THEORY, DEVELOPMENT AND TESTING OF A NOVEL SIX DEGREES OF FREEDOM SENSOR USING OPTICAL INTERFEROMETRY

A thesis submitted in fulfilment of the requirements for the degree of Doctor of Philosophy

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Declaration

I certify that except where due acknowledgement has been made, the work is that of the author alone; the work has not been submitted previously, in whole or in part, to qualify for any other academic award; the content of the thesis/project is the result of work which has been carried out since the official commencement date of the approved research program; any editorial work, paid or unpaid, carried out by a third party is acknowledged; and, ethics procedures and guidelines have been followed.

Robert Masterton Smith

13 June 2015
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CONTENTS

1 INTRODUCTION ...................................................................................................................... 1

1.1 BACKGROUND AND OBJECTIVES ..................................................................................... 2

1.2 RESEARCH QUESTIONS ......................................................................................................... 3

1.3 OVERVIEW OF METHODOLOGY .......................................................................................... 4

   1.3.1 Theoretical analysis of radiant flux and irradiance on a rectangular photodetector
due to interferometer fringing ................................................................................................. 4

   1.3.2 Utilising image sensors to capture mirror tilt angle and tilt axis angle ......................... 4

   1.3.3 Deriving cube mirror position vectors and overcoming translation errors .................. 4

   1.3.4 Design of mirror tilt and tilt axis measurement system ............................................. 5

   1.3.5 Experiment Rig ............................................................................................................... 5

   1.3.6 Experimentation .............................................................................................................. 6

1.4 OVERVIEW OF RESULTS FOUND AND DESCRIBED IN EACH CHAPTER ......................... 6

   1.4.1 Overview of the mathematical analysis ............................................................................ 6

   1.4.2 Determining the position vectors of the cube mirror .................................................... 6

   1.4.3 Mirror tilt and tilt axis calibration system ....................................................................... 6

   1.4.4 Experiment rig testing ...................................................................................................... 7

   1.4.5 Experimentation data capture and analysis .................................................................... 7

1.5 OVERVIEW OF DISCUSSION ............................................................................................... 7

1.6 OVERVIEW OF CONCLUSION ............................................................................................. 7

1.7 FOR FURTHER STUDY - OPTIMISING THE RESOLUTION OF THE INTERFEROMETER OUTPUT ... 7

2 LITERATURE REVIEW .............................................................................................................. 9

2.1 INTRODUCTION .................................................................................................................... 9

2.2 THEORETICAL ANALYSIS OF INTERFEROMETER WAVE FRONT TILT AND FRINGE RADIANT FLUX ON A RECTANGULAR PHOTODETECTOR ................................................................. 9

2.3 6 DOF DISPLACEMENT SENSOR ......................................................................................... 11

3 THEORETICAL ANALYSIS OF FRINGE LINE RADIANT FLUX ON A RECTANGULAR PHOTODETECTOR ................................................................................................................................. 15

3.1 INTRODUCTION .................................................................................................................... 15

3.2 METHODOLOGY .................................................................................................................... 16
Preface

3.2.1 Mathematical analysis.............................................................................................................. 16
3.3 RESULTS ....................................................................................................................................... 38
  3.3.1 Influence of \( \theta \) on the radiant flux \( \Phi_c \) ........................................................................... 38
  3.3.2 Influence of \( x \) and \( y \) on the fringe count........................................................................ 45
  3.3.3 Fringe count speed............................................................................................................... 47
  3.3.4 Fringe transition speed....................................................................................................... 50
  3.3.5 Radiant flux vs. sinusoidal wave front angle and translation.............................................. 51
3.4 SUMMARY..................................................................................................................................... 51

4 UTILISING IMAGE SENSORS TO CAPTURE MIRROR TILT ANGLE AND TILT AXIS ANGLE......... 54
  4.1 INTRODUCTION ...................................................................................................................... 54
  4.2 BASIC OPERATION OF THE MICHELSON INTERFEROMETER ............................................... 55
  4.3 DETERMINING FRINGE LINE ORIENTATION AND SPACING ............................................... 55
  4.4 SUMMARY................................................................................................................................ 56

5 DERIVING CUBE MIRROR POSITION VECTORS................................................................. 57
  5.1 INTRODUCTION ...................................................................................................................... 57
  5.2 METHODOLOGY .................................................................................................................... 57
    5.2.1 Arranging 3 Michelson Interferometers orthogonally to derive the components of the position vectors of a cube mirror .................................................................................. 57
    5.2.2 Mathematical analysis for deriving the cube mirror position vectors for tilt axis angle and tilt angle .............................................................................................................. 59
    5.2.3 Deriving translation direction using only image sensors.................................................. 69
    5.2.4 Eliminating translation error when using a cube mirror ................................................... 71
    5.2.5 Overcoming position vector indeterminates ..................................................................... 74
  5.3 RESULTS ..................................................................................................................................... 76
    5.3.1 Method 1........................................................................................................................... 81
    5.3.2 Method 2........................................................................................................................... 86
  5.4 SUMMARY................................................................................................................................ 89

6 DESIGN OF MIRROR TILT AND TILT AXIS MEASUREMENT SYSTEM .................. 91
  6.1 INTRODUCTION ...................................................................................................................... 91
7 EXPERIMENT RIG

7.1 INTRODUCTION ............................................................................................................. 112

7.2 METHODOLOGY ........................................................................................................... 113

7.2.1 XYZ Translation Stage ............................................................................................. 113

7.2.2 Tilt/Rotation Stage .................................................................................................. 113

7.2.3 Base ......................................................................................................................... 114

7.2.4 Adapter Plate ........................................................................................................... 115

7.2.5 Tilt/Rotation Stage Bracket 1 .................................................................................. 115

7.2.6 Uprights ................................................................................................................ 116

7.2.7 Optical Lever Fixed Component Mounting Bracket ................................................. 116

7.2.8 Tilt/Rotation Stage Bracket 2 ................................................................................ 118

7.2.9 Interferometer Fixed Component Mounting Bracket .............................................. 119

7.2.10 Z-Axis Interferometer Webcam Bracket and Webcams ........................................... 120

7.2.11 IFCM Bracket Support ........................................................................................ 121

7.2.12 Aligning the Optical Levers .................................................................................. 122

7.2.13 Displacement sensor interferometers .................................................................... 124

7.2.14 Experiment set up ................................................................................................ 128

7.3 RESULTS .................................................................................................................... 130

7.3.1 Aligning the interferometers with the cube mirror .................................................. 130

7.4 SUMMARY .................................................................................................................. 134

8 EXPERIMENTATION ........................................................................................................ 136

8.1 INTRODUCTION .......................................................................................................... 136
Preface

8.2 METHODOLOGY ........................................................................................................................................137

8.2.1 Procedure .............................................................................................................................................138

8.3 RESULTS ..................................................................................................................................................148

8.3.1 Standard linear regression ...............................................................................................................148

8.3.2 Residual standard deviation .............................................................................................................149

8.4 SUMMARY ...............................................................................................................................................156

9 DISCUSSION .............................................................................................................................................157

10 CONCLUSION ...........................................................................................................................................160

11 FOR FURTHER STUDY ............................................................................................................................163

11.1 Optimising the resolution of the interferometer output ..............................................................163

11.1.1 Radiant flux normalisation using an apodising filter .................................................................163

11.1.2 Radiant flux normalisation by pixel output normalisation .........................................................164

11.1.3 Optimising the required number of pixel rows and columns .....................................................165

12 REFERENCES ............................................................................................................................................168

