite span, and gives an alternative explanation for induced drag saving associated with this configuration [Demasi et al., 2015a]. However, this does not mean that the tip-vortex is eliminated.
Studies considering an implementation as a commercial transport long-range aircraft originate from the work of Miranda [1972] and Lange et al. [1972, 1974], whereas its general aerodynamic characteristics were investigated by Henderson and Huffman [1975] and Gall and Smith [1987]. In the recent past, the concept has attracted research again and led to several conceptual studies i.e. by Frediani [2005]. Additional emphasis was laid on non-linearities emerging from its structural over-constrained characteristics [Cavallaro and Demasi, 2013; Demasi et al., 2013; Cavallaro et al., 2012; Kim et al., 2008; Blair et al., 2005]. Investigations into aero-structural aspects were conducted to explore the largely unknown trade-off between both disciplines for commercial aircraft by Jansen et al. [2010], Jansen and Perez [2010], Jemitola and Fielding [2012] and Salam and Bil [2015, 2012a,b]. Related to the strong coupling between aerodynamics and structural weight that results from the closed wing arrangement, this is found to be key to enable an efficient design.

In contrast to earlier research effort and Frediani [2005], the latter studies favor a short-range application. This is motivated by a larger reduction potential, due to the increased fraction of flight conditions involving high lift coefficients and hence induced drag compared to the long-range scenario. Moreover, the available fuel capacity is usually problematic and constrains a long-range application. Salam and Bil [2015] showed that structural aspects, in particular the span dictates the overall performance, which is consistent with the findings by Jansen and Perez [2010], noting that the trade-off between aerodynamic

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7http://www.nasa.gov/topics/aeronautics/features/greener_aircraft.html
efficiency and structural weight shifts towards structure. The reduced span results in considerable weight compared to a planar reference, while maintaining near equivalent aerodynamic efficiency [Jansen and Perez, 2010]. This is also in good agreement with the generally anticipated requirement for a light weight wing design for short-range purpose [Frota and Vigneron, 2005]. It may thus be expedient to conduct comparative studies based on equivalent induced drag rather than span.

Further research effort has been directed into the investigation of the influence of planform parameters and the arrangement of lifting surfaces on the induced drag or aerodynamic efficiency [Demasi et al., 2016, 2015a,b, 2014; Hicken and Zingg, 2010; Kang et al., 2009a,b; Mamla and Galinski, 2009; Rizzo, 2007; Demasi, 2007, 2006; Ahmed and Archer, 2000]. While enforcing static longitudinal stability constraints, the impact of specific planform parameter on the aerodynamic efficiency was explored by Andrews and Perez [2015] and Gagnon and Zingg [2015]. They show that the drag increase associated with the non-equivalent lift share among both wings is less severe (if any) than anticipated, supporting findings by Demasi et al. [2015a].

The biplane concept is a classical example to facilitate an induced drag reduction. Early representatives incorporated thin airfoil sections and thus aimed to exploit its structural advantages. They however suffered from major parasite drag penalties due to struts and cables required to attain structural integrity. With the advent of structurally efficient cantilevered wings, the approach became obsolete, also due to penalties in fuel capacity and viscous drag.

![Figure 2.8: Tandem wing ultra-long endurance UAV ADCOM United 40](http://defense-update.com/20130720_russia_to_buy_drones_from_uae.html)
Recently, staggered biplane concepts (tandem wing) have been reconsidered i.e. by Bramesfeld and Malik [2015], Maqsood and Go [2013] and Moschetta and Thipyopas [2007] for the purpose of a micro aerial vehicle (MAV), which are typically limited in span due to operating in confined spaces. The impact of the vertical arrangement on the aerodynamic characteristics, in particular the induced drag was concerned by Demasi et al. [2016, 2015a,b], Broering and Lian [2010], Rizzo [2007], Demasi [2006], Traub [2001], DeYoung [1980], Diehl [1934], Prandtl [1924] and Munk [1923b]. Stagger effects were investigated i.e. by Traub [2001], Selberg [1983], Addoms and Spaid [1975] and Norton [1921]. A tandem concept of an ultra-long endurance unmanned aerial vehicle (UAV) is shown in Figure 2.8.

The joined-wing concept resembles a tandem arrangement. Its swept lifting surfaces are joined, most commonly at the tips to form a diamond-like shape in both the top and front views. A comprehensive introduction into this topic is given in Wolkowitch [1986]. Preliminary experimental investigations were conducted by Cahill and Dexter [1954] and Henderson and Huffman [1975]. Later the aerodynamic and structural characteristics of joined-wing configurations were intensively analyzed i.e. by Samuels [1982], Selberg and Cronin [1986], Kroo et al. [1988] and Hirokazu et al. [1988]. This also accompanied the development of the NASA joined-wing research aircraft [Smith et al., 1987].

![Figure 2.9: Joined-wing high altitude long endurance UAV Xianlong.](http://china-defense.blogspot.de/2013/01/photos-of-day-xianglong-soaring-dragon.html)

Benefits claimed for joined-wing configuration are related to induced drag savings, but also refer to structural aspect providing a stiff and lightweight design. Aero-structural
optimized designs that employ more inboard joint locations were found to achieve drag reductions of 5% at fixed weight compared to a trimmed, equivalent planar concept [Kroo et al., 1988].

With respect to a commercial application the joined-wing configuration has so far not emerged from a conceptual status [Foong and Djojodihardjo, 2012; Sun et al., 2008], but has been applied to several unmanned aerial vehicles as exemplary illustrated in Figure 2.9.

The c-wing configuration extends the concept of a winglet by introducing an additional, horizontal surface directed inboard, which results in a c-like shape. Based on potential-flow methodology and an evolutionary optimization algorithm subjected to span and height constraints, this arrangement was first obtained by Gage [1995]. It offers similar induced drag savings than the box wing while reducing viscous drag penalties [Jansen et al., 2010]. Although this has led to some research interest [Isikveren et al., 2012; McMasters and Kroo, 1998], compared to a conventional winglet or a span extension by means of a raked wing-tip, performance gains cannot be confirmed based on aero-structural investigations [Jansen et al., 2010; Verstraeten and Slingerland, 2009].

Figure 2.10: C-wing concept by Bauhaus Luftfahrt.\textsuperscript{10}

\textsuperscript{10}http://www.bauhaus-luftfahrt.net/presse-medien/ila-2012/08-der-ce-liner
2.4 Related Theory

2.4.1 The Stagger Theorem

The stagger theorem is of fundamental importance within (linear) potential-flow methodology. Based on lifting-line theory and provided that the circulation distribution on each individual surface is held constant, Munk [1923a] showed that the streamwise separation between the lifting elements can be varied arbitrarily without changing the total induced drag of the system. It is also referred to as Munk’s first theorem. As an example, for constraint circulation distributions, the induced drag of the unswept lifting-line is equivalent to the swept lifting-line as illustrated schematically in Figure 2.11.

However, formal difficulties with the application of this concept exist. Due to the representation with a single bound vortex, the wake shed at the trailing edge is commonly not equivalent to the quarter-chord condition assumed by classical lifting-line theory [Lowson, 1990]. In the case of a planform with non-zero trailing edge sweep, the wake shape depends on the incidence angle, forming a non-planar wake trace when projected onto the Trefftz plane in the freestream direction, as illustrated in Figure 2.12. This results in a larger effective span, which implies span efficiency factors larger than based on the quarter-chord condition.

![Figure 2.11: Schematic illustration of Munk’s stagger theorem, adapted from Gall [1984].](image)
Also Kroo and Smith [1990] note that a derivation of the stagger theorem based on the lifting-line concept is unnecessarily restrictive and present a more general approach by means of momentum conservation. They show that the induced drag is defined only by the shape of the wake and its vorticity distribution, rather than the shape of the lifting-line. This indicates some difference towards lifting-line theory, but is still within linear potential-flow theory [Smith, 1995]. Moreover for highly non-planar systems, maintaining equivalent circulation distribution while translating lifting surfaces in the streamwise direction, requires adjustment of the incidence angle of the individual lifting sections accordingly. This subsequently alters the wake shape and hence the induced drag. Conclusively, the induced drag must be considered to be a function of the longitudinal arrangement.

2.4.2 The Wake Substitution Concept

Aiming to compute the influence of wake shape on Trefftz plane induced drag, Smith [1995] developed the hypothesis of streamwise wake substitution. For this purpose, an intermediate surface is placed perpendicular to the freestream velocity vector between the lifting element and the Trefftz plane. Downstream of this partition, the force-free wake can be replaced by a drag-free projection of the wake trace extending in the freestream direction.

The implication of the partition location on the induced drag are of high importance for the present study. Assuming that streamwise variations of the deflection angle and curvature of the vortex filaments may be neglected, substitution can be made in close proximity to the lifting system [Smith, 1995]. As the intermediate surface is required to be
positioned perpendicular to the freestream velocity vector, the downstream intersection of
lifting system and partition surface defines the extent of the possible replacement of the
force-free wake.

In the case of a planform with an unswept trailing edge, the complete force-free wake
and its influence can successfully be removed. For systems where portions of the force-
free wake remain after the substitution, higher-order effects on the induced drag are likely
to exist [Smith, 1995]. Even though the impact may be limited for simple planar wings,
highly non-planar systems like the box wing can incorporate large longitudinal separations
of lifting surfaces, which provide an increased potential to exhibit non-linear wing-wake
interactions.

2.4.3 The Biplane Theorem

Employing the lifting-line theory, Munk [1923b] and Prandtl [1924] derived expressions
to calculate the induced drag of an elliptically-loaded biplane with arbitrary vertical gap.
Further contribution was made by Diehl [1934]. For a system with equal projected spans
$b$, elliptical spanwise loadings and assuming a freestream-fixed wake, the induced drag can
be estimated, according to Prandtl [1924], by Equation 2.12.

\[
D_i = \frac{2}{\pi \cdot \rho_{\infty} \cdot V_{\infty}^2} \cdot \left( \frac{L_1^2}{b^2} + 2 \cdot \sigma \cdot \frac{L_1 \cdot L_2}{b^2} + \frac{L_2^2}{b^2} \right)
\]  

(2.12)

The total induced drag is the sum of the induced drag created by each isolated lifting
surface and a mutual induction. For the special case of an unstaggered system of equal
projected spans, the mutual induced drag of upper (1) and lower wing (2) is equivalent.
Due to the downwash of the upper wing, the lift vector of the lower wing is rotated in the
streamwise direction, resulting in an additional induced drag component. Based on small
angle approximations, the mutual induced drag is:

\[
D_{i, 1,2} = \int \frac{w_{1.2}}{V_{\infty}} dL_2
\]

(2.13)

The induced downwash velocity is given by:
\[ w_{1,2} = \frac{2 \cdot L_1}{\pi \cdot V_\infty \cdot b^2} \cdot z \] (2.14)

In Prandtl [1924], Equation 2.13 is computed using planimetry measurements for different vertical gaps, describing the mutual induced drag by the interference factor \( \sigma \) according to Equation 2.15.

\[ D_{i,1,2} = D_{i,2,1} = \frac{2 \cdot \sigma}{\pi \cdot \rho_\infty \cdot V_\infty^2 \cdot b^2} \cdot \left( \frac{L_1 \cdot L_2}{b^2} \right) \] (2.15)

For an unstaggered system of equal projected spans the interference factor solely depends on the height-to-span ratio and can be approximated by Equation 2.16.

\[ \sigma \approx \frac{1.00 - 0.66 \cdot (h/b)}{1.05 + 3.70 \cdot (h/b)} \] (2.16)

For equivalent spans, the lift division between both wings providing the minimum induced drag is found by substituting \( L_2 = L \cdot r \) and \( L_1 = L \cdot (r - 1) \) in Equation 2.13. Differentiating with respect to the lift division factor \( r \) yields:

\[ r = \frac{1}{2} \] (2.17)

Thus, the lift must be distributed equally between both wings. Inserting this result into Equation 2.13 gives the minimum induced drag an elliptical spanwise loading:

\[ D_{i, \text{min}} = \frac{L^2}{\pi \cdot \rho_\infty \cdot V_\infty^2 \cdot b^2} \cdot \frac{2}{1 + \sigma} \] (2.18)

The maximum span efficiency can be estimated using Equation 2.19. It is obvious, that with increasing vertical gap, the span efficiency improves as well.

\[ e_{\text{Bi, ell}} = \frac{2}{1 + \sigma} \] (2.19)

Extending the biplane concept, Prandtl [1924] presents the box wing configuration, termed Best Wing System, which represents the limiting case of a multiplane with an
infinite number of lifting elements. For given lift and height-to-span ratio, the box wing is found to achieve the least induced drag of all systems. Its optimum span efficiency factor can be approximated with Equation 2.20 and is only dependent on the height-to-span ratio. Unfortunately no further details of the derivation are provided in Prandtl [1924].

\[
e_{\text{Bw, opt}} \approx \frac{1.04 + 2.81 \cdot (h/b)}{1.00 + 0.45 \cdot (h/b)}
\]  

(2.20)

Nevertheless, similar relations for the biplane or the box wing have been presented [Pistoleti, 1932; DeYoung, 1980; Demasi, 2007; Rizzo, 2007; Frediani and Montanari, 2009; Demasi et al., 2014, 2015a,b]. Cone [1962] theoretically investigated closed lifting arcs such as circles, ellipses, ovals or rectangles and described the variation of the span efficiency by means of the camber factor, defined as the ratio of minor to major axis. Related work has been carried out by Demasi et al. [2014], including determination of optimum circulation distributions for a variety of lifting arcs and c-wing concepts. Among others, it is demonstrated, that a quasi-closed c-wing presents the optimal induced drag of an equivalent closed system.

With regards to biplanes, Demasi [2006] stressed the fact that the optimum lift distribution is elliptical only if the distance between the wings approaches zero or infinity, which leads to slightly different span efficiency estimates compared to the classical theory, as apparent from Figure 2.13. Moreover it is demonstrated, that for zero vertical gap, the biplane span efficiency is equivalent to the optimum planar monoplane result. For an infinite vertical separation both wings can be treated individually, which results in a doubling in span efficiency [Demasi, 2006; Rizzo, 2007]. Both findings deviate from the approximation presented by Equation 2.19, which is valid according to Prandtl [1924] for height-to-span ratios between \((h/b) = 1/15\) to \((h/b) = 1/2\).

Subsequent research effort by Demasi et al. [2015a,b] concerned a similar formal difficulty with regards to the asymptotic behavior of Equation 2.20 for box wings in the case of an infinite vertical gap. Although it is demonstrated that the asymptotic value is incorrect, the approximation is in excellent correlation with DeYoung [1980] and Frediani
and Montanari [2009] and sufficient to assess the span efficiency of the box wing for any practical range of height-to-span ratios ((h/b) ≈ 0.10 – 0.30).

In the case of the box wing, Demasi et al. [2015a,b] for the first time formally derived the optimal span loading over both horizontal wings and vertical joints, qualitatively confirming the findings of von Karman and Burgers [1935] and the assumption made by Frediani and Montanari [2009]. For small vertical separations of practical interest, the optimal lift distribution providing minimum induced drag is approximately composed of a constant and an elliptical part for both horizontal wings, whereas on the vertical wings a butterfly-type shape is adapted. According to the general condition for minimum induced drag under constrained lift [Munk, 1923a], the optimal normalwash velocity varies with the cosine of the dihedral angle Θ of the wing.

\[ w_n = w_i \cdot \cos(\Theta) \]  

This results in induced velocities identically zero at the joints and a constant downwash over horizontal wings [Frediani and Montanari, 2009]. However, Demasi et al. [2015a,b] showed in addition that the shape of the optimum distribution depends on the height-to-span ratio. In particular, with increasing height-to-span ratio, the distribution on horizontal wings is affected by a constant term \( A_0 \), whereas on vertical wings a linear term \( B_0 \).
becomes dominant. For a height-to-span ratio of \((h/b) = 0.20\) the optimal circulation distribution is described in good approximation according to Demasi et al. [2015b] by:

- **Horizontal wings**
  \[
  \Gamma_{opt} \approx \left( A_0 + \sqrt{1 - \eta^2} \cdot (A_1 + A_2 \cdot \eta^2) \right) \cdot e_{Bw, \, opt} \tag{2.22}
  \]

- **Vertical wings**
  \[
  \Gamma_{opt} \approx \left( B_0 \cdot \xi + B_1 \cdot \xi^3 \right) \cdot e_{Bw, \, opt} \tag{2.23}
  \]

with the constants \(A_0 = 0.1660\), \(A_1 = 0.2920\), \(A_2 = 0.0129\), \(B_0 = 0.7870\), \(B_1 = 3.8290\) and \(e_{Bw, \, opt} = 1.46\).

Moreover it is demonstrated in Demasi et al. [2015a], that the optimum circulation is not unique. Acknowledging that induced drag is directly related to the gradients of circulation, it is shown that a constant amount of circulation, which is by definition gradient-free, can be added without induced drag penalty. This is of great practical benefit to satisfy static longitudinal stability while maintaining constant aerodynamic efficiency [Andrews and Perez, 2015].

![Diagram of optimal circulation distribution](image)

**Figure 2.14:** Optimal circulation distribution for box wing configurations with a height-to-span ratio of \((h/b) = 0.20\) according to Demasi et al. [2015b].

Based on lifting-line theory, Munk [1923b] presented an approach to compute the effect of longitudinal separation on the lift under the assumption of an elliptical spanload. Similar expressions were obtained by Diehl [1934]. For small values of streamwise stagger \((|St| \leq 1/3 \cdot c)\), the increase or decrease in lift for two wings of equivalent span can be approximated according to Munk [1923b] as:
\[ \Delta C_L = \pm 2 \cdot C_L \cdot \frac{1}{\Lambda} \left( \frac{1}{\epsilon_{Bi, ell}} - 0.5 \right) \cdot \frac{b}{R} \cdot \frac{St}{b} \]  

(2.24)

The term \( \left( \frac{1}{\epsilon_{Bi, ell}} - 0.5 \right) \cdot \frac{b}{R} \) is referred to as the Munk factor and describes the aerodynamic induction. Its magnitude depends on the height-to-span ratio of the system as shown in Figure 2.15 and is similar to a change in effective angles of attack by equal amounts but opposite directions [Munk, 1923b].

![Figure 2.15: The Munk Factor versus the height-to-span ratio (h/b) based on tabulated values given by Munk [1923b].](image)

The quantity \( R \) gives the distance between lifting surface and streamwise wake, which is required in the computation of the aerodynamic induction. With increasing vertical gap, the amount of lift increase or decrease is diminished due to the reduced mutual interference between the lifting elements, but is independent of the direction of stagger. The total lift of the system is not changed. Kang et al. [2013] have found this in disagreement with several practical applications and proposed an empirically derived biplane lift equation to account for the direction of streamwise separation on the total lift.

\[ C_L = (\left( -0.0072 \cdot St + 0.0145 \right) \cdot h + (0.018 \cdot St + 0.0499)) \cdot \alpha \cdot \left( 0.75 + \frac{1.5}{\Lambda} \right) \]  

(2.25)

However, in contrast to Equation 2.24, the derivation of Equation 2.25 is based on the real, force-free wake shape and equivalent geometric rather than aerodynamic equivalent vertical gaps. It is thus not correct to compare lift estimates from both equations directly.
Based on Equations 2.16 and 2.20 and further acknowledging the implications given by the wake substitution concept [Smith, 1995], two key design parameter appear relevant with regards to the induced drag of biplane and box wing concepts. These are in particular:

- The height-to-span ratio \((h/b)\)

- The stagger factor \(St\)

The height-to-span ratio gives the relative vertical distance between the trailing edges of the lifting surfaces. The longitudinal stagger factor describes the horizontal separation between trailing edges normalized by the reference chord length and is considered positive in the positive \(x\)-direction. Both, the height-to-span ratio and the stagger factor are described in the body-fixed \(xyz\)-reference frame.

In contrast to other research effort, it is herein distinguished between (i) the geometric height-to-span ratio \((h/b)\), (ii) the freestream height-to-span ratio \((h/b)_{\infty}\) and (iii) the effective height-to-span ratio \((h/b)_{\text{eff}}\). The freestream height-to-span ratio is defined as the relative vertical extent of the system perpendicular to the freestream velocity vector, whereas the effective height-to-span ratio gives the relative aerodynamic gap of the trailing wake trace. Similarly, the freestream stagger factor \(St_{\infty}\) is defined as the relative streamwise separation between trailing edges. Referred to these definitions, additional discussion is provided within the further scope of this work where appropriate.
CHAPTER 3

Theoretical Background

This chapter presents the relevant governing equations of fluid mechanics and gives a brief review of inviscid computational methodologies. The implemented computational approaches, a commercial CFD-code to solve the Euler equations and a higher-order potential-flow method are discussed more in detail. Emphasis is further made on techniques enabling an accurate prediction of the induced drag, which is of particular concern for the Euler-flow method.

3.1 Governing Equations of Fluid Mechanics

A short discussion of governing equations and their characteristics is given in this section. For a detailed derivation including a more thorough discourse reference is made i.e. to Oertel et al. [2006] or Anderson [2001, 1995]. The connection between various approximations to the general governing equations of fluid mechanics, the Navier-Stokes equations, is depicted in Figure 3.1. These basic equations, containing the fundamental physics of fluid flows, describe the conservation of momentum, mass and energy respectively. With regards to the induced drag, an inviscid phenomenon, the Euler and potential equations are of particular interest for the present work.
3.1.1 Equations for Viscous Flow - The Navier-Stokes Equations

A viscous flow includes dissipative phenomena related to friction and thermal conduction, both increasing the entropy of the flow [Anderson, 1995]. Provided the flow is steady and body forces are neglected, the governing equations within the three-dimensional domain are given as follows:

• Conservation of momentum

\[
\rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \tag{3.1}
\]

\[
\rho \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \tag{3.2}
\]

\[
\rho \left( u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \tag{3.3}
\]

• Conservation of mass

\[
\frac{\partial (\rho \cdot u)}{\partial x} + \frac{\partial (\rho \cdot v)}{\partial y} + \frac{\partial (\rho \cdot w)}{\partial z} \tag{3.4}
\]

• Conservation of energy

\[
\rho \left( u \frac{\partial e_l}{\partial x} + v \frac{\partial e_l}{\partial y} + w \frac{\partial e_l}{\partial z} \right) = -p \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \rho \cdot \dot{\Omega} + k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) \tag{3.5}
\]

\[
+ \tau_{xx} \cdot \frac{\partial u}{\partial x} + \tau_{yx} \cdot \frac{\partial u}{\partial y} + \tau_{zx} \cdot \frac{\partial u}{\partial z} + \tau_{xy} \cdot \frac{\partial v}{\partial x} + \tau_{yy} \cdot \frac{\partial v}{\partial y} + \tau_{zy} \cdot \frac{\partial v}{\partial z} \tag{3.5}
\]

\[
+ \tau_{xz} \cdot \frac{\partial w}{\partial x} + \tau_{yz} \cdot \frac{\partial w}{\partial y} + \tau_{zz} \cdot \frac{\partial w}{\partial z} \tag{3.5}
\]
The scalar variables $p$, $\rho$, $e_t$, $T$ and $\dot{q}$ denote the pressure, the fluid density, the total energy, the temperature and the heat flux respectively, whereas the quantities $u$, $v$ and $w$ represent the velocity vector components with respect to the spatial Cartesian coordinates $x$, $y$ and $z$. The components of the viscous stress tensor are given by $\tau$; the parameter $k$ describes the thermal conductivity of the fluid. Within computational fluid dynamics, these equations are known as the Navier-Stokes equations\footnote{This terminology is historically not strictly correct [Oertel et al., 2006]. In a theoretical fluid dynamics perspective, the Navier-Stokes equations only refer to the conservation of momentum.} [Anderson, 1995], forming a system of second-order partial differential equations, which are very difficult to solve analytically. To date, no general closed-form solutions have been found [Anderson, 2001], which hence requires numerical approaches for adequate approximations. Their inertial components are named convection terms. The diffusion terms are related to the stress tensor and to the fluid viscosity.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{connection.pdf}
\caption{Connection between various approximations to the Navier-Stokes equations, adapted from Cummings et al. [2015].}
\end{figure}

### 3.1.2 Equations for Inviscid Flow - The Euler Equations

An inviscid flow is defined as a flow where the dissipative transport phenomena of viscosity and thermal conductivity are neglected [Anderson, 1995]. For a steady, inviscid flow in the three-dimensional domain, the governing equations accord to the Navier-Stokes equations without viscous terms and reduce to:
\textbf{CHAPTER 3: THEORETICAL BACKGROUND}

- Conservation of momentum

\[ \rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} \] (3.6)

\[ \rho \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} \] (3.7)

\[ \rho \left( u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} \] (3.8)

- Conservation of mass

\[ \frac{\partial (\rho \cdot u)}{\partial x} + \frac{\partial (\rho \cdot v)}{\partial y} + \frac{\partial (\rho \cdot w)}{\partial z} \] (3.9)

- Conservation of energy

\[ \rho \left( u \frac{\partial e}{\partial x} + v \frac{\partial e}{\partial y} + w \frac{\partial e}{\partial z} \right) = -p \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \rho \cdot \dot{q} \] (3.10)

This system of equation is termed the Euler equations, forming a set of second-order partial differential equations, which constitute the most comprehensive description of inviscid flows. Historically, the energy equation was not considered.

\subsection*{3.1.3 Equations for Irrotational Flow - The Potential Equations}

An irrotational flow is free of any vorticity and can be described by a velocity potential \( \Phi \), which is a scalar function of the spatial coordinates and gives the velocity by the gradients of \( \Phi \). As the vorticity is equivalent to the curl of the velocity vector, the angular velocity of a fluid element moving along a streamline is thus zero. Provided the flow remains isentropic, this condition is for instance not satisfied within the viscous boundary layer. Consequently, an irrotational flow is inviscid as well.

- Condition of irrotational flow

\[ \xi = \nabla \times \vec{V} = 0 \] (3.11)
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- Velocity potential

\[ \vec{V} = \nabla \Phi \quad (3.12) \]

The potential function enables a substantial simplification in the computation of the velocity field by reducing the amount of unknown variables. Instead of providing solutions for each individual velocity component \((u, v, w)\), their potential can be obtained. This requires only one equation for one unknown. The components of the velocity potential in Cartesian coordinates are defined as follows:

\[ u = \frac{\partial \Phi}{\partial x}, \quad v = \frac{\partial \Phi}{\partial y}, \quad w = \frac{\partial \Phi}{\partial z} \quad (3.13) \]

To derive a conditional expression for the potential function, the velocity components of the continuity equation are replaced according to Equation 3.13. This yields the compressible or full-potential equation, which is a second-order non-linear differential equation.

- Compressible potential-flow equation

\[ \rho \cdot \left( \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \right) + \frac{\partial \Phi}{\partial x} \cdot \frac{\partial \rho}{\partial x} + \frac{\partial \Phi}{\partial y} \cdot \frac{\partial \rho}{\partial y} + \frac{\partial \Phi}{\partial z} \cdot \frac{\partial \rho}{\partial z} = 0 \quad (3.14) \]

Equation 3.14 can be further rearranged to attain a more convenient expression (compare Oertel et al. [2006]). If the flow is considered incompressible as well, the conservation of mass yields:

\[ \frac{\partial (\rho \cdot u)}{\partial x} + \frac{\partial (\rho \cdot v)}{\partial y} + \frac{\partial (\rho \cdot w)}{\partial z} = \nabla \cdot \vec{V} = 0 \quad (3.15) \]

Combining Equations 3.12 and 3.15 results in the potential-flow equation for an incompressible flow. It accords to the Laplace equation and depicts a linear partial differential equation of second-order.

- Incompressible potential-flow equation

\[ \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad (3.16) \]
The fact that the Laplace equation is linear is particularly important. It allows to synthesize a complex flow of practical interest by a superposition of several elementary flows, i.e. a source, a sink or a vortex flow, for which an analytical solution exits. These elementary flows are termed singularities.

The Laplace equation is exact and does not require the assumption of small disturbances [Cummings et al., 2015] and is further mathematically classified as of an elliptic type [Chapra and Canale, 2010]. Thus, a solution needs to be attained only on the flow boundaries, which significantly reduces the computational effort [Berens, 2008]. To produce the correct flow pattern, boundary conditions are imposed and require the flow to be tangent to the body surface (wall boundary condition) and the flow to be undisturbed at an infinite distance away from the body (infinity boundary condition).

3.2 Euler-Flow

To a great extent, the numerical solution of the Euler equations is based on finite volume formulations to solve the three-dimensional Euler equations in their integral form. This is available as an integral part of most commercial, multiple-purpose CFD-codes like \textit{STAR-CCM+} [CD-Adapco, 2013] and \textit{Fluent} [ANSYS, 2016], stems from non-commercial distributions like \textit{OpenFOAM} [The OpenFOAM Foundation, 2016] or \textit{SU2} [Palacios et al., 2013] or by research institutions like the \textit{TAU}-code [Schwamborn et al., 1999] or the \textit{FUN3D} flow-solver [Biedron et al., 2015]. Dedicated Euler-flow solver exist like the high-fidelity inviscid analysis package \textit{Cart3D} [Melton et al., 1995]. Although the Euler equations describe inviscid flow most precisely, an accurate induced drag estimation is of concern as to be discussed in Section 3.2.1.

3.2.1 Implemented Euler-Flow Method

The three-dimensional Euler equations in their integral form are solved by means of the commercial, cell-centered finite volume method \textit{STAR-CCM+}, version 8.04.010. The spatial discretization of the convective flux terms involves a second-order upwind scheme. Gradient computations are based on a hybrid Gauss least-squares method along with a
Venkatakrishnan gradient reconstructing limiter approach [Venkatakrishnan, 1993]. The segregated-flow model is used. The present method is based on a collocated arrangement of flow variables [Ferziger and Perić, 2002], using a Rhie-Chow interpolation [Rhie and Chow, 1983] for the pressure-velocity coupling to avoid non-physical pressure oscillations [Moguen et al., 2011] combined with a SIMPLE-type algorithm [Patankar, 1980] to control the overall solution process. Linear equation systems for the momentum, the energy and the pressure correction equation are solved using an algebraic multigrid solver (AMG) combined with a bi-conjugate gradient stabilized method to accelerate individual convergence and robustness.

**Lift and Induced Drag Estimation**

The lift is simply computed by means of a surface pressure integration. The accurate induced drag prediction is however challenging. In parts, the problem is associated with the application of the surface pressure integration technique, a nearfield approach, to extract the induced drag. Difficulties emerge due to the approximation of the curved surface of the lifting element with flat facets [Hunt et al., 1999], which leads to inaccuracies caused by the cancellation of opposing pressure forces close in magnitude [Nikfetrat et al., 1992]. Although refining may introduce some improvement, it does not generally remove the problem of spurious entropy creation, an artificial phenomenon related to numerical viscosity, but increases the computational effort. Artificial viscosity may be introduced explicitly for stability reasons in the case of central space schemes [Bourdin, 2002], or implicitly as a result of numerical smoothing in regions with high velocity gradients on inadequate dense grids (truncation error) [Giles and Cummings, 1999]. Both affect the surface pressure distribution, especially in the stagnation area near the leading edge and the recovery region close to the trailing edge [van Dam and Nikfetrat, 1992]. The creation of artificial viscosity in the flowfield is not exclusively dependent on the grid density, but also on its quality [Ueno et al., 2013]. Whereas the influence on the lift prediction can be neglected, the impact on the induced drag is significant [van Dam et al., 1991]. An alternative method is thus required to enable accurate induced drag prediction.

In principle, an estimation of aerodynamic forces is possible based on a body or fluid
flow perspective [van Dam, 1999], which is a consequence of Newton’s third law of motion. Employing the principle of momentum conservation on a fluid-fixed control volume, the forces imposed by an enclosed body on the fluid bound within the control volume can be determined. This is referred to as a farfield approach and is discussed in detail by Destarac and van der Vooren [2004].

Figure 3.2: Schematic illustration of the control volume, the transverse plane (TP) and the correction volume $\Omega_{wake}$ enclosing the wake region to compute the spurious entropy drag contribution.

Differences in the application emerge with respect to the definition of the control volume and the subsequently performed integration process, which is dependent on the present flow problem and computational methodology. The estimation can involve an integration over the control volume surfaces [Hue and Esquieu, 2011; van Dam and Nikfetrat, 1992] or its volume [Gariépy et al., 2013; Vos et al., 2010]. Provided that the upstream and lateral control volume boundaries can be placed at a location far away from the lifting element, where the flow is considered undisturbed, the induced drag is computed by means of a surface integration over the downstream transverse plane (TP) only. This approach is commonly termed a wake integral method, but corresponds to the classical Trefftz plane analysis if the downstream boundary is placed far enough from the lifting element for streamwise velocity variations to vanish. The approach was employed extensively within Euler-flow methodology [Monsch et al., 2007a,b; Chao and van Dam, 2006; Bourdin, 2002; Hunt et al., 1999; Nikfetrat et al., 1992; van Dam et al., 1991] and circumvents most of
the issues associated with the integration of the surface pressure distribution.

