ABSTRACT
In this paper we present a simple and intuitive way to analyze different aspects of digital Delta and Sigma-Delta modulators. The analysis is based on studying the spectrum of the signals at different stages of the modulator. Linear FM (LFM) signals are used as test signals for different configurations. The evaluation of systems is simplified to "spectrum matching" between the input signal sequence and the output sequence of the quantizer. "Noise shaping" is also analyzed for the verification of the linear model assumption.

1. INTRODUCTION
Sigma-Delta (ΣΔ) modulation based analog-to-digital (A/D) conversion technology is a cost effective alternative for high-resolution applications. Oversampling eases analog filter design, and also generates a spectrum with quantization noise pushed towards higher frequencies which are inaudible in audio applications. Over the last few years ΣΔ analogue-to-digital converters (ADCs) and digital-to-analogue converters (DACs) have become widely available, particularly for low-frequency applications such as high-fidelity audio and speech processing, metering applications, and voiceband data telecommunications [1] [2] [3].

Digital ΣΔ systems are easy for implementation and analysis. Analysis of ΣΔ modulation in the z-domain conventionally involves the assumption of a linear model in which the quantizer is modelled as Additive White Gaussian Noise (AWGN) source. In some circumstances this white noise assumption is not valid [4]. The input signal applied normally is a sinusoid or a DC. We present here an analysis by applying a Linear FM (LFM) signal as the input signal. The LFM possesses all possible frequencies with the same magnitude in the bandwidth of interest. All signals used in our analysis will be discrete sequences.

The oversampling process in ΣΔ modulators improves the resolution of a Nyquist rate data converter. This improvement is achieved by sampling the input signal at a significantly faster rate than the Nyquist rate. The ratio between the sampling rate and two times the signal bandwidth is defined as the oversampling ratio (OSR) [4]

\[
\text{OSR} = \frac{f_s}{2f_b}
\]  

where \(f_s\) is the sampling rate and \(f_b\) is the signal bandwidth. Apparently, OSR = 1 means sampling at the Nyquist rate. A system with OSR = 1 is generally called a Nyquist-rate system, while a system with OSR >> 1 is called an oversampling system. In this analysis of ΣΔ systems, the bandwidth of the LFM applied signal is approximately 2 Hz and the sampling rate is 100 Hz, hence OSR = 25.

A discrete-time LFM is given by [5]

\[
x[n] = \cos(\omega_0n^2)
\]

whose instantaneous frequency (IF) is \(2\omega_0n\).
The \(z\)-transform of a sequence \(x[n]\) is defined as [5]

\[
X(z) = \mathcal{Z}(x[n]) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}.
\]

The \(z\)-transform evaluated on the unit circle corresponds to the discrete Fourier transform (DFT). As for practical digital Δ or ΣΔ systems, all signals are finite-duration sequences, it is convenient to find their spectra by DFT. We plot the signals in the continuous-time domain for demonstration purposes.

2. SPECTRAL ANALYSIS OF THE DELTA MODULATOR

![Fig. 1. First-order digital Δ modulator and demodulator.](image)

Consider the 1-bit digital Δ modulator (encoder) and demodulator (decoder) shown in Fig. 1. A sinusoid of normalized frequency is applied to the system as a test signal. Fig. 2 shows the signals in the time domain and their corresponding spectra evaluated at the upper half of the unit circle by \(z\)-transform for the configuration shown in Fig. 1. Using this oversampled ΣΔ technique, the sinusoid is represented as a

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**SPECTRAL ANALYSIS OF DELTA AND SIGMA-DELTA MODULATORS USING LINEAR FM INPUT SIGNALS**

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Fig. 2. Quantizing a sinusoid by the \( \Delta \) modulator: \( x[n] \) is the input signal, \( \hat{x}[n] \) is its estimate, and \( y[n] \) is the output single-bit stream. The corresponding spectra are also shown.

Fig. 3. Quantizing an LFM signal by the \( \Delta \) modulator: \( x[n] \) is the input signal, \( \hat{x}[n] \) is its estimate, and \( y[n] \) is the output single-bit stream. The corresponding spectra are also shown.

Fig. 4. First-order digital adaptive \( \Delta \) modulator and demodulator.

to the spectrum of \( x[n] \) as compared to non-adaptive \( \Delta \) system over the baseband, but the distortion is large, thus an integrator is still necessary at the decoder.

The above analysis shows that the \( \Delta \) modulator is sensitive to the rate of change of the signal.

3. SPECTRAL ANALYSIS OF THE \( \Sigma \Delta \) MODULATOR

\( \Delta \) modulation requires two integrators for the modulation and demodulation processes as shown in Fig. 1. Since integration is a linear operation, the second integrator can be moved before the modulator and becomes a preprocessing integrator. This will then be the configuration of \( \Sigma \Delta \) modulator with two integrators. Again, based on the linear property of integration, the two integrators in Fig. 1 can be combined into a single integrator. A general configuration of a 1-bit \( \Sigma \Delta \) modulator is shown in Fig. 6.

