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Quantum and classical chaos in kicked coupled Jaynes-Cummings cavities

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We consider two Jaynes-Cummings cavities coupled periodically with a photon hopping term. The semiclassical phase space is chaotic, with regions of stability over some ranges of the parameters. The quantum case exhibits dynamic localization and dynamic tunneling between classically forbidden regions. We explore the correspondence between the classical and quantum phase space and propose an implementation in a circuit QED system.

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I. INTRODUCTION

The Jaynes-Cummings (JC) Hamiltonian is the canonical model for atom-light interactions, describing a confined bosonic mode interacting with a two-level system (qubit). This is sufficient to describe a wide range of phenomena in cavity quantum electrodynamics (QED). Systems of coupled JC cavities, the Jaynes-Cummings-Hubbard (JCH) systems, have been suggested for a diverse range of optical applications such as an optical analog for the Josephson junction [1] and Q-switching [2]. Networks of JC systems have also been predicted to exhibit phase transitions [3–5].

Improvements in the realization of photonic cavities in the laboratory have made possible exploration of Jaynes-Cummings systems [6–8] in the strong-coupling regime in a variety of platforms. A current implementation of interest is in circuit QED, where a superconducting optical resonator is capacitively coupled to a Cooper-pair box. This is equivalent to a single cavity mode of the electromagnetic field coupling to a two-level atom. The advantage of circuit QED is that coherence times and atom-field coupling much greater than that can be achieved with visible and near-infrared systems. This makes circuit QED a potential medium for quantum computing, and it already has been used to implement a two-qubit Shor algorithm [9].

The original proposals for observing quantum phase transitions in JCH systems [3–5] called for large numbers of identical systems. Constructing large arrays of cavities which are sufficiently coherent and identical poses a significant challenge. Exploiting long coherence times can allow some analogous effects to be studied by trading large-scale phenomena for small-scale, long-time phenomena. For example, there is an isomorphism between the periodically kicked rotor and the Anderson tight binding model [10]. The Anderson model predicts localization for particles in a disordered lattice, and for dimension greater than three it exhibits a second-order phase transition between metallic and superfluid phases. This has been recently demonstrated in the time domain as a kicked system with cold atoms [11].

We examine the dynamics of a pair of periodically coupled kicked JC systems using both quantum and semiclassical treatments. For two kicked coupled JC systems the semiclassical dynamics are nonintegrable with a complicated phase space composed of regular and chaotic regions. The quantum case exhibits similar structure, which converges to the classical case as the number of excitations in the system increases.

II. MODEL

The JC Hamiltonian, in the rotating wave approximation, is

\[ H_{JC} = \Delta \sigma^+ \sigma + \beta (\sigma^+ a + a \sigma^-), \]  

with \( \sigma \) (\( a \)) the atomic (bosonic) annihilation operator, \( \Delta \) the atom-photon detuning, and coupling energy \( \beta \), and where we set \( \hbar = 1 \). \( H_{JC} \) commutes with the total excitation number operator, \( L = a^d a + \sigma^+ \sigma \) [17]. Therefore the total excitations in the cavity, \( l \), is a good quantum number.

In the bare basis, the eigenstates are

\[ |+, l\rangle = \sin \theta_l |g, l\rangle + \cos \theta_l |e, l - 1\rangle, \]  
\[ |-, l\rangle = \cos \theta_l |g, l\rangle - \sin \theta_l |e, l - 1\rangle, \]  

where \( \tan \theta_l = 2 \beta \sqrt{l} / (\Delta + 2 \chi) \),

\[ H_{JC} |\pm l\rangle = (\pm \chi(l) - \Delta / 2) |\pm l\rangle, \]  

and

\[ \chi(l) = \sqrt{\beta^2 l + \Delta^2 / 4} \]

is the generalized Rabi frequency. Note the \( \sqrt{l} \) dependence in interaction strength. The anharmonic energy spectrum is the source of much interesting behavior: In JC cavities it leads...
the period between kicks and will be shown in the following. Incommensurate energies result in dynamic localization, as photon nonlinearity. In the system under consideration, the dimension of the expectation value of the Heisenberg equations of motion is a periodic delta function with period $T$, and so we do not dwell on this case. For all $\tau \gg 1/\omega_0$, the rotating wave approximation is valid.

The three dimensionless parameters, $\kappa$, $\tau$, $T/\beta$, and $\Delta \beta$, are sufficient to specify the dynamics of $H$. For simplicity we consider only the quasiresonant case, $\Delta \sim 0$, where the key features of the system are most easily elucidated. This makes $\sin \theta_t = \cos \theta_t = 1/\sqrt{2}$ in Eq. (2).

The coupling term breaks the individual excitation conservation of each JC system, but it commutes with the total $L = L_1 + L_2$, and thus we can consider cases of total excitation number individually. For a single excitation, $L = 1$, the excitation oscillates between cavities trivially, with frequency $\kappa \tau$, and so we do not dwell on this case. For all $L > 1$ we find rich behavior with signatures of quantum chaos. However, here we confine ourselves to $L = 2$ in the quantum case and the semiclassical equivalent. Although the dimension of Hilbert space is just eight, many of the features of quantum chaos are already present, and it is this case which will be most accessible experimentally.