Farfield techniques are not necessarily restricted to induced drag prediction. Based on Reynolds-averaged Navier-Stokes (RANS) computations, an estimation of viscous and wave drag contributions is possible, which enables a decomposition of drag into its physical components. This is of particular interest within the design phase [van Dam et al., 1995], providing useful insight into the sources of drag creation, which is seen as a major advantage over the common nearfield technique. The method was employed with great success by Ueno et al. [2013], Gariépy et al. [2013], Hue and Esquieu [2011] and Vos et al. [2010] within the NASA drag prediction workshop².

However, a critical obstacle in the application of the wake integration technique is associated with the creation of spurious entropy drag due to artificial viscosity and grid coarsening downstream of the wing, numerically smearing the vortical wake. Induced drag decays as the location of the integration plane is moved downstream, while spurious entropy drag shows the opposing trend [van Dam, 1999]. To alleviate this effect, the integration can be performed at a very short distance away from the lifting element [Monsch et al., 2007a,b; Chao and van Dam, 2006; Hunt et al., 1999; Nikfetrat et al., 1992; van Dam et al., 1991]. Whereas this appears practical for simple wings, highly non-planar systems like the box wing can incorporate a large longitudinal separation of lifting elements, shifting the integration plane to a more downstream location and causing unacceptable induced drag decay.

To correct farfield induced drag, Snyder and Povitsky [2014] suggest to use a vorticity confinement technique [Steinhoff and Underhill, 1994]. This involves the addition of a source term to counteract numerical diffusion and a heuristic confinement parameter to define its strength. A further refined confinement approach employs a scaled confinement parameter to account for grid size effects. The particular vorticity confinement technique is available in STAR-CCM+, version 9.04.010 and has been tested thoroughly for a simple planar lifting system. It can be confirmed, that the streamwise decay of induced drag is successfully reduced [Snyder and Povitsky, 2014], but the selection of confinement parameter is still found to noticeably affect the solution. Although a systematic variation

²http://aiaa-dpw.larc.nasa.gov/
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of the confinement parameter was performed, it was not possible to attain induced drag predictions of sufficient accuracy independent of the streamwise location of the transverse plane. The vorticity confinement technique is hence refrained from further consideration within the present study.

To attain the correct farfield induced drag, the approach by Destarac and van der Vooren [2004] is employed in the following. It separates the axial velocity defect into components correlating to reversible or irreversible flow phenomena. In an inviscid and subsonic flow, spurious entropy drag represents the only irreversible source, whereas induced drag is related to the reversible process of adding transverse kinetic energy to the flow, downstream of the lifting element [Bourdin, 2002]. Employing thermodynamic properties, the axial velocity defect due to reversible phenomena can be described as follows:

$$\Delta u^* = u - V_\infty + V_\infty \cdot \left( \frac{1}{\kappa - M_\infty^2} \cdot \frac{\Delta s}{R} \cdot \frac{\Delta H}{V_\infty^2} \right)$$  \hspace{1cm} (3.17)

Employing the principle of momentum conservation and locating the upstream and lateral bounding surfaces of the control volume shown in Figure 3.2 at an infinite distance away from the lifting surface, the induced drag on a transverse plane (TP), located at an arbitrary downstream position is given by:

$$D_i = \int \int_{S_{TP}} (\vec{f}_i \cdot \vec{n}) \, dS$$  \hspace{1cm} (3.18)

with the vector $\vec{f}_i$:

$$\vec{f}_i = \frac{\rho_\infty}{2} \cdot \begin{pmatrix} v^2 + w^2 - (1 - M_\infty^2) \cdot \Delta u^*^2 \\ -2 \cdot v \cdot \Delta u^* \\ -2 \cdot w \cdot \Delta u^* \end{pmatrix}$$  \hspace{1cm} (3.19)

Provided the plane is placed perpendicular to the freestream velocity, the contributions from the $y$- and $z$-component of the vector $\vec{f}_i$ vanish. Far downstream of the lifting element, the flow can be considered sufficiently invariant in the streamwise direction. If Mach
number effects are neglected in addition, this yields the classical Trefftz plane expression [Kroo and Smith, 1990] according to Equation 3.20.

\[
\vec{f}_i = \frac{\rho_\infty}{2} \begin{pmatrix} u^2 + w^2 \\ 0 \\ 0 \end{pmatrix}
\]  

(3.20)

It may be further convenient to express Equation 3.19 by means of the streamwise vorticity and the cross-flow stream function, first obtained by Maskell [1973] and later by Wu et al. [1979]. The expression is exact even for small streamwise separations of lifting element and transverse plane as discussed by Giles and Cummings [1999]. However, a computation based on the stream function within STAR-CCM+ is found to be involved, particularly as a special treatment is required at symmetry conditions [Hunt et al., 1999]. The application of Equation 3.19 is thus preferred herein, but must be conducted over a relatively large area to sufficiently capture velocity components [van Dam, 1999]. To prevent a contamination by non-physical effects at the boundaries [Hunt et al., 1999], outer cells are excluded from the integration.

The total force produced due to irreversible phenomena can be computed by a volume integration over the entire flow domain. To correct the induced drag decay, the integration must contain only the trailing wake bound between the trailing edge of the wing and the transverse plane.

\[
D_{Sp} = \iiint_{\Omega_{\text{wake}}} f_{Sp} d\Omega_{\text{wake}} = \iiint_{\Omega_{\text{wake}}} \left( \nabla \cdot \left( \rho \cdot \Delta \vec{u} \cdot \vec{V} \right) \right) d\Omega_{\text{wake}}
\]  

(3.21)

with the irreversible axial velocity defect \( \Delta \vec{u} \):

\[
\Delta \vec{u} = V_\infty \cdot \sqrt{1 + 2 \frac{\Delta H}{V_\infty^2} - \frac{2}{(\gamma - 1) \cdot M_\infty^2} \left( \left( e^\frac{\Delta S}{\gamma} \right)^{\frac{\gamma + 1}{\gamma - 1}} - 1 \right) - V_\infty}
\]  

(3.22)

However, the estimation of the irreversible axial velocity defect based on Equation 3.22 is problematic as noted by Gariépy et al. [2013]. The radical may become negative at
some location within the domain and prevent a computation. Assuming negligible enthalpy variations, the formulation found by Méheut and Bailly [2008] can be used alternatively, to assure the axial velocity defect is finite and well-defined. A further possibility is indicated by Vos et al. [2010] and shows that:

\[ \nabla \vec{f}_\text{Sp} = -\nabla \vec{f}_i \]  

(3.23)

As discussed by Destarac and van der Vooren [2004], this approach is preferred, as the vector \( \vec{f}_i \) exhibits a smoother distribution within the computational domain. This finally leads to the expression for the spurious entropy drag in the wake:

\[ D_{Sp} = \iiint_{\Omega_{wake}} (-\nabla \vec{f}_i) \, d\Omega_{wake} \]  

(3.24)

The corrected induced drag is then found by:

\[ D_{i, cor} = D_i + D_{Sp} \]  

(3.25)

3.3 Potential-Flow

3.3.1 Review of Computational Methods and Related Techniques

A brief review of existing computational methods based on potential-flow theory, limited to the three-dimensional case, is given. An excellent and more thorough discussion is provided in Cummings et al. [2015], with emphasis on vortex-lattice and panel methods.

Lifting-Line Method and its Extensions

The classical lifting-line model [Prandtl, 1923] and its modifications [Multhopp, 1938; Weissinger, 1947] have been assimilated into several numerical codes i.e. by Durston [1993], Eppler and Schmidt-Göller [1990] and Anderson et al. [1980] or more recently by Phillips and Snyder [2000] and Rasmussen and Smith [1999]. Despite its simplicity, reasonably accurate induced drag predictions can be obtained for unswept wings with large aspect
The system of superimposed horseshoe vortices is preferably arranged along the quarter-chord position, although other locations can be adapted and are equivalently suitable. A placement at the trailing edge is especially favorable with regards to the computation of wake roll-up and its effects on induced drag [Eppler and Schmidt-Göller, 1990]. Employing principle functions, i.e. Fourier polynomials as evident in Rasmussen and Smith [1999], to describe the bound spanwise circulation distribution results in a continuous vorticity representation of the trailing sheet and eliminates most of the numerical issues related to the usage of discrete vortex filaments. However, this can become impractical for more general lifting systems, as the complexity of principle function increases substantially.

The extended lifting-line method by Weissinger [1947] facilitates a computation for swept wings with low aspect ratios and introduces collocation points to satisfy the flow tangency condition with an appropriate circulation distribution. These points are located at the three-quarter line to produce the same pitching moment as a flat plate, which however conflicts with the Kutta condition at the trailing edge [Bramesfeld, 2006] and leads to an inaccurate induced velocity prediction close to the trailing edge [Smith, 1995]. In principle, this problem also persists for more advanced vortex-lattice methods, but is greatly alleviated by using a sufficient amount of chordwise elements.

Actually, the streamwise vorticity develops along the chord, so that the full trailing vorticity is reached at the trailing edge [Kroo, 2000]. To improve the spatial representation, multiple, discrete lifting-lines are arranged along the chordwise direction, expanding the concept of the lifting-line model [Wieghardt, 1939; Scholz, 1950]. This approach has been transferred into a numerical model i.e. by Horstmann [1987], introducing the use of elementary wings with parabolic circulation distributions to enable the modeling of geometrically complex lifting systems. Extending this concept further by employing a continuous chordwise vorticity distribution, leads to the lifting-surface method [Truckenbrodt, 1953; Multhopp, 1955; Wagner, 1969], which has been synthesized into computational codes i.e. by Lan [1974] and Schulten [1995] for the purpose of propeller performance calculation. A review of related problems in the application of the numerical lifting-surface theory is given in Landahl and Stark [1968]. In general the method provides accurate prediction of
induced forces, but its versatility is limited due to the arising complexity of the principle functions describing the span and chordwise distributions [Morino and Kuo, 1974].

**Vortex-Lattice Methods**

Vortex-lattice methods (VLM) were one of the first computational approaches practically utilized to predict the aerodynamic characteristics of wings. The NASA-code *VLM4.997* by Herbert and Lamar [1982a,b] has established common standards for vortex-lattice based predictions [Cummings et al., 2015]. Numerous developments, like the general vortex-lattice methods *AVL* by Drela and Youngren [2013] are released under general public license. Due to this excellent availability, but also related to their computational efficiency and ability to accurately predict the induced drag of simple planar lifting systems, vortex-lattice methods are still in wide-spread usage for the purpose of early design and optimization.

![Figure 3.3: Arrangement of horseshoe vortices for vortex lattice methods.](image)

Vortex-lattice methods usually neglect the thickness distribution of the lifting element and are thus oriented towards lifting effects, which includes the creation of induced drag. They are also termed panel methods of zeroth-order [Schmidt-Göller, 1992]. The classical implementation involves a discretization of the planform by means of a lattice of quadrilateral panels, which are placed on the mean (camber) surface. The bound vortex filament of the horseshoe vortex is placed on the quarter-chord line, whereas the collocation point
is located at three-quarter chord line, mid-span of each panel, as shown in Figure 3.3. The trailing wake is assumed rigid and aligned with the freestream direction. The strength of each horseshoe vortex to satisfy the flow tangency conditions at the collocation points is found by solving a system of linear equations. With regards to other methodologies employing continuously distributed singularities, their discretization is comparably coarse and requires special treatment. However, this facilitates a high flexibility in modeling complicated geometries. Aerodynamic forces are obtained by applying the Kutta-Joukowsky theorem at the wing, or in the case of the induced drag by means of a Trefftz plane analysis.

Panel Methods

The initial development of panel methods, which are essentially an extension of vortex-lattice methods, originates from the work of Hess and Smith [1962]. An elaborate discussion on the theory of panel methods and their application is provided i.e. by Cummings et al. [2015] and by Katz and Plotkin [1991].

Compared to vortex-lattice methods, they account for the thickness distribution by arranging the singularities across discretized portions of the lifting element’s outer contour, termed a panel, rather than along the mean surface. These panels consist typically of source-doublet-vorticity distributions that can represent geometries of nearly arbitrary shape [Erickson, 1990]. The trailing wake is usually modeled by means of discrete vortex filaments [Schmidt-Göller, 1992]. The distribution approach allows one to distinguish between lower-order methods employing a singularity distribution of constant strength over each panel, whereas a higher-order method refers to a linear or parabolic singularity distribution or the usage of curved panels.

Well-known representatives of this class of methods are the commercial, lower-order method VSAERO [Maskew, 1982], the higher-order formulation PanAir [Magnus and Epton, 1990] or the advanced lower-order approach PMARC [Ashby et al., 1988], which can consider unsteady flows and resolve trailing wake roll-up effects.

The estimation of aerodynamic forces is mostly performed by a surface pressure integration. While this yields good results with regards to the lift, an accurate prediction of the induced drag is problematic [Letcher, 1989; Towne et al., 1983] and has hence led
to a reconsideration of the classical Trefftz plane analysis. However, in conjunction with force-free wake models, Trefftz plane based computations were shown highly sensitive to minor errors in the computed wake shape and have to be applied with care [Smith and Kroo, 1993].

**Trailing Wake Model**

In principle, three possible wake surrogate models can be incorporated within potential-flow methodology. Usually, the wake representation is motivated by the simple, rigid and streamlined model. It originates from Prandtl’s classical lifting-line theory [Prandtl, 1923], which replaces the true wake with a straight vortex sheet or discrete vortex filaments leaving the trailing edge (or quarter-chord line) in the freestream direction. The fact that it is hence drag-free and permits force contributions only perpendicular to the trailing wake actually enables an accurate induced drag prediction using farfield methodology [Smith and Kroo, 1997]. Although this shape is rather non-physical, its widespread application gives indication of its ability to predict the induced drag with sufficient accuracy. Generally, it is argued that the roll-up process occurs very slowly, so that the nearfield velocities, having the most dominant influence on the bound vorticity and therewith on the induced drag, are similar for the streamlined and the true wake shape [Smith, 1995]. In contrast to a force-free representation, the wake shape is known a-priori, greatly reducing the computational effort.

Another common engineering practice aligns the wake with the trailing edge bisector or tangential to the mean surface representation. This is referred to as a body-fixed wake model and leads to erroneous calculations based on farfield velocities, especially when the wake is non-planar [Kroo and Smith, 1990]. Compared to the freestream-fixed model it does potentially represent the wake properties more accurately in close proximity to the trailing edge (Kutta-condition), but supports longitudinal forces and is hence generally incorrect.

In order to account for the impact of higher-order effects introduced by wake deflection and roll-up, the computed wake must be aligned with the local velocity vector. The resulting wake is hence force-free. In contrast to the drag-free approximation, its shape is
not known a-priori. An iterative procedure is required to compute its rolled-up and force-free shape. This is commonly performed either by spatial-marching like done in Smith [1995], Mortara and Maughmer [1993], Eppler and Schmidt-Göller [1990], Nagati et al. [1987] and Plotkin and Yeh [1986] or computational more efficiently [Katz and Plotkin, 1991] by a time-stepping method as in Bramesfeld [2006] or Lamarre and Paraschivoiu [1992]. Compared to a freestream-fixed representation the computational effort increases.

Lift and Induced Drag Estimation

Within potential-flow methodology, a variety of approaches are available to calculate the lift and induced drag generated by a lifting element. Most commonly, these are based on the application of the Kutta-Joukowsky theorem, the surface pressure integration technique as well as the Trefftz plane analysis. Both former methodologies are referred to as nearfield methods, whereas the Trefftz plane analysis depicts a farfield approach that requires either a drag-free wake aligned with the freestream velocity vector or a force-free representation [Kroo and Smith, 1990].

The prediction of lift is straightforward and can be accomplished by nearfield techniques with sufficient accuracy. Vortex-lattice methods, usually apply the Kutta-Joukowski theorem to the bound vortex filament of each element using the freestream velocity. If induced lift contributions are considered, the induced components in the freestream direction are included. In the case of panel-methods, a surface pressure integration is convenient. Since the streamwise wake shed by a planar wing is basically lift-free, computation of farfield-lift may also be acceptable [Kroo, 2000]. For non-planar systems, the wake is not necessarily lift-free.

An accurate prediction of induced drag is more involved. In principle, induced drag can be computed by applying the Kutta-Joukowski law to the bound vorticity and calculating the downwash velocity directly at the lifting element. For panel methods, the induced drag can alternatively be determined by integrating the pressure distribution on the surface of the wing. However, even in higher-order formulations using second-order approximation of the Bernoulli equation and linear interpolating schemes over each panel, this method has shown to be extremely sensitive to the paneling density [Mortara and Maughmer,
1993]. The fact that the lift can be computed quite accurately suggests that the error is not related to the estimation of the circulation distribution [Schmidt-Göller, 1992], but to insufficient coverage of large pressure gradients in the vicinity of the leading edge and cancellation of small opposing pressure forces close in magnitude [Smith, 1995]. Even with panel densities well beyond practical limits, converged results can hardly be obtained [Schmidt-Göller, 1992]. Moreover, errors are planform-dependent and vary with panel density and distribution law as well as with the angle of attack as demonstrated by Smith [1995].

The Trefftz plane analysis is not affected by these problems and can be applied in conjunction likewise with vortex-lattice, panel or any other potential-based methodology. It is enabled by the principle of momentum conservation and the non-dissipative nature of potential-flow itself, ensuring that all the vorticity is confined in a vortex sheet which is shed from the trailing edge [Kroo, 2000]. In contrast to Euler-flow, the induced drag does not decay with the downstream location of the transverse plane and thus does not require a correction approach.

![Figure 3.4: Schematic illustration of the control volume and the Trefftz plane (TP).](image)

Moving the control volume dimensions to infinity, only the contribution of the downstream aft plane as shown in Figure 3.4, termed Trefftz plane, must be considered. In the case of the freestream-fixed wake model, no streamwise perturbation velocities due to the trailing vortex system are induced and the induced drag is attained according Equation
3.20, provided the Trefftz plane is placed perpendicular to the freestream velocity vector. Further rearranging Equation 3.20 using the velocity potential yields:

\[ D_i = \frac{\rho_\infty}{2} \iint_{S_{TP}} \left( \frac{\partial \Phi^2}{\partial y} + \frac{\partial \Phi^2}{\partial z} \right) dS = \frac{\rho_\infty}{2} \iint_{S_{TP}} (\nabla \Phi \cdot \nabla \Phi) \ dS \]  

(3.26)

Substituting the integrand in Equation 3.26 as follows:

\[ \nabla \Phi \cdot \nabla \Phi = \nabla \cdot (\Phi \nabla \Phi) - \Phi \nabla^2 \Phi \]  

(3.27)

and as \( \nabla^2 \Phi = 0 \) [Smith, 1995] outside the wake:

\[ D_i = \frac{\rho_\infty}{2} \iint_{S_{TP}} (\nabla \cdot (\Phi \nabla \Phi)) \ dS \]  

(3.28)

By applying the Gauss theorem the surface integral of Equation 3.28 can be transferred into a contour integral enclosing the wake trace:

\[ D_i = \frac{\rho_\infty}{2} \oint_{C_{TP}} ((\Phi \nabla \Phi) \cdot \vec{n}) \ dl \]  

(3.29)

The normal component of the crossflow divergence term gives the normal velocity across the wake and permits to replace the contour integral of Equation 3.29 by a line integral along the wake trace.

\[ D_i = \frac{\rho_\infty}{2} \int_{\text{wake}} (\Delta \Phi \cdot \frac{\partial \Phi}{\partial n}) \ dl \]  

(3.30)

The local potential jump across the wake trace is given by \( \Delta \Phi \) and equivalent to the circulation \( \Gamma \) at this spanwise location [Kroo and Smith, 1990]. The partial derivative \( \frac{\partial \Phi}{\partial n} \) represents the normal velocity component \( V_n \). In contrast to the tangential velocity the normal component remains finite and continuous. This allows one to rearrange Equation 3.30 and yields the classical expression for the computation of the Trefftz plane drag using freestream-fixed wakes.
\[ D_i = \frac{\rho \infty}{2} \int_{\text{wake}} (\Gamma \cdot V_n) \, dl \] (3.31)

Although the computation of farfield induced drag is most commonly performed based on the straight, freestream-fixed wake model, the correct, relaxed-wake shape is theoretical equally applicable [Kroo and Smith, 1990]. However, for any practical usage, difficulties emerge based on the original formulation of the Trefftz plane analysis and force-free wake shapes. Deviations from the exact physical, force-free wake shape, which can be easily caused by sparse paneling, especially in the spanwise direction, considerably affect the accuracy of induced drag predictions. For planar systems, Smith [1995] demonstrated that the predicted induced drag is very sensitive to minor errors in the computed wake shape. The problem is further intensified by the fact that geometrical deviations generally increase as the vortex sheet proceeds downstream. In order to comply with the common Trefftz plane approach, the estimation is performed in the farfield, hence at a location of highest geometrical and numerical uncertainty, potentially introducing significant errors [Schmidt-Göller, 1992].

### 3.3.2 Higher-Order Potential-Flow Method

The higher-order potential-flow method \textit{FreeWake} by Bramesfeld [2006] is used to derive most of the herein presented potential-flow results. It is based on the multiple lifting-line method by Horstmann [1987]. As an essential feature, the method introduces distributed vorticity elements (DVE) to discretize the lifting system and its wake. Using distributed vorticity elements results in a continuous wake vortex sheet, avoiding most of the singularity problems associated with the common trailing wake representation based on vortex filaments. Particularly in the context of wake-relaxation, this is seen as an advantage and provides a numerically more robust behavior [Bramesfeld, 2006].

The higher-order potential-flow model was employed for a range of different inviscid flow related problems, i.e. in Krebs and Bramesfeld [2016], Bissonnette and Bramesfeld [2016], Cole et al. [2013], Bramesfeld and Malik [2015] and Bramesfeld and Maughmer [2008a] with great success. Its applicability has been recently extended by a derivative
(WindDVE) for the purpose of horizontal-axis wind turbines by Maniaci [2013] and Basom and Maughmer [2011]; Basom [2010].

**Distributed Vorticity Element**

Distributed vorticity elements (DVE) are employed to represent the lifting system and its trailing wake. These trapezoidal sub-elements are planar, infinitely thin and consist of a vortex filament along its leading and trailing edge connected by a vortex sheet as shown in Figure 3.5. As noted by Anderson [2001], placing these elements along the mean surface of the wing has real physical significance and is more than just a mathematical device. In a viscous flow, the boundary layer entails large velocity gradients, which produce a substantial amount of vorticity.

\[
\Gamma_{LE}(\eta) = A + B\eta + C\eta^2
\]

\[
\Gamma_{TE}(\eta) = -A - B\eta - C\eta^2
\]

**Figure 3.5:** Illustration of a distributed vorticity element, adapted from Bramesfeld [2006].

A local, element-fixed reference frame \((\xi\eta\zeta)\) is introduced to described the DVE properties and dimensions. It originates from the collocation point, which is located at the mid-chord and mid-span position of the DVE. The vorticity of the sheet is aligned with the local flow direction, whereas the sheet is located in \(\xi\eta\)-plane with the \(\zeta\)-axis oriented normal to the element. The vortex filaments have quadratic spanwise circulation distributions of equal strength but opposite orientation, whereas the vortex sheet has a linear...
chordwise vorticity distribution to maintain the Helmholtz theorem in the streamwise direction. In contrast to both side edges of the element, which remain parallel to the \( \xi \)-axis, the leading and trailing edge vortex filaments can incorporate sweep angles, \( \phi_{LE} \) and \( \phi_{TE} \), with respect to the \( \eta \)-axis. The span of the particular distributed vorticity element amounts \( 2 \times \eta_i \), whereas its chord length is given by \( 2 \times \xi_i \) along the mid-span location.

To form a distributed vorticity element of finite chord length, two co-planar, streamwise staggered vortex filaments and two associated semi-infinite vortex sheets, both of opposite strength are superimposed and cancel aft of the trailing edge filament as illustrated in Figure 3.6. Consequently, its velocity induction is composed of the influences of leading and trailing vortex filaments and two semi-infinite vortex sheets. For incompressible flows, this is enabled by means of the Biot-Savart law [Anderson, 2001], which provides a relation between the induced velocity and a known vorticity distribution. According to Horstmann [1987], the velocity that a vortex filament segment induces at an arbitrary location \( P_0(\xi_0, \eta_0, \zeta_0) \) can be computed based on an analytic solution as follows:

\[
\vec{V}_{i, \text{filament}}(\xi_0, \eta_0, \zeta_0) = \int_{-\eta_i}^{\eta_i} \frac{A + B \cdot \eta + C \cdot \eta^2}{4 \cdot \pi \cdot r_1^4} \cdot \begin{pmatrix} -\zeta_0 \\ \zeta_0 \cdot \tan(\phi) \\ \xi_0 - \eta_0 \cdot \tan(\phi) \end{pmatrix} d\eta 
\]

(3.32)

The velocity induced by the semi-infinite vortex sheet is given by:

\[
\vec{V}_{i, \text{sheet}}(\xi_0, \eta_0, \zeta_0) = \int_{-\eta_i}^{\eta_i} \frac{B + 2 \cdot C \cdot \eta}{4 \cdot \pi \cdot (\eta_0 - \eta)^2 + \zeta_0^2} \cdot \left( \frac{\xi_0 - \eta \cdot \tan(\phi)}{r_1} + 1 \right) \cdot \begin{pmatrix} 0 \\ \zeta_0 \\ \eta_0 - \eta \end{pmatrix} d\eta 
\]

(3.33)

where \( r_1 \) is:

\[
r_1 = \sqrt{(\xi_0 - \eta \cdot \tan(\phi))^2 + (\eta_0 - \eta)^2 + \zeta_0} 
\]

(3.34)
A direct application of Equation 3.32 and Equation 3.33 is however problematic, as
integrands exhibit singularities, i.e. at the center of the vortex filaments. Although these
do typically not cause numerical issues [Basom and Maughmer, 2011], a special singularity
treatment is required for the vorticity sheet.

\[
\begin{align*}
\Gamma_{LE}(\eta) &= A + B\eta + C\eta^2 \\
\gamma(\eta) &= B\eta + 2C\eta \\
\Gamma_{TE}(\eta) &= -A - B\eta - C\eta^2 \\
\gamma(\eta) &= -B\eta - 2C\eta \\
\end{align*}
\]

(a) Leading edge vortex filament and semi-infinite vortex sheet.
(b) Trailing edge vortex filament and semi-infinite vortex sheet.
(c) Distributed vorticity element of finite chord length.

Figure 3.6: Illustration of a distributed vorticity element of finite chord length composed of vortex filaments along its leading and trailing edge and two semi-infinite vortex sheets, adapted from Bramesfeld [2006].

Vortex Sheet Singularity Treatment

The vortex sheet induces a tangential velocity component in the \( \eta \)-direction and a compo-
ment normal to its \( \xi \eta \)-plane. Across the sheet itself, induced velocities are preserved, but a
discontinuous change in the tangential component exists, which is equivalent to the local
sheet strength [Anderson, 2001]. Directly on the sheet the tangential velocity is hence
undefined and set to zero for numerical purposes. Otherwise it remains finite.

The normal induced velocity at the side edges of the vortex sheet are of particu-
lar concern, approaching infinity and complicating a numerical implementation process
[Bramesfeld and Maughmer, 2008a]. In the case a neighboring co-planar sheet exists, a
finite net velocity results at the common side edge, provided the vorticity is continuous across both elements. This is because the velocities induced by two neighboring sheets are of equivalent (infinite) magnitude but opposite direction. Although this presents no problems analytically, it cannot be accommodated from a numeric perspective [Basom and Maughmer, 2011]. To circumvent this issue, Bramesfeld [2006] introduces additional singularities at the side edges of each individual vortex sheet to achieve self-induced velocity of finite magnitude. Although this modifies the individual self-induced contribution, the combined induced velocity remains unaffected. The magnitude of the added singularity is described by the parameter \( k_{se} \), which gives the strength in multiples of the minimum vortex sheet half-span. An equivalent procedure is employed for a swept leading edge to attain finite and numerical well-behaved individual velocities.

At the tip of a lifting surface and its wake edge, where no neighboring vortex sheet exits, the normal induced velocity still approaches infinity. This is however only of concern if the computation is performed exactly at the edge itself, which is in particular required during the wake relaxation process. Here, the induced velocities at the side edge are used to displace the wake. An additional singularity is hence introduced at the side edge of the wake to enforce finite normal induced velocities. As no neighboring vortex sheet exits, the added singularity canceling is unbalanced, which can affect the remaining flowfield. In Bramesfeld [2006], a singularity parameter of \( k_{se} = 0.01 \) is found to yield a smooth overall velocity distribution with only minor tip-related impact.

**Lifting Surface Representation**

Distributed vorticity elements can used to represent the lifting surface by arranging several DVEs across the chord and span at its mean surface. The example wing provided in Figure 3.7 is modeled by two chordwise and three spanwise elements, placing the leading edge vortex filament of the particular DVE at the quarter-chord location of each sub-wing element. This causes an overlap of lifting surface and wake element and assures the flow leaves the trailing edge smoothly according to the Kutta condition (compare i.e. [Anderson, 2001]). The collocation point is thus effectively located at the three-quarter chord line of the lifting element.
The determination of circulation coefficients ($A$, $B$ and $C$) involves the solution of a system of linear equations, satisfying three boundary conditions. These are in particular, (i) the flow tangency condition at the collocation point, (ii) a continuous circulation and (iii) a continuous vorticity distribution across two neighboring elements.

**Trailing Wake Representation**

The trailing wake is also composed of distributed vorticity elements. Under steady conditions with constant streamwise vorticity distributions, the leading and trailing edge filaments can be omitted [Bramesfeld, 2006], forming a continuous vortex sheet. This avoids many of the singularity issues common to conventional potential-flow methods.

A time-stepping method is used to evolve the wake. As the wing progresses forward with each time-step $\Delta t$, a new spanwise row of distributed vorticity elements is emitted into the wake from the trailing edge. The vorticity within the wake is given by the circulation strength of the most downstream elements on the lifting surface, located along the trailing edge. During the first iteration step, a spanwise row of semi-infinite vortex sheets are shed into the freestream direction to attain a stable initial solution and accelerate convergence [Bramesfeld, 2006].

Two wake models are available within the present method. The common freestream-
fixed wake and a relaxed, force-free representation.

To produce a force-free wake, the mid-chord points at the side edges of each distributed vorticity element are iteratively displaced during each time-step with the local induced flowfield. The wake elements are allowed to stretch and compress in span. This requires an adjustment of their vorticity distribution to maintain equivalent total circulation according to Equation 3.35, to comply with the Helmholtz theorem.

\[
\frac{1}{2} \int_{-\eta_i}^{\eta_i} \Gamma(\eta) \, d\eta = \text{const.} \quad (3.35)
\]

Particularly in the context of wake relaxation, the application of distributed vorticity elements is a major advantage over methodologies using discrete vortex filaments and solid-core models to avoid numerical issues associated with the induced velocities that become infinite as the center is approached. Solid-core models, contradict the assumption of irrotational flow and introduce a solution dependency on the choice of the core radii. Furthermore, these models also do not resolve issues such as vortex pairing or the intersecting of two filaments in the wake, which becomes important when considering wakes from multiple lifting surfaces.

Figure 3.8 exemplary shows the freestream-fixed and relaxed-wake shape for a planar, linear tapered wing after performing 20 time-steps for an angle of attack of \( \alpha = 4.0^\circ \). The planform is adapted from Chao and van Dam [2006] and is characterized by an aspect ratio of \( \Lambda = 7.0 \) and a taper ratio of \( \lambda = 0.5 \).

![Figure 3.8: Tapered wing adapted from Chao and van Dam [2006] either with a freestream-fixed, (a), or a relaxed-wake shape, (b), after 20 time-steps at an angle of attack of \( \alpha = 4.0^\circ \).](image-url)
Lift and Induced Drag Estimation

Lift and induced drag forces can be determined by means of the Kutta-Joukowsky law in its three-dimensional form. Figure 3.9 shows a fixed vortex filament segment of infinitesimal length $d\mathbf{s}$, i.e. located at the trailing edge of the distributed vorticity element.

![Figure 3.9: Three-dimensional Kutta-Joukowsky law, adapted from Eppler and Schmidt-Göller, 1990.](image)

The segment of strength $\Gamma$, which is exposed to the flow velocity of $\mathbf{V}$, experiences a force $d\mathbf{R}$. According to Eppler and Schmidt-Göller [1990], the resulting force vector is given as follows:

$$d\mathbf{R} = \rho \cdot (\mathbf{V} \times \Gamma d\mathbf{s})$$  \hspace{1cm} (3.36)

with

$$\mathbf{V} = \mathbf{V}_\infty + \mathbf{V}_i = \begin{pmatrix} u \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} u_i \\ v_i \\ w_i \end{pmatrix}$$  \hspace{1cm} (3.37)

Based on the aerodynamic coordinate system provided in Figure 3.9, the force vector can be expressed in components, which yields the lift and induced drag as its $z_a$ and $x_a$-component respectively.

$$L = \int_{d\mathbf{s}} \rho \cdot \Gamma \cdot ((u + u_i) \, dy_a - v_i \, dx_a)$$  \hspace{1cm} (3.38)

$$D_i = \int_{d\mathbf{s}} \rho \cdot \Gamma \cdot (v_i \, dz_a - w_i \, dy_a)$$  \hspace{1cm} (3.39)
Applying the three-dimensional Kutta-Joukowsky law in conjunction with the present higher-order potential-flow methodology, the lift can be computed along the vortex filaments of the distributed vorticity element. As apparent from Equation 3.38, the lift is created by a freestream and an induced velocity component. This facilitates an individual computation, which is expedient with regards to a numerical implementation, but also enables investigation of both contributions separately. An analytical solution is used to integrate the lift force caused by the freestream velocity, whereas a numerical approach is employed for lift forces related to induced effects.

