A SUMMATION FORMULA FOR SEQUENCES INVOLVING FLOOR AND CEILING FUNCTIONS

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ABSTRACT. A closed form expression for the Nth partial sum of the pth powers of $\|\sqrt{n}\|$ is obtained, where $\| \cdot \|$ denotes the nearest integer function. As a consequence, a necessary and sufficient condition for the divisibility of $n$ by $\|\sqrt{n}\|$ is derived together with a closed form expression for the least nonnegative residue of $n$ modulo $\|\sqrt{n}\|$. In addition an identity involving the zeta function $\xi(s)$ and the infinite series $\sum_{n=1}^{\infty} 1/\|\sqrt{n}\|^{s+1}$ for real $s > 1$ is also obtained.

1. Introduction. In a recent paper, see [3], the author examined the problem of determining a closed form expression for those sequences $\langle b_m \rangle$ formed from an arbitrary sequence of real numbers $\langle a_n \rangle$ in the following manner. Let $d \in \mathbb{N}$ be fixed, and for each $m \in \mathbb{N}$ define $b_m$ to be the $m$th term of the sequence consisting of $nd$ occurrences in succession of the terms $a_n$, as follows:

$$a_1, \ldots, a_1, a_2, \ldots, a_2, a_3, \ldots, a_3, \ldots.$$  \(d, a_1 \text{ terms} \) \(2d, a_2 \text{ terms} \) \(3d, a_3 \text{ terms} \)

For example, if $a_n = n$ and $d = 1$ then the resulting sequence $\langle b_m \rangle$ would be

$$1, 2, 2, 3, 3, 4, 4, 4, 4, \ldots.$$ 

Specifically, the problem described above required the construction of a function $f : \mathbb{N} \to \mathbb{N}$ such that $b_m = a_{f(m)}$. As was shown in [3] the required function $f(\cdot)$ can easily be described in terms of a combination of floor and ceiling functions, that is the functions defined as $[x] = \max\{n \in \mathbb{Z} : n \leq x\}$ and $\lceil x \rceil = \min\{n \in \mathbb{Z} : x \leq n\}$ respectively. In particular, for the sequence in (1), we have that $b_m = a_{f(m)}$ where

$$f(m) = \left\lfloor \sqrt{\frac{2m}{d}} + \frac{1}{2} \right\rfloor.$$
In this note we continue our examination of those sequences defined in (1) by deriving a summation formula for the $N$th partial sum $S_N = \sum_{m=1}^{N} b_m$. Our goal here will be to deduce, as a consequence of the aforementioned formula, a closed form expression for the partial sum of the $p$th powers of $\|\sqrt{n}\|$, where $\|x\|$ denotes the nearest integer to $x$. In particular, as special cases it will be shown that

$$\sum_{n=1}^{N} \frac{1}{\|\sqrt{n}\|} = \frac{N}{\|\sqrt{N}\|} + \|\sqrt{N}\| - 1$$

$$\sum_{n=1}^{N} \|\sqrt{n}\| = \frac{\|\sqrt{N}\|}{3} (3N + 1 - \|\sqrt{N}\|^2).$$

As will be seen, the method used to establish (3) and (4) is quite different from that employed in establishing a closed form expression for $\sum_{n=1}^{N} \lfloor\sqrt{n}\rfloor$ as demonstrated in [1, p. 86]. In addition, as a consequence of (3), a necessary and sufficient condition will be derived for the divisibility of $N$ by $\|\sqrt{N}\|$, together with a closed form expression for the least nonnegative residue of $N$ modulo $\|\sqrt{N}\|$.

2. Main result. We begin with a technical result which will help facilitate the calculation of the $N$th partial sum of the sequences defined in (1).

**Lemma 2.1.** Suppose $(a_n)$ is an arbitrary sequence of real numbers, and let $d \in \mathbb{N}$. Then, for the sequence $(b_m)$ defined in (1), we have

$$(5) \quad S_N = \sum_{m=1}^{N} b_m = \left( N - \frac{d}{2} (f(N) - 1)f(N) \right) a_{f(N)} + d \sum_{n=1}^{f(N)-1} na_n,$$

where $f(\cdot)$ is the function in (2).