12.1 OTHER REFERENCES ............................................................................................................................171
LIST OF TABLES

TABLE 1: Position vectors of a model cube mirror from initial position to final position via 5 steps of yaw, roll and pitch, and used for subsequent analysis. ........................................ 80

TABLE 2: Method 1 components for XY, XZ, YX and YZ due to ±J expression. .............................. 81

TABLE 3: Position vectors X, Y and Z derived from model cube mirror tilt angle and tilt axis angles. ........................................................................................................................... 82

TABLE 4: Components of position vectors X and Y creating 16 permutations. Permutation 7 being the only one that is orthogonal and of unit length. .............................................. 83

TABLE 5: Permutation filter calculations. ................................................................. 84

TABLE 6: Eight possible solutions to the cube mirror position vectors. ................................. 87

TABLE 7: The dot product of permutations of vectors X, Y and Z, their common logarithm and the product of all their values................................................................. 88

TABLE 8: Possible X, Y and Z cube mirror position vectors with the Y interferometer slightly misaligned ..................................................................................................................... 139

TABLE 9: Cube mirror components when in perfect alignment. .............................................. 140

TABLE 10: Transition matrix to convert from misaligned reference frame to Cartesian coordinate system. .......................................................... 140

TABLE 11: Components of the steering mirror normal vectors. ............................................. 141

TABLE 12: Components of the laser beam vectors. .............................................................. 142

TABLE 13A): Test 4, Steps 1 – 4, determining the magnitude of Y interferometer misalignment. ................................................................................................................................. 144

TABLE 14: Cube mirror position vectors from step 6 of each test. ........................................... 151

TABLE 15: Optical lever position vectors from step 12 of each test......................................... 152

TABLE 16: Linear regression, r-squared and residual standard deviation of cube mirror and optical lever components (Euler angles pitch and yaw). ......................................................... 153

TABLE 17A): Zero position test, Y interferometer slightly misaligned, Steps 1 to 4.............. 173

TABLE 18A): Test 1, Steps 1 – 4, determining the magnitude of Y interferometer misalignment. ................................................................................................................................. 177

TABLE 19A): Test 2, Steps 1 – 4, determining the magnitude of Y interferometer misalignment. ................................................................................................................................. 181

TABLE 20A): Test 3, Steps 1 – 4, determining the magnitude of Y interferometer misalignment. ................................................................................................................................. 185

TABLE 21A): Test 5, Steps 1 – 4, determining the magnitude of Y interferometer misalignment. ................................................................................................................................. 189
Table 22A): Test 6, Steps 1 – 4, Determining the magnitude of Y interferometer misalignment. Page 193

Table 23A): Test 7, Steps 1 – 4, Determining the magnitude of Y interferometer misalignment. Page 197

Table 24A): Test 8, Steps 1 – 4, Determining the magnitude of Y interferometer misalignment. Page 201

Table 25A): Test 9, Zero position, Steps 1 – 4, Determining the magnitude of Y interferometer misalignment. Page 205

Table 26A): Test 10, Steps 1 – 4, Determining the magnitude of Y interferometer misalignment. Page 209

Table 27A): Test 11, Steps 1 – 4, Determining the magnitude of Y interferometer misalignment. Page 213

Table 28A): Test 12, Steps 1 – 4, Determining the magnitude of Y interferometer misalignment. Page 217

Table 29A): Test 13, Steps 1 – 4, Determining the magnitude of Y interferometer misalignment. Page 221

Table 30A): Test 14, Steps 1 – 4, Determining the magnitude of Y interferometer misalignment. Page 225

Table 31A): Test 15, Zero position, Steps 1 – 4, Determining the magnitude of Y interferometer misalignment. Page 229
LIST OF FIGURES

FIGURE 1: MICHELSON INTERFEROMETER........................................................................................................... 16
FIGURE 2: WAVE FRONTS 1 AND 2 WITH MIRROR M2 TILTED AT ANGLE $\theta/2$ ABOUT THE Z-AXIS. ...... 17
FIGURE 3: INTERFEROGRAM OF PROJECTED WAVE FRONTS IN FIGURE 2. ...................................................... 18
FIGURE 4: NORMALISED RADIANT FLUX VS. VARYING $\theta$ WITH $y = 0$, $\lambda = 680$ nm, $\Delta D_H = 0$, INTEGRAL WIDTH $s = 1$ mm, FOR INTEGRAL BOUNDARIES $x_2 = 0.5$ mm & $x_1 = -0.5$ mm AND $x_2 = 1$ mm & $x_1 = 0$ mm. .......................................................................................................................... 21
FIGURE 5: NORMALISED RADIANT FLUX AGAINST WAVE FRONT ANGLE; A) NODE POINTS; B) WAVE FRONT ANGLE AT 99% RADIANT FLUX; $s = 10$ $\mu$m, 100 mm, AND 1 mm; $\lambda = 680$ nm. ............ 39
FIGURE 6: CONTOUR PLOT OF EQUAL WAVE FRONT ANGLES OF 99% RADIANT FLUX AS A FUNCTION OF THE SIDE LENGTH $s$ OF A SQUARE PHOTODETECTOR AREA AND LASER WAVE LENGTH $\lambda$. .................. 40
FIGURE 7: NORMALISED RADIANT FLUX VS. WAVE FRONT ANGLE $\alpha$); AT $x = 0$, $s/2$, $s$, $2s$ AND B); AT $x = 0$ AND 10s. IN BOTH EXAMPLES $s = 0.001$ m AND $\lambda = 680$ nm; SECONDARY NODES ARE MARKED WITH GREEN DOTS IN THE FIRST CYCLE UP TO THE 2ND NODE POINT IN A). ................................................................. 41
FIGURE 8: NORMALISED RADIANT FLUX ACROSS AN $\alpha$-RANGE OF 4 MM AT DIFFERENT WAVE FRONT ANGLES $\alpha$ (IN DEGREES); THE DASHED GREEN LINES INDICATE THE $\alpha$-POSITION OF THE BLUE RADIANT FLUX CURVE SHOWN IN FIGURE 7(A) ($s = 1$ mm, $x = s/2$, $s$, $2s$); THE DASHED PURPLE LINE INDICATES THE RADIANT FLUX LEVEL (INTERSECTIONS OF GREEN DASHED LINES AND PURPLE RADIANT FLUX CURVE) AT THE FIRST TRIPLE SECONDARY NODE (AT $\theta = 0.02597^\circ$); THE 1ST MINIMUM REFERS TO THE RADIANT FLUX CURVE AT $x = 0$ (FIGURE 7(A)). ................................................................. 41
FIGURE 9: NORMALISED RADIANT FLUX AGAINST WAVE FRONT ANGLE AT TWO DIFFERENT $x$; $\lambda = 680$ nm, $y = 0.02$ m............................................................................................................................................. 42
FIGURE 10: A) NORMALISED RADIANT FLUX VS. WAVE FRONT ANGLE AT EXAMPLES OF $y$ (M) AND B) NORMALISED RADIANT FLUX VS. $x$; $s = 0.00001$ m, $y = 0$, 0.0002, 0.002, AND 0.02 m; B) SHOWS THE POSITION OF THE CENTRE FRINGE (I.E., FRINGE NUMBER 0) RADIANT FLUX AT ANGLE $\alpha + 1^\circ$ AND MOVEMENT OF THE FRINGE PATTERN WITH INCREASING $y$ (NOTE THAT AMPLITUDE RANGE AND FRINGE DENSITY ARE INDEPENDENT OF $y$).................................................................................................................. 43
FIGURE 11: NORMALISED RADIANT FLUX VS. WAVE FRONT ANGLE ($\alpha = 0^\circ$ TO $0.2^\circ$) AND DISTANCE FROM THE BEAM CENTRE ($x = 0$ mm TO 5 mm) AT $y = 0$ mm, AND $\lambda = 680$ nm; A): $s = 1$ mm & $y$ = 20 mm, $\lambda = 680$ nm; B): $s = 0.5$ mm & $y$ = 20 mm, $\lambda = 680$ nm; C): $s = 0.1$ mm & $y$ = 20 mm, $\lambda = 680$ nm; D): $s = 0.01$ mm & $y$ = 20 mm, $\lambda = 680$ nm; E): $s = 1$ mm & $y = 1$ mm, $\lambda = 680$ nm; F): $s = 0.5$ mm & $y = 1$ mm, $\lambda = 680$ nm; G): $s = 1$ mm & $y = 1$ mm, $\lambda = 1,550$ nm; H): $s = 0.5$ mm & $y = 1$ mm, $\lambda = 1,550$ nm.................................................................................................................................................. 45
FIGURE 12: THE EFFECT OF VARIABLE $y$ ON THE FRINGE COUNT; A) $x = 0.001$ m, $y = 0.02$ m, $\lambda = 680$ nm, $\Delta D_H = 0$; B) $x = 0.001$, $y = 0.573$ m, $\lambda = 680$ nm, $\Delta D_H = 0$. ......................................................................................................................... 45
FIGURE 13: FRINGE COUNTING AS A FUNCTION OF $x$ AND $\theta$; $y = 0.30$ m, $\lambda = 680$ nm, $\Delta D = 0$; THE FRINGE COUNT IS SHOWN ON THE RIGHT SIDE, FOR $x = 0$, 0.001, AND 0.0015 m; "0°" = PEAK OF CENTRE FRINGE; POSITIVE AND NEGATIVE FRINGE NUMBERS REFER TO NEGATIVE AND POSITIVE $x$, RESPECTIVELY, I.E., TO LEFT AND RIGHT SIDES OF THE CENTRE FRINGE LINE "0°". .............................................................................................................................. 46
FIGURE 14: FRINGE COUNT WITH CONCURRENT SINUSOIDAL VARYING $\theta$ AND $\Delta D$, WITH $\lambda = 680$ nm, FOR: A) $x = 0.001$ m, $y = 0.1$ m, $\theta_{MAX} = 0.5^\circ$, $\Delta D = 50$ $\mu$m, $f = 5$ Hz; B) $x = 0.002$ m, $y = 0.1$ m,
\[ \theta_{\text{max}} = 0.5^\circ, \Delta D M = 50 \ \mu M, F = 5 \ Hz; \ c) x = 0.001 \ M, y = 0.3 \ M, \theta_{\text{max}} = 0.5^\circ, \Delta D M = 50 \ \mu M, F = 5 \ Hz; \ d) x = 0.001 \ M, y = 0.1 \ M, \theta_{\text{max}} = 0.8^\circ, \Delta D M = 50 \ \mu M, F = 5 \ Hz \]
Preface