The (streamwise) vorticity and therefore the induced drag corresponds to the transverse kinetic energy, which is entirely shed from the trailing edge and confined in the subsequent wake. This allows one to lump the distributed circulation into a single line at the trailing edge. The induced drag is thus estimated by taking the cross product between the circulation that is shed into the wake at the trailing edge and the velocity induced by the wake at this spanwise location. It is referred to as Eppler’s trailing edge analysis [Eppler and Schmidt-Göller, 1990]. Its application to a panel method is discussed in Mortara and Maughmer [1993]. Estimating the induced drag at the trailing edge is especially favorable with regards to the relaxed-wake model, as an accurate prediction by means of a Trefftz plane analysis is avoided.
Investigation of Planar Reference Systems

The higher-order potential-flow method and the implemented Euler-flow farfield approach are initially tested. Their ability to provide accurate induced drag predictions is assessed and validated by means of a set of planar reference systems. The computed span efficiency factors are compared to theoretical values based on lifting-line theory and projections from other references.

4.1 Test Cases

4.1.1 The Elliptical and Crescent Wing

Two planar systems were considered as initial test cases. Their induced drag characteristics have been extensively investigated within other references previously. These are the classical elliptical wing, with an unswept, straight quarter-chord line and a planform with an unswept, straight trailing edge, referred to as crescent wing. According to Equation 4.1, both platforms, which are depicted in Figure 4.1, are characterized by an elliptical chord distribution along the wingspan.

\[ c(\eta) = c_r \cdot \left(\sqrt{1 - \eta^2}\right) \]  

(4.1)

The crescent and elliptical wing can further be described by the normalized x-coordinate of the tip trailing edge position \( x_t = (x/c_r) \) [Smith, 1995]. The tip trailing edge position
CHAPTER 4: INVESTIGATION OF PLANAR REFERENCE SYSTEMS

for the elliptical wing is given by \( x_t = 0.25 \), whereas for the crescent wing the tip trailing edge position occurs at \( x_t = 1.00 \). In order to avoid numerical issues, the span is usually clipped close to the wing-tip, at a relative half-span location of \( \eta = 0.4995 \), to attain a non-zero tip chord length [van Dam and Nikfetrat, 1992]. The geometric properties of these planforms are summarized in Table 4.1 and consequently slightly deviate from the exact theoretical shape. An equivalent procedure is applied by Smith [1995] and by van Dam and Nikfetrat [1992] using an Euler-flow model. Both wings are untwisted; their sectional shape is represented by symmetric airfoils with a maximum thickness-to-chord ratio of \((t/c) = 0.12\) (NACA 0012) and a sharp trailing edge.

![Figure 4.1: Planar reference systems.](image)

The leading edge \( x \)-coordinate \( x_{le} \) is defined by Equation 4.2.

\[
x_{le}(\eta) = x_t \cdot \left(1 - \sqrt{1 - \eta^2}\right)
\]  

(4.2)

The selection of these systems was motivated by preceding research effort, in particular by van Dam [1987], who evaluated induced drag characteristic for a family of elliptical wings, that are characterized by different tip trailing edge positions. These investigations were conducted at an angle of attack of \( \alpha = 4.0^\circ \), under subsonic flow conditions and employed a low-order panel method and an iterative wake-relaxation procedure. Reported induced drag savings, especially for substantially sheared-back tip geometries (i.e. \( x_t = 1.5 \)), were later found of artificial nature [Smith and Kroo, 1990; DeHaan, 1990; Lam and Maull, 1993; Mortara and Maughmer, 1993]. This was caused by induced drag calculations
using a surface pressure integration. Nevertheless, the crescent wing is an important special case within the field of research on induced drag prediction.

\[
\begin{array}{cccccc}
\Lambda & c_{ref}, \, m & c_r, \, m & c_t, \, m & b, \, m & S_{ref}, \, m^2 \\
6.982 & 0.785 & 1.000 & \approx 0.051 & 5.491 & 4.318 \\
\end{array}
\]

**Table 4.1:** Common geometric properties of the elliptical and crescent wing.

The elliptical wing originates from the lifting-line theory [Prandtl, 1923] and is commonly known to achieve the theoretically minimum induced drag of all single planar wings for given span and lift. Within the lifting-line theory, the bound circulation is lumped into a single, unswept lifting-line at the quarter-chord location. For the elliptical wing, this results in an elliptical spanwise circulation and a flat and straight trailing wake. However, if the longitudinal extent of the elliptical wing is considered, i.e. by arranging several lifting-lines along the chord, the spanwise circulation is not distributed uniformly and causes a non-elliptical loading and hence larger induced drag than based on the lifting-line theory [Smith, 1995].

The wake of the elliptical wing using the quarter-chord location is not equivalent to the wake leaving from its trailing edge. For any non-zero angle of attack, a non-planar wake is shed from the trailing edge, whereas opposed to that, the wake remains planar for the quarter-chord condition. This is problematic, because the induced drag depends on the wake shape and the distribution of vorticity in the wake [Kroo and Smith, 1990]. Thus, various wake shapes lead to differences in induced drag, although deviations are likely marginal in this particular case. If a high accuracy is required, this prevents a critical investigation of theoretical predictions based on the lifting-line theory against present inviscid computational methodologies by means of the elliptical wing.

Due to its straight and unswept trailing edge, the wake of the crescent wing remains planar for any angle of attack and provides an exact geometrical representation of the quarter-chord bound trailing wake assumed by the lifting-line theory. Based on a higher-order panel method in conjunction with a freestream-fixed wake, the crescent wing was found to exhibit a more nearly elliptical spanwise loading than the classical elliptical wing.
The crescent wing therefore produces minimum induced drag in very close agreement with lifting-line theory [Smith, 1995].

### 4.1.2 The Split-Tip Wing

In addition to the elliptical and crescent wing, a split-tip wing adapted from Smith [1995] was analyzed. The system depicted in Figure 4.2 is planar, untwisted and employs NACA 0012 airfoils sections with a sharp trailing edge. Its geometric properties are summarized in Table 4.2.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$c_{ref}$, m</th>
<th>$c_r$, m</th>
<th>$c_t$, m</th>
<th>$b$, m</th>
<th>$S_{ref}$, $m^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.000</td>
<td>0.900</td>
<td>1.000</td>
<td>0.400</td>
<td>6.000</td>
<td>5.400</td>
</tr>
</tbody>
</table>

**Table 4.2:** Geometric properties of the split-tip wing adapted from Smith [1995].

The system was specifically designed to produce a non-planar wake that promotes wing-wake interactions between forward and aft tip to intensify higher-order effects on induced drag. In contrast to other studies on wing-tip sails [Spillman and Allen, 1978; Zimmer, 1983], the non-planar character of the wake does not stem from dihedral effects, but is induced by the inclination of the planform to the freestream direction. Under subsonic flow conditions and at an angle of attack of $\alpha = 9.0^\circ$, Smith [1995] ascertained induced drag savings of about 5% using a freestream-fixed wake model and approximately

---

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11% based on a hybrid computation of the rolled-up wake geometry compared to the theoretical optimum for planar monoplanes. The system therefore provides an important aspect of the validation approach for the higher-order potential-flow method and the application of the relaxed-wake model, as well as for the Euler-based simulation.

4.2 Computational Implementation

4.2.1 Euler-Flow Method

The simulation models established herein utilized the commercial CFD-code \textit{STAR-CCM+}, version 8.04.010 to solve the three-dimensional Euler equations. Following the flow conditions of van Dam and Nikfetrat [1992] and Bourdin [2002], the simulation was conducted at a subsonic freestream Mach number of $M_\infty = 0.20$.

Flowfield and Geometrical Representation

A rectangular-shaped flow domain was created, which ranges from 15 reference chords upstream to 50 reference chords downstream. The lateral and vertical dimensions are given by 15 and $2 \times 15$ reference chords respectively, as depicted in Figure 4.3. Compared to previous research effort, i.e. van Dam and Nikfetrat [1992], relatively large farfield dimensions were selected to avoid distortions by the flowfield boundaries.

Figure 4.3: Schematic illustration of the flowfield, transverse plane (TP) and the correction volume $\Omega_{\text{wake}}$ enclosing the wake region to compute the spurious entropy drag contribution.
Equivalent to the geometrical properties given in Table 4.1 and 4.2, half-models were established. The present flow conditions allow exploitation of symmetry conditions to reduce the cell count and the computational effort. According to Smith [1995], the cross-sectional shape is represented by NACA 0012 airfoil section with a sharp trailing edge. To produce this contour while maintaining an equivalent thickness-to-chord ratio, the original NACA 0012 airfoil section with a blunt trailing edge [Abbott and Von Doenhoff, 1959] is rescaled. This avoids problems with the Kutta-condition in conjunction trailing edges of finite thickness [van Dam, 1999].

The geometrical modeling of the elliptical and crescent wing involved the creation of a multiple-section surface. Required cross-sections were distributed by a half-cosine function to produce a larger number of supporting points in regions of significant leading edge curvature and variation of chord length near the tip. This assures a sufficient geometrical fidelity. The chord distribution was computed by means of Equation 4.1. The wing-tip was closed with a flat patch.

In the case of the split-tip wing, the geometrical modeling is straightforward. However, a blended fairing was required to attain a smooth transition from the main wing to the tip surface elements. Although the impact of the fairing on the induced drag of the system may be limited, it should be noted that geometrical uncertainties are evident compared to the case considered by Smith [1995]. Unfortunately, no geometric details of the fairing, in particular about its relative spanwise location and extent are reported. This required one to perform an approximation using existing but incomplete drawings (compare Figure 51 in Smith [1995]). For the present study, the fairing was considered to range from a relative half-span location of \( \eta = 0.317 \) to \( \eta = 0.333 \), joining upper and lower surface tangentially at its respective ends. Both wing-tip were closed with a flat patch.

Freestream conditions were introduced at the farfield boundaries of the domain, except for its lateral side at \( y = 0.0 \), where symmetry conditions apply. The wing surface was treated as a solid wall, incorporating a no-slip setting as physical viscosity effects (surface friction) are inherently not included. The freestream velocity vector was imposed perpendicular to the upstream boundary of the flow domain. To produce an angle of attack, the wing was incline to the freestream direction by a rotation along its trailing edge.
Computational Grid

A trimmed-cell approach was selected to create a predominately hexahedral, Cartesian-type grid with cut-cells on functional geometry boundaries. This grid type is usually well-suited for the analysis of external flow problems, such as for cars, ships and aircraft, that are characterized by a distinctive preference direction of the flow, to be aligned with the grid [Maley, 2012]. The approach was originally developed by Purvis and Burkhalter [1979] and later by Wedan and South [1983] in conjunction with numerical solutions of the transonic full-potential equations and extended to an Euler-based application by Clarke et al. [1986]. It allows one a fast, near-automated grid generation of high quality to be performed [Ingram et al., 2003]. This is especially of advantage given the considerable amount of simulation-runs performed within this study.

Commonly, the grid generation process uses an initial Cartesian background grid that divides the flow domain with uniform hexahedral elements, which are allowed to cut through functional boundaries of an immersed body [Fidkowski, 2007]. In close proximity to these functional boundaries, cut-cells are split and refined into smaller hexahedral elements of a desired dimension, i.e. based on local refinement strategies. Cells, primarily located inside of the body are subsequently removed, whereas elements intersecting the body surface are trimmed or cut at their edge penetration points with the contour to produce a conforming grid.

Local, curvature-based surface and volume grid refinement strategies were applied to
improve the numerical resolution and hence accuracy in solution-critical areas of the flow domain. For example at the leading edge, due to the existence of large velocity gradients, a curvature-based refined was utilized. Volume-based refinements were introduced in the wake region downstream of the lifting element, to minimize the numerical smearing of the wake. Figure 4.4 exemplary illustrates the farfield and surface grid for the split-tip wing.

Lift and Induced Drag Estimation

The span efficiency factor is used to enable a comparison among different lifting systems and computational methodologies according to Section 2.2.3. As apparent from Equation 4.3, this involves the estimation of the lift and induced drag coefficient. The definition of the span efficiency based on Equation 4.3 is convenient for monoplanes and equivalent to the expression provided by Equation 2.8. However, the latter is preferred in the case of multiple-surface lifting systems such as box wings and biplanes.

\[
e = \frac{C_{D_{\text{r, ell}}}}{C_{D_i}} = \frac{C_L^2}{\pi \cdot \Lambda \cdot C_{D_i}} \tag{4.3}
\]

Pursuant to the discussion provided in Section 3.2.1, lift coefficients were computed based on a surface pressure integration, whereas the induced drag was predicted using the farfield method of Destarac and van der Vooren [2004]. To correct the streamwise decay of induced drag, the computation of the spurious entropy drag \(D_{Sp}\) considers the fluid volume \(\Omega_{\text{wake}}\) that encloses the trailing wake region. The volume was established based on geometric threshold conditions. It originates from a distance of \(c_{\text{ref}} = 0.05\) reference chords downstream of the trailing edge, to avoid an intersection with the lifting element and proceeds streamwise to coincide with the transverse plane (TP), which is located at standardized distance of \(d_{TP} = 40 \cdot c_{\text{ref}}\). The addition of spurious entropy drag and transverse plane induced drag gives the corrected induced drag estimate according to Equation 3.25.
4.2.2 Higher-Order Potential-Flow Method

The higher-order potential-flow model [Bramesfeld, 2006] (FreeWake) was used to compute the induced drag and related quantities of the present planar test cases. The method employs either a freestream-fixed or a relaxed-wake approach. Independent of the configuration, a total number of 60 time-steps was performed to securely satisfy convergence, which is verified in Figure 4.10. A time-step width of \( \Delta t = 0.25 \cdot \frac{c_{\text{ref}}}{V_\infty} \) was used according to Bramesfeld [2006] to evolve the wake.

In contrast to the Euler-flow simulation, the freestream Mach number equals \( M_\infty = 0.00 \), as a compressibilty correction is not available within the present potential-flow methodology. The contribution by finite airfoil thickness and higher freestream Mach number results in larger lift and induced drag estimates for the Euler-flow-reference compared to the higher-order potential-flow model. However, the effect on the span efficiency is negligible. This was demonstrated by Smith [1995] for the crescent wing using a full-potential code. The maximum relative error in span efficiency within the Mach number range of \( M_\infty = 0.01 \) and \( M_\infty = 0.30 \) is below \( \Delta \varepsilon_{\text{rel}} = 0.1\% \).

**Geometrical Representation**

The wings were represented based on their mean surface; the thickness distribution was neglected. Inflow conditions were symmetric with respect to the global \( yz \)-reference frame. This allows the application of symmetry conditions to minimize the computational effort. Boundary conditions between distributed vorticity elements (DVE) were set to assure equal vorticity and vorticity gradient between neighboring elements [Bramesfeld, 2006]. At the wing-tip the circulation was defined zero, whereas at the symmetry plane zero vorticity was assumed. The elliptical and crescent planform were modeled according to Equation 4.1 and 4.2 by a finite number of spanwise sections.

An implementation of the split-tip wing within the higher-order potential-method is currently problematic and not possible correctly without a modification of the source code. This appears to be generally feasible similar to Horstmann [1987], but requires a special forking-law between main wing and both tips to enforce boundary conditions.
Unfortunately, this could not be achieved due to time constraints. However, at least for the relaxed-wake model, an approximate procedure was employed with good success. The split-tip wing was represented by two isolated, staggered lifting surfaces of equivalent chord. This required to position the upstream main wing element slightly above the subsequent element ($\approx 0.01 \cdot c_r$) to prevent an impingement of the upstream wake, causing numerical instabilities.

**Computational Grid**

In contrast to the Euler-simulation, only the wing and its trailing wake need to be considered by arranging equally-spaced distributed vorticity elements across the chord and spanwise direction. A specific distribution approach is not necessary to ensure proper spanwise discretization [Bramesfeld, 2006]. The parabolic initial function that represents the quadratic circulation distribution on the DVE largely minimizes discretization errors, which predominately occur close to the wing-tip (compare Figure 2 in Horstmann [1987]). An example of the discretized crescent wing is shown in Figure 4.5 for both wake models.

![Figure 4.5: Discretization of the crescent wing based on the freestream-fixed wake, (a), and the relaxed-wake, (b), after 20 timesteps at an angle of attack of $\alpha = 4.0^\circ$.](image)

**Lift and Induced Drag Estimation**

As discussed in Section 3.3.2, the lift computation was performed along the vortex filaments of the distributed vorticity element and includes induced lift effects. Similarly, the induced drag was calculated along the trailing edge based on the Kutta-Joukowsky theorem.
4.3 Spatial Convergence Study

4.3.1 Euler-Flow Method

![Figure 4.6](image.png)

**Figure 4.6**: Variation of the span efficiency factor $e$ after satisfying the convergence criterion for the crescent wing.

A set of continuously refined grids was produced by means of a constant refinement factor $r = 1.5$ to investigate the spatial grid convergence. Dependent on the test case, this results in cell counts between approximately $8 \times 10^5$ and $85 \times 10^6$. For each grid level, an individual converged solution was attained based on an asymptotic criterion requiring a variation of the farfield induced drag coefficient of less than $\Delta C_{Di} = 1 \times 10^{-5}$ for 100 consecutive iterations. Although the application of such an asymptotic criterion using an engineering quantity of interest was found problematic for farfield induced drag predictions [Gariépy et al., 2011], no significant variations, even for the accuracy demands of the current study, were ascertained beyond the point of convergence for any further iteration steps. This is confirmed in Figure 4.6 for the crescent wing. The variation in span efficiency compared to the estimate based on the convergence criterion amounts approximately $\Delta e_{rel} \approx 0.03\%$ for more than 2000 additional iteration steps.

The solution was considered independent of the grid element size for variations of the span efficiency factor of less than $\Delta e = 0.003$ within three consecutive grid levels. This is successfully achieved for all test cases, as evident from Figure 4.7, depicting the span efficiency factor versus the non-dimensional grid element size, $h_{grid}$. However, a non-monotonic convergence behavior is observed, even for the three finest grids involved,
CHAPTER 4: INVESTIGATION OF PLANAR REFERENCE SYSTEMS

Figure 4.7: Spatial convergence behavior of the span efficiency factor for the crescent and elliptical wing, (a), and the split-tip wing, (b), based on the Euler-flow model.

Provided the correction volume $\Omega_{\text{wake}}$ remains bound between the trailing edge and the transverse plane, the utilized farfield estimation by Destarac and van der Voozen [2004] theoretically permits induced drag predictions independent of the streamwise location of the transverse plane. For the crescent wing, a study was conducted to confirm the effectiveness of this correction approach.

Figure 4.8 illustrates the impact of various streamwise locations $d_{TP}$ on the uncorrected induced drag coefficient $C_{Di}$ and the spurious entropy drag coefficient $C_{D, Sp}$. In good agreement with literature, i.e. Bourdin [2002], the uncorrected induced drag coefficient is found to decay downstream of the lifting element, which results in an artificial increase in span efficiency. The spurious entropy drag coefficient exhibits an opposed trend and increases with the downstream location. The addition of uncorrected induced drag and spurious entropy drag coefficient yields the corrected induced drag coefficient $C_{Di}$, that is confirmed insensitive on the streamwise location of the transverse plane.

4.3.2 Higher-Order Potential-Flow Method

The influence of the chordwise and spanwise element number on the span efficiency factor is depicted in Figure 4.9. In the case of the elliptical and crescent wing, the chordwise density
is found of minor impact. For element counts larger than \( n_c = 6 \), the span efficiency quickly converges, regardless of the wake model. This is in-line with the findings by Bramesfeld and Maughmer [2008].

In contrast to that, the split-tip wing requires a considerable larger amount, in particular in the case of a relaxed trailing wake. Chordwise element counts, equivalent to the elliptical and crescent wing, lead to fluctuations in span efficiency, unless the density is increased beyond approximately \( n_c = 10 \) for the freestream-fixed and \( n_c = 16 \) for the relaxed-wake model. This most probably results from the approximation of the planform by means of two separate staggered wings, causing close encounter and hence strong mutual interaction of the wake shed by the forward main wing with its equivalent rear part. Compared to the relaxed-wake model, the freestream-fixed wake alignment facilitates a larger vertical distance of the upstream wake and the downstream main wing element. This likely translates into less interference and a lower element demand.

In general, the effect of spanwise element variations is found more substantial, which compares well with results of Bramesfeld and Maughmer [2008]. An element count of at least \( n_c = 24 \) is required for all three systems to attain a spatially converged solution. The influence of the wake-model on the convergence behavior is confirmed generally weak. For the split-tip wing, fluctuations in span efficiency are evident for comparable sparse spanwise panelings, which are however alleviated with increasing density.
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Figure 4.9: Influence of the chordwise and spanwise element number $n_c$ and $n_s$ on the span efficiency factor $\varepsilon$ for the elliptical, (a-b), the crescent, (c-d), and the split-tip wing, (e-f).
For the crescent wing, the temporal convergence history of the span efficiency factor is shown in Figure 4.10. It is apparent that convergence is attained within about 20 time-steps for both wake models. Performing a total amount of 60 time-steps is thus sufficient and conservative to assure a converged solution.

![Figure 4.10: Temporal convergence behavior for the crescent wing.](image)

### 4.4 Computational Results and Discussion

#### 4.4.1 Spanwise Load Distribution

Independent of the methodology or wake model involved, the crescent wing is found to produce a more near-elliptical spanload than the elliptical wing, as confirmed in Figure 4.11. Deviations from the ideal elliptical shape are only concentrated in the outboard region of the wing and lead, under equivalent flow conditions, to a larger span efficiency associated with the crescent wing. This is apparent from Table 4.3 and in consistency with Smith [1995].

#### 4.4.2 Computed Span Efficiency Factors

Computed span efficiency factors are summarized in Table 4.3 and compared to predictions from other references. In the case of the elliptical and crescent wing, an excellent agreement is achieved between the higher-order potential-method and the Euler-flow reference. This is true despite Euler-flow computations were performed at a freestream Mach
number of $M_\infty = 0.20$ and consider the finite thickness distribution of the airfoil sections. Estimates closely approach the theoretical value of $e = 1.000$. The present results compare well to Smith [1995] and DeHaan [1990] based on potential-flow methodology, and are in good general consistency with Euler-flow predictions by Bourdin [2002] and van Dam and Nikfetrat [1992].

However, the relative deviation in span efficiency between freestream-fixed and relaxed-wake estimates is of concern. Predictions relying on the higher-order potential-flow method show a notable difference by about $\Delta e_{rel} \approx 0.7\%$ in span efficiency for the crescent wing, compared to a value of $\Delta e_{rel} \approx 0.1\%$ according to Smith [1995]. Due to its straight and unswept trailing edge, only very limited contribution by higher-order wake effects is anticipated, resulting in span efficiency predictions largely independent of the applied wake model. This is reflected more consistently by Smith [1995], although larger deviations are evident with regards to the theoretical value and the Euler-flow reference. For the elliptical wing, Smith [1995] indicates the computed span efficiency to be considerably affected by the selected wake model approach in consistency with the wake substitution concept, which leads to a deterioration in span efficiency associated with the relaxed-wake model. This is in contrast to the prediction based on the higher-order potential-flow method, that indicates close agreement of estimates and a slight increase in span efficiency associated with the relaxed-wake shape.

![Graphs](image)

(a) Relaxed-wake model.  
(b) Euler-flow model.

**Figure 4.11:** Comparison of the normalized spanload of the elliptical and crescent wing based on the relaxed-wake model, (a), and the Euler-flow model, (b).
For the split-tip wing, the relaxed-wake estimate shows excellent consistency with Smith [1995], but over-predict the span efficiency with regards to the Euler-flow reference. Euler-flow results by Hicken and Zingg [2010] for a similar configuration are in good in agreement with present findings. With regards to the freestream-fixed estimate, a substantially improved efficiency is associated with the higher-order potential-flow method compared to Smith [1995]. This is likely due to the approximation applied for the split-tip wing planform, causing a modification of the freestream-fixed wake shape. As the wake is aligned with the freestream direction, a non-planar wake is created between the upstream and downstream main wing element, which is in contrast to Smith [1995] and ultimately leads to a larger span efficiency factor. The relaxed-wake model leaves the trailing edge aligned with the local flow and passes over the downstream main wing element in close vertical proximity.

<table>
<thead>
<tr>
<th></th>
<th>FreeWake</th>
<th>STAR-CCM+</th>
<th>[Smith, 1995]</th>
<th>[DeHaan, 1990]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$e_{\text{fixed}}$</td>
<td>$e_{\text{relaxed}}$</td>
<td>$e$</td>
<td>$e_{\text{fixed}}$</td>
</tr>
<tr>
<td>Elliptical wing</td>
<td>0.994</td>
<td>0.996</td>
<td>0.992</td>
<td>0.985</td>
</tr>
<tr>
<td>Crescent wing</td>
<td>0.993</td>
<td>1.000</td>
<td>0.998</td>
<td>0.991</td>
</tr>
<tr>
<td>Split-tip wing</td>
<td>1.089</td>
<td>1.115</td>
<td>1.105</td>
<td>1.048</td>
</tr>
</tbody>
</table>

**Table 4.3:** Comparison of computed span efficiency factors of planar references.

As a general remark it should be noted, that relaxed-wake estimates of Smith [1995] and DeHaan [1990] were attained by means of a Trefftz plane analysis, which is favored over a surface pressure integration but it is still problematic. For planar systems, Smith and Kroo [1993] showed that the predicted induced drag is very sensitive to the spanwise paneling and minor errors in the computed force-free wake shape. The higher-order potential-method computes the induced drag at the trailing edge [Eppler and Schmidt-Göller, 1990], which is considered as a decisive advantage in the accurate induced drag prediction.

Based on the present study, the higher-order potential-flow method and the Euler-
based farfield estimation have successfully demonstrated their ability to attain accurate
induced drag projections in good consistency with relevant theory and other references.
Preliminary Investigation

As a common representative of the class of highly non-planar systems, the current preliminary investigation considers the induced drag prediction of two simplified box wing configurations. In particular, issues occurring in its accurate computation employing the higher-order potential-flow method are discussed. Based on related flowfield characteristics, the plausibility of the solution and possible reasons causing deviations are assessed and identified. This includes an analysis of wake traces on a downstream partition surface and distributions of the streamwise vorticity, as well as an alternative computation of the induced drag by means of a Trefftz plane analysis.

5.1 Test Cases

Two simplified box wing configurations with single wing aspect ratios of $\Lambda_{1,2} = 6.0$ and positive or negative stagger factor were investigated. Their individual planforms are depicted in Figure 5.1; further geometric properties are summarized in Table 5.1. The systems are composed of two equivalent wings of constant chord length, which are connected by two joints at their wing-tips to form a closed, box-like and continuous surface. Both horizontal wings are unswept, which does not represent a realistic scenario for commercial aircraft. Nevertheless, because sweep alters the overall longitudinal wing arrangement, it potentially distorts the investigation of stagger effects on the induced drag prediction. Moreover, lifting surfaces do not incorporate camber, incidence or twist. To attain high
aerodynamic efficiencies, a careful design and selection of these parameters is of course required [Addoms and Spaid, 1975], but not an objective of the current study. Apart from that, camber, incidence and twist may affect the relative sensitivity of induced drag and system angle of attack in dependency of stagger, likely preventing general conclusions. The individual wing dimensions are further motivated by preceding research effort on planar wing induced drag characteristics by van Dam [1987] and Smith [1995]. Closed lifting surface configurations with similar geometric characteristics were considered by Gall and Smith [1987] and Kang et al. [2009a,b].

![Isometric view of the box wing planforms with a geometric height-to-span ratio of \((h/b) = 0.20\).](image)

**Figure 5.1**: Isometric view of the box wing planforms with a geometric height-to-span ratio of \((h/b) = 0.20\).

<table>
<thead>
<tr>
<th>(A_{1,2})</th>
<th>(c_{ref}, \text{m})</th>
<th>(b, \text{m})</th>
<th>(S_{ref}, \text{m}^2)</th>
<th>((h/b))</th>
<th>(St)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.0</td>
<td>1.0</td>
<td>6.0</td>
<td>12.0</td>
<td>0.20</td>
<td>±3.0</td>
</tr>
</tbody>
</table>

**Table 5.1**: Geometric properties of box wing configurations.

A meaningful selection of the stagger factor and the height-to-span ratio is particularly important. To identify values of practical and engineering interest, streamwise and vertical separations concerned in relevant research were reviewed. The objective is to enable more general conclusions about the impact of the geometrical arrangement on the induced drag and its prediction, despite the limited geometric complexity of the systems that are considered. In addition, both parameters must facilitate an investigation of higher-order wake and wake surrogate effects. As inferred from the wake substitution concept [Smith, 1995],

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a sufficient amount of streamwise separation is advisable to promote and intensify their contribution. This is congruent with more practical considerations, as in general, staggered box wings better reflect commercial aircraft scenarios. This is because the streamwise separation of wings enables the maintenance of a high aerodynamic efficiency, while enforcing static longitudinal stability and trim constraints [Andrews and Perez, 2015].

With regards to the height-to-span ratio, it is apparent that smaller vertical gaps are more likely to cause non-linear wing-wake interference. Although these interferences can be exploited to attain induced drag savings beyond linear theory, smaller height-to-span ratios result in overall reduced span efficiency factors for given span and lift [Prandtl, 1924; Munk, 1923b; Pistolesi, 1932; Cone, 1962; DeYoung, 1980; Demasi, 2007; Rizzo, 2007; Frediani and Montanari, 2009; Demasi et al., 2014, 2015a,b]. Larger vertical separations are thus favorable to maximize the span efficiency, but diminish the sensitivity of the induced drag and its prediction to higher-order wake effects.

Staggered box wings or biplanes with unswept horizontal lifting surfaces were analyzed i.e. by Bramesfeld and Malik [2015], Gall and Smith [1987] and Norton [1921] and concerned a negative separation of \( St = -1.0 \). The parametric investigations conducted by Kang et al. [2009a,b] involved stagger factors ranging from \( St = -1.5 \) to \( St = 1.5 \). Geometrically more complex systems were considered within the subsequent references. Defining stagger as the streamwise distance of the root quarter-chord locations between the forward and aft wing, Andrews and Perez [2015] analyzed the impact of stagger of up to \( St = +2.0 \) on aircraft efficiency. Similar, Salam and Bil [2015] describe the longitudinal arrangement by means of the leading edge root chord separation between both wings and investigated a range of approximately \( St \approx +3.0 \) to \( St \approx +5.0 \). The parametric study by Mamla and Galinski [2009] encompasses values of about \( St \approx -6.0 \) to \( St \approx +5.0 \) based on an equivalent definition of the stagger factor, whereas the systems concerned by Lange et al. [1974] and by Henderson and Huffman [1975] are characterized by a stagger factor of \( St \approx +5.0 \). The initial box wing geometry presented in Gagnon and Zingg [2015] uses a comparably large root chord separation of about \( St \approx +10.0 \). However, it must be acknowledged that lifting surfaces presented by the latter references are composed of wings with contrary sweep angles. The actual amount of stagger is therefore dependent on the
spanwise location, usually resulting in a larger streamwise separation at the wing root than compared to the tip. The average stagger factor $St_{avg}$ is thus smaller than indicated by the root chord separation. In the case of Andrews and Perez [2015], Gagnon and Zingg [2015], Lange et al. [1974] and Henderson and Huffman [1975], the average stagger amounts $St_{avg} \approx +1.0$, $St_{avg} \approx +6.0$, $St_{avg} \approx +3.0$ and $St_{avg} \approx +3.0$ respectively. With regards to these geometrically more complex systems, it becomes apparent that positive stagger factors are preferred.

This preference is certainly related to the extent of realizable vertical separations, which is closely coupled to the longitudinal arrangement of the system. For positive stagger and based on aerodynamic considerations, the upper wing may be attached on top of the vertical stabilizer. This facilitates a larger vertical gap and results in a larger span efficiency compared to an equivalent negative stagger, where the upper wing is typically forced to be situated on top of the fuselage. In addition, downwash effects induced by the upstream wing are likely to be alleviated compared to a negative-staggered arrangement. However at equivalent vertical gaps, negative-staggered systems attain larger span efficiencies for positive angels of attack [Munk, 1923b; Selberg, 1983; Selberg and Cronin, 1986; Kang et al., 2009a,b; Mamla and Galinski, 2009]. Viscous drag and structural weight increase constrain the vertical gap dimensions [Andrews and Perez, 2015]. Existing conceptual research has concerned a wide spectrum of possible vertical separations. Within parametric studies, height-to-span ratios in the range of $(h/b) = 0.06$ to $(h/b) = 0.5$ were considered by Andrews and Perez [2015]. Kang et al. [2009a,b] analyzed configurations with height-to-span ratios between $(h/b) = 0.08$ to $(h/b) = 0.33$. The investigations by Bramesfeld and Malik [2015], Gagnon and Zingg [2015], Lange et al. [1974] and Gall [1984] considered vertical gaps of $(h/b) = 0.10$, $(h/b) = 0.20$, $(h/b) = 0.27$ and $(h/b) = 0.16$ respectively.