**Proof.** For the sequence defined in (1), we have $b_m = a_n$, whenever $n(n-1)d/2 < m \leq n(n+1)d/2$, that is, $f(m) = n$ when

$$m \in I_n = \left[ \frac{n(n-1)}{2} \right] d + 1 \quad \left[ \frac{n(n+1)}{2} \right] d.$$

Now defining the mapping \( S : \mathbb{N} \to \mathbb{N} \) by \( S(N) = \max\{n \in \mathbb{N} : N \not\in \bigcup_{r=1}^{n} I_r\} \) and noting that each interval \( I_n \) contains \( nd \) integers, observe the following

\[
S_N = \sum_{n=1}^{S(N)} \sum_{r \in I_n} a_{f(r)} + \sum_{r \in I_{S(N)+1}} a_{f(r)}
\]

(6)

\[
= \sum_{n=1}^{S(N)} n \, da_n + \sum_{r \in I_{S(N)+1}} a_{S(N)+1}.
\]

Our task is thus reduced to determine a closed form expression for \( S(N) \) in terms of \( N \) and so evaluate the second summation in (6). Suppose \( N \in I_n \) for some \( n \in \mathbb{N} \), then by definition of \( I_n \),

\[
\frac{n(n-1)}{2} d < N \leq \frac{n(n+1)}{2} d.
\]

(7)

Now if, for some \( x \in \mathbb{R}^+ \) we have \( n_1 < x \leq n_2 \) for \( n_1, n_2 \in \mathbb{N} \), then \( n_1 + 1 \leq \lceil x \rceil \leq n_2 \). Consequently, from the inequality in (7) we have \( n(n-1) + 1 \leq \lceil 2N/d \rceil \leq n(n+1) \). However, as \( \sqrt{n(n+1)} < n + 1/2 \) and \( n - 1/2 < \sqrt{n(n-1)} + 1 \), one in turn deduces that

\[
n < \sqrt{\lceil 2N/d \rceil} + \frac{1}{2} < n + 1.
\]

Thus, we have \( n = f(N) \) and so \( S(N) = f(N) - 1 \). Finally as the number of integers \( r \in I_{S(N)+1} = I_{f(N)} \) with \( r \leq N \) is given by

\[
N - f(N)(f(N) - 1) \frac{d}{2} - 1 + 1,
\]

one sees that the second summation in (6) is equal to

\[
\sum_{r \in I_{f(N)}, r \leq N} a_{f(N)} = \left( N - \frac{d}{2} (f(N) - 1)f(N) \right) a_{f(N)}.
\]

Hence (6) yields (5) as required. \( \square \)
Before establishing the main result it should be noted that the mapping $x \mapsto \|x\|$ is strictly, by definition, multi-valued at $x = (2n + 1)/2$, where $n \in \mathbb{N}$, since $(2n + 1)/2$ lies a distance of $1/2$ units from $n$ and $n + 1$. In such cases the convention, as in [2, p. 78], is to set $\|(2n + 1)/2\| = n + 1$. However this ambiguity does not arise for the mapping $N \mapsto \|\sqrt{N}\|$, where $N \in \mathbb{N}$, as $\sqrt{N} \neq (2n + 1)/2$ for any $n \in \mathbb{N}$. We now prove our main result for summing the $p$th powers of $\|\sqrt{n}\|$, from which (3) and (4) will follow as a corollary.

**Theorem 2.1.** Suppose $p \in \mathbb{R}$, then

$$
(8) \sum_{n=1}^{N} \|\sqrt{n}\|^p = (N - (\|\sqrt{N}\|-1) \|\sqrt{N}\|) \|\sqrt{N}\|^p + 2 \sum_{n=1}^{\|\sqrt{N}\|-1} n^{p+1}.
$$

In particular, when $p \in \mathbb{N}$, then

$$
\sum_{n=1}^{N} \|\sqrt{n}\|^p = (N - (\|\sqrt{N}\|-1) \|\sqrt{N}\|) \|\sqrt{N}\|^p
$$

$$
+ \frac{2}{p+2} \sum_{k=0}^{p+2} \left( \begin{array}{c} p+2 \\ k \end{array} \right) B_k \|\sqrt{N}\|^{p+2-k},
$$

where $B_k$ denotes the $k$th Bernoulli number.