A. Semiclassical dynamics

We derive the classical equations of motion by taking the expectation value of the Heisenberg equations of motion (see, for example, [21]). Between kicks each system evolves separately as

$$\langle \dot{a} \rangle = \dot{E} = -i \beta \dot{S},$$

$$\langle \dot{\sigma} \rangle = \dot{S} = i \Delta \dot{S} + i \beta E \dot{S} z,$$  \( \langle \dot{\sigma}_z \rangle = S^z = 2 \beta i (S^+ S^- - S^z E), \)

where $E$, the electric field, and $S$, vectors on the Bloch sphere, are now classical quantities. For no detuning the uncoupled equations of motion are equivalent to that of a pendulum with momentum $E$ and $S_z = \cos \theta$, the height of the bob. This motion has two constants of motion,

$$N_i = |E_{i,2}|^2 + \frac{1}{2} (S_{z1} + 1),$$

$$S_z^2 + 4 S^+ S^- = 1.$$  \( \text{While this has an analytical solution in terms of elliptical functions, in practice it is easier to numerically integrate.} \)

The kick is given by the map

$$\begin{pmatrix} E_1 \\ E_2 \end{pmatrix}_{n+1} = \begin{pmatrix} \cos \kappa' & \sin \kappa' \\ -\sin \kappa' & \cos \kappa' \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}_n.$$  \( \text{The kicked hopping leads to nonintegrable dynamics, so that the only constant of motion is now $N_1 + N_2 = N$. In general this results in a chaotic phase space; however, for some values of $\kappa$ and $T$ there will be regions in which the motion is semiregular. These regions are described by KAM (Kolmogorov-Arnold-Moser) theory [22]. In an unperturbed system the path in the $d$-dimensional phase space in action-angle variables lies on the surface of a $d$-torus. If the periods in each dimension are sufficiently incommensurate then the system is confined near a deformed torus for small perturbations. The system becomes increasingly chaotic as the perturbation is turned up, leading to destruction of some tori. The phase space is then a chaotic sea with islands of stability which are topologically separate, from the chaos as well as each other. Eventually the perturbation destroys all these regions and the dynamics becomes fully chaotic.} \)

The centers of stability that survive the longest are usually found around short periodic orbits. In this kicked system, however, there are in general no single-period orbits, making the motion difficult to determine the precise point at which the phase space becomes fully chaotic. However, numerical simulations for the $N = 2$ case indicate that for small $\kappa \tau$ the most persistent KAM tori are around $N_{1,2} = \sqrt{2} \sin (\kappa \tau)^2, N_{2,1} = \sqrt{2} \cos (\kappa \tau)^2$ (Fig. 2(b)]. That is, in these four regions of phase space the energy in the system remains localized to a single cavity. As each period $\kappa \tau$ is increased these regions become leaky (cantori) and eventually disappear, after which the phase space is fully chaotic.

The value of $\kappa \tau$ at which the system becomes chaotic is dependent on $T$. The period for a small electric field in a cavity is $2 \pi \tau$; when $\beta \tau$ is resonant with this the KAM tori are destroyed with much smaller $\kappa \tau$. Unlike other kicked systems, this system is still regular for some $\kappa \tau$ at the resonances due to the nonlinear nature of the perturbation that each cavity sees. The range of parameters in which this mode occurs is shown in Fig. 3(a), where the destabilizing effect of the resonances can be seen around $\beta \tau = 2 \pi \tau$. We can also consider the
The dynamics of a kicked system can be studied through the eigenstates \( \psi_i \) of \( U \). On application of \( U \) the Floquet states pick up eigenphase \( e^{i\lambda_i} \). Thus the problem is equivalent to a time-invariant Hamiltonian. This allows the calculation of the long-term behavior of the system.

The quantum equivalent of KAM tori can be understood as dynamic localization [23]: States which are initially in the localized regions have exponentially suppressed diffusion into chaotic areas of phase space.

If some state \( \psi \) is well represented by a small number of basis states, \( \psi_i \), we may consider \( \psi \) to be localized to some degree. This can be quantified with the participation number \( P(\psi) \) [24]:

\[
P(\psi) = \left( \sum_i |\langle \psi_i | \psi \rangle|^2 \right)^{-1},
\]

which we have normalized by the total dimension \( d \) of the space. \( P \) is 1/d when \( |\langle \psi_i | \psi \rangle| = 1 \) for some \( i \) and 1 when \( \psi \) projects evenly onto the \( \psi_i \). One can consider this to be a indication of quantum ergodicity [25].

While \( P \) is dependent on the choice of basis (i.e., we can always choose some basis with \( \psi \) as a basis), comparing the eigenstates of the unperturbed Hamiltonian to the perturbed best represents the degree of mixing [26]. We therefore take the \( \kappa = 0 \) eigenstates as the basis, and increasing \( \kappa \) leads to Floquet states with increasing \( P \).