Stagger factors of $St = \pm 3.0$ between lifting surfaces are used herein. These are considered to provide a realistic representation of the possible average range of streamwise separations, enabling a more general usability of results and aim to specifically promote and intensify higher-order wake effects. With regards to a height-to-span ratio of engineering interest, a value of $(h/b) = 0.20$ is selected for the present work. It is again emphasized that the height-to-span ratio is defined as the vertical gap between the trailing edges of
both wings. This is in contrast with other definitions given in the literature based on the quarter-chord or outer airfoil contour separation [Kang et al., 2009a,b; Hicken and Zingg, 2010].

5.2 Computational Implementation

The computations were performed as described in the procedure detailed in Chapter 4.2. The following section is thus limited to a brief summary and description of certain differences emerging from the inherent geometrical properties of the box wing configuration.

5.2.1 Euler-Flow Method

The rectangular-shaped flow domain extends 15 reference chords upstream and 50 reference chords downstream of the lower wing trailing edge, whereas the vertical and lateral dimensions are given by 25 and 2 × 20 reference chords respectively, as shown in Figure 5.2. Symmetry conditions were applied, to reduce the cell count and consequently the computational effort. The simulations were conducted at a subsonic freestream Mach number of \( M_\infty = 0.20 \), equivalent to the planar reference cases. Lifting systems were established according to the geometrical properties given in Table 5.1. Their streamwise cross-sections are represented by a single, thin and symmetric airfoil with a maximum thickness-to-chord ratio of \( (t/c) = 0.04 \). The intention here is to minimize thickness and Mach number effects and thus to improve the agreement with the higher-order potential-flow method, which considers a freestream Mach number of \( M_\infty = 0.00 \) and neglects the thickness contribution. Airfoil sections were modeled with a sharp trailing edge. The freestream velocity vector direction was imposed perpendicular to the upstream boundary of the flow domain and aligned with the predominately hexahedral grid. To incline the freestream velocity vector, the entire lifting system was rotated along its lower wing trailing edge. Local surface and volume-based grid refinement strategies were employed to improve the numerical resolution and accuracy in solution-critical areas of the flow domain as illustrated in Figure 5.3.

The estimation of lift coefficients was based on a surface pressure integration, whereas
the induced drag was computed by the wake integration technique [Destarac and van der Vooren, 2004] discussed in Section 3.2.1. The approach was validated in conjunction with planar reference systems in Chapter 4. To compute the contribution of spurious entropy drag $D_{Sp}$ in the induced drag estimation, a correction volume $\Omega_{\text{wake}}$ enclosing the wake region was constructed. The volume originates in close proximity downstream of the wing’s individual trailing edges ($\approx 0.05 \cdot c_{\text{ref}}$) and proceeds a distance of $d_{TP}$ downstream to coincide with a transverse plane as illustrated in Figure 5.2.

**Figure 5.2:** Schematic illustration of the flowfield, transverse plane (TP) and the correction volume $\Omega_{\text{wake}}$ enclosing the wake region to compute the spurious entropy drag contribution.

**Figure 5.3:** Implemented predominately hexahedral grid in $STAR-CCM+$ for the box wing configuration with a positive stagger factor of $St = +3.0$. 

(a) Farfield grid.  
(b) Wing surface grid.
5.2.2 Higher-Order Potential-Flow Method

The thickness distribution of the lifting systems was neglected. Therefore, conforming models based on mean surface representations were implemented, using a freestream-fixed or relaxed-wake wake approach as depicted in Figure 5.4. Symmetry conditions were used to minimize the computational expense. To evolve the wake, a time-step width of $\Delta t = 0.25 \cdot \left( \frac{c_{\text{ref}}}{V_\infty} \right)$ was used according to Bramesfeld [2006]. A total amount of 60 iteration steps were performed to assure convergence. Transition conditions were set to maintain equal vorticity and vorticity gradient among neighboring distributed vorticity elements.

![Freestream-fixed wake.](a) ![Relaxed-wake.](b)

*Figure 5.4:* Discretization of the lifting surfaces, the freestream-fixed wake, (a), and the relaxed-wake, (b), after 20 time-steps at system angle of attack of $\alpha = 8.0^\circ$.

The neighboring DVEs at the junction between horizontal and vertical wings enclose a large lateral angle (i.e. are not co-planar), which possibly causes non-physical velocity peaks at the side-edge due to incomplete singularity canceling [Basom and Maughmer, 2011]. This is a particular problem in the wake-relaxation procedure, where the side-edge velocities are employed to relax the wake. To alleviate these numerical issues, a blending element was introduced between horizontal and vertical wings. Although this procedure appears only necessary for the relaxed-wake model case, blending elements were also used for the freestream-fixed wake to assure comparability within potential-flow results.

A compressibility-correction is not available within the present potential-flow method. To assess contributions by Mach number effects, a study was initiated using the higher-order panel method *PanAir* [Magnus and Epton, 1990]. This computational method

\[A\text{ comparison towards the higher-order potential-flow method is given in Section 7.1.5}\]
Figure 5.5: Assessment of the relative deviation in computed lift coefficient $C_L$, induced drag coefficient $C_{D_i}$ and span efficiency factor $e$ due variations in freestream Mach number $M_\infty$ based on the higher-order panel method (*PanAir*).

represents the trailing vorticity by discrete freestream-fixed filaments and incorporates a Prandtl-Glauert [Anderson, 2001] correction to account for compressibility effects. The thickness distribution was selected equivalent to the Euler-flow test case. Results for the positive-staggered configuration at a system angle of attack of $\alpha = 8.0^\circ$ are compiled in Figure 5.5. Similar to Smith [1995] for the crescent wing, the impact of the Mach number on the span efficiency is found to be very limited. The deviation in span efficiency due to the inequality in freestream Mach number amounts approximately $\Delta e_{rel} \approx 0.08\%$ in span efficiency, which is considered acceptable. Nevertheless, the deviation in lift and induced drag coefficient are more pronounced.

### 5.3 Spatial Convergence Study

#### 5.3.1 Euler-Flow Method

The spatial convergence was investigated applying a constant grid refinement factor of $r = 1.5$. This did produce cell counts between approximately $8 \times 10^6$ and $61 \times 10^6$. An asymptotic convergence criterion, equivalent to Section 4.3.1, was established. The induced drag variation was required to be less than $\Delta C_{D_i} = 1 \times 10^{-5}$ for 100 consecutive iterations for each individual grid level.

The convergence behavior of the span efficiency factor at an angle of attack at $\alpha =$
8.0°, including a Richardson extrapolation [Roache, 1997], is plotted against the non-dimensional grid element size $h_{grid}$ in Figure 5.6. In contrast to planar systems, a monotonic convergence behavior is evident, permitting an extrapolation. The ratio of the grid convergence index $R_{CGI}$ based on the three finest consecutive grid levels is approximately one. This indicates that the solution is well within the asymptotic range of convergence and a Richardson extrapolation can be applied successfully to obtain the continuum value [Roache, 1997]. Overall, the spatial convergence is found to be excellent. The relative deviation between the extrapolated continuum value and the span efficiency based on the finest grid (level 1) constitutes approximately $\Delta e_{rel} \approx 0.3\%$ for the positive-staggered and $\Delta e_{rel} \approx 0.4\%$ for the negative-staggered system. Details of the convergence study are summarized in Table 5.2.

For both stagger factors, the impact of the downstream location of the transverse plane on the span efficiency factor was studied to verify the effectiveness of the applied induced drag estimation and correction technique within the present scope. As indicated in Figure 5.7 for an angle of attack of $\alpha = 8.0^\circ$, the span efficiency factor is virtually independent of the streamwise location of the transverse plane. Actually, span efficiency slightly decreases for more downstream locations, which is equivalent to an increase in induced drag and opposed to the anticipated general behavior of the uncorrected induced
Table 5.2: Spatial convergence behavior of the span efficiency factor $e$ for a system angle of attack of $\alpha = 8.0^\circ$.

<table>
<thead>
<tr>
<th>Grid level</th>
<th>$h_{grid}$</th>
<th>Cell count</th>
<th>$e_{St=+3.0}$</th>
<th>$\Delta e_{St=+3.0}$</th>
<th>$e_{St=-3.0}$</th>
<th>$\Delta e_{St=-3.0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.23</td>
<td>$8 \times 10^6$</td>
<td>1.403</td>
<td>1.2%</td>
<td>1.537</td>
<td>1.5%</td>
</tr>
<tr>
<td>2</td>
<td>0.15</td>
<td>$21 \times 10^6$</td>
<td>1.395</td>
<td>0.6%</td>
<td>1.526</td>
<td>0.7%</td>
</tr>
<tr>
<td>1</td>
<td>0.10</td>
<td>$61 \times 10^6$</td>
<td>1.391</td>
<td>0.3%</td>
<td>1.521</td>
<td>0.4%</td>
</tr>
<tr>
<td>0</td>
<td>0.00</td>
<td>-</td>
<td>1.386</td>
<td>-</td>
<td>1.515</td>
<td>-</td>
</tr>
</tbody>
</table>

Figure 5.7: Impact of the downstream location of the transverse plane $x_{TP}$ on the span efficiency factor $e$ for a system angle of attack of $\alpha = 8.0^\circ$.

drag, as demonstrated for planar reference systems in Figure 4.8. The decay of induced drag is hence over-compensated; its prediction based on more downstream locations considered conservative. The absolute streamwise variation in span efficiency accumulates to approximately $\Delta e \approx 0.003$ for the positive-staggered and $\Delta e \approx 0.002$ for the negative-staggered case, which is equivalent to a relative deviation of less than $\Delta e_{rel} \approx 0.1\%$ with respect to the most downstream location.

5.3.2 Higher-Order Potential-Flow Method

The solution was considered to be converged if the variation in span efficiency of two successive refinement steps was less than $\Delta e = 0.003$. The span efficiency factor is found to
be weakly dependent on the chordwise element density, as evident from Figure 5.8(a-b) for the freestream-fixed wake approach. In the case of the relaxed-wake model given in Figure 5.8(c-d), the number of chordwise elements gains slightly more influence, particularly for angles of attack larger than \( \alpha \approx 5.0^\circ \). A chordwise element count of \( n_c = 12 \) is found to provide converged results independent of stagger or wake model approach. The influence of the spanwise element density, as depicted in Figure 5.9, is found to be in general more distinctive than in the chordwise direction. Especially in case of the negative-staggered system and the relaxed-wake approach, a considerably larger element count of \( n_s = 20 \) is required. Otherwise, \( n_s = 12 \) are found to discretize lifting surfaces sufficiently.

5.4 Computational Results and Discussion

5.4.1 Computed Span Efficiency Factors

The computed span efficiency factors versus the system angle of attack are presented in Figure 5.10a for the positive-staggered and Figure 5.10b for the negative-staggered system. Substantial differences in span efficiency are evident among freestream-fixed and relaxed-wake model estimates, as well as compared to the Euler-flow reference, especially at larger system angles of attack. In the case of positive stagger, the potential-flow method over-predicts the span efficiency factor, whereas for negative stagger the system efficiency is under-predicted relative to the Euler-based solution. The maximum relative error to the Euler-flow reference is associated with the positive-staggered system and the freestream-fixed wake model and accumulates to \( \Delta e_{rel} \approx 13.3\% \) in span efficiency. For the negative-staggered system, a similar error level exists with a maximum of about \( \Delta e_{rel} \approx -12.1\% \) in span efficiency, however applying to the relaxed-wake model.

For the potential-method, the estimated correlation of the span efficiency factor and the system angle of attack is of critical concern. Compared to the Euler-flow reference, an inverse dependency is predicted. In particular for positive stagger, the freestream-fixed wake model indicates a linear increase in span efficiency with the angle of attack. Opposed to that, an estimation based on the Euler-flow reference shows a non-linear correlation, which is characterized by a weak variation for angles of attack smaller than \( \alpha \approx 3.0^\circ \) and
Figure 5.8: Influence of the chordwise element number \( n_c \) on the span efficiency factor \( e \) versus the system angle of attack \( \alpha \) for the freestream-fixed, (a-b), and the relaxed-wake model, (c-d).

For angles of attack beyond the optimum, a continuous decrease in span efficiency is observed. For larger angles of attack, span efficiency factors increase and hence considerably deviate from the Euler-flow result. In the case of negative stagger, an almost linear increase of the span efficiency with the system angle of attack is predicted using the Euler-flow reference. This conflicts with potential-flow results, which decrease with the system angle of attack.
5.4.2 Assessment of Trailing Wake Flowfield Properties

Provided Euler-based estimates reflect the induced drag characteristics correctly, reasonable doubts exist with regards to the ability of the higher-order potential-flow method to predict the induced drag of the present box wing configurations correctly. An investigation
was conducted to explore the methodological or physical reasons associated with the pronounced deviations encountered for the potential-based method and to thoroughly assess the plausibility of the Euler-flow results more in general. It was specifically aimed to relate span efficiency predictions to basic flowfield properties, in particular the wake shape and its vorticity distribution, in order to substantiate results or to disclose methodological or physical errors. This is enabled by means of related theorems, especially the biplane theorem [Prandtl, 1924; Munk, 1923b], the stagger theorem in its more general form given by Kroo and Smith [1990] and the wake substitution concept [Smith, 1995]. These theorems were detailed in Section 2.4; their key statements shall be shortly recapitulated here for the purpose of convenience.

Based on lifting-line theory [Prandtl, 1923], a proportional correlation exists between the optimum span efficiency factor of the box wing and its vertical extent. This correlation was first approximated by [Prandtl, 1924] (compare Equation 2.20) and confirmed sufficiently accurate for the range of practical height-to-span ratios. [Demasi et al., 2014, 2015a,b]

Without restricting the derivation of the stagger theorem to lifting-line theory, it becomes apparent that the induced drag depends only on the wake shape and the distribution
of vorticity in the wake [Kroo and Smith, 1990]. A direct comparison of the wake shape and its vorticity is hence qualified to assess the solution more in detail, but is prevented by the complex structure of the rolled-up wake and lacks from a sufficient valuation basis. Conclusions regarding the relative induced drag or span efficiency factor however can be made indirectly, applying the wake substitution concept [Smith, 1995]. As discussed in Section 2.4.2, a transverse partition surface can be located downstream of the lifting element to replace the physical, force-free wake with a streamwise projection. This can be done without affecting farfield induced drag [Smith, 1995]. The extent of streamwise substitution is limited by the intersection with the downstream trailing edge of the system, in essence implying, that the wake trace at the partition surface defines the induced drag [Smith, 1995].

Further acknowledging that the height-to-span ratio of the system is a measure for the system efficiency, it is hypothesized, that the vertical extent of the wake trace at the partition surface, referred to as the effective height-to-span ratio \((h/b)_{\text{eff}}\), can be utilized to assess the span efficiency of present highly non-planar systems. For non-zero angles of attack, the effective height-to-span ratio is in general different compared to the geometric height-to-span ratio. The effective height-to-span ratio depends on the system angle of attack and the stagger factor. Already Munk [1923b] noted that the aerodynamic induction is determined by the position of the layer of unsteadiness of the potential-flow behind the wing (the wake trace) and its direction. It is thus astonishing that existing studies do not account for the effective height-to-span ratio, its dependency on the stagger factor and its relation to the span efficiency.

**Wake Trace on Partition Surface**

The location of the transverse partition surfaces is illustrated in Figure 5.11 and coincides with the most downstream trailing edge. The analysis of the wake trace is thus effectively limited to the wake shed from the upstream trailing edge and in parts from the vertical wing. It should be noted that the partition surface is not positioned perpendicular to the freestream direction. To assure an improved illustration of the wake traces and enable a comparison between the positive and negative-staggered system, the plane is aligned with
the body-fixed $z$-direction. Although this does not exactly comply with the theory of wake substitution, it is considered sufficient to provide meaningful insight.

![Diagram](image)

**Figure 5.11:** Transverse partition surface location for a positive stagger factor of $St = +3.0$, (a), and a negative stagger factor of $St = -3.0$, (b).

An exemplary investigation was performed for a system angle of attack of $\alpha = 8.0^\circ$, as depicted in Figure 5.12. Based on a visual assessment, a strong geometrical similarity is evident among the relaxed-wake trace and the Euler-flow reference. This gives initial indication that the relaxed-wake shape, resulting from local alignment with the (induced) velocities downstream, and hence the associated flowfield are reasonable and similar to the Euler-flow reference, which contrasts the poor agreement in computed span efficiency.

Employing the effective height-to-span ratio as a valuation basis, it becomes apparent that the effective vertical gap differs considerably in dependency of the employed wake model. For the freestream-fixed wake model, the effective vertical gap is constant across the span, where it is a function of the spanwise location in the case of the relaxed-wake model, permitting free-wake deformations, and the Euler-flow reference. The effective height-to-span ratio is further found to depend on the stagger factor of the system. This effect originates from the orientation of the wake trajectories at positive angles of attack. As the system is progressively inclined to the freestream velocity vector, the effective height-to-span ratio, is continuously reduced in the case of positive stagger and respectively increased for negative stagger, as already discussed by Munk [1923b].

Consistent with the Euler-flow reference, the application of the freestream-fixed and relaxed-wake model results in a reduction of the effective vertical gap for the positive-
staggered system and an increase in the case of the negative-staggered system for positive system angles of attack. Based on that, an equivalent correlation of the span efficiency and the angle of attack is plausible and anticipated, however not predicted using the higher-order potential-flow method.

This fact can be further substantiated by an approximation of the optimum span efficiency factor based on Equation 2.20 and an average value of the effective vertical gap at the partition surface perpendicular to the freestream direction. It is referred to as $\epsilon_{Bw, \text{opt}}$ and additionally provided in Figure 5.12, quantitatively supporting the qualitative observation made using the effective height-to-span ratio and contradicting existing results. Independent of the wake model, the approximation consistently indicates a larger span efficiency associated with the negative-staggered system and an improved agreement for equivalent stagger compared to the Euler-flow reference.

**Distribution of the Streamwise Vorticity**

Contour plots of the streamwise vorticity $\gamma_x$ at a system angle of attack of $\alpha = 8.0^\circ$ are given for the freestream-fixed, the relaxed-wake model and the Euler-flow reference in Figure 5.13. In contrast to the investigation of wake traces, the computation of cross-flow induced velocities is performed on a transverse plane, located at a distance of one reference chord length downstream of the system, to avoid numerical issues due to intersecting with the trailing edge. Although this does not congruently correlate vorticity contours to wake traces at the partition surface, it still meaningful represents flowfield characteristics related to the induced drag of the system. This is additionally supported acknowledging, that under steady-state conditions evident herein, the absolute streamwise vorticity remains constant in the streamwise direction within potential-flow. For the Euler-flow reference, the streamwise decay of vorticity related to artificial viscosity effects is considered limited, because of the high element density in this region of the flow domain.

Independent of stagger, the contours based on the relaxed-wake and the Euler-flow reference show reasonable similarity. The highest absolute vorticity value is accumulated in the tip region of the wing-junctions, or in close proximity to the rolled-up vortex cores. For the relaxed-wake model, the vortex core centers are smeared, or even two core locations
Figure 5.12: Wake traces at the partition surface for a system angle of attack of $\alpha = 8.0^\circ$.
exist. This is likely due to the blending element at the wing-junction but possibly also related to numerical leakage effects of the wake's distributed vorticity elements in that region. Nevertheless, for the relaxed-wake model and the Euler-flow reference, the agreement of the vortex core locations is acceptable. As a consequence of the spanwise flow pattern introduced by the upward or downward orientation of the vertical lifting surfaces, vortex cores are found to be shifted either inboards for the upper wing, whereas the lower wing vortex proceeds in the opposite direction, according well with theory [Maughmer, 2003]. This effect cannot be resolved by the freestream-fixed wake model. Consistent with the assessment of wake traces, vorticity contours of course similarly represent differences in effective height-to-span ratio in dependency on the stagger factor. No evidence of significant non-physical distortion of the flowfield exists.

5.4.3 Trefftz Plane Analysis

The preceding investigation of the trailing wake flowfield properties has provided evidence that the wake traces and the vorticity distributions associated with the Euler-based solution consistently represent the computed correlation of the span efficiency factor, the longitudinal stagger and the system angle of attack. A reasonable similarity exists between the wake traces and the streamwise vorticity distributions of the Euler-flow reference and the relaxed-wake approach.

For the selected potential-flow method, the approximations of the span efficiency factor based on linear theory have demonstrated an improved accordance compared to the Euler-flow reference for equivalent stagger. It was further shown that opposed to the computation of induced drag, the approximations using the wake trace consistently reflect efficiency advantages of the negative-staggered system. This implies, that the inherent flowfield and induced velocities are similar to the Euler-flow reference and free from physical or numerical errors sufficient to cause significant deviations in the computation of the span efficiency factor. It was thus assumed that the errors are caused by methodological issues, probably related to the computation of induced drag based on the trailing edge analysis [Eppler and Schmidt-Göller, 1990] in conjunction with the present box wing configurations.
Figure 5.13: Contours of the streamwise vorticity $\gamma_x$ at a distance of one reference chord length $c_{ref}$ downstream of the system trailing edge for a system angle of attack of $\alpha = 8.0^\circ$. 

(a) Freestream-fixed wake, $St = +3.0$.

(b) Freestream-fixed wake, $St = -3.0$.

(c) Relaxed-wake, $St = +3.0$.

(d) Relaxed-wake, $St = -3.0$.

(e) STAR-CCM+, $St = +3.0$.

(f) STAR-CCM+, $St = -3.0$. 

CHAPTER 5: PRELIMINARY INVESTIGATION
In an attempt to further explore and substantiate the assumption of methodological-dependent errors related to the computation of induced drag within the higher-order potential-flow method, a Trefftz plane analysis was established and exemplary utilized together with the relaxed-wake model to provide alternative induced drag estimates. The estimation approach originally follows from the stagger theorem [Munk, 1923a] or can be derived in a less restrictive manner applying the principle of momentum conservation [Kroo and Smith, 1990], as shown in Section 3.3.1. In theory the Trefftz plane is located at an infinite distance downstream of the lifting element. For any practical case, the plane is placed several spans away from the system, to ensure that the impact of the bound vortex has died out and only cross-flow contributions need to be considered.

Figure 5.14: Dimension of the Trefftz plane located at a distance of one reference chord length $c_{\text{ref}}$ downstream of the system trailing edge.

To minimize the impact of geometrical uncertainties (compare Section 3.3.1), the present study positions the transverse plane in close proximity downstream of the lifting surfaces as illustrated in Figure 5.14. However, this requires the streamwise perturbation velocities to be taken into account and prevents the reduction to the classical line integral but involves a numerical surface integration of induced velocities. For force-free wakes, streamwise perturbation velocities are generally produced in the transverse plane, as the wake trajectory is not aligned with the freestream direction, but are often omitted to enable
the line integration and can be neglected anyway, even for induced drag prediction with high accuracy demands [Smith, 1995]. This is not correct if the plane is located close to the lifting element, where the bound vorticity induces significant streamwise perturbations. To enable accurate induced drag computation in the nearfield, the evaluation of the complete surface integral as given by Equation 5.1 is required [Smith, 1995].

\[ D_i = \int \int_{S_{TP}} \left( \vec{f}_i \cdot \vec{n} \right) dS \]  

(5.1)

with the vector \( \vec{f}_i \):

\[
\vec{f}_i = \frac{\rho_\infty}{2} \cdot \begin{pmatrix}
    u^2 + w^2 - u^2 \\
    -2 \cdot v \cdot u \\
    -2 \cdot w \cdot u
\end{pmatrix}
\]

(5.2)

For methodologies employing discrete trailing vortex filaments, this approach can still become impractical. In order to resolve velocities adequately, the singular vortex centers must be approached so closely that the model can become invalid [Mortara and Maughmer, 1993]. This problem is largely avoided for the higher-order potential-flow method herein due to the continuous vortex sheet representing the vortical wake, although in the plane of the sheet the tangentially induced velocity is undetermined [Bramesfeld, 2006].

To capture all significant streamwise perturbations, velocity computations and integration enclose a large area [van Dam, 1999], which ranges 25 reference chords in the spanwise direction and ±25 reference chords in the vertical direction, as depicted in Figure 5.14. The sampling approach involved a nested refinement strategy to limit the computational expense but to sufficiently resolve velocities gradients close to the lifting element. The inner region extends 5 reference chords in the spanwise direction and ±6 reference chords vertically, employing a grid spacing of 0.05 reference chords, in contrast to a spacing
of 0.25 reference chords in the outer region. The velocity estimation leads to prolonged computation times compared to the trailing edge analysis and is hence not suitable on a routine basis.

Computed span efficiency factors using the Trefftz plane analysis are summarized in Table 5.3 for a system angle of attack of $\alpha = 8.0^\circ$ and compared to estimates based on the trailing edge analysis and the Euler-flow reference (grid level 1) for positive and negative stagger. It is found that the Trefftz plane span efficiency predictions considerably improve the consistency with the Euler-flow reference independent of stagger. The relative error in span efficiency is reduced by about $\Delta e_{rel} \approx 7.7\%$ for the positive-staggered system and by approximately $\Delta e_{rel} \approx 6.7\%$ for the negative-staggered system. As the Trefftz plane estimation is directly based on computations of the induced velocities within the trailing wake flowfield, this clearly implies, that the flowfield is reasonable. It is hence likely, that errors are due to methodological issues, probably related to the computation of the induced drag at the trailing edge.

<table>
<thead>
<tr>
<th>Estimation approach</th>
<th>$e_{St=+3.0}$</th>
<th>$\Delta e_{St=+3.0}$</th>
<th>$e_{St=-3.0}$</th>
<th>$\Delta e_{St=-3.0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trailing edge analysis</td>
<td>1.533</td>
<td>10.1%</td>
<td>1.419</td>
<td>-6.7%</td>
</tr>
<tr>
<td>Trefftz plane analysis</td>
<td>1.359</td>
<td>-2.4%</td>
<td>1.521</td>
<td>0.0%</td>
</tr>
<tr>
<td>\textit{STAR-CCM+}</td>
<td>1.392</td>
<td>-</td>
<td>1.521</td>
<td>-</td>
</tr>
</tbody>
</table>

\textbf{Table 5.3:} Comparison of the span efficiency factors $e$ based on the trailing edge analysis and Trefftz plane analysis for the relaxed-wake model compared to \textit{STAR-CCM+} at a system angle of attack of $\alpha = 8.0^\circ$. 

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Inferred by preliminary study results, an investigation was initiated to explore and clarify on potential issues related to the induced drag estimation at the trailing edge in conjunction with the presented box wing configurations. Both, a methodological or programming error were taken into consideration. Subsequent to an intensive examination of the computational code, a methodological simplification in the implemented estimation technique, not permissible for box wing configurations and other highly non-planar concepts, was identified as the source of error. In-line with relevant theory, a modified approach is presented and tested herein.

6.1 Theoretical Derivation

As discussed in Section 3.3.2, the induced drag is calculated along the trailing edge based on the Kutta-Joukowsky theorem by taking the cross product between the circulation that is shed into the wake and the velocity induced by the wake at this spanwise location. For each trailing edge DVE, this involves a numerical integration (Simpson’s rule with overhanging parts [Chapra and Canale, 2010]), which further requires a determination of induced velocities at three spanwise locations along the trailing edge.

For lifting systems incorporating trailing edge sweep, the normal velocities induced by the distributed vorticity sheet of the wake become singular along the leading edge and require a special treatment [Bramesfeld and Maughmer, 2008]. Utilizing the stagger the-
orem of Munk [1923a] and provided the spanwise circulation distribution is maintained, the induced drag remains finite and equivalent to that of an unswept system. During the discretization procedure, the trailing edge sweep is therefore effectively removed while a constant spanwise circulation distribution is retained, forming an unswept but longitudinally staggered system.

\[ \Gamma_i \quad \Gamma_{ii} \]

\[ A \quad B \quad C \]

**Figure 6.1:** Three vortex systems that ultimately produce identical induced drag, adapted from Bramesfeld [2006]

To compute the mutual induction by several, longitudinally arranged distributed vorticity elements, the trailing edge points of the induced element must be shifted into the plane of the inducing DVE’s trailing edge points. According to the stagger theorem, the projection must be performed in the streamwise direction, actually aligned with the freestream-fixed wake trajectory, which is theoretically exact for this particular case. Despite existing formal difficulties with the stagger theorem (compare Section 2.4.1), this is practical, as the bound circulation is lumped into a single line at the trailing edge.

Nevertheless, the classical derivation by Munk [1923a] is unnecessarily restricted by inherent theory [Kroo and Smith, 1990], the lifting-line concept and the straight, freestream-fixed wake vortex sheet. Generally, the case of force-free wakes is not included, which is of particular concern with regards to the projection of the trailing edge points. Omitting these restrictions, a more general derivation of the Stagger Theorem is obtained by means of the principle of momentum conservation. Kroo and Smith [1990] showed that the induced drag depends only on the geometric shape of the wake and its vorticity distribution. If limited to the lifting-line concept the classical statement of the stagger theorem by Munk
CHAPTER 6: MODIFIED TRAILING EDGE ANALYSIS

[1923a] is re-attained.

Establishing the trailing edge analysis in conjunction with arbitrary wake shapes consequently requires a correlation between the projection of the trailing edge points and the wake shape and its trajectory. In this context, the wake trajectory is characterized by its longitudinal inclination with regards to the body-fixed reference frame. For a freestream-fixed wake, the inclination is equivalent to the system angle of attack, which results in a projection aligned with the streamwise direction in compliance with the stagger theorem. For the case of a force-free wake, an equivalent approach is proposed.

This can also be inferred from [Munk, 1923b], noting that the aerodynamic induction is described by the position of the vortex sheet and its direction, as discussed in Section 2.4.1. For present highly non-planar concepts, the vertical distance between the vortex sheets corresponds to the effective height-to-span ratio and is a measure for the relative induced drag of the system. It is therefore sensible to align the projection to match the effective height-to-span ratio accordingly.

6.2 Computational Implementation and Discussion

In its original implementation, the projection approach employed the trailing edge condition of the most aft surface DVE, actually shifting the trailing edge points in the body-fixed direction, as depicted in Figure 6.2. In contrast to the preceding theoretical derivation, the projection is incorrectly aligned with the lifting surface. In the case of highly non-planar lifting concepts, the projection approach is unable to account for any variation in effective height-to-span ratio imposed by the system angle of attack or wake model, which ultimately results in erroneous predictions of mutual induction effects and hence induced drag. A projection in the body-fixed direction is thus not adequate.

However, as demonstrated in Chapter 4, the body-fixed alignment is sufficiently accurate for planar monoplane wings. This may hold valid, for lifting systems with moderate trailing edge sweep, as long as the non-planar character of the wake, originating from the combination of sweep and angle of attack is comparatively small.

The modified technique discussed herein utilizes the leading edge condition of the
most upstream post-trailing edge wake element, which is equivalent to the trailing edge condition of the most aft surface DVE. In contrast to the original approach, this does not require one to make an assumption of the local wake inclination, but it can be determined from the surface normal of the respective most upstream post-trailing edge wake element. This is expedient to enable the projection starting from the first wake iteration step, which is especially of concern for the relaxed-wake model. Within the provided source code, the approach referred to as *Eppler-PTWE* has been existing before but remained inoperative. The particular section of the source code is provided in Appendix A.

![Graph a](image1.png)  
![Graph b](image2.png)

**Figure 6.2:** Comparison of the original and modified projection approaches shifting the trailing edge points of the induced DVE onto the plane of the inducing DVE for the freestream-fixed wake, (a), and relaxed-wake, (b).

As illustrated in Figure 6.2, the projection is executed by means of the local spanwise inclination $\epsilon_{\text{wake}}$ of the particular post-trailing edge wake element. For a freestream-fixed wake, the inclination of the wake element is constant in the spanwise and streamwise direction. It is equivalent to the system angle of attack and shifts the trailing edge points of the induced element in the streamwise direction, either upstream or downstream. In the case of the relaxed-wake, the wake DVEs are aligned with the local velocity vector. Their longitudinal inclination varies with the spanwise and streamwise location and is in general not equivalent to the system angle of attack.

The modified technique has enabled a considerably better agreement of computed span efficiency factors with Euler-based predictions, which will be discussed in Chapter 7.
and Chapter 8. However, it should be recognized that the modified approach is considered only strictly accurate for the freestream-fixed wake model. In the case of the relaxed-wake model, the wake DVE inclination is dependent on the streamwise location. A projection involving the most upstream post-trailing edge wake element can hence only approximate the local inclination at the projection plane, which may demonstrate some limitation of this induced drag estimation methodology in principle.