Figure 58: Aligning the optical laser beam for concentricity with laser module........124
Figure 59: Schematic diagram of the interferometer laser driver................................125
Figure 60: Laser assembly (exploded view)............................................................126
Figure 61: Laser assembled.....................................................................................126
Figure 62: Photograph of experiment set up............................................................129
Figure 63: Manually aligning X and Y interferometer beamsplitters with IFCM bracket upside down.................................................................131
Figure 64: Best alignment of the X, Y and Z interferometers with the cube mirror, A) X interferogram, B) Y interferogram, C) Z interferogram...............................131
Figure 65: Zero position of the projected optical lever beams X', Y' and Z' compared with X, Y and Z where they were designed to be with best alignment of the X, Y and Z interferometers with the cube mirror..................................................134
Figure 66: Test 4, A) X interferometer interferogram, B) Y interferometer interferogram, C) Z interferometer interferogram, D) Optical lever beam positions. .................................................................147
Figure 67: Yaw and pitch reference frames, A) XYZ, B) YZX, C) ZXY..........................148
Figure 68: Laser beam with Gaussian profile...........................................................164
Figure 69: Bull’s Eye™ apodising neutral density filter.............................................164
Figure 70: Image sensor with single row & column of pixels....................................166
Figure 71: Interferogram.........................................................................................166
Figure 72: Fringe lines aligned with Max/Min instances............................................166
Figure 73: Misalignment of fringe lines with Max/Min instances...............................166
Figure 74: Fringe lines centred on image sensor.....................................................167
Figure 75: Alternative tilt axis angle.......................................................................167
Figure 76: Additional horizontal/vertical pixel array.................................................167
Figure 77: Misalignment of fringe lines...................................................................167
Figure 78: Zero position test, A) X interferometer interferogram, B) Y interferometer interferogram, C) Z interferometer interferogram, D) Optical lever beam positions. ........................................................................176
Figure 79: Test 1, A) X interferometer interferogram, B) Y interferometer interferogram, C) Z interferometer interferogram, D) Optical lever beam positions. ........................................................................180
Figure 80: Test 2, A) X interferometer interferogram, B) Y interferometer interferogram, C) Z interferometer interferogram, D) Optical lever beam positions. ........................................................................184
Preface

Figure 81: Test 3, A) X interferometer interferogram, B) Y interferometer interferogram, C) Z interferometer interferogram, D) Optical lever beam positions. .............................................................................................................

Figure 82: Test 5, A) X interferometer interferogram, B) Y interferometer interferogram, C) Z interferometer interferogram, D) Optical lever beam positions. .............................................................................................................

Figure 83: Test 6, A) X interferometer interferogram, B) Y interferometer interferogram, C) Z interferometer interferogram, D) Optical lever beam positions. .............................................................................................................

Figure 84: Test 7, A) X interferometer interferogram, B) Y interferometer interferogram, C) Z interferometer interferogram, D) Optical lever beam positions. .............................................................................................................

Figure 85: Test 8, A) X interferometer interferogram, B) Y interferometer interferogram, C) Z interferometer interferogram, D) Optical lever beam positions. .............................................................................................................

Figure 86: Test 9, A) X interferometer interferogram, B) Y interferometer interferogram, C) Z interferometer interferogram, D) Optical lever beam positions. .............................................................................................................

Figure 87: Test 10, A) X interferometer interferogram, B) Y interferometer interferogram, C) Z interferometer interferogram, D) Optical lever beam positions. .............................................................................................................

Figure 88: Test 11, A) X interferometer interferogram, B) Y interferometer interferogram, C) Z interferometer interferogram, D) Optical lever beam positions. .............................................................................................................

Figure 89: Test 12, A) X interferometer interferogram, B) Y interferometer interferogram, C) Z interferometer interferogram, D) Optical lever beam positions. .............................................................................................................

Figure 90: Test 13, A) X interferometer interferogram, B) Y interferometer interferogram, C) Z interferometer interferogram, D) Optical lever beam positions. .............................................................................................................

Figure 91: Test 14, A) X interferometer interferogram, B) Y interferometer interferogram, C) Z interferometer interferogram, D) Optical lever beam positions. .............................................................................................................