**Proof.** We first show that $\|x\| = \lfloor x + 1/2 \rfloor$ for every $x \in \mathbb{R}^+$. Indeed suppose $\|x\| = n$, taking the largest if two are equally distant. Setting $n = x + \theta$ with $-1/2 < \theta < 1/2$, observe $\lfloor x + 1/2 \rfloor = n + \lfloor -\theta + 1/2 \rfloor = n$ since $0 \leq -\theta + 1/2 < 1$. Consequently, $\|\sqrt{m}\| = \lfloor \sqrt{m} + 1/2 \rfloor$ and so from (2) we deduce that the sequence $\langle \|\sqrt{m}\|^p \rangle$ corresponds to the sequence $\langle b_m \rangle$ defined in (1), with $a_n = n^p$ and $d = 2$. Hence, in this instance, we see that (5) reduces to (8) as required. Finally if $p \in \mathbb{N}$, then the second equality follows immediately from the identity

$$
\sum_{k=0}^{m-1} k^m = \frac{1}{m+1} \sum_{k=0}^{m} \left( \begin{array}{c} m+1 \\ k \end{array} \right) B_k n^{m+1-k}.
$$

We now examine (8) in the case when $p = \pm 1$. 
Corollary 2.1.

\[ \sum_{n=1}^{N} \| \sqrt{n} \|^{p} = \begin{cases} N \| \sqrt{N} \|^{-1} + \| \sqrt{N} \| - 1 & \text{for } p = -1 \\ (\| \sqrt{N} \|/3)(3N + 1 - \| \sqrt{N} \|^{2}) & \text{for } p = 1. \end{cases} \]

Proof. Setting \( p = -1 \) in (8), observe that

\[ \sum_{n=1}^{N} \| \sqrt{n} \|^{-1} = (N - (\| \sqrt{N} \| - 1) \| \sqrt{N} \|) \| \sqrt{N} \|^{-1} + 2 \sum_{n=1}^{\| \sqrt{N} \| - 1} 1 \]

\[ = N \| \sqrt{N} \|^{-1} + \| \sqrt{N} \| - 1. \]

Similarly, setting \( p = 1 \) in (8) and recalling \( \sum_{r=1}^{n} r^{2} = n(n + 1)(2n + 1)/6 \), one arrives, after some simplification, at the second formula. \( \square \)

Using the summation formula in (3) we can deduce the following divisibility property.

**Corollary 2.2.** Suppose \( N \in \mathbb{N} \). Then \( \| \sqrt{N} \| \) divides \( N \) if and only if either \( N = \| \sqrt{N} \|^{2} \) or \( N = \| \sqrt{N} \| (\| \sqrt{N} \| + 1) \). Moreover, the least nonnegative residue of \( N \) modulo \( \| \sqrt{N} \| \) is given by

\[ N - \| \sqrt{N} \|^{2} + \frac{\| \sqrt{N} \|}{2} \left( (-1)^{\lceil (N/\| \sqrt{N} \|) - \| \sqrt{N} \| + 1 \rceil} \right. \\
\left. + 2(-1)^{1/2(\lceil (N/\| \sqrt{N} \|) - \| \sqrt{N} \| + 1 \rceil)} - 1 \right). \]

Proof. From the summation formula in (3) it is immediate that \( \| \sqrt{N} \| \) divides \( N \) if and only if \( \sum_{n=1}^{N} 1/\| \sqrt{n} \| \) is an integer. Recalling that \( \| \sqrt{m} \| = \lfloor \sqrt{m} + 1/2 \rfloor \), we deduce that the sequence \( \langle \| \sqrt{m} \|^{-1} \rangle \) corresponds to the sequence \( \langle b_{m} \rangle \) defined in (1), with \( a_{n} = 1/n \) and \( d = 2 \). Consequently, from (6) we find that

\[ \sum_{n=1}^{N} \frac{1}{\| \sqrt{n} \|} = 2 (\| \sqrt{N} \| - 1) + \sum_{r \in I_{\| \sqrt{N} \|} \cap [1, \| \sqrt{N} \|]} \frac{1}{\| \sqrt{N} \|}, \]

(9)
and so our task is reduced to determining those $N \in I_{\|\sqrt{N}\|}$ for which the summation on the righthand side of (9) is integer valued. Now since the interval $I_{\|\sqrt{N}\|}$ contains $2\|\sqrt{N}\|$ integers we see that

$$\frac{1}{\|\sqrt{N}\|} \leq \sum_{r \in I_{\|\sqrt{N}\|}} \frac{1}{r} \leq \frac{1}{\|\sqrt{N}\|}. $$

Furthermore, as the number of integers $r \in I_{\|\sqrt{N}\|}$ with $r \leq N$ is equal to $N - \|\sqrt{N}\|(\|\sqrt{N}\| - 1)$, we conclude that the summation in question assumes the integer values of 1 and 2 if and only if

$$N - \|\sqrt{N}\|(\|\sqrt{N}\| - 1) = \|\sqrt{N}\| \text{ and } 2\|\sqrt{N}\|, $$

respectively. Hence, $\|\sqrt{N}\|$ divides $N$ if and only if $N = \|\sqrt{N}\|^2$ or $N = \|\sqrt{N}\|(\|\sqrt{N}\| + 1)$.