6.3 Computational Validation

6.3.1 Planar Validation Case

The initial computational validation of the modified projection approach concerned the crescent wing, which was presented in Section 4.1.1. Computed span efficiency factors attained by means of the original and the modified approach, using the freestream-fixed and the relaxed-wake model are summarized in Table 6.1. In addition, estimates based on a Trefftz plane analysis, equivalent to the procedure described in Section 5.4.3 and Euler-flow predictions are provided. The relative deviation in span efficiency $\Delta e$ refers to the theoretical value of $e = 1.000$ based on lifting-line theory. The agreement between predictions employing the original and the modified formulation is found to be excellent, which confirms the anticipated limited effect of the actual projection approach on induced drag estimation for simple planar monoplane wings at low angles of attack. Although the original approach contradicts stagger theorem, it can be considered as a sufficiency accurate approximation in this particular case. Therefore, the conclusions drawn in Chapter 4 utilizing the original projection are unaffected and prevail valid.

For the freestream-fixed wake model, this is quite obvious, because the crescent wing does likely not exhibit wake or angle of attack-induced effects. This is certainly due to its planform being planar, its straight, un-swept trailing edge and its limited streamwise extent compared to the highly non-planar systems concerned herein. The wake shape and thus the induced drag are independent of the angle of attack. Neglecting the minor contribution by the rolled-up tip-vortex, similar is true for the relaxed-wake model case, as inferred by the wake substitution concept [Smith, 1995].
Table 6.1: Comparison of the span efficiency factors $\epsilon$ for the crescent wing based on original and modified projection approach for the freestream-fixed and relaxed-wake model at an angle of attack of $\alpha = 4.0^\circ$.

<table>
<thead>
<tr>
<th>Computational-methodology</th>
<th>Estimation technique</th>
<th>$\epsilon_{\text{fixed}}$</th>
<th>$\Delta\epsilon_{\text{fixed}}$</th>
<th>$\epsilon_{\text{relaxed}}$</th>
<th>$\Delta\epsilon_{\text{relaxed}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{FreeWake}$</td>
<td>Trailing edge analysis, original</td>
<td>0.993</td>
<td>-0.7%</td>
<td>1.000</td>
<td>0.0%</td>
</tr>
<tr>
<td></td>
<td>Trailing edge analysis, modified</td>
<td>0.997</td>
<td>-0.3%</td>
<td>1.000</td>
<td>0.0%</td>
</tr>
<tr>
<td></td>
<td>Trefftz plane analysis</td>
<td>-</td>
<td>-</td>
<td>0.998</td>
<td>-0.2%</td>
</tr>
<tr>
<td>$\text{STAR-CCM+}$</td>
<td>Farfield analysis</td>
<td>-</td>
<td>-</td>
<td>0.998</td>
<td>-0.2%</td>
</tr>
</tbody>
</table>

6.3.2 Highly Non-Planar Validation Case

The simplified box wing configurations presented in Section 5.1 were investigated to confirm the computational validity of the modified projection approach and to demonstrate the capability to predict the induced drag with an improved accuracy compared to the original formulation. Relative induced drag estimates were obtained for the positive ($St = +3.0$) and negative-staggered system ($St = -3.0$) at a system angle of attack of $\alpha = 8.0^\circ$ using the freestream-fixed and relaxed-wake model. Computational results of this study are summarized in Table 6.2 for the positive-staggered system and in Table 6.3 for the negative-staggered system respectively.

The modified projection approach is found to considerably improve the consistency with the Euler-flow reference, in particular for the relaxed-wake model. Estimates obtained in Section 5.4.3 based on Trefftz plane analysis are confirmed. For the positive-staggered system, the relative error in computed span efficiency is reduced by about $\Delta\epsilon_{rel} \approx 8.4\%$ for the freestream-fixed wake and by about $\Delta\epsilon_{rel} \approx 9.1\%$ for the relaxed-wake model compared to the Euler-flow reference. In the case of the negative-staggered system, a similar magnitude of error reduction is evident.
### CHAPTER 6: MODIFIED TRAILING EDGE ANALYSIS

#### Computational-methodology Estimation technique $\varepsilon_{\text{fixed}}$ $\Delta \varepsilon_{\text{fixed}}$ $\varepsilon_{\text{relaxed}}$ $\Delta \varepsilon_{\text{relaxed}}$

<table>
<thead>
<tr>
<th>FreeWake</th>
<th>Trailing edge analysis, original</th>
<th>1.560</th>
<th>12.1%</th>
<th>1.533</th>
<th>10.1%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Trailing edge analysis, modified</td>
<td>1.340</td>
<td>-3.7%</td>
<td>1.377</td>
<td>-1.0%</td>
</tr>
<tr>
<td></td>
<td>Trefftz plane analysis</td>
<td>-</td>
<td>-</td>
<td>1.359</td>
<td>-2.4%</td>
</tr>
<tr>
<td>STAR-CCM+ (grid level 1)</td>
<td>Farfield analysis</td>
<td>-</td>
<td>-</td>
<td>1.391</td>
<td>-</td>
</tr>
</tbody>
</table>

**Table 6.2:** Comparison of the span efficiency factors $\varepsilon$ for the positive-staggered system ($St = +3.0$) based on original and modified projection approach for the freestream-fixed and relaxed-wake model at an angle of attack of $\alpha = 8.0^\circ$.

<table>
<thead>
<tr>
<th>FreeWake</th>
<th>Trailing edge analysis, original</th>
<th>1.421</th>
<th>-6.6%</th>
<th>1.419</th>
<th>-6.7%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Trailing edge analysis, modified</td>
<td>1.531</td>
<td>0.7%</td>
<td>1.523</td>
<td>0.2%</td>
</tr>
<tr>
<td></td>
<td>Trefftz plane analysis</td>
<td>-</td>
<td>-</td>
<td>1.521</td>
<td>0.0%</td>
</tr>
<tr>
<td>STAR-CCM+ (grid level 1)</td>
<td>Farfield analysis</td>
<td>-</td>
<td>-</td>
<td>1.521</td>
<td>-</td>
</tr>
</tbody>
</table>

**Table 6.3:** Comparison of the span efficiency factors $\varepsilon$ for the negative-staggered system ($St = -3.0$) based on original and modified projection approach for the freestream-fixed and relaxed-wake model at an angle of attack of $\alpha = 8.0^\circ$. 

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In conclusion, the implementation of the modified projection approach for the induced drag computation at the trailing edge is found to enable accurate induced drag prediction for highly non-planar concepts in the first place.
Induced Drag Prediction for Box Wings

The accurate prediction of induced drag and associated quantities for box wing configurations is concerned herein. Estimations are mainly based on the higher-order potential-flow method employing the modified collocation point projection described in the previous chapter and the Euler-flow reference. With regards to higher-order wake effects, the study explicitly distinguishes between contributions predominately induced by the system angle of attack and those related to variations of the height-to-span-ratio and the stagger factor. In addition, the computational expense of each methodology is determined.

7.1 Angle of Attack-Induced Wake Effects

For two box wing configurations of reduced geometric complexity and opposed longitudinal stagger, the present section explores the reasons and impact of higher-order wake and wake surrogate effects on accurate induced drag prediction in dependency on the system angle of attack. Taking into account a positive and a negative-staggered system furthermore permits investigations into the correlations with the longitudinal arrangement of the system. Trailing flowfield properties are analyzed to substantiate findings. Moreover, the accuracy of the higher-order potential-flow method is assessed against the Euler-flow reference. Predictions based on other potential-flow techniques are presented additionally to provide a comparison with more usual computational implementations. The principle methodological procedure that is adapted herein resembles the common analysis case, in-
SECTION 7.1: ANGLE OF ATTACK-INDUCED WAKE EFFECTS

tending to predict the aerodynamic performance of a configuration with given geometrical dimensions, but contrasts the parametric investigation of gap and stagger-related effects to be discussed in Section 7.2.

7.1.1 Test Cases

For the current investigation, the simplified configurations presented in Section 5.1 were revisited. The positive ($St = +3.0$) and equivalent negative-staggered system ($St = -3.0$) with a height-to-span ratio of $(h/b) = 0.20$ are illustrated in Figure 7.1. Their geometrical properties were discussed in Section 5.1. Selected key design parameters are considered to reflect values of practical and engineering interest. The computational implementation within the higher-order potential-method and the Euler-flow reference was retained as detailed in Section 5.2 and is therefore omitted at this point.

![Figure 7.1: Isometric view of box wing planforms with a geometric height-to-span ratio of $(h/b) = 0.20$.](image)

(a) Positive stagger factor $St = +3.0$

(b) Negative stagger factor $St = -3.0$

7.1.2 Spatial Convergence Study

A spatial convergence study was already performed in Section 5.3. Nevertheless, with regards to the higher-order potential-flow method, the implementation of the modified trailing edge analysis ($Eppler-PTWE$) demands the reconsideration of the spatial convergence behavior. This is obvious, especially for the relaxed-wake model, acknowledging that in contrast to the original approach, the modified collocation point projection employs the inclination of the most upstream post-trailing edge wake DVE. The induced drag predic-
tions are therefore potentially more sensitive to the wake shape, which may translate into an increased element count, particularly in the spanwise direction. Based on subsequent studies, this can be confirmed for the positive-staggered system.

The impact of the element density on the estimation of the span efficiency factor using the modified trailing edge analysis was studied varying the number of chordwise and spanwise elements individually for each system. Figure 7.2 illustrates the convergence dependency of the span efficiency factor on the chordwise paneling versus the system angle of attack for the positive and negative-staggered system respectively. The convergence criterion is fulfilled for span efficiency variations of less than $\Delta e = 0.003$ between two successive refinement steps.

Compared to the relaxed-wake model, the freestream-wake approach requires generally less elements to produce convergent predictions, which is essentially related to the limited geometrical complexity of the freestream-fixed wake. For the positive as well as for the negative-staggered system, a total number of $n_c = 12$ elements in the chordwise direction is found sufficient. Relaxed-wake convergence characteristics for the positive-staggered system are primarily driven by efficiency projections at larger angles of attack. These require an improved spatial resolution of $n_c = 16$ in the chordwise direction. In the case of the negative-staggered system, convergence properties at larger angles of attack are less important. Minor peaks in the distribution, emerging for angles of attack above $\alpha \approx 5.0$, are alleviated quickly with increasing element count. A total number of $n_c = 14$ chordwise elements provides converged span efficiency factors.

The convergence dependency of the span efficiency factor on the spanwise paneling is summarized in Figure 7.3. Similar to the chordwise variation, element density effects are more pronounced for the positive than for the negative-staggered case; the freestream-fixed wake model requires generally less elements than the relaxed-wake approach. For the positive-staggered system, the convergence characteristics at larger system angles of attack are once more critical. In the case of the freestream-fixed model, a spanwise element count of $n_s = 10$ is found to provide converged span efficiency factors, whereas $n_s = 16$ and $n_s = 14$ respectively are necessary for the relaxed-wake model.
### 7.1.3 Temporal Convergence Study

Among others, the computational efficiency of potential-flow methods arises from the fact that only the lifting surfaces and their trailing wakes are involved in the discretization process [Berens, 2008]. In contrast to the common practice, the higher-order potential-flow model utilizes a continuous vorticity sheet to represent the trailing wake, which is evolved iteratively by a time-stepping method. This is necessary to produce a force-free wake, but also applied in the case of the freestream-fixed wake. Thus, a complete numerical description of the trailing wake involves a discretization of its spanwise and longitudinal extent.
The spanwise element count in the wake is equivalent to those of the preceding lifting surface. Its impact on the span efficiency factor was therefore already covered in Section 7.1.2. The longitudinal discretization of the wake is described by the selected time-step width $\Delta t$, which represents the longitudinal dimension of a wake element, emitted from the trailing edge as the wing progresses continuously upstream. Whereas the selection of the time-step width is considered to have only limited impact on the span efficiency factor for the freestream-fixed wake model case, the shape of the relaxed-wake, and hence the induced drag of the system are affected. The wake must deform freely; a too wide time-step results in a wake that is not force-free and artificially stiff. Opposed to that, an
excessive reduction of the time-step width unnecessarily increases the computational effort and creates practical limits. According to Bramesfeld [2006], a time-step width equivalent to one quarter of the reference chord relative to the freestream velocity yields consistent results. However, the analysis of the present lifting systems may require different time-step widths.

To qualify the geometrical distortion of the relaxed-wake by time-step effects and to enable a valuation basis for its selection, two figures of merit or error indicators were introduced to determine the congruence of the relaxed-wake with the wake shape based on the Euler-flow reference. These are in particular the root-mean-square error of the relative Euclidean distance \(d_{\text{euc}}/b\) and the root-mean-square error of the roll-up angle deviation \(\Delta \nu\) between a relaxed-wake element and the Euler-flow wake shape at equivalent spanwise locations. Both, the Euclidean distance and the roll-up angle deviation, were computed over a set of outboard sections of the wake between \(\eta = 0.44\) and \(\eta = 0.50\), which are characterized by significant roll-up effects, bound between upstream and downstream trailing edge. The investigation concerns time-step widths between \(\Delta t = 0.10\) and \(\Delta t = 0.40\).

For a system angle of attack of \(\alpha = 8.0^\circ\), the impact of the selected time-step width on the span efficiency factor, the root-mean-square error of the relative Euclidean distance and the roll-up angle deviation are depicted in Figure 7.4 for positive and negative stagger. In general, time-step width effects are substantially more distinctive for the positive-staggered system, with an overall relative variation of about \(\Delta e_{\text{rel}} \approx 3.2\%\) in computed span efficiency compared to the Euler-flow reference. This is also evident from error indicators, showing an overall higher sensitivity to the selected time-step width in the case of positive stagger. For the negative-staggered system, the effect on the span efficiency factor is found to be less than \(\Delta e_{\text{rel}} \approx 0.1\%\). This reduced sensitivity is considered to reflect a limited impact by the wake shape, which is likely related to the smaller mutual induction caused by the larger effective vertical separation between upstream wake and downstream wing. A unique selection of an appropriate time-step width is however involved, as individual minima exist for both error indicators, which requires an adequate compromise. A time-step width of one quarter of the reference chord relative to the freestream velocity has been selected for
both stagger factors, which also facilitates a direct comparison to preliminary results of Chapter 5.

![Graphs showing RMSE and span efficiency factor vs time-step width](image)

**Figure 7.4:** Impact of the time-step width $\Delta t$ on the RMSE of the Euclidean distance ($d_{eucl}/b$), (a), the RMSE of the roll-up angle deviation $\Delta \nu$, (b), and the span efficiency factor $e$, (c), at a system angle of attack of $\alpha = 8.0^\circ$.

### 7.1.4 Computational Results and Discussion

#### Spanwise Load Distribution

The spanwise load distributions based on the higher-order potential-flow method and the Euler-flow reference, for the positive and negative-staggered system, are depicted in Figure 7.5 and expressed relative to the maximum value of the optimum distribution described by Demasi et al. [2015a,b].
For the present vertical separation, the loadings of both horizontal wings resemble a superposition of a constant and an elliptical part, whereas on the vertical wings, the distribution can be approximated by a constant and a cubic term [Demasi et al., 2015a,b]. Compared to this optimum distribution, the loading on horizontal wings is offset, in particular reduced on downstream lifting surfaces, due to the longitudinal stagger and results in an unequal lift division. Consequently, also the loading on vertical wings is affected. Although an uneven load distribution between the two horizontal surfaces does not necessarily preclude the achievement of minimum induced drag, [Demasi et al., 2014, 2015a,b], the shown loadings do not comply with the optimum distribution. This is inherently related to the selected geometrical characteristics of the planform, incorporating no twist and constant chord length. Especially in more outboard regions of the wing, this causes larger than optimum loadings.

The distributions based on the freestream-fixed and relaxed-wake model are found virtually identical, suggesting very limited impact by wake surrogate models on the spanwise loading. Reasonable similarity exists compared to the Euler-flow distribution, although notable deviations are encountered in between. These are attributed to thickness and Mach number effects, causing a larger loading for the Euler-flow model especially on horizontal wings.

**Computed Span Efficiency Factor**

The computed span efficiency factors versus the system angle of attack are presented in Figure 7.6. Compared to the previous analysis in Section 5.4, present potential-flow efficiencies demonstrate a significantly improved agreement with the Euler-flow reference, regardless of the longitudinal arrangement, the wake representation or the angle of attack. In dependency of the stagger factor, the general correlation of the span efficiency factor and the angle of attack is found in agreement with Euler-flow predictions and the assessment of the effective height-to-span ratios of the wake traces presented in Section 5.4.2. This improvement is attributed to the implementation of the modified trailing edge analysis.

In addition, Trefftz plane estimates are found in acceptable consistency and support the principle correctness of the present potential-flow predictions. Nevertheless, differences
CHAPTER 7: INDUCED DRAG PREDICTION FOR BOX WINGS

Figure 7.5: Spanwise load distribution on lower wing, (a-b), vertical wing, (c-d), and upper wing, (e-f), at a system angle of attack of $\alpha = 8.0^\circ$. 

<table>
<thead>
<tr>
<th>Graph</th>
<th>Description</th>
<th>$St$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>Lower wing (upstream),</td>
<td>+3.0</td>
</tr>
<tr>
<td>(b)</td>
<td>Lower wing (downstream),</td>
<td>−3.0</td>
</tr>
<tr>
<td>(c)</td>
<td>Vertical,</td>
<td>+3.0</td>
</tr>
<tr>
<td>(d)</td>
<td>Vertical,</td>
<td>−3.0</td>
</tr>
<tr>
<td>(e)</td>
<td>Upper wing (downstream),</td>
<td>+3.0</td>
</tr>
<tr>
<td>(f)</td>
<td>Upper wing (upstream),</td>
<td>−3.0</td>
</tr>
</tbody>
</table>

Legend:
- Freestream-fixed wake
- Relaxed-wake
- STAR-CCM
- Optimum spanload
SECTION 7.1: ANGLE OF ATTACK-INDUCED WAKE EFFECTS

(b) Negative stagger factor

\[ St = -3.0. \]  

\[ \alpha, \text{ deg} \]  

\[ \psi \]

\[ \alpha, \text{ deg} \]

(a) Positive stagger factor

\[ St = +3.0. \]  

\[ \alpha, \text{ deg} \]  

\[ \psi \]

\[ \alpha, \text{ deg} \]

**Figure 7.6:** Computed span efficiency factor \( e \) for a positive stagger factor of \( St = +3.0 \), (a), and a negative stagger factor of \( St = -3.0 \), (b), versus the system angle of attack \( \alpha \).

In span efficiency projections exist between the selected computational methodologies and wake surrogate models.

In the case of the positive-staggered system, the span efficiency factor decreases with the system angle of attack. This complies with the progressive reduction in effective height-to-span ratio and thus efficiency with increasing positive angle of attack. The relaxed-wake model under-predicts Euler-based span efficiencies, but provides overall best agreement. The relative error in span efficiency compared to the Euler-solution amounts approximately \( \Delta e_{rel} \approx -1.0\% \) on average. Although a similar correlation of the span efficiency and the angle of attack is obtained for the freestream-fixed wake model, considerable differences exists compared to relaxed-wake estimates and the Euler-flow reference. Especially at larger angles of attack, the freestream-fixed wake model under-estimates the span efficiency factor. This is caused by the deflection and roll-up of the force-free wake, which alters the effective height-to-span ratio of the system and therefore the span efficiency. The relative deviation in span efficiency between freestream-fixed and relaxed-wake model estimates at equivalent angles of attack quantifies the extent of higher-order wake effects. The positive-staggered system is particularly prone to higher-order wake effects, because of the vertical
contraction of wake trajectories, leading to smaller effective height-to-span ratios and thus
closer interactions between trailing wakes and lifting surfaces. For example for a system
angle of attack of $\alpha = 8.0^\circ$, the relative error in span efficiency constitutes approximately
$\Delta e_{rel} \approx 2.7\%$. The relaxed-wake model is therefore necessary, especially at larger angles
of attack, to enable accurate induced drag predictions for the positive-staggered system
based on potential-flow methodology.

For the negative-staggered system, potential-flow estimates increase with the system
angle of attack, in accordance with predictions based on the Euler-flow reference and the
evaluation of characteristic wake traces. Span efficiency projections using the relaxed-wake
model show good consistency with the Euler-flow reference. The deviation of freestream-
fixed wake and relaxed-wake estimates is small with respect to the case of positive stagger.
The relative error due to higher-order wake effects is less significant than for the positive-
staggered system and amounts, at an angle of attack of $\alpha = 8.0^\circ$, about $\Delta e_{rel} \approx 0.5\%$
in span efficiency. For the negative-staggered system the freestream-fixed wake model is
therefore acceptable.

The present results confirm that the selected wake model approach can affect the in-
duced drag prediction and its accuracy. Significant positive stagger, commonly preferred
within practical application scenarios in combination with high positive system angles
of attack, which typically represent takeoff and initial climb out flight conditions, are
major contributing factors that promote higher-order wake effects for box wing configura-
tions. With respect to aircraft performance, an accurate induced drag prediction during
these flight conditions is particularly important and facilitates considerable efficiency gains
[Kroo, 2000].

Box wing concepts are especially favorable for short or medium range applications
[Jansen and Perez, 2010], which expands the effective fraction of the takeoff and initial
climb out phase and hence the benefit of induced drag reductions. To enable accurate in-
duced drag and performance projections under these circumstances, an Euler-flow method
or a potential-based relaxed-wake model is mandatory.

Negative stagger facilitates a larger span efficiency factor than equivalent positive
stagger for any positive angle of attack. This is related to the larger effective height-to-span
ratio associated with the negative-staggered system [Munk, 1923b; Selberg, 1983; Selberg and Cronin, 1986; Kang et al., 2009a,b; Mamla and Galinski, 2009]. For commercial aircraft scenarios, the geometric height-to-span ratio is restricted in the case of negative stagger, as usually only the height of the fuselage can be utilized to separate wings vertically. Nevertheless, the effective vertical gap and the span efficiency is progressively increased for positive angles of attack.

Acknowledging that spanwise load distributions of present configurations are non-optimal, it is sensible to assess the plausibility of computed span efficiency factors with regards to the linear theoretical optimum efficiency based on the approximate relation provided by Equation 2.20. For equivalent freestream height-to-span ratios \((h/b)_\infty\), computed and theoretical optimum span efficiency factors are depicted in Figure 7.7. In the case of the negative-staggered system and independent of the system angle of attack or computational methodology, smaller than optimum span efficiency factors are evident, consistent with non-optimal loadings described in Figure 7.5. Limited to small angels of attack, this is also correct for the positive-staggered system, whereas predictions at larger angles of attack actually exceed theoretical optimum efficiencies. For the relaxed-wake model and the

\[ \begin{align*}
\text{(a)} & \text{ Positive stagger factor } \\
& St = +3.0. \\
\text{(b)} & \text{ Negative stagger factor } \\
& St = -3.0.
\end{align*} \]

Figure 7.7: Comparison of computed span efficiency factor \(e\) and optimum span efficiency factor \(e_{Bw, \text{opt}}\) based on biplane theory for a positive stagger factor of \(St = +3.0\), (a), and a negative stagger factor of \(St = -3.0\), (b), versus the system angle of attack \(\alpha\).
Euler-flow reference, this can be attributed in parts to higher-order wake effects, however, a different explanation is sought for the freestream-fixed wake.

![Graphs showing comparison of span efficiency factor and optimum span efficiency factor](image)

**Figure 7.8:** Comparison of computed span efficiency factor $e$ and optimum span efficiency factor $e_{Bw, \text{opt}}$ based on biplane theory without induced lift contributions for a positive stagger factor of $St = +3.0$, (a), and a negative stagger factor of $St = -3.0$, (b), versus the system angle of attack $\alpha$.

The approximate relation given by Equation 2.20 is based on the lifting-line concept, which neglects induced lift effects [Schmidt-Göller, 1992]. These are caused by streamwise velocity inductions due to the non-planar character of the lifting system. Induced lift effects progressively gain impact as the angle of attack or amount of stagger is increased, but do not affect induced drag [Schmidt-Göller, 1992]. Opposed to negative stagger, positive streamwise separations result in a positive induced lift contribution and thus in an overall lift increase. This is because the upstream wing generally carries more lift than the downstream wing, independent of the direction of stagger, which results in an overall greater influence by the upstream wing induction [Eppler, 1997]. The induction is further dependent on the vertical joint orientation. For the positive-staggered system, the joint of the upstream wing is oriented in the upward direction, which results in an outboard spanwise flow component, subsequently increasing the streamwise velocity and lift [Eppler, 1997; Verstraeten and Slingerland, 2009]. In contrast to that, the downward orientation of
the joint for the upstream wing of the negative-staggered system reduces the streamwise flow component and induces negative lift.

The higher-order potential-method permits the decomposition of freestream and induced lift contributions. Excluding induced lift, computed span efficiency factors are given in Figure 7.8. For the positive-staggered system, freestream-fixed span efficiency estimates result in smaller than optimum efficiencies, in consistency with present non-optimal loadings. Relaxed-wake efficiency factors that are larger than the linear optimum based on Equation 2.20 are found beyond angles of attack of approximately $\alpha \approx 5.0^\circ$. This results from higher-order wake effects. In the case of the negative-staggered system, neglecting induced lift effects increases the span efficiency, but still does lead to smaller than optimum estimates. The relative deviation in span efficiency between the freestream-fixed and the relaxed-wake model is not affected by induced lift.

**Wake Traces on Partition Surface**

To gain insight into the physical reasons related to higher-order wake effects, an investigation of wake traces was conducted. From Figure 7.9 it is evident that wake deflection is the predominant contribution causing higher-order wake effects in the case of the positive-staggered system. This is because the force-free wake shapes of the relaxed-wake model and the Euler-flow reference are aligned with the local flowfield, which leads to an on average larger effective height-to-span ratio and hence efficiency than compared with a freestream-fixed wake. In addition, the vertical wing effectively mitigates roll-up effects at the tip and also induces a spanwise velocity component. This increases the effective span by shifting the tip-vortex slightly outboards [Maughmer, 2003], reducing induced drag.

For the negative-staggered system, the influence of wake deflection results in smaller effective height-to-span ratios than for the freestream-fixed wake, potentially leading to smaller span efficiencies as well. In contrast to the positive-staggered case, the contribution of the wake roll-up of the tip-vortex is more pronounced and results in two opposing effects. The roll-up increases the effective height-to-span ratio especially in the tip region where the effect of vertical separation is most effective [Lowson, 1990]. But also, related to the downward orientation of the vertical wing, this actually produces a spanwise contraction.
of the wake [Maughmer, 2003], which reduces the effective span and increases induced
drag. Despite substantial differences in average effective height-to-span ratio between the
freestream-fixed wake, the relaxed-wake and the Euler-flow reference, the opposing effects
of wake roll-up and deflection ultimately result in similar span efficiency estimates.

7.1.5 Predictions by other Potential-Flow Methodologies

Predictions by other potential-flow methodologies, in particular the general vortex-lattice
method AVL [Drela and Youngren, 2013], the multiple-lifting line formulation LiftingLine
[Horstmann, 1987] and the higher-order panel method PanAir [Magnus and Epton, 1990]
are presented. These results are partly contained in Schirra et al. [2014c] and are intended
to provide a comparison with more common potential-flow methods. Induced drag esti-
mates were obtained by means of Trefftz plane analyses. The lift was determined directly
at the wing, either applying the Kutta-Joukowsky law, or a surfaces pressure integration
in the case of the higher-order panel method. For both stagger factors, converged span
efficiency factors are depicted in Figure 7.10.

Compared to the Euler-flow reference, substantial deviations in computed span effi-
ciency factors exist for the vortex-lattice method and the multiple-lifting line technique.
Both incorrectly predict an increase in span efficiency for the positive-staggered system
and a decrease for the negative-staggered system respectively with increasing system angle
of attack. Deviations are especially pronounced at larger angles of attack and essentially
attributed to wake surrogate effects. With regards to the vortex-lattice method, Drela and
Youngren [2013]1 note that the freestream direction must be at reasonably small angles
to the body-fixed x-axis, because of the parallel orientation of the trailing vorticity. A
similar methodological procedure is employed within the multiple-lifting line formulation
by Horstmann [1987].

These trailing wake trajectories actually represent a body-fixed wake model, that
is aligned with the mean surface of the lifting system. As discussed earlier, this wake
placement results in a wake that is not drag-free and supports longitudinal forces, which
prevents an accurate induced drag predictions based on farfield velocities (Trefftz plane).

1Compare manual provided with reference
SECTION 7.1: ANGLE OF ATTACK-INDUCED WAKE EFFECTS

Figure 7.9: Wake traces at the partition surface for a system angle of attack of $\alpha = 8.0^\circ$. 

(a) FreeWake: Freestream-fixed wake, $St = +3.0$. 

(b) FreeWake: Freestream-fixed wake, $St = -3.0$. 

(c) FreeWake: Relaxed-wake, $St = +3.0$. 

(d) FreeWake: Relaxed-wake, $St = -3.0$. 

(e) STAR-CCM+, $St = +3.0$. 

(f) STAR-CCM+, $St = -3.0$. 

\[ \epsilon_{Bw, \text{opt}} \approx 1.326 \]

\[ \frac{(h/b)_{\text{eff}}}{\eta} = \text{const} \]

\[ \epsilon_{Bw, \text{opt}} \approx 1.598 \]

\[ \frac{(h/b)_{\text{eff}}}{\eta} = \text{const} \]

\[ \epsilon_{Bw, \text{opt}} \approx 1.380 \]

\[ \frac{(h/b)_{\text{eff}}}{\eta} = f(\eta) \]

\[ \epsilon_{Bw, \text{opt}} \approx 1.532 \]

\[ \frac{(h/b)_{\text{eff}}}{\eta} = f(\eta) \]

\[ \epsilon_{Bw, \text{opt}} \approx 1.396 \]

\[ \frac{(h/b)_{\text{eff}}}{\eta} = f(\eta) \]

\[ \epsilon_{Bw, \text{opt}} \approx 1.538 \]

\[ \frac{(h/b)_{\text{eff}}}{\eta} = f(\eta) \]
CHAPTER 7: INDUCED DRAG PREDICTION FOR BOX WINGS

(a) Positive stagger factor  
\( St = +3.0 \)

(b) Negative stagger factor  
\( St = -3.0 \)

Figure 7.10: Computed span efficiency factor \( c \) for a positive stagger factor of \( St = +3.0 \), (a), and a negative stagger factor of \( St = -3.0 \), (b), versus the system angle of attack \( \alpha \).

This is especially true when the wake is non-planar [Kroo and Smith, 1990]. Alternatively computing the induced drag directly at the lifting elements introduces an own set of issues, which should be avoided in particular by means of a farfield analysis.

Moreover, with this wake surrogate model it is not possible to account for variations of the effective height-to-span ratio induced by the stagger or the system angle of attack. Its application with regards to present box wing configurations or similar highly non-planar lifting systems is therefore considered generally problematic, which limits the applicability of both computational methods. This may have affected induced drag predictions based on the general vortex-lattice method AVL by previous research efforts, i.e. by Salam and Bil [2015] and Demasi et al. [2015b], which is especially true, if the correlation of span efficiency and system angle of attack is explicitly considered, as in Mamla and Galinski [2009] and Kang et al. [2009a,b].

It is remarked, that based on the body-fixed wake model, an improved prediction of the span efficiency may be obtained by re-scaling the vertical dimension of the system to match the effective height-to-span ratio. Actually, this produces a wake trace equivalent to
the freestream-fixed case, but still results in a miss-alignment of the trailing wake compared to the freestream direction.

Because of its freestream-fixed wake, predictions using the higher-order panel method show an improved agreement with regards to the general correlation of the span efficiency factor and the system angle of attack. Nevertheless for the positive-staggered system, estimates substantially larger than given by optimum conditions exist for equivalent freestream height-to-span ratios \((h/b)_{\infty}\). Assuming that similar to the higher-order potential-flow method, non-optimal spanwise loadings are evident, this is considered to be related to induced lift effects and the lift contribution due to airfoil thickness. Sections with a thickness-to-chord ratio of \((t/c) = 0.04\), equivalent to the Euler-flow model were utilized. In the case of the negative-staggered system, deviations compared to other methodologies are of acceptable magnitude; predictions remain below the theoretical optimum, but exceed Euler-flow projections.

### 7.2 Gap and Stagger-Induced Wake Effects

A parametric study was conducted to explore the impact by gap and stagger-induced wake effects on the lift and induced drag characteristics for box wing configurations of limited geometrical complexity. To facilitate an investigation into the principal impact of gap and stagger variations and to avoid overlayed effects induced by the system angle of attack, lifting systems with similar aerodynamic rather than geometric design parameters were considered.