Figure 92: Test 15, A) X interferometer interferogram, B) Y interferometer interferogram, C) Z interferometer interferogram, D) Optical lever beam positions. .............................................................................................................
# LIST OF ABBREVIATIONS AND ACRONYMS

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>ADC</td>
<td>Analogue to Digital Convertor</td>
</tr>
<tr>
<td>CAD</td>
<td>Computer Aided Design</td>
</tr>
<tr>
<td>CCW</td>
<td>Counter clock-wise</td>
</tr>
<tr>
<td>CW</td>
<td>Clock-wise</td>
</tr>
<tr>
<td>DoF</td>
<td>Degree of Freedom</td>
</tr>
<tr>
<td>JPEG</td>
<td>Joint Photographic Experts Group</td>
</tr>
<tr>
<td>OPD</td>
<td>Optical Path Difference</td>
</tr>
<tr>
<td>PSD</td>
<td>Position Sensitive Detectors</td>
</tr>
<tr>
<td>QPD</td>
<td>Quadrature Phase Decoder</td>
</tr>
<tr>
<td>SG</td>
<td>Strain Gauge</td>
</tr>
<tr>
<td>WMV</td>
<td>Windows Media Video</td>
</tr>
</tbody>
</table>
# List of Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Position vector $\mathbf{A}$, $(A_x, A_y, A_z) = (1, 0, 0)$</td>
</tr>
<tr>
<td>$A^v$</td>
<td>Position vector $\mathbf{A}$ after 5 steps of yaw, roll and pitch of model cube mirror $(A_x^v, A_y^v, A_z^v)$</td>
</tr>
<tr>
<td>$A_x$</td>
<td>x-axis component of position vector $\mathbf{A}$</td>
</tr>
<tr>
<td>$A_y$</td>
<td>y-axis component of position vector $\mathbf{A}$</td>
</tr>
<tr>
<td>$A_z$</td>
<td>z-axis component of position vector $\mathbf{A}$</td>
</tr>
<tr>
<td>B</td>
<td>Position vector $\mathbf{B}$, $(B_x, B_y, B_z) = (0, 1, 0)$</td>
</tr>
<tr>
<td>$B^v$</td>
<td>Position vector $\mathbf{B}$ after 5 steps of yaw, roll and pitch of model cube mirror $(B_x^v, B_y^v, B_z^v)$</td>
</tr>
<tr>
<td>$B_x$</td>
<td>x-axis component of position vector $\mathbf{B}$</td>
</tr>
<tr>
<td>$B_y$</td>
<td>y-axis component of position vector $\mathbf{B}$</td>
</tr>
<tr>
<td>$B_z$</td>
<td>z-axis component of position vector $\mathbf{B}$</td>
</tr>
<tr>
<td>C</td>
<td>Position vector $\mathbf{C}$, $(C_x, C_y, C_z) = (0, 0, 1)$</td>
</tr>
<tr>
<td>$C^v$</td>
<td>Position vector $\mathbf{C}$ after 5 steps of yaw, roll and pitch of model cube mirror $(C_x^v, C_y^v, C_z^v)$</td>
</tr>
<tr>
<td>$C_x$</td>
<td>x-axis component of position vector $\mathbf{C}$</td>
</tr>
<tr>
<td>$C_y$</td>
<td>y-axis component of position vector $\mathbf{C}$</td>
</tr>
<tr>
<td>$C_z$</td>
<td>z-axis component of position vector $\mathbf{C}$</td>
</tr>
<tr>
<td>$c$</td>
<td>Speed of light $(3 \times 10^8 \text{ ms}^{-1})$</td>
</tr>
<tr>
<td>$C_D$</td>
<td>Decay constant</td>
</tr>
<tr>
<td>$d_1$</td>
<td>Optical path distance from beamsplitter to fixed mirror M1</td>
</tr>
<tr>
<td>$d_2$</td>
<td>Optical path distance from beamsplitter to movable mirror M2</td>
</tr>
<tr>
<td>$\Delta d_i$</td>
<td>The change in distance in metres of the laser spot at the projection surface cast by the optical lever</td>
</tr>
<tr>
<td>$\Delta d_x$</td>
<td>Horizontal distance between fringe lines</td>
</tr>
<tr>
<td>$\Delta d_y$</td>
<td>Vertical distance between fringe lines</td>
</tr>
<tr>
<td>$\Delta d_m$</td>
<td>Total distance between the fixed and moving mirror or the change in distance of the moving mirror from one position to the next (m)</td>
</tr>
<tr>
<td>$\Delta d_{\text{max}}$</td>
<td>Peak amplitude of translation of the mirror M2</td>
</tr>
<tr>
<td>$\Delta d_f$</td>
<td>Orthogonal distance between fringe lines (m)</td>
</tr>
<tr>
<td>$E$, $E$</td>
<td>Electric field vector, electric field $(\text{Vm}^{-1})$</td>
</tr>
<tr>
<td>$E_0$</td>
<td>Peak amplitude of the electric filed vector</td>
</tr>
<tr>
<td>$E_{\text{sum}}$, $E_{\text{sum}}$</td>
<td>Sum of electric field vectors, sum of the electric field</td>
</tr>
<tr>
<td>$E_{\text{sum}}^*$</td>
<td>Complex conjugate of $E_{\text{sum}}$</td>
</tr>
<tr>
<td>$F_D$</td>
<td>Decay function</td>
</tr>
</tbody>
</table>
Preface

$F_x$ Tangent value of the tilt axis angle $\varphi_x$

$F_y$ Tangent value of the tilt axis angle $\varphi_y$

$F_z$ Tangent value of the tilt axis angle $\varphi_z$

$f$ Frequency of oscillation (Hz)

$G_x$ Direction cosine of position vector $\mathbf{X}$

$G_y$ Direction cosine of position vector $\mathbf{Y}$

$G_z$ Direction cosine of position vector $\mathbf{Z}$

$H$ Variable consisting of terms $F_x, F_y, F_z, G_x, G_y, G_z$

$I$ Irradiance of the electric field (Wm$^{-2}$)

$i$ Unit vector along the $x$-axis

$I$ Vector of incident laser beam of unit length

$I'$ Projection vector of vector $\mathbf{I}$ on normal vector $\mathbf{N}$

$I''$ Rejection vector of vector $\mathbf{I}$

$i_x$ $x$-axis component of vector $\mathbf{I}$

$i_y$ $y$-axis component of vector $\mathbf{I}$

$i_z$ $z$-axis component of vector $\mathbf{I}$

$j$ Unit vector along the $y$-axis

$J$ Variable consisting of terms $F_x, F_y, F_z, G_x, G_y, G_z$

$k$ Unit vector along the $z$-axis

$k$ Wave number $= \frac{2\pi}{\lambda}$

$k$ Wave vector of the monochromatic wave front

$k_1$ Wave vector of wave front 1

$k_2$ Wave vector of wave front 2

$L_{xy}$ Erroneous translation measured in $x$-direction due to cube mirror translation in $y$-direction

$L_{xz}$ Erroneous translation measured in $x$-direction due to cube mirror translation in $y$-direction

$L_x$ Cube mirror translation in $x$-direction

$L_{tx}$ Total translation of the cube mirror in the $x$-direction

$L_{yx}$ Erroneous translation measured in $y$-direction due to cube mirror translation in $x$-direction

$L_{yz}$ Erroneous translation measured in $y$-direction due to cube mirror translation in $z$-direction

$L_y$ Cube mirror translation in $y$-direction

$L_{ty}$ Total translation of the cube mirror in the $y$-direction

$L_{zx}$ Erroneous translation measured in $z$-direction due to cube mirror translation in $x$-direction

$L_{zy}$ Erroneous translation measured in $z$-direction due to cube mirror translation in $y$-direction
$L_z$ Cube mirror translation in $z$-direction