We again implement a digital version of Fig. 6 and show that a \( \Sigma \Delta \) modulation system is a better solution for spectral matching. \( \Sigma \Delta \) modulators encode the integral of the signal and thus their performance is insensitive to the rate of change of the signal.

Fig. 7 shows that the spectrum of the LFM \( x[n] \) is
Fig. 5. Quantizing an LFM by adaptive Δ modulator.

Fig. 6. First-order digital 1-bit ΣΔ modulator and demodulator.

squashed to lower frequencies by the preprocessing integrator and thus we have a situation similar to that in Fig. 1 and Fig. 2. The spectrum of the encoder output \( y[n] \) matches the spectrum of the input of the encoder \( x[n] \) over the bandwidth of interest, and hence only a LPF is needed in the decoder. The ΣΔ modulator is less sensitive to the quantization step and the sampling rate.

It should be also interesting to investigate the spectra of the ΣΔ modulator with one integrator as in Fig. 6. As shown in Fig. 8, there is no "spectrum squash" observed and the bandwidth of the output of the integrator is almost the same as that of the input signal \( x[n] \). As in the case of two-integrator ΣΔ, the spectrum of the input signal \( x[n] \) matches that of the quantizer output \( y[n] \) inside the bandwidth of interest, thus the decoder only needs a LPF to recover the input signal.

4. NOISE SHAPING OF THE ΣΔ MODULATOR

The noise shaping ability of an oversampling ΣΔ modulator allows the input signal of interest (baseband) to pass essentially unfiltered through the modulator but high-pass filters the quantization noise. For the ease of analysis of the important characteristics such as noise shaping, the nonlinearity of the quantizer in the ΣΔ modulator is approximated by an analytical linear model. There are arguments about the validity of the white noise linear model [4]. We investigate under what condition the linear model is valid. In the literature, Fig. 6 is approximated by a linear model to make the analysis tractable, i.e., the quantizer is linearized by using an input-independent additive white noise model, and the modulator output is given by [4] [6]:

\[
Y(z) = X(z)z^{-1} + E(z)(1 - z^{-1})
\]  

(5)

where \( X(z), Y(z), \) and \( E(z) \) are the \( z \)-transforms of the input, the output, and the quantization error, respectively. If we let \( H_x = z^{-1} \) and \( H_e(z) = (1 - z^{-1}), \) the output is just a delayed version of the signal plus quantization noise that has been shaped by a first-order \( z \)-domain differentiator or a high-pass filter. This process is known as the "noise shaping". The corresponding time-domain version of the modulator output is

\[
y[n] = x[n - 1] + e[n] - e[n - 1]
\]  

(6)

where the \( e[n] - e[n - 1] \) term is the first-order difference of \( e[n] \).

The transfer function \( H_e(z) = (1 - z^{-1}) \) is also called noise transfer function (NTF) \( N(z) = (1 - z^{-1}), \) and the magnitude of the NTF can be found out by letting \( z = e^{j2\pi f/f_s} \) and we have

\[
|N(f)| = 2\sin(\pi f/f_s).
\]  

(7)

From Eq. 5 and Eq. 6, we obtain \( \mathcal{Z}(e[n] - e[n - 1]) = \mathcal{Z}(y[n] - x[n - 1]) = E(z)(1 - z^{-1}) \). This provides a way to evaluate the spectrum of \( E(z)(1 - z^{-1}) \) from the difference of the input \( x[n - 1] \) and the output \( y[n] \). Again, we use the LFM as the input signal \( x[n] \). Fig. 9 shows a comparison of the magnitude spectra for \( N(f) \) and simulated power spectral density (PSD) of \( E(z)(1 - z^{-1}) \) for the ΣΔ modulator shown in Fig.
6. Apparently, the match between the two curves in shape means $E(z) = \text{constant}$, and hence $e[n]$ is a white noise process. In our simulation with an LFM input, increasing the duration of the LFM means a better match of the two curves (up to a scaling factor). The PSD of $(y[n] - x[n-1])$ in Fig. 9 is obtained with a sample size of $2^{12}$ and OSR = 25. The noise shaping is observed as the noise over the band of interest is significantly attenuated and is high-pass filtered outside the band of interest. In other words, the noise introduced by the quantization is pushed to higher frequencies which can be easily filtered out by a low pass filter (LPF) in DAC. Simulation also shows that with a higher OSR, the noise over the band of interest will be further attenuated. We conclude that at least in this case the linear model with the quantization error modelled as a white noise source is a valid assumption.

5. CONCLUSION

A linear FM (LFM) signal is applied as a test signal at the input of digital $\Sigma$ and $\Sigma\Delta$ modulators to reveal the relationship of the spectra at different stages. By squashing the spectrum of the signal using the $\Sigma\Delta$ modulator, a simpler decoder and a better performance is achieved for the $\Sigma\Delta$ modulator. We see that for the same parameters such as the sampling frequency and quantization step, the $\Sigma\Delta$ modulator outperforms the adaptive $\Delta$ modulator.

The noise shaping ability and the validity of the linear model of the $\Sigma\Delta$ modulator are also investigated. Similar principle may be applied to other configurations of the $\Sigma\Delta$ modulator in future works.

6. REFERENCES