The angle of attack and stagger modify the freestream and effective vertical gap for equivalent geometric height-to-span ratios. For positive angles of attack, negative stagger results in performance gains over equivalent positive stagger, as shown in Section 7.1. For this parametric study, the variation of the horizontal separation of lifting surfaces is therefore aligned with the freestream direction, equivalent to the stagger theorem [Munk, 1923a]. This is referred to as the freestream stagger \(St_{\infty}\), whereas the freestream height-to-span ratio \((h/b)_{\infty}\) gives the vertical separation of the trailing edges perpendicular to the freestream direction.
CHAPTER 7: INDUCED DRAG PREDICTION FOR BOX WINGS

7.2.1 Test Cases and Design Parameter

The parametric investigation of the freestream stagger factor $St_\infty$ and the freestream height-to-span ratio $(h/b)_\infty$ considered a set of simplified box wing configurations at an angle of attack of $\alpha = 4.0^\circ$ under subsonic flow conditions. Their common geometric properties are based on the lifting systems presented in Section 7.1. Both horizontal wings with single wing aspect ratios of $\Lambda_{1,2} = 6.0$ are unswept and do not incorporate taper, camber or twist. Dependent on the freestream stagger factor and height-to-span ratio, the sweep angle of vertical wings was adjusted to form a closed lifting system.

<table>
<thead>
<tr>
<th>Design parameter</th>
<th>Lower bound</th>
<th>Upper bound</th>
<th>Step size</th>
</tr>
</thead>
<tbody>
<tr>
<td>$St_\infty$</td>
<td>-3.0</td>
<td>+3.0</td>
<td>1.0</td>
</tr>
<tr>
<td>$(h/b)_\infty$</td>
<td>0.10</td>
<td>0.40</td>
<td>0.05 (0.10)$^2$</td>
</tr>
</tbody>
</table>

Table 7.1: Investigated design parameter range.

The investigated parameter range is given in Table 7.1 and considered streamwise stagger factors between $St_\infty = -3.0$ and $St_\infty = +3.0$ in increments of $\Delta St_\infty = 1.0$. The freestream height-to-span ratio was varied between $(h/b)_\infty = 0.10$ and $(h/b)_\infty = 0.40$ in increments of $\Delta (h/b)_\infty = 0.05$. Following the discussion provided in Section 5.1, the current parameter range is considered to cover values of practical and engineering interest. The impact of gap and stagger variations on the geometric properties of the system are depicted in Figure 7.11, not showing all intermediate steps for the sake of clarity.

A spatial or temporal convergence study was not performed for the exploration of gap and stagger-related effects, neither for the higher-order potential-flow method nor for the Euler-flow reference. The discretization found in Section 7.1.2 and Section 5.3.1 is suggested to apply for the present investigation. This assumption is supported by acknowledging the comparable low system angle of attack, resulting in a less complex flow pattern and a reduced element demand. The present discretization approach was thus considered conservative.

$^2$In the case of the Euler-flow reference.
Figure 7.11: Vertical and longitudinal arrangement of lifting surfaces within the parametric investigation.


7.2.2 Computational Results and Discussion

Computed Span Efficiency Factor

For constant freestream height-to-span ratios, the impact of streamwise stagger on the span efficiency factor is compiled in Figure 7.12. Independent of the computational methodology, the span efficiency of the system is affected by the stagger factor, which originates from the correlation of the streamwise separation and the effective height-to-span ratio between the upstream wake and the downstream trailing edge. This is also true for freestream-fixed estimates, but does not present a formal contradiction to the classical description of the stagger theorem because the circulation distribution was not constrained. Dependent on stagger, induced lift contributions affect the span efficiency in addition.

In contrast to the investigation based on identical geometric height-to-span ratios, systems with positive stagger are found to gain larger span efficiencies than equivalent negative systems. The contribution by mutual wing-wing and wing-wake induced effects is diminished with increasing freestream height-to-span ratios. Although this limits the sensitivity of the span efficiency to stagger, the vertical gap affects the streamwise location that yields maximum span efficiency. With increasing freestream height-to-span ratios, the stagger providing highest span efficiency shifts from the largest positive separation involved to smaller streamwise stagger factors. This is confirmed independently by the higher-order potential-flow model and the Euler-flow reference.

In general, both wake models predict a similar correlation of the span efficiency and streamwise stagger factor, which is in general accordance with the Euler-flow reference. However, the freestream-fixed model under-predicts the span efficiency compared to the relaxed-wake for any positive-staggered system, but is in acceptable agreement for an equivalent negative. The deviations progressively increase with the amount of stagger, in particular for positive streamwise separations. For a freestream height-to-span ratio of \((h/b)_\infty = 0.20\), the deviations accumulate to about \(\Delta e_{rel} \approx 0.9\%\) in span efficiency for a positive stagger factor of \(St = +3.0\), compared to \(\Delta e_{rel} \approx 0.3\%\) for an equivalent negative stagger of \(St = -3.0\). These discrepancies in span efficiency originate from wake deflection and roll-up effects, but are less significant than in Section 7.1.4. This is due to
the smaller angle of attack of $\alpha = 4.0^\circ$. Nevertheless, in-line with Section 7.1.4, higher-order wake effects are found to be particularly enabled by positive stagger. Smaller vertical separations cause closer interactions between the upstream wake and the downstream wing and intensifies these effects.

Similar to findings of Section 7.1.4, an estimation based the relaxed-wake model improves the consistency with the Euler-flow reference, independent of stagger or height-to-span ratio. For unstaggered systems, computed span efficiency factors are essentially independent of the applied wake model, computationally verifying implications given by the wake substitution concept [Smith, 1995]. An entire substitution of the force-wake is feasible, implying negligible impact by wake roll-up or deflection. Under these conditions, the freestream-fixed wake model is certainly sufficient to permit induced drag prediction in good agreement with the Euler-flow reference.

### Wake Traces on Partition Surface

An analysis of the wake traces at the partition surface was performed to substantiate the findings of the previous Section 7.2.2 and to explore their physical reasons. For a freestream height-to-span ratio of $(h/b)_\infty = 0.20$ and a freestream stagger factor of $St_\infty = +3.0$ and $St_\infty = -3.0$ respectively, wake traces are given in Figure 7.13 for the freestream-fixed and the relaxed-wake model, as well as for the Euler-flow reference. The partition surface coincides with the most downstream trailing edge as depicted in Figure 5.11. However, in contrast to Section 7.1.4, the surface is positioned perpendicularly to the freestream direction, in compliance with Smith [1995]. This alignment is correct within theory and expedient in the present case, to assess the span efficiency factor by means of the effective vertical gap, while maintaining equivalent freestream height-to-span ratio and shifting lifting surfaces in the streamwise direction.

In general, present findings confirm those of Section 7.1.4, qualitatively indicating excellent geometric consistency of wake shapes derived from relaxed-wake model and Euler-flow reference. The freestream-fixed trace is inherently different and, resulting from the alignment of the partition surface, congruent with the system trailing edge. Despite this is fairly obvious even without an assessment of wake traces, it provides meaningful insight
Figure 7.12: Span efficiency factor $e$ versus the stagger factor $St_\infty$ for various freestream height-to-span ratios $(h/b)_\infty$.
SECTION 7.2: GAP AND STAGGER-INDUCED WAKE EFFECTS

into the correlation of span efficiency and stagger factor based on the freestream-fixed wake model. The wake trace and therefore the effective height-to-span ratio are independent of the stagger factor. Therefore, the streamwise variation in span efficiency cannot be attributed to wake alignment effect, but is related to the vorticity distribution in the wake or its integral quantity the circulation, that is ultimately linked to the (induced) lift. A constraint circulation or vorticity hence leads, in accordance with classical stagger theorem [Munk, 1923a], to equivalent induced drag or span efficiency, provided measures taken to maintain circulation, i.e. wing twist, do not alter the shape of the trailing edge or the subsequent wake. This emphasizes of the limited applicability of the classical stagger theorem beyond the lifting-line theory. In particular for the relaxed-wake and the Euler-flow reference the wake shape is dependent on the streamwise location.

The principal mechanisms that cause higher-order wake impact accord to those identified in Section 7.1.4. For positive stagger, the deflection of the force-free wake increases the effective height-to-span ratio compared to the freestream-fixed wake. Additionally, the flowfield induced by the vertical wing shifts the rolled-up tip-vortex outboards, which increases the effective span and thus reduces the induced drag. In the case of negative stagger, the effective height-to-span ratio is diminished compared to a freestream-fixed approach, due to the wake deflection. The roll-up of the tip-vortex introduces two opposed effects that increase the effective height-to-span ratio at the tip but also decrease the effective span. This finally leads to a span efficiency very similar compared to the freestream-fixed wake model.

Streamwise Vorticity Distribution

For the maximum and minimum freestream stagger factor involved, the streamwise vorticity distributions, one reference chord downstream of the system trailing edge are given in Figure 7.14, for a freestream height-to-span ratio of \((h/b)_{\infty} = 0.20\). Reasonable agreement exists between contours based on the relaxed-wake model and the Euler-flow reference. The spanwise vortex core locations are in good consistency and reflect the anticipated spanwise flow pattern induced by the orientation of the vertical wing, shifting the vortex outwards on the lower wing and inwards on the top wing [Maughmer, 2003]. Contours
(a) *Free Wake*: Freestream-fixed wake, $St_\infty = +3.0$.

(b) *Free Wake*: Freestream-fixed wake, $St_\infty = -3.0$.

(c) *Free Wake*: Relaxed-wake, $St_\infty = +3.0$

(d) *Free Wake*: Relaxed-wake, $St_\infty = -3.0$.

(e) *STAR-CCM*+, $St_\infty = +3.0$.

(f) *STAR-CCM*+, $St = -3.0$.

**Figure 7.13:** Wake traces at the partition surface for a constant freestream height-to-span ratio of $(h/b)_\infty = 0.20$. 

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are displaced downwards according to the wake trajectories. Due to the comparable low system angle of attack of $\alpha = 4.0^\circ$, the wake deflection and roll-up is less pronounced, limiting the discrepancies encountered for the freestream-fixed wake model.

**Induced Drag and Lift Coefficient**

The span efficiency factor enables a fair comparison of the aerodynamic efficiency among different lifting systems and computational methodologies, but prevents detailed investigations of the individual contribution of induced drag and lift. In contrast to the procedure involving effects induced by the system angle of attack in Section 7.1, the impact of the streamwise stagger factor on the induced drag and lift coefficient of the particular system was analyzed explicitly. This is to gain additional insight into the reasons causing higher-order effects and deviations between the two methodologies. In Figure 7.15a, the correlation of the induced drag coefficient and the stagger factor, for example, is presented for a system with a freestream height-to-span ratio of $(h/b)_\infty = 0.20$. An overall increased induced drag level is obtained based on the Euler-solution. This is caused by the influence of the finite thickness distribution of the lifting element ($(t/c) = 0.04$) and larger freestream Mach number, translating into larger lift (compare Figure 7.15b) and induced drag coefficient as well.

Potential-flow estimates indicate that the unstaggered system produces minimum induced drag. By contrast, minimum induced drag conditions predicted by the Euler-flow reference are evident for a positive stagger of $\text{St}_\infty = +1.0$, although a virtually equivalent value is attained for the unstaggered system. From its minimum, the induced drag generally increases with larger amounts of stagger, either in the positive or negative direction, varying substantially within small stagger increments close to the minimum location.

With regards to the minimum induced drag estimate, the maximum relative variation amounts approximately $\Delta C_{\text{Di}, \text{rel}} \approx 16\%$ ($\text{St}_\infty = -1.0$) based on the Euler-flow reference, $\Delta C_{\text{Di}, \text{rel}} \approx 8\%$ ($\text{St}_\infty = -2.0$) utilizing the relaxed-wake model and $\Delta C_{\text{Di}, \text{rel}} \approx 9\%$ ($\text{St}_\infty = 2.0$) employing the freestream-fixed wake. Compared to the unstaggered system, the higher-order potential-flow method describes a near symmetric correlation between the induced drag coefficient and the freestream stagger factor. In fact it is not symmetric, but
(a) *FreeWake*: Freestream-fixed wake, $St_{\infty} = +3.0$.

(b) *FreeWake*: Freestream-fixed wake, $St_{\infty} = -3.0$.

(c) *FreeWake*: Relaxed-wake, $St_{\infty} = +3.0$

(d) *FreeWake*: Relaxed-wake, $St_{\infty} = -3.0$.

(e) *STAR-CCM+*, $St_{\infty} = +3.0$.

(f) *STAR-CCM+*, $St_{\infty} = -3.0$.

**Figure 7.14:** Contours of the streamwise vorticity $\gamma_x$ at a distance of one reference chord length $c_{ref}$ downstream of the system trailing edge for a constant freestream height-to-span ratio of $(h/b)_{\infty} = 0.20$. 

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Figure 7.15: Induced lift effects versus the freestream stagger factor $St_\infty$ for a constant freestream height-to-span ratio of $(h/b)_\infty = 0.20$.

differences between equivalent streamwise positive and negative separations are much less pronounced than for the Euler-flow reference. Based on the relaxed-wake shape, the induced drag of a positive-staggered system is smaller than of an equivalent negative, which is in principle accordance with Euler-flow results.

In terms to the estimated dependency of the span efficiency factor and streamwise
stagger factor given in Figure 7.15d, present findings warrant further investigation. This is especially true in the case of the freestream-fixed wake model. The predicted induced drag savings associated with negative stagger potentially conflict with span efficiency gains for the positive-staggered system. Also induced drag estimates based on the relaxed-wake model are of concern, as their projected induced drag savings for positive-staggered systems are less substantial compared to the Euler-flow reference. This partly contradicts the excellent agreement in computed span efficiency. Recalling the definition of the span efficiency factor given in Section 2.2.3, it is convenient to explore the effect of streamwise stagger on the lift coefficient for clarification. This is also of benefit to assess the impact of the thickness distribution and Mach number on the lift coefficient with respect of the streamwise stagger more thoroughly, which causes deviations between the Euler-flow reference and the higher-order potential-method.

The lift coefficient versus the streamwise stagger factor is depicted in Figure 7.15b for a system with a freestream height-to-span ratio of \( (h/b)_\infty = 0.20 \). In principle, the correlation of the lift coefficient and the stagger factor exhibits some similarities to the characteristic trend for the induced drag coefficient in Figure 7.15a. For the range of positive stagger factors, good qualitative agreement is obtained with potential-flow results by Kang et al. [2009a,b]. The minimum lift coefficient is found for the unstaggered system, or in the case of the Euler-flow reference in close proximity to it. An excellent consistency exists between lift coefficients based on the freestream-fixed and relaxed-wake model. The impact by higher-order wake effects on the estimation of the lift coefficient therefore proves to be extremely limited and can be neglected. This is also true for induced lift effects presented in Figure 7.15c. Because of this, it is sufficient to conclude, that deviations in computed span efficiency or induced drag between freestream-fixed and relaxed-wake model estimates are not related to lift or induced lift contributions, but are actually caused by higher-order wake effects on induced drag.

Despite similarities compared to the Euler-flow reference, the correlation of the lift coefficient and streamwise separation, as predicted by the higher-order potential-flow model, is of concern. In contrast to Euler-flow lift projections, potential-flow results indicate an improved lift performance associated with positive stagger. However, lift gains for
the positive-staggered system are in-line with larger induced drag levels, based on the freestream-fixed wake shape. In case of the relaxed-wake model, the estimated correlation of lift coefficient and induced drag aids to improve the agreement in computed span efficiency.

Although the contribution of induced lift effects is extremely small and does not affect induced drag [Eppler, 1997], it is of importance, increasing or decreasing lift progressively for a positive or negative-staggered system. Neglecting induced lift effects in compliance with lifting-line theory [Schmidt-Göller, 1992] results in lift coefficients, virtually independent of the direction of stagger, which is in good accordance with Equation 2.24 and the biplane theorem. As depicted in Figure 7.15d, this surely affects the correlation of span efficiency and stagger factor. In the case of the freestream-fixed wake model, this translates into a reversed dependency of the span efficiency and stagger factor, indicating efficiencies gains for negative-staggered systems. For the relaxed-wake, a trend in agreement with the Euler-flow reference is preserved.

The lift division between both horizontal wings is depicted in Figure 7.16 versus the streamwise stagger factor. Independent of its vertical arrangement, a considerable larger amount of lift is produced by the upstream wing. This characteristic agrees with estimates computed from measurements of the individual downwash angle in Kang et al. [2009a,b] and is consistently predicted by the higher-order potential-method and Euler-flow reference. For non-zero stagger factors, the Euler-flow reference predicts a ratio of lift division between upstream and downstream wing of \( \left( \frac{C_{L,us}}{C_{L,ds}} \right)_{St+} = 1.8 \) for positive stagger and \( \left( \frac{C_{L,us}}{C_{L,ds}} \right)_{St-} = 1.4 \) in the case of negative stagger respectively. This is in contrast to the potential-method, where a near constant ratio of lift division of about \( \left( \frac{C_{L,us}}{C_{L,ds}} \right) = 1.5 \) is evident, independent of the direction of stagger. Generally, the unequal lift division can be attributed to the flowfield induced by the upstream wing, which causes a reduction of the effective angle of attack and lift on the subsequent wing. Even though the inflow conditions of the upstream lifting surface are undisturbed, deviations between both methodologies exist. These are however less substantial than on the downstream wing. In comparison to the potential-flow method, the Euler-flow reference predicts larger lift coefficients for negative stagger factors, whereas in contrast smaller lift
coefficients result for any positive arrangement involved. This is certainly related to the more complex flowfield induced on the downstream wing, but does not provide sufficient reasoning to explain deviation among both methodologies.

Although the relative separation among lifting surfaces is large with regards to classical airfoil cascades, an explanation is approached by considering their primary effects. According to Smith [1975], the effects causing cascades can be distinguished as follows:

- Slat effects
- Circulation effects
- Dumping effects
- Off-the-surface pressure recovery
- Fresh boundary layer effect

For inviscid computations, only inviscid slat and circulation effects become relevant, whereas remaining contributions are related to viscous, boundary layer effects and are neglected by the higher-order potential-method and the Euler-flow reference. Slat effects induce an upstream distortion of the flow. The velocities due to the circulation of the upstream lifting element run counter to those in close proximity of the downstream leading edge and

**Figure 7.16:** Lift division between upstream, (a), and downstream, (b), wing versus the stagger factor $St_\infty$ for a constant freestream height-to-span ratio of $(h/b)_\infty = 0.20$. 
so effectively alleviate pressure peaks [Smith, 1975], resulting in smaller lift. Circulation effects refer to a downstream flow distortion and describe the lift increase of an upstream element, due to the circulation of a preceding surface. This effectively causes the flow to approach the upstream trailing edge at a larger effective angle of attack, which leads to larger circulation on the upstream wing and lift as well.

Both effects can be represented by potential-flow and Euler-methods, but require taking into account the relative thickness of the lifting element and its pressure distribution or contour velocities to simulate slat effects [Berens, 2008]. In addition, the thickness contribution should generally reinforce the circulation effect, as noted by Smith [1975]. In this context, the higher-order potential-flow method is therefore generally not capable of computing the impact of slat effects on the lift and therewith on the induced drag and may further suffer from less accurate prediction of the circulation effect. Although it is not possible to separate individual contributions, in particular by a comparison among involved methodologies, it appears plausible that the impact of the thickness contribution alters the relative sensitivity between upstream and downstream lift share in dependency on the longitudinal arrangement.

Acknowledging the present lift division, induced lift effects can be assessed more thoroughly. Generally, these are caused by the vertical wing surface inducing streamwise velocities and thus vertical lift forces. Because of its larger total lift, these are more dominant on the upstream wing, resulting in an overall greater influence of the induction [Eppler, 1997]. Induced lift gains for the positive-staggered system are therefore considered to be associated with the upward orientation of the vertical wing. This induces an outboard directed, spanwise flow component (compare Figure 7.13), subsequently increasing the streamwise velocity and lift [Eppler, 1997; Verstraeten and Slingerland, 2009]. In contrast to that, the downward orientation of the vertical wing for a negative-staggered system reduces the streamwise flow component and creates negative induced lift.

The impact by the given range of freestream height-to-span ratios on the lift coefficient is compiled in Figure 7.17 against the streamwise stagger factor. Independent of the methodology or stagger factor involved, an increasing vertical gap is confirmed to translate into larger lift coefficients as well. It is consistently shown that despite the considerable
differences in the correlation of streamwise arrangement and lift coefficient between the potential-flow method and the Euler-based simulation discussed previously, the lift becomes increasingly independent of the stagger factor at larger vertical gaps. In particular for the Euler-flow reference, the predicted lift loss associated with the unstaggered system is continuously diminished with increasing freestream height-to-span ratios. This is likely related to reduced aerodynamic interactions between lifting surfaces. For the range of positive stagger factors, predictions are found in good qualitative agreement with Kang et al. [2009a,b].

**Figure 7.17:** Lift coefficient $C_L$ versus the stagger factor $St_\infty$ for various freestream height-to-span ratio $(h/b)_\infty$.
The main application area of the higher-order potential-flow method employed herein is the accurate induced drag prediction during early design stages. It is meant to fill the gap between the inexpensive linear potential-methodology with a freestream-fixed wake and the computationally intensive Euler-flow model. As indicated in previous sections, the relaxed-wake model provides span efficiency estimates with an accuracy that is consistent with Euler-based predictions. Nevertheless, the computational expense of the wake relaxation process needs to be considered in dependency of the present flow-problem.

The computational expense of each methodology was therefore determined at an angle of attack of $\alpha = 8.0^\circ$. The evaluation did only consider the elapsed computation time required by the flow-solver instead of the accumulated CPU time per process. Also any pre-processing effort, such as required for the grid generation, was neglected. Although this provides an advantage for the Euler-based simulation, it enables a more realistic comparison and is conservative with regards to the potential-flow model.

The study concerned the computational expense for the positive-staggered system as presented in Section 5.1. For the higher-order potential flow-method, the system involves a slightly finer spatial discretization (compare Section 7.1.2), which translates into an increased computational overhead compared to the negative-staggered system. According to the temporal investigation in Section 7.1.3, a time-step width of one quarter of the reference chord length relative to the freestream velocity was utilized.

As apparent from Figure 7.18, span efficiency factors converge in about 20 time-steps for both wake models. The CPU time required to attain these results is provided in Figure 7.18b. Due to the time-stepping approach, the number of elements in the wake increase progressively, as does the subsequent computational effort. The CPU time required for the relaxed-wake model is increased compared to the freestream-fixed wake model, as wake elements must be aligned with the local velocity vector at every time-step. In addition, the relaxed-wake model solution requires a larger element count to produce converged results. From Table 7.2 it becomes evident that although the computational time for the relaxed-wake is more than four times larger than for the freestream-fixed wake model, it
is nevertheless by about two orders of magnitudes faster than the Euler-solution, although exploiting parallel computing techniques.

![Graph](image)

**Figure 7.18:** Convergence behavior for a positive stagger factor of \( St = +3.0 \) and a system angle of attack of \( \alpha = 8.0^\circ \).

### Table 7.2: CPU time comparison for a positive stagger factor of \( St = +3.0 \) and a system angle of attack of \( \alpha = 8.0^\circ \).

<table>
<thead>
<tr>
<th>Estimation methodology</th>
<th>( e )</th>
<th>( \Delta e )</th>
<th>computation time, min</th>
<th>cores</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>Free Wake</em>: Freestream-fixed wake</td>
<td>1.340</td>
<td>-4%</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td><em>Free Wake</em>: Relaxed-wake</td>
<td>1.377</td>
<td>-1%</td>
<td>17</td>
<td>1</td>
</tr>
<tr>
<td><em>Star-CCM</em>+: (Grid level 1)</td>
<td>1.391</td>
<td>-</td>
<td>( \sim 1800 )</td>
<td>11</td>
</tr>
</tbody>
</table>

The Euler-flow induced drag estimates, with an accuracy equivalent to that of the relaxed-wake model case, were obtained on the finest grid (level 1). An estimation based on a coarser grid (level 2) may be acceptable for smaller system angles of attack. Despite the simulation being executed in parallel, the overall computation time based on the convergence criterion introduced in Section 5.3.1 is dramatically increased. Compared to the relaxed-wake model case, a more than two orders of magnitudes greater computation time
is required, which likely prevents a direct application in conceptual design.

Shifting the Trefftz plane further upstream closer to the trailing edge reduces the amount of iterations and speeds-up the solution. Provided longitudinal velocity components are considered, a virtually constant span efficiency estimate is obtained, as demonstrated in Section 5.3.1. Compared to a location of 40 reference chords downstream of the lifting element, a placement at about two reference chords reduces the computational overhead by approximately 20%.

A placement even closer to the lifting elements is generally possible, but should exclude a region of approximately one reference chord downstream of the lifting elements to avoid numerical issues during the integration process. Nevertheless, even when employing these numerical efficiencies, the Euler calculations take considerably more computational effort than the potential-flow method, with only marginal differences in prediction outcomes.

7.4 Application to a Commercial Box Wing Aircraft Concept

The impact of higher-order wake and wake surrogate effects is demonstrated by applying the higher-order potential-flow method to a box wing configuration providing more realistic planform properties for a commercial aircraft concept. This extended test case scenario particularly intends to substantiate the general validity of findings and conclusions based on the simplified box wing configurations with regards to wake-related effects.

7.4.1 Test Case

The positive-staggered system, with a single wing aspect ratio of $\Lambda_{1,2} = 9.6$ and a taper ratio of $\lambda_{1,2} = 0.25$ respectively, is illustrated in Figure 7.19. Its general planform properties were derived from Gagnon and Zingg [2015] and are summarized in Table 7.3. The system does not incorporate wing twist and its representation is based on a mean surface approximation. Because of transonic effects, commonly occurring under cruise flight conditions, both horizontal lifting surfaces are characterized by opposite leading edge sweep angles.
CHAPTER 7: INDUCED DRAG PREDICTION FOR BOX WINGS

<table>
<thead>
<tr>
<th>$\lambda_{1,2}$</th>
<th>$\lambda_{1,2}$</th>
<th>$c_{ref}$,</th>
<th>$b$,</th>
<th>$S_{ref}$,</th>
<th>$\phi_{le, 1}$,</th>
<th>$\phi_{le, 2}$,</th>
<th>$(h/b)_{avg}$</th>
<th>$St_{avg}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.6</td>
<td>0.25</td>
<td>0.625</td>
<td>6.0</td>
<td>7.5</td>
<td>40.0</td>
<td>-23.0</td>
<td>0.20</td>
<td>+6.0</td>
</tr>
</tbody>
</table>

Table 7.3: Planform properties for a commercial box wing aircraft concept similar to Gagnon and Zingg [2015].

This also facilities a large streamwise separation to satisfy longitudinal stability and trim constraints. The vertical lifting surfaces are swept to join the wing-tips. The height-to-span ratio at the root amounts $(h/b)_r = 0.22$ and reduces towards the tip to a value of $(h/b)_t = 0.18$ due contrary wing dihedral. This may be required in a practical application to attain sufficient ground clearance in the case of a lower wing-mounted engine or more generally because of lateral stability considerations. The average stagger factor is approximately $St_{avg} = 6.0$. According to Bramesfeld [2006], a time-step width of $\Delta t = 0.25 \cdot (c_{ref}/V_\infty)$ was utilized.

Figure 7.19: Planform geometry for a commercial box wing aircraft concept similar to Gagnon and Zingg [2015].
7.4.2 Computational Results and Discussion

Spanwise Load Distribution

The spanwise load distributions computed by the higher-order potential-flow method are depicted in Figure 7.20 and expressed relative to the maximum value of the optimum distribution \( C_{n0, \text{opt}} \) [Demasi et al., 2015b].

![Graphs showing spanwise load distribution on lower, vertical, and upper wings](image)

(a) Lower wing (upstream).

(b) Vertical wing.

(c) Upper wing (downstream).

**Figure 7.20:** Spanwise load distribution on lower wing, (a), vertical wing, (b), and upper wing, (c), at a system angle of attack of \( \alpha = 8.0^\circ \).

Due to the inherent geometrical characteristics of the planform, present distributions are non-optimal and resemble a more triangular spanwise loading. The centroid of the distribution is shifted inboards, which results in a larger loading in the root region, espe-
especially on the more lift-effective upstream wing. By contrast, the load on the vertical wing is reduced. The impact by wake surrogate models on the spanwise loading is negligible.

**Computed Span Efficiency Factors**

Computed span efficiency factors versus the system angle of attack are presented in Figure 7.21. Independent of the wake surrogate model, the span efficiency is predicted to decrease with the angle of attack. This is plausible acknowledging that the effective height-to-span ratio is progressively reduced.

![Figure 7.21: Computed span efficiency factor $e$ versus the system angle of attack $\alpha$.](image)

Based on the freestream-fixed wake model, the correlation between span efficiency factor and system angle of attack is actually linear, which is also found true for the relaxed-wake model up to an angle of attack of approximately $\alpha \approx 5.0^\circ$. The slope however changes for larger angles of attack. Similar to findings involving the positive-staggered system in Section 7.1.4, deviations exist between estimates based on the freestream-fixed and relaxed-wake model. These are attributed to higher-order wake effects and found to increase with the system angle of attack. At an angle of attack of $\alpha = 8.0^\circ$, higher-order wake effects are found to contribute approximately $\Delta e_{rel} \approx 2.5\%$ in span efficiency.

Computational results for the present, more realistic planform properties are in-line with findings based on the simplified box wing configuration incorporating positive stagger. This supports the conclusion that a relaxed-wake model is required more generally to enable accurate induced prediction for box wing configurations incorporating considerable
amounts of positive stagger.
Induced Drag Prediction for Biplanes

Following the investigation of the previous chapter, the induced drag characteristics of simple biplane configurations, in particular with regards to higher-order wake and wake surrogate effect are explored. It is intended to substantiate findings and conclusions based on the box wing concept to expand the general validity of hereon drawn conclusions. In addition, the effect by relaxed-wake parameters on accurate potential-flow prediction is analyzed. The computational expense of each methodology is evaluated.

8.1 Angle of Attack-Induced Wake Effects

Equivalent to Section 7.1, the present section explores the reasons and impact of higher-order wake and wake surrogate effects on accurate induced drag prediction in dependency on the system angle of attack. As subject of investigation, two biplane configurations of reduced geometric complexity and opposed longitudinal stagger are considered. This enables the investigation of correlations between angle of attack, as well as to the longitudinal arrangement of the system. The trailing flowfield properties are investigated to substantiate findings; the accuracy of the higher-order potential-flow method is assessed against the Euler-flow reference. Moreover, because of the the finite wing span, the impact of the added canceling singularity parameter in conjunction with relaxed-wakes is determined.
8.1.1 Test Cases

The planforms of both simplified biplane configurations concerned herein are illustrated in Figure 8.1. Their geometric properties are equivalent to box wing configurations presented in Section 5.1 and are characterized by a single wing aspect ratio of $\Lambda_{1,2} = 6.0$, a geometric height-to-span ratio of $(h/b) = 0.20$ and stagger factors of $St = +3.0$ and $St = -3.0$ respectively. The lifting surfaces are of constant chord length and do not incorporate camber, sweep or twist. In contrast to the box wing concept, the present lifting systems are composed of two separate wings of finite span. This required additional considerations in the scope of the current study. Relative induced drag estimates were obtained for an angle of attack range of $\alpha = 1.0^\circ$ to $\alpha = 9.0^\circ$ under subsonic flow conditions using the Euler-flow reference and the higher-order potential-method. The computational approach presented in Chapter 5 for either methodology is fully adapted. A discourse on the computational implementation is hence omitted.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8.png}
\caption{Isometric view of biplane planforms with a geometric height-to-span ratio of $(h/b) = 0.20$.}
\end{figure}

8.1.2 Wing-Tip Shape

Unlike the continuous lifting surface of box wings, the individual wings of the present configuration are characterized by a finite span. With regards to the Euler-model, lifting elements are of finite thickness, which requires the connection of the upper and lower surface panels to obtain a closed body. In principle, this is rather trivial to accomplish, i.e. by means of a flat patch. However, in the case of an Euler-flow model, the design of the
wing-tip can considerably affect the induced drag of the system. In contrast to potential-flow theory, vorticity can be shed from the side-edge of the wing. While this is a real phenomenon in viscous flows, its existence in an Euler-simulation is artificial [Barton and Pulliam, 1986] and related to the inherent numerical viscosity. This potentially alters the trailing wake shape, which, together with its vorticity distribution, determine the induced drag [Smith, 1995]. A design, promoting severe side-edge separations, is therefore generally not adequate to enable a comparison with present potential-based estimates. To prevent or considerably alleviate such separations, an appropriate wing-tip shape is desired.

The principle impact of two different wing-tip shapes, a sharp and a well-rounded, was explored. However, this study did not intend to provide a comprehensive and self-contained investigation into the impact of wing-tip shapes on the induced drag.

For the sharp wing-tip at an angle of attack of $\alpha = 8.0^\circ$, the side-edge separation has a noticeable effect on the distribution of the section lift coefficient $C_l$ as shown in Figure 8.2c. Its impact also extends towards more inboard sections of the wing and results in an overall increased lift coefficient. From the streamlines in Figure 8.2a it can be seen that the separation at the tip already occurs in close proximity to the leading edge, creating a non-planar trailing wake further downstream and reducing induced drag. This, together with the increased lift coefficient, translates into a more than 2% larger span efficiency factor compared to the blended wing-tip. Considering that it is basically the vertical extent of the system at the tip that affects the induced drag [Lowson, 1990], the decrease in induced drag can be verified based on the vertical extent of the wake trace at the individual wing trailing edge. Qualitatively comparing both tip-shapes, the sharp design results in a larger vertical gap at the tip and thus in a smaller induced drag. This is in good accordance with Hicken and Zingg [2010], presenting a similar result for two optimized planar wing-tip shapes. In conclusion, the sharp wing-tip is not suitable to enable a comparison with potential-based estimates.