$L_{Tz}$ Total translation of the cube mirror in the $z$-direction

$l$ Distance from the optical lever to the projection surface

M1 Fixed mirror

M2 Moving mirror

$m$ Non-integer form of the integer node numbers $n_p$

$N$ Mirror normal vector of unit length

$N_S$ Static mirror normal vector of unit length

$\mathbf{N}_{SM}$ Steering mirror normal vector of unit length

$\mathbf{N}_T$ Tilting mirror normal vector of unit length

$\mathbf{N}'_T$ Virtual tilting mirror normal vector of unit length

$\mathbf{N}_{y}$ The third optical lever position vector derived from the cross product of the two orthogonal tilting mirror normal vectors

$n$ Number of reflections through the optical lever system

$n_f$ Integer enumerating the optical path difference $2\Delta d_m$ in terms of whole wavelengths when the wave front angle = 0, the $n^{th}$ fringe line at a point $(x, y)$ on the plane of the photodetector

$n_m$ Integer enumerating the instances in time the fringe count speed is maximum

$n_p$ Integer of the $n^{th}$ primary node

$n_s$ Integer of the $n^{th}$ secondary node

$n_0$ Integer enumerating the instances in time the fringe count speed is zero

$n_{RI}$ Refractive index of the medium through which the light is travelling

$n_{sx}$ $x$-axis component of static mirror normal vector $\mathbf{N}_S$

$n_{sy}$ $y$-axis component of static mirror normal vector $\mathbf{N}_S$

$n_{sz}$ $z$-axis component of static mirror normal vector $\mathbf{N}_S$

$n_{smx}$ $x$-axis component of steering mirror normal vector $\mathbf{N}_{SM}$

$n_{smy}$ $y$-axis component of steering mirror normal vector $\mathbf{N}_{SM}$

$n_{smz}$ $z$-axis component of steering mirror normal vector $\mathbf{N}_{SM}$

$n_{tx}$ $x$-axis component of tilting mirror normal vector $\mathbf{N}_T$

$n_{ty}$ $y$-axis component of tilting mirror normal vector $\mathbf{N}_T$

$n_{tz}$ $z$-axis component of tilting mirror normal vector $\mathbf{N}_T$

$n'_{tx}$ $x$-axis component of virtual tilting mirror normal vector $\mathbf{N}'_T$

$n'_{ty}$ $y$-axis component of virtual tilting mirror normal vector $\mathbf{N}'_T$

$n'_{tz}$ $z$-axis component of virtual tilting mirror normal vector $\mathbf{N}'_T$

$n_0$ Integer enumerating the instances in time the fringe count speed is zero

$n_{max}$ Integer enumerating the instances in time the fringe count speed is maximum
n_x  Translation fringe count in the x-direction
n_y  Translation fringe count in the y-direction
n_z  Translation fringe count in the z-direction
n_{i_x}  x-axis component of mirror normal vector N
n_{i_y}  y-axis component of mirror normal vector N
n_{i_z}  z-axis component of mirror normal vector N
r  Unit vector of the electric field in 3 dimensional space
R_x  Rejection vector of cube mirror position vector X
R_y  Rejection vector of cube mirror position vector Y
R_z  Rejection vector of cube mirror position vector Z
R_{1,2,3...11,etc.}  Optical lever reflection vectors 1 – 11 etc.
R'_{1,2,3...11,etc.}  Projection vector of vectors R_{1,2,3...11,etc} on associated mirror normal vector
R''_{1,2,3...11,etc.}  Rejection vector of vectors R_{1,2,3...11,etc}
\mathbf{r}_{11x}  x-axis component of reflection vector \mathbf{R}_{11}
\mathbf{r}_{11y}  y-axis component of reflection vector \mathbf{R}_{11}
\mathbf{r}_{11z}  z-axis component of reflection vector \mathbf{R}_{11}
\mathbf{r}_{12x}\mathbf{n}  x-axis component of normalised reflection vector \mathbf{R}_{12}
\mathbf{r}_{12y}\mathbf{n}  y-axis component of normalised reflection vector \mathbf{R}_{12}
\mathbf{r}_{12z}\mathbf{n}  z-axis component of normalised reflection vector \mathbf{R}_{12}
s  Width of photodetector (m)
T_x  Tilt axis vector, which is the cross product of vectors X and \mathbf{i}
T_y  Tilt axis vector, which is the cross product of vectors Y and \mathbf{j}
T_z  Tilt axis vector, which is the cross product of vectors Z and \mathbf{k}
t  Time (s)
u  Unit vector defining the direction of propagation of the wave front
X_x  x-axis component of position vector X
X_y  y-axis component of position vector X
X_z  z-axis component of position vector X
X'_x  x-axis component of cross product position vector X'
X'_y  y-axis component of cross product position vector X'
X'_z  z-axis component of cross product position vector X'
X  Position vector X, (X_x, X_y, X_z)
X'  Cross product position vector X', (X'_x, X'_y, X'_z)
\mathbf{\hat{x}}  Unit vector in x-direction
x_1  Represents the first and second position respectively of the fringe lines
$x_1$ Location of one side of the first photodetector
$x_2$ Location of the other side of the first photodetector
$x_3$ Location of one side of the second photodetector
$x_4$ Location of the other side of the second photodetector

$Y_x$ $x$-axis component of position vector $Y$
$Y_y$ $y$-axis component of position vector $Y$
$Y_z$ $z$-axis component of position vector $Y$

$Y'_x$ $x$-axis component of cross product position vector $Y'$
$Y'_y$ $y$-axis component of cross product position vector $Y'$
$Y'_z$ $z$-axis component of cross product position vector $Y'$

$Y$ Position vector $Y$, $(Y_x, Y_y, Y_z)$
$Y'$ Cross product position vector $Y'$, $(Y'_x, Y'_y, Y'_z)$

$\hat{y}$ Unit vector in $y$-direction

$z$ Vertical side length of the photodetector

$Z_x$ $x$-axis component of position vector $Z$
$Z_y$ $y$-axis component of position vector $Z$
$Z_z$ $z$-axis component of position vector $Z$

$Z'_x$ $x$-axis component of cross product position vector $Z'$
$Z'_y$ $y$-axis component of cross product position vector $Z'$
$Z'_z$ $z$-axis component of cross product position vector $Z'$

$Z$ Position vector $Z$, $(Z_x, Z_y, Z_z)$
$Z'$ Cross product position vector $Z'$, $(Z'_x, Z'_y, Z'_z)$

$\hat{z}$ Unit vector in $z$-direction

$\alpha_x$ Tilt angle of the cube mirror $x$-axis face with respect to the $x$-axis
$\alpha_y$ Tilt angle of the cube mirror $y$-axis face with respect to the $x$-axis
$\alpha_z$ Tilt angle of the cube mirror $z$-axis face with respect to the $x$-axis

$\beta_x$ Tilt angle of the cube mirror $x$-axis face with respect to the $y$-axis
$\beta_y$ Tilt angle of the cube mirror $y$-axis face with respect to the $y$-axis
$\beta_z$ Tilt angle of the cube mirror $z$-axis face with respect to the $y$-axis

$\gamma_x$ Tilt angle of the cube mirror $x$-axis face with respect to the $z$-axis
$\gamma_y$ Tilt angle of the cube mirror $y$-axis face with respect to the $z$-axis
$\gamma_z$ Tilt angle of the cube mirror $z$-axis face with respect to the $z$-axis