Figure 3(b) shows the average participation number of the Floquet states over a range of \( \kappa \) and \( \beta T \) for a system with two excitations. We denote the subspace of states with two excitations in the same cavity as \( \psi_2 \), and likewise the states with one excitation in each cavity as \( \psi_1 \). The regions where \( P \) is small correspond to states with both excitations in the same cavity being dynamically separated from states with excitations in both cavities (i.e., an approximate symmetry of \( U_f \)).

The suppression is destroyed by resonances which occur at

\[
T = \frac{n\pi}{\kappa}, \quad n = 1, 2, ..., \left\lfloor \frac{\beta T}{\pi} \right\rfloor.
\]

At these values the phase accrued after each period is 0, and so there is no destructive interference. This implies that it is indeed dynamical localization suppressing dispersion in the system. For example, when \( T = \frac{\pi}{\kappa} \), the states in \( \psi^2 \) pick up no relative phase to states with \( E = 0 \). This removes the interference suppressing transmission into these states and destroys the localization.

In Fig. 3(b) we can see, for the atomic limit, that the dependence of localization on the parameters correspond qualitatively to the semiclassical case, though with important differences. The frequency at which the classical cavities oscillate depends continuously on the energy in the cavity, and in general it is different from the Rabi frequency of the quantum case; these two only coincide in the limit \( l \to \infty \). Thus, the locations of resonances are different in the two regimes.

Note also that, in contrast to the classical case, the resonance removes the localization for arbitrarily small \( \kappa \). Resonances
in the classical case are not sharp, due to the energy-dependent frequencies.

For time-independent systems, chaos can be studied via the statistics of energy levels; however, in periodic systems, the eigenphases of the unitary operator are not observable. However, the observables of a chaotic system are ergodic. That is, the mean of some experientially observable quantities over an ensemble of random states is identical to the mean of $\hat{\sigma}$. However, the observables of a chaotic system are ergodic. The statistics of energy levels; however, in periodic systems, the eigenphases of the unitary operator are not observable.

The local transition occurs around $\kappa \tau \approx 0.1$ and for the delta-function-kicked approximation to be valid we need the pulse time $\tau \ll 1/\beta$. For the coupling strengths cited here, this requires a pulse time of $\tau \approx 10^{-10}$ s and, therefore, $\kappa$ of order 1 GHz. Between pulses $\kappa$ must be of the same order as the decoherence rate (i.e., $\sim 1$ MHz) such that the dispersion due to the constant intercavity coupling is small over the time of the experiment. Thus a sequence of $\sim 100$ kicks could be applied within the coherence time. We have seen that this is long enough to observe dynamic tunneling and localization or delocalization by including the decoherence and dephasing explicitly in the simulation.

The tunable hopping term could be achieved using an intermediate qubit coupling such as in [16,20]. In such schemes the effective coupling is of order

$$\kappa_{\text{eff}} \sim \beta_{13} \beta_{23}/\Delta_3,$$

where $\beta_{13}$, $\beta_{23}$, and $\Delta_3$ are the coupling strengths of each resonator to the intermediate qubit and its detuning, respectively, and $\Delta_3 \gg \beta$. This requires the coupling to the intermediate qubit to be significantly greater than the other couplings. The detuning can be controlled in situ, allowing the coupling to be switched on and off.

Spectroscopic measurements can be used to determine the final state [8]. Although there will be significant interaction with the environment, the only final states of interest are those that still have two excitations. One can therefore largely remove the effects of atomic relaxation and photon dissipation with a postselection scheme, given a temperature smaller than the characteristic energies of the system. Dephasing terms will still be relevant; however, these are generally ignorable over the time frames considered [6].
IV. CONCLUSIONS

The phenomena discussed have been observed in other systems, such as dynamic tunneling and localization in cold atoms [27,28]. Circuit QED allows direct control over many system parameters and direct measurement of the state of the system. This can be used, for example, to study the effect of noise by controlling the detuning parameter in situ.

As circuit QED is proving to be an important field, with a wide range of possible applications, understanding chaotic behavior in these systems will be crucial. An experimental realization of the system seems quite possible, although it is not without challenges, specifically in achieving a sufficiently large intercavity coupling. It would allow the study of the rich behavior that can be expected in coupled Jaynes-Cummings systems and open up new regimes for investigating quantum chaos.

We have presented a simple model which exhibits a transition from localization to ergodicity and dynamic tunneling. Importantly, we see this behavior even for small Hilbert space dimension, and, although interesting behavior can be seen for any number of excitations above two, the lowest case most clearly conveys the aspects we have emphasized. Furthermore, the two-excitation case will most likely be the easiest to implement experimentally. Constantly improving control in circuit QED systems means that it will be possible to study the higher dimensional cases. This could potentially allow a novel means for probing the transition between classical and quantum chaos.

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