The side-edge separation is successfully minimized by applying a well-rounded wing-tip that smoothly joins the upper and lower surface. For an angle of attack of $\alpha = 8.0^\circ$, the streamlines around the upstream wing in Figure 8.2b remain attached to the tip until the trailing edge, given no indication of any major premature flow separation. The vertical
extent of the wake trace is more similar to potential-flow theory and sheds from the trailing edge. This is emphasized by the spanwise distribution of the section lift coefficient in Figure 8.2d. Although minor peaks in the distribution are still evident in close proximity to the wing-tip, deviations are substantially minimized compared to the sharp wing-tip case. The blended design was therefore considered to provide an adequate reference baseline.

**Figure 8.2:** Influence of the wing-tip shape on the side-edge separation on the upstream wing for a system angle of attack of $\alpha = 8.0^\circ$.

### 8.1.3 Spatial Convergence Study

**Euler-Flow Method**

The spatial convergence was investigated by applying a constant grid refinement factor of $r = 1.5$ according to the procedure described in Section 5.3.1. Cell counts range from approximately $3 \times 10^6$ to $58 \times 10^6$ based on a predominately hexahedral grid. For an angle of attack at $\alpha = 8.0^\circ$, the convergence behavior of the span efficiency factor is plotted against the non-dimensional grid element size in Figure 8.3. Further details are summarized in
Table 8.1. A Richardson extrapolation [Roache, 1997] was performed successfully for the positive-staggered system, indicating excellent spatial convergence. The relative deviation between the extrapolated continuum value and the span efficiency based on the finest grid (level 1) constitutes approximately \( \Delta e_{\text{rel}} \approx 0.1\% \). The grid convergence index \( R_{\text{GCI}} \) ratio using the three finest grid levels is approximately one, which indicates that the solution is well within the asymptotic range of convergence. For the negative-staggered system, a non-monotonic convergence behavior is experienced. The maximum deviation in span efficiency for the three finest grid levels involved amounts approximately \( \Delta e = 0.01 \). Although a clear converging trend is recognized, an additional refinement step may be required, but is prevented by insufficient computational performance available.

![Spatial convergence behavior of the span efficiency factor for a system angle of attack of \( \alpha = 8.0^\circ \).](image)

**Figure 8.3:** Spatial convergence behavior of the span efficiency factor \( e \) for a system angle of attack of \( \alpha = 8.0^\circ \).

**Higher-Order Potential-Flow Method**

Element density effects were studied for the freestream-fixed and relaxed-wake model, in order to assess their impact on the prediction of the span efficiency factor for both stagger factors and the angle of attack range from \( \alpha = 1.0^\circ \) to \( \alpha = 9.0^\circ \). As evident from Figure 8.4, the dependency on the chordwise element count, which was varied between \( n_c = 4 \) and \( n_c = 14 \), is found to be weak in general. A chordwise element count of \( n_c = 8 \) and \( n_c = 10 \) was selected for the freestream-fixed and relaxed-wake model respectively.
Table 8.1: Spatial convergence behavior of the span efficiency factor $e$ for a system angle of attack of $\alpha = 8.0^\circ$.

The impact of the spanwise element density on the span efficiency factor is more pronounced and depicted in Figure 8.5. For the freestream-fixed wake model a spanwise element density of $n_s = 16$ is found to produce converged results in the case of the positive staggered system, whereas only $n_s = 12$ are required for the negative-staggered system. A significant larger number of elements of $n_s = 26$ is necessary for the positive-staggered system using the relaxed-wake model. In the case of the negative-staggered system, $n_s = 14$ elements in the spanwise direction are required. The larger spanwise element count of the relaxed wake-model, especially in the case of positive stagger, is likely to originate from the geometrical complexity of the rolled-up vortex structures of the relaxed-wake shape, requiring an improved spatial discretization.

8.1.4 Relaxed-Wake Parameter Impact

At the side-edge of a wake, where no spanwise neighboring element exists, the induced velocities approach infinity without adequate treatment. Thus, a canceling singularity is added to keep velocities finite and numerically valid [Bramesfeld, 2006]. The magnitude of the added singularity is described by the parameter $k_{se}$, which gives the strength in multiples of the minimum vortex sheet half-span $\eta_{min}$. The addition of a canceling singularity at the side-edge facilitates the numerical implementation of a continuous vortex sheet, but introduces a dependency of the induced velocities on the magnitude of the canceling
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Figure 8.4: Influence of the chordwise element number $n_c$ on the span efficiency factor $e$ versus the system angle of attack $\alpha$ for the fixed-wake, (a-b), and the relaxed-wake model, (c-d).

parameter. This is problematic in the case of the relaxed-wake model, where the velocities at the side-edge are used for the relaxation process. Depending on the magnitude of the singularity parameter, the roll-up behavior of the wake is altered, which consequently affects the induced drag of the system. An appropriate selection of the singularity parameter limits the side-edge velocities to provide a numerically stable solution, but should avoid a restriction of the roll-up process. A value of $k_{se} = 0.010$, which equals 1% of the shortest distributed vorticity element half-span, $\eta_{min}$, is suggested [Bramesfeld, 2006].

The shape of the relaxed-wake is affected by the choice of the value of the canceling singularity parameter, $k_{se}$, and the time-step width, $\Delta t$. Any distortion of the solution by
these parameters is undesirable, but cannot be avoided completely. Although the effect may be limited for planar monoplane wings, present configurations may be more sensitive to minor changes in the computed wake shape. Thus, an evaluation of the parameter impact is crucial to provide accurate induced drag estimates. To evolve the relaxed-wake correctly, the tip-vortex roll-up needs to be captured adequately by a sufficient small time-step width, accounting for the relatively high velocities at the side-edge of the vortex sheet, which occur despite the added singularity canceling. The wake must deform freely; a too
wide time-step results in a wake, that is not force-free and artificially stiff. An excessive reduction of the time-step width unnecessarily increases the necessary computation time and creates a practical limit. Dependent on the quality requirements of the induced drag prediction, a trade-off between accuracy and computational efficiency may have to be considered.

Therefore, a parametric study of the canceling singularity parameter and the time-step width was conducted to investigate their principal impact on the wake shape and on the induced drag estimation. In compliance with the wake substitution concept, only the wake of the upstream wing, bound between upstream and downstream trailing edge, was evaluated. Parameters were varied according to the range given in Table 8.2 for a system angle of attack of $\alpha = 8.0^\circ$. To characterize the roll-up behavior of the wake, the roll-up angle serves as a suitable indicator. It is defined as the enclosed angular difference between the surface normal of the respective wake distributed vorticity element and the wing-fixed $\zeta$-axis on the $\eta\zeta$-plane.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Lower bound</th>
<th>Upper bound</th>
<th>Step size</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_{se}$, $\eta_{min}$</td>
<td>0.005</td>
<td>0.050</td>
<td>(0.005) 0.010</td>
</tr>
<tr>
<td>$\Delta t$, $c/V_\infty$</td>
<td>0.05</td>
<td>0.30</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Table 8.2: Investigated relaxed-wake parameter range.

Based on a constant time-step width of $\Delta t = 0.20$, the influence of the singularity parameter on the roll-up angle of the tip wake element $\nu_{tip}$ versus the stagger factor, $St$, is shown in Figure 8.6c for the positive-staggered system and in Figure 8.6d for the negative-staggered system respectively. Independent of the stagger, it is found that increasing the singularity parameter damps the roll-up and effectively shifts it away from the upstream trailing edge. Of course this reduces the impact of the roll-up on the upstream wing [Schmidt-Göller, 1992], but also alters the shape of the wake further downstream and thus its influence on the downstream wing.

For the positive-staggered system, the streamwise distance required to perform an
initial rotation of \( \nu_{\text{tip}} = 180^\circ \) varies between approximately \( \Delta St = +0.75 \) and \( \Delta St = +1.25 \) for the smallest \( k_{se} = 0.005 \) and largest \( k_{se} = 0.050 \) value of the singularity parameter respectively. A generally more downstream location is evident in the case of the negative-staggered system, which indicates slower and less significant roll-up effects. Depending on the value of the singularity parameter, the distance required to perform an initial rotation of \( \nu_{\text{tip}} = 180^\circ \) varies between \( \Delta St = -2.5 \) and \( \Delta St = -0.5 \). This distance is indicated to slightly increase further downstream for consecutive rotations, which is equivalent to a deceleration of the roll-up process. For a singularity parameter of \( k_{se} = 0.005 \) and \( k_{se} = 0.010 \), a full rotation is completed within the bounds of upstream and downstream trailing edge in the case of the positive-staggered system.

The impact of the time-step width for a constant value of the singularity parameter of \( k_{se} = 0.020 \) on the roll-up angle of the tip wake element is illustrated in Figures 8.6(a-b) for the positive and negative-staggered system. Close to the upstream trailing edge, until a downstream location of approximately \( St = +0.5 \) for the positive and \( St = -2.5 \) for the negative system, the roll-up angle is essentially unaffected by the time-step width. Beyond these streamwise locations, the amount of roll-up is progressively damped with increasing time-step width, similar to the singularity parameter. For a time-step width of \( \Delta t = 0.05 \) and \( \Delta t = 0.10 \) a full rotation is performed between upstream and downstream trailing edge in the case of the positive-staggered system. Time-steps wider than these lead to a wake with a larger artificial stiffness.

The span efficiency factor of the system is affected by the selected values for the singularity parameter and the time-step width as shown in Figures 8.6(e-f). A dominant impact on the span efficiency is observed by to the singularity parameter. Its influence is substantially more distinctive for the positive than for the negative-staggered case. This is supposed to originate from the smaller effective height-to-span ratio causing a stronger interaction of trailing wake and downstream lifting element and hence a higher sensitivity of induced drag to geometric deviations in computed wake shape. The maximum absolute variation in span efficiency caused by the singularity parameter accumulates to \( \Delta e = 0.060 \) for the positive-staggered system and \( \Delta e = 0.005 \) for the negative-staggered system respectively. In contrast to that, the influence of the time-step width is more pronounced.
Figure 8.6: Impact of the time-step width $\Delta t$, (a-b), and the singularity parameter $k_{se}$, (c-d), on the roll-up angle at the wake tip element $\nu_{tip}$ and on the span efficiency factor $e$, (e-f).
SECTION 8.1: ANGLE OF ATTACK-INDUCED WAKE EFFECTS

for the negative than for the positive-staggered case, however on a less significant level than for the singularity parameter. The absolute variation in span efficiency amounts $\Delta e = 0.010$ for the positive-staggered system, which is found virtually independent of the singularity parameter involved. For the negative-staggered system, the span efficiency varies by about $\Delta e = 0.020$ at maximum for the smallest value of the singularity parameter of $k_{se} = 0.005$.

It is also interesting to note that depending on the stagger factor of the system, the impact of singularity parameter and time-step width on the span efficiency factor is diametrically opposed. Whereas for the positive-staggered system a small value of the singularity parameter results in a small span efficiency as well, a small singularity parameter is associated with a comparable high span efficiency for the negative-staggered system. Likewise, a small time-step width facilitates an increase in span efficiency for the positive-staggered systems and a decrease in the case of the negative-staggered system.

The parametric study also reveals that at least for the positive-staggered system, no convergence is achieved based on the considered range of relaxed-wake parameters. For the range of parameters tested, the span efficiency varies considerably. The ability to provide accurate induced drag predictions therefore depends heavily on an appropriate selection of parameters. This is problematic and has also be noted by Basom and Maughmer [2011], who suggest to employ more inboard velocities of the DVE during the relaxation process. The approach was adapted in an attempt to circumvent at least the issue related to the selection of the added singularity parameter. However it was found, that the shape of the wake becomes dependent on the inboard location instead. To enable an appropriate selection of relaxed-wake parameters, a relaxed-wake shape congruence study was performed to obtain an 'optimum' parameter set. The approach is discussed in the preceding section.

**Relaxed-Wake Shape Congruence**

For the relaxed-wake, the congruence with the wake shape based on the Euler-flow reference was evaluated to obtain an 'optimum' set of values for the singularity parameter and time-step width. The expediency of this approach becomes obvious, realizing that besides the vorticity, the wake shape characterizes the induced drag [Smith, 1995].
enable a qualification of those effects, two figures of merit or error indicators were used
to determine the congruence of both wake shapes, i.e. the root-mean-square error of the
relative Euclidean distance, $d_{euc}/b$, and the root-mean-square error of the roll-up angle de-
viation, $\Delta \nu$, between a relaxed-wake element and the Euler-flow wake shape at equivalent
spanwise stations. Both, the Euclidean distance and the the roll-up angle deviation, were
computed over a set of outboard sections of the wake between $\eta = 0.44$ and $\eta = 0.50$,
that are characterized by significant roll-up effects. The result of this comparative study
is summarized in Figure 8.7. Independent of the stagger factor and time-step width, the
error of the Euclidean distance and the amount of the roll-up angle deviation reduce along
with a decreasing singularity parameter to their respective minimum. For a constant time-
step width, a distinct minimum exists in most of the cases. Beyond this point the trend
reverses and the error increases again.

With regard to the negative-staggered system it can be generally noted that the devi-
ation in wake shape is on a considerably lower level, as evident from both error indicators.
Moreover, the dependency of geometrical deviations is shown to be less dependent of the
selection of relaxed-wake parameters than compared with the positive-staggered system.
This is considered to be related to its larger effective height-to-span ratio causing weaker
aerodynamic interaction between wing and trailing wake.

For the positive-staggered system, two parameter combinations were identified to pro-
duce a virtually equivalent minimum error based on both indicators. That is a time-step
width of $\Delta t = 0.05$ and a singularity parameter of $k_{se} = 0.020$ and a time-step width of
$\Delta t = 0.10$ and a singularity parameter of $k_{se} = 0.010$. Although both combinations of
parameters potentially yield induced drag estimates of equivalent accuracy, the latter is
preferred since it reduces the computation time that is required to attain converged results.
As inferred by the wake substitution concept, the wake must be relaxed at least until the
most downstream trailing edge. A smaller time-step width thus requires a larger number
of iteration steps and hence a larger computational overhead.

In the case of the negative-staggered system however, individual minima exist for
each error indicator. A singularity parameter of $k_{se} = 0.020$ and a time-step width of
$\Delta t = 0.20$ is considered to provide a sufficient compromise, also acknowledging that both
error indicators are relatively insensitive to parameter selection compared to the positive-staggered case.

![Graphs showing RMSE comparison](image)

**Figure 8.7:** Impact of the singularity parameter $k_{se}$ for constant time-step widths $\Delta t$ on the RMSE of the Euclidean distance $d_{euc}/b$, (a-b), and the RMSE of the roll-up angle deviation $\Delta \nu$, (c-d), compared to the Euler-based reference over the outboard spanwise sections of the wake between $\eta = 0.44$ to $\eta = 0.50$.

To complement the wake shape congruence study and to demonstrate the effect of wake parameters qualitatively, wake traces for the positive-staggered system that result for the smallest and largest root-mean-square error ($\Delta t = 0.10$, $k_{se} = 0.010$ and $\Delta t = 0.50$, $k_{se} = 0.050$, respectively) are compared to the wake trace based on the Euler-flow reference in Figures 8.8(a-b). It is clearly shown, that deviations among both shapes are primarily

...
related to the tip region of the wake. This supports the validity of the present approach to evaluate the congruence of wake shapes based on a set of outboard sections. Compared to the Euler-flow reference, the wake roll-up is sufficiently captured in the minimum error case, whereas the wake development is overly restricted in the maximum error case. This result is additionally supported by the contour plot of the relative Euclidean distance in Figures 8.8(c-d) and the roll-up angle deviation in Figures 8.8(e-f). Differences in the computed wake are merely concentrated in the tip region of the wake and vary with the streamwise location. More inboard sections are generally less affected. For the maximum error case, major deviations occur for streamwise locations larger than approximately $St = +1.5$.

An equivalent result is evident for the negative-staggered system in Figures 8.9(a-f). According to the investigated impact of relaxed-wake parameters on both error indicators, deviations in the computed wake shape of the minimum and maximum error case are less significant.

### 8.1.5 Computational Results and Discussion

#### Spanwise Load Distribution

The spanwise load distributions using the higher-order potential-flow method and the Euler-flow reference, for the positive and negative-staggered system, are shown in Figure 8.10. The loadings are given relative to the maximum value of the elliptical distribution $C_{n0, ell}$ assumed by Prandtl [1924]. For any finite vertical separation, this does not represent optimal conditions as indicated by Demasi [2006], but is sufficient to assess present spanwise loadings with regards to span efficiency predictions.

Distributions show good agreement between the computational methodologies utilized. The longitudinal stagger results in an unequal lift division, contradicting minimum induced drag conditions. Equivalent to the box wing configurations in Section 7.1.4, the upstream wing is found more lift-efficient. Inherent to the geometrical planform characteristics incorporating no twist and constant chord length, present loadings do not represent an elliptical spanwise distribution.
Figure 8.8: $St = +3.0$: Wake traces at the partition surface, (a-b), contours of the relative Euclidean distance ($d_{\text{euc}}/b$), (c-d), and contours of the roll-up angle deviation $\Delta \nu$, (e-f), bound between the upstream trailing edge and the partition surface.
STAR-CCM+ FreeWake

(a) Minimum error case:
k_{se} = 0.020, \Delta t = 0.20.

(b) Maximum error case:
k_{se} = 0.050, \Delta t = 0.050.

(c) Minimum error case:
k_{se} = 0.020, \Delta t = 0.20.

(d) Maximum error case:
k_{se} = 0.050, \Delta t = 0.050.

(e) Minimum error case:
k_{se} = 0.020, \Delta t = 0.20.

(f) Maximum error case:
k_{se} = 0.050, \Delta t = 0.050.

Figure 8.9: \( St = -3.0 \): Wake traces at the partition surface, (a-b), contours of the relative Euclidean distance \((d_{euc}/b)\), (c-d), and contours of the roll-up angle deviation \(\Delta \nu\), (e-f), bound between the upstream trailing edge and the partition surface.
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Figure 8.10: Spanwise load distribution on lower wing, (a-b), and upper wing, (c-d), at a system angle of attack of $\alpha = 8.0^\circ$.

**Computed Span Efficiency Factor**

Computed span efficiency factors versus the system angle of attack are shown in Figure 8.11 for the positive and negative-staggered system, employing the relaxed-wake parameter combinations obtained in the preceding Section 8.1.4. Dependent on the stagger factor, both wake models correctly predict a negative or positive slope of the span efficiency factor with increasing system angle of attack. Considering the progressive reduction of the effective height-to-span ratio with the system angle of attack for the positive-staggered system, and the reversed trend for the negative-staggered system, the present results are
consistent with the theory on the effective height-to-span ratio and findings of Section 7.1.4 based on box wing configurations.

Nevertheless, considerable deviations exist between estimates using the freestream-fixed and relaxed-wake model or the Euler-flow reference, in particular for the positive-staggered system. The freestream-fixed wake model under-estimates the span efficiency factor compared to the Euler-flow reference based on the finest grid (level 1), especially at larger angles of attack. This is attributed to higher-order effects, related to the roll-up and deflection of the trailing wake, which are not considered by the simple freestream-fixed wake model. Effects gain impact as the system angle of attack increases and amount at an angle of attack of \( \alpha = 8.0^\circ \) about \( \Delta e_{rel} \approx 3.8\% \) in span efficiency. For the relaxed-wake model (\( \Delta t = 0.10, k_{se} = 0.010 \)), estimates show an excellent agreement with the Euler-flow reference. The maximum relative deviation is less than \( \Delta e_{rel} \approx 0.5\% \) on average or \( \Delta e_{rel} \approx 0.1\% \) at an angle of attack of \( \alpha = 8.0^\circ \). This constitutes a major improvement compared to the freestream-wake model case.

For the negative-staggered system, the freestream-fixed wake model slightly over-predicts the span efficiency by about \( \Delta e_{rel} \approx 0.5\% \) on average compared to the relaxed-wake model (\( \Delta t = 0.20, k_{se} = 0.020 \)) and the Euler-flow references. The accuracy is only weakly dependent on the selected wake model approach. The effective higher-order wake impact on the span efficiency, derived from the relative difference in span efficiency between both wake models is less significant than for the positive-staggered system and amounts \( \Delta e_{rel} \approx 0.8\% \) at an angle of attack of \( \alpha = 8.0^\circ \). This is assumed to be related to the larger effective height-to-span ratio for the negative-staggered system.

For equivalent freestream height-to-span ratios \( (h/b)_\infty \), the maximum span efficiency factor based on elliptical span loadings \( e_{Bi, el} \) is given in Figure 8.11. In agreement with non-optimal loadings, smaller span efficiency factors are estimated for the negative-staggered system independent of the computational methodology. Due to higher-order wake effects, relaxed-wake and Euler-flow predictions exceed linear predictions despite non-optimal loadings for the positive-staggered system at larger angles of attack.

Qualitatively, there is good agreement of the present results compared to those of Section 7.1.4. This clearly demonstrates, that a freestream-fixed wake model approach
SECTION 8.1: ANGLE OF ATTACK-INDUCED WAKE EFFECTS

is not necessarily appropriate to enable an accurate induced drag estimation for highly non-planar concepts such as box wings or biplanes characterized by considerable amounts of positive stagger, especially at larger angles of attack. To enable accurate induced drag predictions under these conditions, a potential-based relaxed-wake model or an Euler-flow simulation is mandatory.

Figure 8.11: Computed span efficiency factor $e$ for a positive stagger factor of $St = +3.0$, (a), and a negative stagger factor of $St = -3.0$, (b), versus the system angle of attack $\alpha$.

Wake Trace on Partition Surface

Characterizing the induced drag, the shapes of the wake trace at the partition surface are shown in Figure 8.12 for the positive and negative-staggered system at a system angle of attack of $\alpha = 8.0^\circ$. The considerable deviations of the span efficiency factors that are apparent between the predictions using the freestream-fixed and relaxed-wake model, especially for the positive-staggered system, are also reflected by the wake traces, which are fundamentally different in shape. An assessment regarding the span efficiency of the system is again made by means of an evaluation of the effective height-to-span ratio of the wake trace on the partition surface.

A constant spanwise height-to-span ratio is obtained for the freestream-fixed wake model, whereas the height-to-span ratio varies along the span for the relaxed-wake model.
CHAPTER 8: INDUCED DRAG PREDICTION FOR BIPLANES

and the Euler-flow reference. With the exception of the tip region, the effective height-to-span ratio for the Euler-flow reference and the relaxed-wake model are noticeably larger compared to the freestream-fixed wake approach, which eventually facilitates larger span efficiency factors in the case of the positive-staggered system. For the negative-staggered system the effect is reversed, leading to an increased effective height-to-span ratio and efficiency for the freestream-fixed wake model than compared with the relaxed-wake trace or the Euler-flow reference.

This is supported by computing the spanwise mean value of the effective height-to-span ratio, to approximate the span efficiency factor based on Equation 2.19 and given as $e_{Bi, ell}$ in Figure 8.12. Approximated span efficiency factors are found to be in reasonable agreement with estimates computed in Section 8.1.5. Good consistency is found between the relaxed-wake model and the Euler-flow reference for equivalent stagger, whereas larger deviations are apparent for the freestream-fixed model. In the case of the positive-staggered system, the span efficiency is smaller than compared with the relaxed-wake model or the Euler-flow reference, whereas for the negative-staggered system the efficiency is found considerably larger.

The impact of higher-order wake effects originates from the deflection of the wake and its roll-up. In the case of a positive-staggered system, the deflection of the wake causes a larger effective height-to-span ratio except for the wing-tip, leading to larger efficiency as well. The roll-up at the tip slightly reduces the effective span and considerably diminishes the effective vertical gap at the tip. This is considered a critical parameter [Lowson, 1990] and thus partly compensates gains due to wake deflection. For the negative-staggered system, the trend is reversed. Whereas wake deflection results in a height-to-span ratio smaller than for the freestream-fixed model, the roll-up of the wake at the tip facilitates a large effective vertical gap at the tip. It is concluded that although higher-order wake effects are present for the negative-staggered system, opposed wake effects and the overall larger vertical separation result in a span efficiency that is quite similar despite fundamental differences in computed wake shape between the freestream-fixed and relaxed-wake model.
SECTION 8.1: ANGLE OF ATTACK-INDUCED WAKE EFFECTS

Figure 8.12: Wake traces at the partition surface for a system angle of attack of $\alpha = 8.0^\circ$. 

(a) FreeWake: Freestream-fixed wake, $St = +3.0$. 

(b) FreeWake: Freestream-fixed wake, $St = -3.0$. 

(c) FreeWake: Relaxed-wake, $St = +3.0$. 

(d) FreeWake: Relaxed-wake, $St = -3.0$. 

(e) STAR-CCM $+$, $St = +3.0$. 

(f) STAR-CCM $+$, $St = -3.0$. 

$\epsilon_{Bi, ell} \approx 1.237$ 

$h/b_{eff} = \text{const}$ 

$\eta (z/b)$ 

$\eta (z/b)$ 

$\epsilon_{Bi, ell} \approx 1.413$ 

$h/b_{eff} = \text{const}$ 

$\eta (z/b)$ 

$\eta (z/b)$ 

$\epsilon_{Bi, ell} \approx 1.281$ 

$h/b_{eff} = f(\eta)$ 

$\eta (z/b)$ 

$\eta (z/b)$ 

$\epsilon_{Bi, ell} \approx 1.386$ 

$h/b_{eff} = f(\eta)$ 

$\eta (z/b)$ 

$\eta (z/b)$ 

$\epsilon_{Bi, ell} \approx 1.278$ 

$h/b_{eff} = f(\eta)$ 

$\eta (z/b)$ 

$\eta (z/b)$ 

$\epsilon_{Bi, ell} \approx 1.389$ 

$h/b_{eff} = f(\eta)$ 

$\eta (z/b)$ 

$\eta (z/b)$
CHAPTER 8: INDUCED DRAG PREDICTION FOR BIPLANES

Streamwise Vorticity Distribution

The streamwise vorticity distribution are shown in Figure 8.13. For the positive-staggered system, contours of the streamwise vorticity of the relaxed-wake model and the Euler-flow reference show good agreement with regards to their characteristic shape, intensity and vortex core location relative to the system trailing edge. The vortex cores based on the relaxed-wake are, however, defused, especially in the case of the upstream wing. This is possibly related to numerical leakage effects of the wake distributed vorticity elements in that region. Despite the simplicity of the freestream-fixed wake model, vortex core locations are found to be in reasonably good agreement with the Euler-flow reference.

In the case of the negative-staggered system, acceptable consistency is evident between contours based on the relaxed-wake shape and the Euler-flow model. Whereas the location of the vortex cores still closely agrees, their shape is notably diffused, which may be attributed to the selected sparse spanwise element density and explains the encountered deviation in computed tip vortex intensity between the relaxed-wake model and the Euler-flow reference, particularly in the case of the upstream wing. Contours based on the freestream-fixed wake model surely differ significantly, which especially becomes apparent based on an evaluation of the vortex core locations.

8.2 Gap and Stagger-Induced Wake Effects

This parametric study investigates the impact by gap and stagger-induced wake effects on biplane lift and induced drag characteristics and intends to support the findings and conclusions based on the box wing concept. The methodological procedure detailed in Section 7.2 was adapted.

8.2.1 Test Cases and Design Parameter

The geometric properties of the systems under investigation accord to those provided for the box wing concept in Section 7.2, but exclude the vertical lifting surface. The study was conducted at a system angle of attack of $\alpha = 4.0^\circ$, exploring the design parameter range consistent with Section 7.2.1 and as summarized in Table 8.3. To prevent side-edge
Figure 8.13: Contours of the streamwise vorticity $\gamma_x$ at a distance of one reference chord length $c_{ref}$ downstream of the system trailing edge for a system angle of attack of $\alpha = 8.0^\circ$. 
separations, the well-rounded wing-tip shape was utilized for Euler-based predictions. A spatial or temporal convergence study was not performed. A discretization equivalent to Section 8.1.3 was used. This is considered conservative, because convergence in Section 8.1.3 was successfully attained at a larger system angle of attack of $\alpha = 8.0^\circ$.

<table>
<thead>
<tr>
<th>Design parameter</th>
<th>Lower bound</th>
<th>Upper bound</th>
<th>Step size</th>
</tr>
</thead>
<tbody>
<tr>
<td>$St_\infty$</td>
<td>-3.0</td>
<td>+3.0</td>
<td>1.0</td>
</tr>
<tr>
<td>$(h/b)_\infty$</td>
<td>0.10</td>
<td>0.40</td>
<td>0.05 (0.10)$^1$</td>
</tr>
</tbody>
</table>

Table 8.3: Investigated design parameter range.

With regards to relaxed-wake parameters, the ‘optimum’ values found in Section 8.1.4 were employed. In particular for any positive stagger, computations were preformed with a singularity parameter of $k_{se} = 0.010$ and a time-step width of $\Delta t = 0.10$, whereas predictions for negative stagger considered a singularity parameter of $k_{se} = 0.020$ and a time-step width of $\Delta t = 0.20$ respectively. In the case of an unstaggered system, the estimated induced drag and related quantities were attained as their mean values based on computations involving both parameter sets. Deviations due to parameter selections are however negligible for the unstaggered case.

8.2.2 Computational Results and Discussion

Computed Span Efficiency Factor

For constant freestream height-to-span ratios, the computed impact of the freestream stagger on the span efficiency factor is depicted in Figure 8.14. Compared to the investigation conducted in Section 7.2, a similar characteristic dependency of the span efficiency and the streamwise stagger factor is apparent for the relaxed-wake model and the Euler-flow reference. At equal freestream height-to-span ratios, positive stagger is found to produce a span efficiency factor larger than equivalent negative stagger, confirming the findings of Section 7.2. Independent of the freestream height-to-span ratio, the absolute highest span

$^1$In the case of the Euler-flow reference.
efficiency is associated with the maximum positive freestream stagger factor considered. This is different to the box wing concept, where the maximum span efficiency shifted from the maximum positive stagger involved towards smaller, positive streamwise separations with increasing vertical gap.

With regards to the freestream-fixed wake, the dependency of the span efficiency factor on the streamwise stagger is virtually symmetric to the unstaggered case. The maximum span efficiency is attained for the unstaggered system. The efficiency initially decays for increasing amounts of stagger, but subsequently slightly increases for larger streamwise separations. This, as well as the general correlation of span efficiency and stagger factor, contradict predictions based on the relaxed-wake and the Euler-flow reference. For freestream height-to-span ratios smaller than \( (h/b)_\infty = 0.25 \), freestream-fixed wake predictions show acceptable consistency with both other methodologies in the case of negative stagger. For any positive-staggered system, larger deviations become evident, which are attributed to the impact of higher-order wake effects.

The agreement between estimates based on the relaxed-wake model and the Euler-flow reference is found to be excellent for freestream height-to-span ratios of \( (h/b)_\infty = 0.10 \) and \( (h/b)_\infty = 0.20 \), whereas discrepancies emerge for height-to-span ratios beyond, especially for negative stagger. For a system incorporating a freestream height-to-span ratio of \( (h/b)_\infty = 0.20 \) and a streamwise stagger factor of \( St_\infty = +3.0 \), the difference between freestream-fixed and relaxed-wake estimates, constituting the contribution by higher-order wake effects, accumulates to approximately \( \Delta e_{rel} \approx 2.0\% \) in span efficiency, whereas the deviation of relaxed-wake predictions compared to the Euler-flow reference constitutes \( \Delta e_{rel} \approx 0.4\% \). For negative stagger, the discrepancies among both wake models and the Euler-flow references amounts less than \( \Delta e_{rel} \approx 0.2\% \) on average, indicating very limited impact by higher-order wake effects on the span efficiency.

The differences in span efficiency encountered for unstaggered systems are of concern. Deviations between relaxed-wake and Euler-flow estimates are of acceptable value and remain relatively constant within the considered range of vertical separations. However, for span efficiency predictions based on the freestream-fixed wake model, discrepancies are evident compared to relaxed-wake and Euler-flow predictions, being considerably more
distinctive than for the box wing configuration in Section 7.2. This partly distorts an assessment of the accuracy and the contribution by higher-order wake effect. For an unstaggered system only very limited contribution by higher-order wake effects is anticipated, implying good agreement between freestream-fixed and relaxed-wake based predictions for unstaggered systems. In principle, higher-order wake effects are considered to decrease with increasing vertical separation as a result of decaying mutual inductions. However, results exhibit an opposing trend.

**Wake Traces on Partition Surface**

For the case of a freestream height-to-span ratio of \((h/b)_{\infty} = 0.20\), the wake traces at the partition surface for the minimum \((St_{\infty} = -3.0)\) and maximum \((St_{\infty} = +3.0)\) freestream stagger factor involved are given in Figure 8.15.