$\delta$ Angle of the tilted mirror in the optical lever
$\delta'$ Angle of the virtual tilted mirror in the optical lever

$\epsilon_x$ Angle of rejection vector $R_x$ with respect to the $y$-axis
$\epsilon_y$ Angle of rejection vector $R_y$ with respect to the $z$-axis
$\epsilon_z$ Angle of rejection vector $R_z$ with respect to the $x$-axis
Preface

$\epsilon_0$  
Permittivity of a vacuum

$\zeta$  
The whole angle between the incident vector $\mathbf{I}$ and the $n^{th}$ reflection vector $\mathbf{R}_n$ of the virtual tilted mirror of the optical lever system

$\theta$  
Angle between two wave fronts (rad)

$\theta_l$  
The angle between incident vector $\mathbf{I}$ and mirror normal vector $\mathbf{N}$

$\theta_{0l}$  
The angle subtended by lengths $l$ and $\Delta d_l$ ($l >> \Delta d_l$) of the optical lever

$\theta_0$  
Angle $\theta$ when $\theta = 0$

$\theta_{1,2,3\ldots,11}$  
Angle $\theta$ at specific instances

$\theta_{\text{max}}$  
Peak amplitude of the wave front angle $\theta$

$\theta_{np}$  
Primary node angle

$\theta_{ns}$  
Secondary node angle

$\theta_{\text{rev}}$  
Wave front angle at which direction of fringe movement reverses as mirror tilt increases (rad)

$\theta_x$  
Angle of the tilted wave front of the $x$ axis interferometer

$\theta_y$  
Angle of the tilted wave front of the $y$ axis interferometer

$\theta_z$  
Angle of the tilted wave front of the $z$ axis interferometer

$\lambda$  
Wavelength (m)

$\xi_{xy}$  
Angle of position vector $\mathbf{X}$ relative to the $x$-axis in the $x$-$y$ plane

$\xi_{xz}$  
Angle of position vector $\mathbf{X}$ relative to the $x$-axis in the $x$-$z$ plane

$\pi$  
$\pi = 3.14159$

$\Phi_e$  
Radiant flux (W)

$\Phi_n$  
Normalised radiant flux

$\Phi_0$  
Maximum radiant flux amplitude where $\Phi_0 = 1$

$\varphi_x$  
Tilt axis angle of the cube mirror $x$-axis face with respect to the $y$-axis

$\varphi_y$  
Tilt axis angle of the cube mirror $y$-axis face with respect to the $y$-axis

$\varphi_z$  
Tilt axis angle of the cube mirror $z$-axis face with respect to the $y$-axis

$\omega$  
Angular frequency $= 2\pi f$
Preface

LIST OF APPENDICES

APPENDIX A .......................................................................................................................... 173
ABSTRACT

This thesis describes the design, development and testing of a novel sensor based on optical interferometry that measures displacement to 6 degree of freedom. It starts with an in-depth mathematical analysis of the behaviour of the radiant flux over the active area of a rectangular photodetector under varying mirror tilt angle, width of photodetector active area, distance of photodetector from beam centre, distance of the photodetector from the optical origin and light source wavelength. Findings not evident in the literature are presented, such as flux behaviour beyond the first modulation amplitude zero and the effect of fringe contraction and expansion on the flux at varying distance from beam centre and optical origin. The outcome of the analysis was crucial for deciding to use image sensors rather than photodetectors for this displacement sensor. Obtaining tilt and rotation information from the interferogram is discussed as well as the orthogonal design of the sensor. The mathematics for deriving the position vectors of displaced mirrors from the fringe analysis is given in detail. An experiment rig was built to test the sensor that included an XYZ translation stage as well as a Tilt/Rotation stage, which together provided a means of displacing the interferometer cube mirror to 6 DoF. The experiment rig integrated a 3 DoF tilt/rotation optical lever system designed specifically to accurately measure pitch, roll and yaw of the 6 DoF displacement sensor. Experimental data showed the optical lever system and 6 DoF sensor to have better than 0.01° correlation for pitch, roll and yaw over the test range of ±0.5° angular displacements. The accuracy and resolution of measuring linear displacement using optical interferometry is already well known, therefore this research adds a novel way of including angular displacement to that capability to provide displacement measurement to 6 DoF using 3 interferometers.
1 INTRODUCTION

This dissertation is structured in the following manner:

Chapter 1 – Overview and research questions

An overview of the theory, development and testing of measuring displacement to six degrees of freedom using three interferometers arranged orthogonally about a cube mirror. Research questions in achieving displacement in this manner are outlined.

Chapter 2 – Literature review

A literature review is given, covering the means of measuring displacement to six degrees of freedom using both interferometry and technologies other than interferometry. This is followed by an overview of how the technology researched in this thesis fills a knowledge gap.

Chapter 3 – Theoretical analysis of radiant flux on a rectangular photodetector

A theoretical analysis of radiant flux from an interferogram projected onto a rectangular photodetector was carried out. The outcome of this analysis showed that a single photodetector presented limitations on the range of angular displacement measurement. An image sensor is therefore used to analyse the interferogram and increase the range of angular displacement measurement.

Chapter 4 – Utilising an image sensor to capture mirror tilt angle and tilt axis angle

A brief description is given of how the relevant data for the displacement sensor can be captured by the image sensor from the interferogram.

Chapter 5 – Deriving the position vectors of the 6 degree of freedom sensor

A detailed description is provided of the arrangement of the three orthogonal interferometers about a cube mirror, together with two methods of deriving the three respective position vectors of the cube mirror with applied angular displacement.

Chapter 6 – Design of a tilt/rotation system to measure accuracy of the 6 degree of freedom sensor

In order to determine the accuracy of tilt/rotation of the cube mirror about each of the orthonormal Cartesian axes outlined in chapter 5, a tilt-and-rotation measuring device needed to be designed. The resulting system, based on using optical levers to amplify the tilt and rotation angles applied to the cube mirror is explained as well as the method of resolving the position vectors of the optical levers.

Chapter 7 – Experiment rig

Having designed both the interferometer displacement sensor and the optical lever tilt-and-rotation measuring device, this chapter outlines how the two systems are integrated onto one experimentation rig to facilitate testing and data capture/analysis concurrently from both systems.
Chapter 8 – Experimentation to determine degree of correlation between 6 degree of freedom sensor and the tilt/rotation measuring system

Having manufactured and assembled the experimentation rig, tilts and rotations were applied to the apparatus and the outputs from both systems were tabulated. Deriving the position vectors from both the cube mirror and optical lever systems for each of the tests, enabled direct comparison and demonstrated the degree of correlation between the two systems mounted on the rig. The accuracy verification is described at the end of the chapter.

Chapter 9 – Discussion

Chapter 9 discusses some of the shortcomings of the research methodology, the effects of the methods on the data obtained and it compares the results of this research with previous research.

Chapter 10 - Conclusion

This chapter contains the concluding statements on the outcomes of the experimentation, in answering the research questions and filling the knowledge gap.

1.1 Background and Objectives

Deflection is the measure a solid body is displaced under a force and/or moment. To determine the magnitude of the force or moment that produces the deflection, the magnitude of the displacement of one part of the solid body relative to another is measured. Several technologies currently exist that measure this relative displacement to determine the load being applied to the solid body. This thesis researches a new method of measuring displacement not covered in the literature that could be used as an alternative means of measuring deflection of solid bodies.