Denote the number of integers $r \in I_{\|\sqrt{N}\|}$ with $r \leq N$ by $R(N)$. After equating (3) with (9) and solving for $N/\|\sqrt{N}\|$, observe from the argument above that the least nonnegative residue of $N$ modulo $\|\sqrt{N}\|$ is equal to $R(N)$, when $1 \leq R(N) < \|\sqrt{N}\|$ and $R(N) - \|\sqrt{N}\|$, when $\|\sqrt{N}\| \leq R(N) < 2\|\sqrt{N}\|$, while zero, when $R(N) = 2\|\sqrt{N}\|$. Thus the desired residue can be calculated from the following formula

$$(10) \quad R(N) - \sigma(N)\|\sqrt{N}\| - 2\phi(N)\|\sqrt{N}\|,$$

where

$$\sigma(N) = \begin{cases} 
0 & 1 \leq R(N) < \|\sqrt{N}\| \\
1 & \|\sqrt{N}\| \leq R(N) < 2\|\sqrt{N}\| \\
0 & R(N) = 2\|\sqrt{N}\|
\end{cases}$$

and

$$\phi(N) = \begin{cases} 
0 & 1 \leq R(N) < 2\|\sqrt{N}\| \\
1 & R(N) = 2\|\sqrt{N}\|
\end{cases}.$$

Via a simple application of the floor function, we see from inspection that the functions $\sigma(N)$ and $\phi(N)$ are given by

$$\sigma(N) = -\frac{1}{2} \left( (-1)^{\lfloor R(N)/\|\sqrt{N}\| \rfloor} - 1 \right)$$

and

$$\phi(N) = -\frac{1}{2} \left( (-1)^{\lfloor R(N)/2\|\sqrt{N}\| \rfloor} - 1 \right).$$
Finally substituting the previous expressions for \(\sigma(N)\) and \(\phi(N)\) into (10) produces, after some simplification, the desired residue formula.

\[\square\]

**Remark 2.1.** If \(N = s^2\) or \(N = s(s + 1)\) for some \(s \in \mathbb{N}\), then in either case \(s = \|\sqrt{N}\|\). Thus, the previous corollary implies that \(\|\sqrt{N}\|\) divides \(N\) if and only if \(N\) is either a square or a product of two consecutive integers.

To close, we establish a curious connection between the zeta function \(\zeta(s)\), for real \(s > 1\), and the infinite series involving terms of the form \(\|\sqrt{n}\|^{-(s+1)}\).

**Corollary 2.3.** Suppose \(s > 1\). Then

\[
\sum_{n=1}^{\infty} \frac{1}{\|\sqrt{n}\|^{s+1}} = 2\zeta(s).
\]

**Proof.** After setting \(p = -(s + 1)\) in (8) we need only show that

\[
(N - (\|\sqrt{N}\| - 1)\|\sqrt{N}\|) \|\sqrt{N}\|^{-(s+1)} = o(1)
\]
as \(N \to \infty\). Now, by definition of the floor and ceiling functions, observe that

\[
\|\sqrt{N}\| = \left\lfloor \sqrt{N} + \frac{1}{2} \right\rfloor = \left\lceil \sqrt{N} + \frac{1}{2} \right\rceil - 1 \geq \sqrt{N} + \frac{1}{2} - 1 = \sqrt{N} - \frac{1}{2}.
\]

Consequently, \((\|\sqrt{N}\| - 1)\|\sqrt{N}\| \geq (\sqrt{N} - 3/2)(\sqrt{N} - 1/2) = N - 2\sqrt{N} + 3/4\), and so \(N - (\|\sqrt{N}\| - 1)\|\sqrt{N}\| \leq 2\sqrt{N} - 3/4\). Thus,

\[
0 < (N - (\|\sqrt{N}\| - 1)\|\sqrt{N}\|) \|\sqrt{N}\|^{-(s+1)} \leq \frac{2\sqrt{N} - (3/4)}{\sqrt{N} - (1/2)^{s+1}} \to 0,
\]
as \(N \to \infty\) since \(s > 1\).

\[\square\]
REFERENCES


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