Regarding the freestream-fixed wake model, the effective height-to-span ratio is constant along the span and independent of the streamwise arrangement. Therefore, the approximation of the span efficiency factor \(e_{Bi, opt}\) based on Equation 2.19 yields equivalent estimates for the range of stagger factors involved. Due to non-optimal spanwise loadings, approximations exceed computations by about \(\Delta e_{rel} \approx 1.0\%\). Because the effective height-to-span ratio is independent of stagger, the predicted dependency of the span efficiency and the stagger factor conclusively refers to variations in the streamwise vorticity or ultimately the lift distribution.

Good qualitative agreement is found for both stagger factors between wake traces based on the relaxed-wake model and the Euler-flow reference. This is also reflected by the good accordance of their approximated span efficiency factors. The effective height-to-span ratio for the positive-staggered system based on the relaxed-wake model and the Euler-flow reference is on average larger than in the case of the freestream-fixed wake model, in agreement with wake-related deviations in span efficiency. The approximation of span efficiency exceeds computed values by approximately \(\Delta e_{rel} \approx 1.2\%\). As it is the vertical gap at the tip, which is most effective due to the accumulation of vorticity there, the local reduction in effective height-to-span ratio related to the rolled-up wake vortex structure, results in actually smaller efficiency gains.
Figure 8.14: Span efficiency factor $e$ versus the freestream stagger factor $St_\infty$ for various freestream height-to-span ratios $(h/b)_\infty$. 

(a) $(h/b)_\infty = 0.10$.

(b) $(h/b)_\infty = 0.15$.

(c) $(h/b)_\infty = 0.20$.

(d) $(h/b)_\infty = 0.25$.

(e) $(h/b)_\infty = 0.30$.

(f) $(h/b)_\infty = 0.35$.

(g) $(h/b)_\infty = 0.40$. 

$\text{Figure 8.14: Span efficiency factor } e \text{ versus the freestream stagger factor } St_\infty \text{ for various freestream height-to-span ratios } (h/b)_\infty.$
For the negative-staggered system, an on average smaller effective height-to-span ratio is evident compared to the freestream-fixed wake model, causing potentially smaller span efficiency based on the relaxed-wake model and the Euler-flow reference as well. However, according to the findings of Section 8.2.2, near-equivalent efficiency predictions are attained. This is in turn attributed to the rolled-up wake vortex structure at the tip, causing a local increase in effective height-to-span ratio and thus span efficiency, compensating the effect of smaller average effective height-to-span ratio.

Streamwise Vorticity Distribution

For a freestream height-to-span ratio of \( (h/b)_{\infty} = 0.20 \), the contours of the streamwise vorticity are shown in Figure 8.16, at a distance of one reference chord length downstream of the system trailing edge for maximum positive \( (St_{\infty} = +3.0) \) and negative \( (St_{\infty} = -3.0) \) stagger involved. Similar to results presented in Section 8.1.5, vorticity contours of the relaxed-wake model and the Euler-flow reference show good agreement in their characteristic shape and vortex core location relative to the system trailing edge. Inherent to the freestream-fixed wake model, the vortex core locations coincide with the respective trailing edge. The streamwise vorticity, as indicated by its maximum amount at the tip, varies with the stagger factor. This causally results in a dependency of the span efficiency factor on the longitudinal separation.

Induced Drag and Lift Coefficient

The induced drag coefficient versus the streamwise stagger factor for a freestream height-to-span ratio of \( (h/b)_{\infty} = 0.20 \) is depicted in Figure 8.17a for both wake models and the Euler-flow reference. The associated dependency of the lift coefficient is given in Figure 8.17b. Overall both distributions describe a characteristic similar to the box wing concept in Section 7.2.2. Deviations in computed induced drag and thus lift between the higher-order potential-flow model and the Euler-flow reference originate from the thickness distribution of the lifting element and larger freestream Mach number for Euler-based simulations. This causes larger lift and induced drag as well. However, involved methodologies consistently predict minimum induced drag and also minimum lift for the unstaggered system.
SECTION 8.2: GAP AND STAGGER-INDUCED WAKE EFFECTS

Figure 8.15: Wake traces at the partition surface for a constant freestream height-to-span ratio of $(h/b)_\infty = 0.20$. 

(a) Freewake: Freestream-fixed wake, $St_\infty = +3.0$.

(b) Freewake: Freestream-fixed wake, $St_\infty = -3.0$.

(c) Freewake: Relaxed-wake, $St_\infty = +3.0$

(d) Freewake: Relaxed-wake, $St_\infty = -3.0$.

(e) STAR-CCM+, $St_\infty = +3.0$.

(f) STAR-CCM+, $St_\infty = -3.0$. 

$(h/b)_{\text{eff}} = \text{const}$

$(h/b)_{\text{eff}} = f(\eta)$

$\epsilon_{\text{Bi, ell}} \approx 1.347$ 

$\epsilon_{\text{Bi, ell}} \approx 1.347$

$\epsilon_{\text{Bi, ell}} \approx 1.364$

$\epsilon_{\text{Bi, ell}} \approx 1.327$

$\epsilon_{\text{Bi, ell}} \approx 1.369$

$\epsilon_{\text{Bi, ell}} \approx 1.325$
Figure 8.16: Contours of the streamwise vorticity $\gamma_x$ at a distance of one reference chord length $c_{ref}$ downstream of the system trailing edge for a constant freestream height-to-span ratio of $(h/b)_\infty = 0.20$. 

(a) Freewake: Freestream-fixed wake, $St_\infty = +3.0$. 
(b) Freewake: Freestream-fixed wake, $St_\infty = -3.0$. 
(c) Freewake: Relaxed-wake, $St_\infty = +3.0$ 
(d) Freewake: Relaxed-wake, $St_\infty = -3.0$. 
(e) STAR-CCM+, $St_\infty = +3.0$. 
(f) STAR-CCM+, $St_\infty = -3.0$. 

$\gamma_x$, $s^{-1}$

0 70 140 210 280 350

0 70 140 210 280 350
According to the dependency of the span efficiency factor and in compliance with Munk [1923b], the correlation of induced drag and lift coefficient are almost symmetric, relative to the unstaggered system for the freestream-fixed wake model. The induced drag is found independent of the direction of stagger. This contrasts predictions based on the relaxed-wake model and the Euler-flow reference, indicating a generally smaller induced drag level for any positive-staggered system within the considered parameter range. This is predominately caused by the deflection of the relaxed-wake, causing larger effective height-to-span ratios. The relative sensitivity of the induced drag to the stagger factor is confirmed most pronounced within small streamwise variations in proximity to the unstaggered system [Kang et al., 2009a,b].

Figure 8.17: Induced lift effects versus the freestream stagger factor \( St_\infty \) for a constant freestream height-to-span ratio of \( (h/b)_\infty = 0.20 \).

In the case of negative stagger, the induced drag prediction is merely independent of the employed wake-model approach. This also applies in general for the lift coefficient, indicating negligible impact by higher-order wake effects and direction of stagger. Based on Euler-flow predictions, a negative-staggered system is found more lift-efficient in accordance with Kang et al. [2009a,b], especially for small streamwise separations. The dependency on the direction of streamwise arrangement is attributed to the contribution by cascade effects as discussed in Section 7.2.2 and cannot be resolved adequately by the
CHAPTER 8: INDUCED DRAG PREDICTION FOR BIPLANES

higher-order potential-flow method.

The computed lift division between the upstream and the downstream wing is presented in Figure 8.18. In good agreement with theory [Smith, 1975], the upstream wing is found to be more lift-efficient than the downstream wing, independent of the computational methodology involved. For an unstaggered system, the lift share becomes balanced. In comparison to the box wing configurations in Section 7.2.2, the relative deviation in lift coefficient is less significant, in particular with regards to the downstream wing.

![Graph](a) Upstream wing.  ![Graph](b) Downstream wing.

Figure 8.18: Lift division between upstream (a) and downstream (b) wing versus the freestream stagger factor $St_\infty$ for a constant freestream height-to-span ratio of $(h/b)_\infty = 0.20$.

The impact of the freestream height-to-span ratio on the lift coefficient is given in Figure 8.19 versus the freestream stagger factor for the freestream-fixed, the relaxed-wake model and the Euler-flow reference. The lift coefficient is found to increase with the freestream height-to-span ratio, independent of the methodology or stagger factor involved. The sensitivity to the streamwise arrangement of lifting elements decreases with increasing freestream height-to-span ratio, in consistency with Kang et al. [2009a,b] and the findings of Section 7.2.2.

8.3 Computational Expense

The critical system with regards to the computational efficiency is associated with the positive-staggered system as presented in Section 8.1.1. This is caused by its spatial
Figure 8.19: Lift coefficient $C_L$ versus the freestream stagger factor $St_\infty$ for various freestream height-to-span ratio $(h/b)_\infty$.

The span efficiency factor is found to converge in about 14 iterations for the freestream-fixed wake model, as shown in Figure 8.20a. Caused by the small time-step width of $\Delta t = 0.10$, approximately 20 iterations must be performed for the relaxed-wake case before convergence is reached. This leads to an increased number of iterations steps to evolve the wake at a sufficiently large distance downstream compared to the freestream-fixed model and a time-step width of $\Delta t = 0.25$.

For the relaxed-wake model, the overall larger element count but also the computational effort to relax the wake, ultimately accumulate in a more than six times larger com-
CHAPTER 8: INDUCED DRAG PREDICTION FOR BIPLANES

![Figure 8.20: Convergence behavior for a system angle of attack of $\alpha = 8.0^\circ$.](image)

**Table 8.4:** CPU time comparison for a positive stagger factor of $St = +3.0$ and a system angle of attack of $\alpha = 8.0^\circ$.

<table>
<thead>
<tr>
<th>Estimation methodology</th>
<th>$e$</th>
<th>$\Delta e$</th>
<th>computation time, min</th>
<th>cores</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>FreeWake:</strong> Freestream-fixed wake</td>
<td>1.225</td>
<td>-3.7%</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td><strong>FreeWake:</strong> Relaxed-wake</td>
<td>1.275</td>
<td>0.2%</td>
<td>13</td>
<td>1</td>
</tr>
<tr>
<td><strong>STAR-CCM+:</strong> Grid level 1</td>
<td>1.274</td>
<td>0.2%</td>
<td>$\sim$1600</td>
<td>11</td>
</tr>
</tbody>
</table>

Computational overhead compared to the freestream-fixed wake model. Nevertheless, the time frame required is considered still within feasible bounds for the purpose of conceptional design or a multidisciplinary design optimization. Accepting a marginal deterioration of the relaxed-wake accuracy, a larger time-step width can facilitate a smaller computational expense. This has already been indicated in Section 8.1.4, noting that in contrast to the singularity parameter, the time-step width only weakly affects the span efficiency. For a time-step width of $\Delta t = 0.25$, the relative error in span efficiency constitutes approximately $\Delta e_{rel} \approx -0.3\%$, which is still excellent but reduces the computational time by about 50% compared to a time-step width of $\Delta t = 0.10$. 

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Similar to the box wing concept, Euler-flow induced drag estimates, with an accuracy equivalent to that of the relaxed-wake model case, are obtained on the finest grid (level 1). An estimation based on a coarser grid (level 2) may be acceptable for smaller angles of attack. The involved computational effort is thus significant compared to the relaxed-wake model approach.
CHAPTER 9

Conclusion

9.1 Summary of Research

The research presented herein concerned the accurate induced drag prediction for highly non-planar lifting systems, in particular for box wing and biplane configurations. For these, commonly utilized linear potential-flow methodologies can become inaccurate, especially for systems incorporating considerable amounts of stagger. This is primarily related to the application of simplified fixed-wake model approaches, which are unable to account for the correlation of the effective height-to-span ratio factor and span efficiency (body-fixed wake) or neglect free-wake deformations such as deflection and roll-up effects (body-fixed and freestream-fixed wake).

The impact by higher-order wake and wake surrogate effects on the induced drag estimation was therefore systematically investigated, in dependency on the geometrical arrangement of the system and the system angle of attack. Comparisons were made to an Euler-flow reference to provide validation and assessment of accuracy associated with present potential-flow techniques and wake surrogate models. Computational predictions were predominately obtained by the higher-order potential-flow method FreeWake, incorporating a freestream-fixed and relaxed-wake model, and an Euler-flow method based on the commercial CFD-code STAR-CCM+. For the latter, a farfield analysis was established to avoid issues related to induced drag calculations based on surface pressure integration.
In addition, other potential-flow techniques were involved to illustrate the effect of an incorrect body-fixed wake placement (*LiftingLine* and *AVL*) and to perform evaluations towards methodologies in more common usage (*PanAir*). Based on well-documented planar lifting systems, a reference baseline was established for the higher-order potential-flow model and the Euler-flow reference. Both successfully provided induced drag estimates in excellent consistency with theoretical values and other references, permitting proof of concept, especially with regards to the implemented Euler-based farfield analysis.

Specific test configurations of limited geometric complexity were derived, which exhibit higher-order wake effects, while representing vertical and horizontal separations of practical and engineering interest. A preliminary investigation campaign for two box wing configurations of opposite stagger ascertained considerable issues with regards to induced drag predictions based on the higher-order potential-flow method. Studying trailing flow-field properties like the wake shape and its vorticity distribution, but also by establishing an alternative induced drag computations (*Trefftz plane analysis*), strong evidence was provided that encountered discrepancies are related to the existing induced drag estimation at the trailing edge. A simplification, causing incorrect projection of induced element’s trailing edge points in the body-fixed direction was identified as the source of error. The modified approach proposed instead aligns the projection with the inclination of the most upstream post-trailing edge wake DVE, enabling accurate induced drag computations for present systems using the higher-order potential-flow method. This is presumably also correct for other highly non-planar lifting system.

Succeeding detailed studies involved box wing and biplane configurations with equivalent geometrical characteristics. Higher-order wake effects, predominately induced by the system angle of attack and those related to variations of the height-to-span-ratio and the stagger factor, were concerned by individual investigations. Among others it was demonstrated, that especially for systems characterized by a substantial amount of positive stagger, higher-order wake effects gain considerable impact as the angle of attack is increased. The plausibility of potential-flow predictions was verified by evaluation of characteristic properties such as the spanwise load distribution, the assessment of computed span efficiency factors with regards to linear theory and efficiency approximations based
on effective height-to-span ratio, as well as by comparisons to the Euler-flow reference. Analyses of wake traces and vorticity distributions were conducted to gain insight into the reasons causing higher-order wake effects and to provide further substantiation. The computational expense associated with the freestream-fixed and relaxed-wake model, as well as the Euler-flow reference was evaluated for identified critical test cases.

In order to support the general validity of wake-related findings and conclusions based on the simplified box wing configuration, the impact of higher-order wake and wake surrogate effects was investigated by applying the higher-order potential-flow method to a box wing configuration proving more realistic planform properties for a commercial aircraft concept. For the considered positive-staggered system, higher-order wake effects were confirmed to have a noticeable impact and required a relaxed-wake model to enable accurate induced drag prediction based on potential-flow methodology.

9.2 Scientific Contribution

9.2.1 Research Questions

The present research has made novel contributions in the field of accurate induced drag prediction for highly non-planar lifting systems in particular for box wing and biplane configurations. In addition to contributions outlined in Section 1.3.4, the following scientific outcome is ascertained with regards to formulated research questions:

RQ-1. What impact has the employed wake model approach on accurate potential-flow induced drag prediction for highly non-planar concepts?

   a) How and to what extent is the induced drag (prediction) for highly non-planar concepts affected by higher-order wake effects?
   b) How do higher-order wake effects depend on the geometrical arrangement of the system and the system angle of attack?

It is confirmed that wake model effects can considerably affect accurate induced drag prediction for the highly non-planar configurations considered within the present research.
Estimates based on the body-fixed wake model (\textit{LiftingLine} and \textit{AVL}) are found generally inappropriate for induced drag estimation involving highly non-planar concepts. Besides supporting longitudinal forces distorting accurate farfield induced drag prediction, this wake placement does permit accounting for the correlation of the effective height-to-span ratio and the system angle of attack. It is shown that this results in erroneous induced drag projections and an entirely incorrect dependency on the angle of attack. Compared to relaxed-wake model estimates, the relative error in computed span efficiency can accumulate up to $\Delta e_{rel} \approx 10.0\%$. Opposed to that, the wake trajectories of the freestream-fixed and relaxed-wake model (\textit{FreeWake}) are aligned either with the freestream direction or the local flowfield downstream of the trailing edge. Thus, both do account for variations of the effective height-to-span ratio, induced by the system angle of attack or streamwise arrangement of lifting surfaces. Although induced drag projections can differ considerably between the freestream-fixed and relaxed-wake model, both models attain consistent results. Whereas the relaxed-wake model is found to substantially improve the accuracy of potential-based induced drag predictions for present highly non-planar lifting systems, the freestream-fixed wake model generally permits to resolve correlations between induced drag, system angle of attack and geometrical arrangement correctly and in accordance with linear theory.

Under certain conditions, higher-order wake effects are confirmed to cause span efficiency gains or losses compared to freestream-fixed wake predictions. Based on wake shape analyses, these can be attributed to the deflection and roll-up of the force-free wake, which alters the effective height-to-span ratio of the system and therewith the induced drag. Wake roll-up predominately affects the vertical separation in close proximity to the tips and additionally modifies the spanwise location of tip-vortices, which ultimately leads to differences in effective wingspan. In dependency on the geometrical arrangement of the system and the angle of attack, higher-order wake effects are found to accumulate up to approximately $\Delta e_{rel} \approx 4.0\%$ in computed span efficiency.

Higher-order wake effects are especially promoted by increasing system angles of attack. However, the sensitivity of the system depends on the amount and direction of the longitudinal separation and the inherent height-to-span ratio. Systems incorporating con-
CHAPTER 9: CONCLUSION

Considerable amounts of positive stagger are found particularly prone to higher-order wake effects. This is due to the vertical contraction of wake trajectories, leading to smaller effective height-to-span ratios and thus closer interactions between trailing wakes and lifting surfaces. For positive-staggered systems at positive angels of attack, computations based on freestream-fixed wake result in induced drag estimates in excess to relaxed-wake and Euler-flow predictions. In contrast to this, the impact by higher-order wake effects is found of much less significance for equivalent negative-staggered systems, indicating comparability good agreement between freestream-fixed and relaxed-wake based estimates. In accordance with the wake substitution concept, the impact by higher-order wake effects on the induced drag characteristics of unstaggered systems is negligible, independent of the height-to-span ratio involved. Opposed to stagger, increasing height-to-span ratios generally diminishes the sensitivity to higher-order wake effects.

RQ-2. With what accuracy can potential-based methodologies predict the induced drag of highly non-planar concepts within the subsonic flow regime compared to an Euler-flow reference?

a) How does this depend on the employed wake model approach, the geometrical arrangement of the system and the system angle of attack?

Within the scope of test conditions considered herein, potential-flow techniques are found capable to predict induced drag with an accuracy that is equivalent to Euler-flow methodology. Nevertheless, this is highly dependent on the employed wake surrogate model, the inherent geometrical arrangement and the system angle of attack. In general, the relaxed-wake model enables improved consistency with the Euler-flow reference, independent of the height-to-span ratio, the stagger factor or system angle of attack. An estimation based on freestream-fixed wake provide adequate accuracy for small positive stagger factors at low angles of attack or for negative stagger factors and large height-to-span ratios more in general. For unstaggered systems, the accuracy associated with potential-flow predictions is independent of the selected wake model approach.
RQ-3. Which computational expense is related to the relaxed-wake compared to a freestream-fixed wake approach and an Euler-flow reference and how does it apply within a design method for highly non-planar concepts?

Analyzing the computational expense, wake-relaxation is found to prolong the computation time compared to the freestream-fixed approach by approximately a factor of four to six, depending on the test case involved. However, relaxed-wake estimates of equivalent accuracy level can be attained by about two orders of magnitudes faster than for the Euler-flow reference. Accurate induced drag predictions for highly non-planar lifting systems based on potential-flow relaxed-wake shape therefore provides an excellent trade-off between the inexpensive linear potential-flow methods and the computationally intensive Euler-flow model and is perfectly suited for an application in early design stages or in a MDO effort.

9.2.2 Additional Scientific Outcome

Additional scientific outcome is based on the following:

- Study findings enable design implications for highly non-planar lifting systems, in particular for box wing and biplane configurations, to attain efficiency advantages by exploiting higher-order wake effects. In particular, for commonly favored positive longitudinal arrangements, wake deflection facilitates larger effective vertical separations at positive angles of attack, which can be utilized to attain an improved aerodynamic performance. Smaller height-to-span ratios can be employed at induced drag levels equivalent to linear predictions, alleviating some principle issues related to box wing designs, such as the increase in wetted area or wing stability or structural weight considerations. Moreover, the knowledge provided by present research permits a preselection of an appropriate computational methodology or wake surrogate model, to provide accurate induced drag prediction in dependency of the given geometrical arrangement and system angle of attack.
• The implementation and validation of the modified trailing edge analysis has extended the applicability of the higher-order potential-flow method. Although the effect is minimal for simple monoplane wings, the accuracy of induced drag estimates for more complex wing arrangements is likely to be improved. This is considered especially true for predictions based on the relaxed-wake shape and systems sensitive to higher-order wake effects. The improved design capabilities for the higher-order potential-flow model, facilitates the development of systems taking advantage of wake effects.

• For the higher-order potential-flow model, the influence of the added singularity canceling and time-step width on the relaxed-wake shape and induced drag of biplane configurations with positive and negative stagger was determined. Both parameters were shown to affect the wake shape and thus induced drag, especially for the positive-staggered system. Based on a wake-shape congruence study, a set of parameters were derived, yielding best geometrical agreement between the relaxed-wake shape and the equivalent wake based on the Euler-flow model. This did ultimately translate in an excellent agreement between relaxed-wake and Euler-flow predictions, but also provides a starting point for improvement of the higher-order potential-flow method.
Appendix

Appendix A

double Induced_DVE_Drag_pwe (const GENERAL info, const PANEL* panelPtr,
                           const DVE* surfacePtr,DVE** wakePtr,
                           const int rightnow, double* D_force)
{
    //This function is the DVE expansion of the function
    //that computes the induced drag at the trailing edge, where
    //the spanwise bound vorticity has been collapsed into a single vortex.
    //The method is discussed more thoroughly in:
    //Eppler and Schmid-Goeller, "A Method to Calculate the Influence of
    //Vortex Roll-Up on the Induced Drag of Wings," Finite Approximations
    //Kutta-Joukowski is being applied to the trailing edge at three points
    //along each trailing edge element. Similarly to the lift computation,
    //Simpson's Rule is used to compute the total drag of each element.
    //Function computes forces/density for each spanwise section of the
    //wing by applying Kutta-Joukowski theorem to either of its edges and
    //its center in order to get the local forces. The total spanwise
// section forces are determined by integrating with Simpson's rule
// with overhang.

// Note: the velocities are not computed directly at the edges, but
// (1 - delta)/2 of the elementary span further towards the center.
// Otherwise, the computed induced velocity would become singular at
// that point due to the next neighboring elementary wing influence.

// input:
// info - general information
// panelPtr - information on panels
// surfacePtr - information on surface DVEs
// wakePtr - information on wake DVEs
// rightnow - current time step

// output:
// CDi - total drag coefficient
// D_force - local drag force/density along span

int panel, i, span, time, k;

int index = 0;  // index of surface DVEs along trailing edge of wing

double A, B, C;  // vorticity distribution coefficient along t.e.
double eta, etas;  // half span of elementary wing l, 90% value of eta
double eD[3], eL[3], eS[3];  // drag, lift, side force direction
double S[3];  // trailing edge vector
double X[3][3];  // points along trailing edge element,
// left X[1], center X[0], right X[2]
double delX[3], Xstar[3];  // delta for correction, corrected X
double u[3], w_ind[3][3];  // delta and total ind. vel. at t.e. edges & center
double gamma1, gammao, gamma2;  // vorticity at trailing edge edges and center
double R1[3], Ro[3], R2[3];  // res. ind. force at trail. edge edges and center
double R[3];  // resultant ind. force/density of element i
double CDi = 0, CLi = 0, CYi = 0;  // total induced drag at trailing edge
double tempA[3], tempS;
int type;  // type of wake DVE
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DVE tempDVE;  // temporary DVE

// loop over panels
for (panel=0; panel<info.npanel; panel++)
{
    // the first index of trailing edge DVE
    index += panelPtr[panel].n * (info.m-1);
    // loop over number of trailing edge DVEs of current panel
    for (i=0; i<panelPtr[panel].n; i++)
    {
        // The trailing edge condition of the most aft surface DVE's
        // is identical to the leading edge condition of the most recent
        // wake DVE (time index "rightnow").

        // drag force direction
eD[0] = surfacePtr[index].U[0];
eD[1] = surfacePtr[index].U[1];

        // the lift direction eL = \{U x [0,1,0]\}/|U x [0,1,0]|,
tempS = 1/sqrt(surfacePtr[index].U[0]*surfacePtr[index].U[0] + surfacePtr[index].U[2]*surfacePtr[index].U[2]);
eL[0] = -surfacePtr[index].U[2]*tempS;
eL[1] = 0;
eL[2] = surfacePtr[index].U[0]*tempS;

        // the side force direction eS = U x eL / |U x eL |
cross(eL, surfacePtr[index].U, tempA);
tempS = 1/norm2(tempA);
scalar(tempA, tempS, eS);

        A = surfacePtr[index].A;
        B = surfacePtr[index].B;
        C = surfacePtr[index].C;

        // Computing the three points along the unswept trailing edge,
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// at which Kutta-Joukowsky is applied
// the left and right points are 20% of half span away from edge
eta = surfacePtr[index].eta;
eta8 = eta*.8;  //0.8 as done for lift computation,

// X1:
Edge_Point(surfacePtr[index].xo,surfacePtr[index].nu,\n  surfacePtr[index].epsilon,surfacePtr[index].psi,\n  surfacePtr[index].phiTE,eta8,surfacePtr[index].xsi,X[1]);
  //Subroutine in wake.geometry.cpp

// X2:
Edge_Point(surfacePtr[index].xo,surfacePtr[index].nu,\n  surfacePtr[index].epsilon,surfacePtr[index].psi,\n  surfacePtr[index].phiTE,eta8,surfacePtr[index].xsi,X[2]);
  //Subroutine in wake.geometry.cpp

//X0 = (X1+X2)/2
vsum(X[1],X[2],tempA);  scalar(tempA,0.5,X[0]);

//computing the normalized vector along the trailing edge
S[0] = X[2][0] - X[1][0];
S[1] = X[2][1] - X[1][1];
tempS= 0.5/eta8;

//initializing induced velocities of current span location
w_ind[1][0] = 0;  w_ind[1][1] = 0;  w_ind[1][2] = 0;
w_ind[0][0] = 0;  w_ind[0][1] = 0;  w_ind[0][2] = 0;
w_ind[2][0] = 0;  w_ind[2][1] = 0;  w_ind[2][2] = 0;

//loop over spanwise elements
//the method computes and adds the induced velocities of
//each spanwise strip in the wake
for (span=0;span<info.nospanelement;span++)
{

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// loop over the three points of trailing edge
for (k=0;k<3;k++)
{
    // point expressed with reference to the 1st post-TE wake DVE
    delX[0] = X[k][0] - wakePtr[rightnow][span].xo[0];
    delX[1] = X[k][1] - wakePtr[rightnow][span].xo[1];

    // transforming delX into local frame of wake DVE at TE
    Glob_Star(delX, wakePtr[rightnow][span].nu, \
               wakePtr[rightnow][span].epsilon, \
               wakePtr[rightnow][span].psi, tempA);
    // function in ref_frame_transform.cpp

    // moving point into plane of leading edge of unswept wake DVE
    tempA[0] = -wakePtr[rightnow][span].xsi;

    // transforming back into global reference frame
    Star_Glob(tempA, wakePtr[rightnow][span].nu, \
              wakePtr[rightnow][span].epsilon, \
              wakePtr[rightnow][span].psi, delX);
    // function in ref_frame_transform.cpp

    // point along TE in plane of unswept TE
    Xstar[0] = delX[0] + wakePtr[rightnow][span].xo[0];

    // loop across wake elements of one spanwise location other than
    // the first post-trailing edge one
    for(time=0;time<=rightnow;time++)
    {
        // assigning temporary DVE that induces on trailing edge
        // as Schmid-Goeller discusses in his dissertation,
        // it has no sweep and belongs to a spanwise strip of wake

        // loop over the three points of trailing edge
        for (k=0;k<3;k++)
        {
            // point expressed with reference to the 1st post-TE wake DVE
            delX[0] = X[k][0] - wakePtr[rightnow][span].xo[0];
            delX[1] = X[k][1] - wakePtr[rightnow][span].xo[1];

            // transforming delX into local frame of wake DVE at TE
            Glob_Star(delX, wakePtr[rightnow][span].nu, \
                       wakePtr[rightnow][span].epsilon, \
                       wakePtr[rightnow][span].psi, tempA);
            // function in ref_frame_transform.cpp

            // moving point into plane of leading edge of unswept wake DVE
            tempA[0] = -wakePtr[rightnow][span].xsi;

            // transforming back into global reference frame
            Star_Glob(tempA, wakePtr[rightnow][span].nu, \
                      wakePtr[rightnow][span].epsilon, \
                      wakePtr[rightnow][span].psi, delX);
            // function in ref_frame_transform.cpp

            // point along TE in plane of unswept TE
            Xstar[0] = delX[0] + wakePtr[rightnow][span].xo[0];

            // loop across wake elements of one spanwise location other than
            // the first post-trailing edge one
            for(time=0;time<=rightnow;time++)
            {
                // assigning temporary DVE that induces on trailing edge
                // as Schmid-Goeller discusses in his dissertation,
                // it has no sweep and belongs to a spanwise strip of wake
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// elements that starts at the point of interest

tempDVE.xo[0] = wakePtr[time][span].xo[0];
tempDVE.xo[1] = wakePtr[time][span].xo[1];

if(time==rightnow) tempDVE.phiLE = 0;
else tempDVE.phiLE = wakePtr[time][span].phiLE;

if(time==rightnow) tempDVE.phiTE = 0;
else tempDVE.phiTE = wakePtr[time][span].phiTE;

tempDVE.nu = wakePtr[time][span].nu;
tempDVE.epsilon = wakePtr[time][span].epsilon;
tempDVE.psi = wakePtr[time][span].psi;

tempDVE.eta = wakePtr[time][span].eta;
tempDVE.xsi = wakePtr[time][span].xsi;

tempDVE.A = wakePtr[time][span].A;
tempDVE.B = wakePtr[time][span].B;
tempDVE.C = wakePtr[time][span].C;

if(time==rightnow) tempDVE.singfct=0;
else tempDVE.singfct = wakePtr[time][span].singfct;

type = 1; // DVE is only a vortex sheet
if(time==0) type = 3; // oldest wake is semi-infin. vort. sheet

// computes induced velocity in X[k] due to DVE tempDVE
Single_DVE_Induced_Velocity(info,tempDVE,Xstar,w,type);

// subroutine in induced_velocity.cpp

w_ind[k][0] += w[0];
w_ind[k][1] += w[1];
w_ind[k][2] += w[2];
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208 } // end loop over time, along a strip in wake
209 } // end loop over k, three point of trailing edge
210 } // end loop over span, across complete wake’s width
211
212 // Integration of induced forces with Simpson’s Rule
213 // Integration requires overhanging edges!!
214 // See also KHH lines 2953 - 2967, A23SIM
215
216 // Kutta-Joukowski at left (1) edge
217 cross(w_ind[1], S, tempA); // w1xS
218 gamma1 = A - B*eta8 + C*eta8*eta8; // gamma1
219 scalar(tempA, gamma1, R1);
220
221 // Kutta-Joukowski at center
222 cross(w_ind[0], S, tempA); // w0xS
223 gamma0 = A;
224 scalar(tempA, gamma0, Ro);
225
226 // Kutta-Joukowski at right (2) edge
227 cross(w_ind[2], S, tempA); // w2xS
228 gamma2 = A + B*eta8 + C*eta8*eta8;
229 scalar(tempA, gamma2, R2);
230
231 // The resulting induced force of element l is
232 // determined by numerically integrating forces across element
233 // using Simpson’s Rule with overhanging parts
234 R[0] = (R1[0] + 4*Ro[0] + R2[0])*eta8/3; // Rx
237
238 // plus overhanging parts
239 R[0] += (7*R1[0] - 8*Ro[0] + 7*R2[0])*(eta - eta8)/3; // Rx
242
243 // the DRAG FORCE/density is the induce force in eD direction
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D_force[i] = dot(R, eD);

// add all partial drag/lift/side values [force/density]
CDi += D_force[i];

index++;  // index of next trailing edge DVE
// loop over number of trailing edge DVEs of current panel
}  // end loop i over trailing edge DVEs of current panel
// end loop panel over number of panels

// non-dimensionalize
tempS = 0.5*info.Uinf*info.Uinf*info.S;
CDi /= tempS;

if (info.sym==1)
{
  CDi *= 2;  // sym. geometry and flow, twice the drag
}
return CDi;

//===============================================
// END Induced_DVE_Drag computation
// Drag along trailing edge - PTWE Version
//===============================================


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