Multi-axis load cell sensors are readily available that measure deflection to 2 and 3 degrees of freedom (DoF). However, sensors that can measure deflection to 6 DoF as a single device are not that common and some of those that are available e.g. [1,2] cost > USD5,000 even for low load measurement (±200 N and ±12 Nm simultaneously) [1]. These devices use strain sensors, which can be of various types arranged around a solid body. The most common strain sensor is the strain gauge (SG) – metal foil, piezo resistive, piezo electric, fibre optic, capacitive – which is bonded to the calibrated solid body in distinct configurations. Being bonded makes the sensors susceptible to excessive mechanical stress that can compromise reliability and accuracy of measurement. In most cases, SG sensors also suffer from crosstalk, an unwanted measurement induced by deflection that is asymmetric to the primary measurement axis/axes. Crosstalk prevalent in these devices, to a large extent, can be compensated for by sensor choice, configuration, mechanical design and signal processing. Nonetheless, it is difficult to isolate crosstalk simultaneously from all 6 measurements of interest.

To overcome the need to bond the sensor to the solid body as well as avoiding crosstalk, strain can be measured using optical techniques, especially optical interferometry. The physics of light lends itself to non-contact applications and the straight line property of light makes it almost immune to crosstalk. Optical interferometry is based on detecting the phase shift between two beams of light and therefore the scale of measurement of displacement is the wavelength of light being used. Resolving the exact phase shift between the two light beams is straightforward, therefore linear displacement can be measured down to nanometres, a resolution far greater than SGs are able to achieve. Also, deformation of SG’s is limited to a small fraction of its physical shape whereas optical
interferometry can be used to accurately measure large linear and angular displacements as well as micro- and nano-scale displacements when it comes to measuring deflection of solid bodies.

Using optical interferometry, there are many possible arrangements and those that are relevant are discussed. Displacement of a solid body can be measured to 6 DoF by arranging 6 Michelson interferometers in a single, double and triple beam arrangement aligned with plane mirrors mounted on 3 orthogonal sides of a cube [3]. A similar arrangement with 5 Michelson interferometers using a double and a triple beam aligned with two orthogonal mirrors and an ancillary probe aligned with the third axis has been proposed [4]. A further method uses laterally sampled white light interferometry comprising a Mirau interferometer for 6 DoF measurement of solid body motion by laterally scanning the interference fringes [5]. A 6 DoF displacement sensor that utilises 4 interferometer systems, each comprising a laser and 2 pairs of diffraction gratings configured between the solid body and a reference frame, is also suggested [6]. In yet another system, deflection of a solid body can be measured using a digital speckle pattern interferometer [7].

Sensors capable of measuring displacement to 6 DoF using optical interferometry are generally limited to laboratory and highly specialised manufacturing applications like position sensing and control systems in silicon wafer lithography. They are specifically designed for displacement measurement rather than deflection measurement of a solid body. Physically, they are large and too cumbersome for measuring displacement or deflection in mass industrial applications. In addition, they are not conducive to displacement or deflection applications requiring miniaturisation or remote applications requiring ultra-low electrical power consumption.

This research focuses on displacement measurement to 6 DoF using a cube mirror such as [3] and [4], however, it only uses 3 orthogonally arranged interferometers about the cube mirror as opposed to 6 and 5 interferometers respectively. This novel approach realises a 6 DoF displacement sensor that has the following crucial attributes:

- utilises features of fringing from the 3 interferometers never done before to resolve 6 DoF – minimises the number of optics and opto-electronics
- uniquely determines translation direction using only the image sensors– reducing the interferometers to absolute basic configuration as well as minimising signal processing
- comprises inexpensive optics and opto-electronics – reducing cost
- a single sensor suited for the smallest to largest applications – maximising adaptability
- capable of occupying small spaces - miniaturisation
- potential for continual enhancement of measurement range, accuracy and sampling rate aligned with Moore’s Law development of the opto-electronics – futureproofing

1.2 Research questions

With 3 Michelson interferometers orthogonally arranged about a cube mirror:

i. How can fringe spacing and fringe slope be used to determine the tilt and rotation of the cube mirror about each Cartesian axis?

ii. How can translation direction be determined solely from fringe spacing, fringe slope, fringe count & fringe direction?
iii. Using a cube mirror inherently generates translation error, how can this be compensated for?

iv. How accurate and sensitive is the proof of concept sensor?

v. How can indeterminates in the mathematical modelling be overcome?

1.3 Overview of methodology

1.3.1 Theoretical analysis of radiant flux and irradiance on a rectangular photodetector due to interferometer fringing

To begin with, a full understanding of the behaviour of the radiant flux of the fringe pattern across a rectangular photodetector was required as this is predominantly the shape of the active area of discrete photodiodes or pixels in an image sensor. In Chapter 3 a mathematical analysis is presented that includes examination of variables specific to this thesis that to the best of the author’s knowledge are not covered in the literature.

1.3.2 Utilising image sensors to capture mirror tilt angle and tilt axis angle

There are four vital elements of data contained within an interferogram that cannot all be captured by a single photodetector, i.e. fringe spacing, fringe line orientation, fringe count and fringe movement direction. To do so requires an array of closely spaced photodetectors such as pixels in an image sensor.

Chapter 4 deals with one method to extract this data from the radiant flux captured by pixels as well as the associated digital signal processing.

1.3.3 Deriving cube mirror position vectors and overcoming translation errors

Having extracted the 4 elements of data from each of the three interferometers, two methods are described in Chapter 5 how the position vectors of the cube mirror can be calculated when the cube mirror undergoes angular displacement. Both methods have inherent indeterminates that create uncertainty when one, or all three, of the interferometers are in perfect alignment. Methods of overcoming this are presented. The mathematic modelling of the cube mirror angular displacement is unique.

When the moving mirror translates, the fringe lines move across the interferogram orthogonally to their slope. Their slope is defined by the tilt axis angle of the moving mirror.

Direction of translation of the moving mirror in a Michelson interferometer cannot be obtained solely from the direction of movement of the fringe lines and this is why phase detection methods such as quadrature phase shift detection, heterodyning or separating the s and p polarised waves are common practice. However, in Chapter 5 it is shown that by using the position vectors of the cube mirror together with direction of fringe movement, the direction of translation can be determined. Determining mirror translation in this way is novel.

The cube mirror has 3 orthogonal mirrored sides. If the cube is tilted 3-dimensionally and it is translated along one of the three Cartesian axes, a fringe count should only occur for
the interferometer on that axis. Due to the other two sides being sloped, the associated interferometers also detect a translation albeit far smaller than that along the first axis. This induces a translation error for the latter two interferometers and it is shown in Chapter 5 how this error can be eliminated. In the literature, this error is not addressed in displacement realisations to 6 DoF using a cube mirror.

1.3.4 Design of mirror tilt and tilt axis measurement system

To test the 6 DoF displacement sensor a means was required to apply precise amounts of tilt and rotation to the cube mirror. To do this, a 3-axis tilt/rotation stage was required, one that provided an actual measurement of any crosstalk induced to adjacent axes when applying tilt adjustment to the third axis. As such a stage could not be obtained for the thesis and as this capability was paramount, an inexpensive tilt/rotation stage was bought and a separate bespoke tilt/rotation measurement system was implemented and mounted on the stage.

Chapter 6 describes how the tilt and rotation measurement system was designed using orthogonally arranged optical levers to amplify rotation about each of the Cartesian axes. Each optical lever projected its laser beam onto a distant screen. The system was based on the principle that a slight change in the inclination of the laser beam can be measured in terms of angle by measuring by how much the beam moved on the screen and measuring the distance between the laser and the screen.

Two mathematical algorithms were developed to resolve the magnitude of tilt and rotation of the tilt/rotation stage in terms of the position vectors of the optical levers, of which the simpler of the two algorithms was adopted for experimentation. As the tilting mirrors of the orthogonal optical levers are immovably fixed to one another, crosstalk induced tilt of the tilt/rotation stage about the second and third axis was concurrently measured when tilt and rotation was applied about the first axis.

Using the position vectors of the optical lever system as a basis, the accuracy of the cube mirror position vectors could be determined.

1.3.5 Experiment Rig

Chapter 7 elaborates on how the experiment rig was designed to apply translation, tilt and rotation to the 6 DoF displacement sensor and to correlate the magnitude of the applied changes to the derived cube mirror position vectors and respective optical lever position vectors.

An XYZ translation stage was used to apply translation along the respective x, y and z axes. A tilt/rotation stage, which was mounted on the translation stage, was used to induce pitch, roll and yaw. The cube mirror and the tilting mirrors of the optical levers were mounted to the tilt/rotation stage by means of brackets and all could therefore be moved to 6 DoF.

The three Michelson interferometers of the 6 DoF sensor were orthogonally mounted about the cube mirrored on three adjacent sides. Translation, tilt and rotation applied to the glass cube simulated displacement of or deflection on a solid body being subjected to compression, tension, torsion, shear and bending. The consequent change in fringe count, fringe line spacing and fringe line inclination of the 3 interferograms were captured by respective webcam image sensors. From these parameters the position vectors of the cube mirror were calculated.
The rig also comprised the optical lever tilt measuring system with their respective beams exiting the sensor projected towards a screen placed a distance away from the experiment rig. The direction of projection of the beams was designed so that they project across to a wall/screen in front of where one sat to operate the experiment. The change of position of the beams on the screen was proportional to the magnitude of pitch, roll and yaw made by the tilt/rotation stage.

The position vectors from the optical lever tilt measuring system were determined from the displacement of the 3 laser beams projected onto the distant screen.

Also described is the method of collimating and focussing the interferometers lasers and the laser driver electronics.

1.3.6 Experimentation

The experiment rig was placed on a sturdy workbench, perpendicular to a wall, approximately 2.2m away from the wall, on which a sheet of A3 graph paper was placed. At the outset, the interferometer displacement sensor and optical lever systems were zero’d. Then a series of sixteen tests were undertaken where tilt and/or rotation of varying magnitudes were applied to the assembly. For each test, the outputs from the two systems were captured and tabulated. For the ninth and last tests, the apparatus was returned to the zero position, to determine whether there had been any drift. Methodology is described in detail in Chapter 8, which also describes step-by-step, how each of the position vectors arising from the two systems was calculated.

1.4 Overview of results found and described in each chapter

1.4.1 Overview of the mathematical analysis

The radiant flux and the fringe count are influenced by a range of parameters, namely the wave front angle \( \theta \), the position \( x \) of the photodetector with respect to the centre line, the laser wave length \( \lambda \), the distance \( y \) between the mirror and the photodetector, and the side length \( s \), i.e., the size of the photodetector, all of which are variable. The influence of these parameters is explained in a systematic way based on the equations derived.

The significant outcome of the mathematical analysis revealed that if discrete photodiodes are used to capture the radiant flux of the fringe pattern, the range of measurement of angular displacement about the Cartesian axes for the 6 DoF sensor is rather limited. To increase this range it was evident that image sensors were required with a pixel pitch less than 10 μm.

1.4.2 Determining the position vectors of the cube mirror

The two derived methods to determine the position vectors of the cube mirror were modelled mathematically in Excel. Virtual rotations were applied of the cube mirror and both resolved its position vectors with > 99.999% accuracy proving the mathematics of both methods functioned properly.

1.4.3 Mirror tilt and tilt axis calibration system

The mathematical model of the optical lever tilt calibration system was created in Excel and tested using a 3D CAD package to perform tilts and rotations of the optical lever system. The accuracy of the model was > 99.5% proving the mathematics of the optical lever system functioned properly.
1.4.4 Experiment rig testing

Having assembled the experiment rig and switching on the interferometers and optical levers, it was found that despite having had the components of the experiment rig machined to high tolerance, there was a slight misalignment of the interferometers about the cube mirror. Not having anticipated this outcome, no means of adjustment of interferometer alignment had been provided for. A work-around method of manually aligning the interferometers orthogonally therefore needed to be devised. The best outcome of this exercise resulted in the y-axis interferometer marginally misaligned with the other two. This misalignment was easily corrected using change-of-basis vector transformation.

1.4.5 Experimentation data capture and analysis

For each of the sixteen tests, the position vectors of the interferometer displacement sensor and the optical lever system were calculated using the respective methodologies described in Chapters 5 and 6. Upon analysis of the data from the two systems using Euler angles, a standard linear regression of the experimentation results found that for an expected gradient of 1, the worst case gradient was 13.3% and the worst case r-squared was 0.9281. From the residual standard deviation analysis, the worst case deviation was a yaw of 7.0432% of full circle due to the vector being close 90° in pitch. The data from the experimentation showed an extremely close correlation of the cube mirror and optical lever position vectors.

1.5 Overview of discussion

Chapter 9 discusses some of the shortcomings of the research methodology, e.g. difficulty aligning the interferometers as no adjustment had been built into the experiment rig, and overcoming one of the interferometers being misaligned. It compares the results of this research with previous research that measures linear and angular displacement by arranging interferometers orthogonally about a cube mirror. It also identifies how this research fills a knowledge gap left by previous research and how application of the technology can fill a particular need. It discusses the effects of the methods used on the data obtained, for example, approximating the optical lever with a virtual mirror and how this may have affected the results.

1.6 Overview of conclusion

Chapter 10 summarises the outcomes from each chapter and how the research questions have been satisfactorily answered. The objective of this thesis was to research, develop and test a novel sensor using optical interferometry to measure displacement to 6 degrees of freedom. Experimentation results showed a close correlation between the 6 DoF sensor and the optical lever system and as a consequence the objectives of the thesis have been fulfilled.

1.7 For further study - optimising the resolution of the interferometer output

The experiment rig and 6 DoF sensor system designed for this research was a simple proof of concept of the technology. This section details how the system can be optimised to:

- improve resolution and accuracy of deflection measurement
• increase sampling rate through Moore’ Law development of the opto-electronics as well as bespoke image sensors to realise MHz sampling to capture ever higher displacement frequencies
• minimise digital signal processing
2 LITERATURE REVIEW

2.1 Introduction
The research has predominantly been based on two aspects;

1. understanding the behaviour of the radiant flux over the active area of a rectangular photodetector under varying mirror tilt angle, width of photodetector active area, distance of photodetector from beam centre, distance of the photodetector from the optical origin and light source wavelength
2. design and development of a sensor that measures linear and angular displacement to 6 DoF based on 3 orthogonally arranged Michelson interferometers

This literature review is separated accordingly.

2.2 Theoretical analysis of interferometer wave front tilt and fringe radiant flux on a rectangular photodetector
The Michelson interferometer [9] shown in Figure 1 has been used extensively in the field of metrology, most famously in the Michelson-Morley experiment [10]. This interferometer configuration is typically used to accurately measure linear displacement by detecting the phase difference between two monochromatic beams of light that overlap in space as well as direction. Light from the laser is split by means of a beamsplitter into a transmitted and reflected beam of equal amplitude. The beamsplitter has a dielectric coating which ideally absorbs very little of the incident light, causes no phase shift between the transmitted and reflected beams nor affects polarisation of the electromagnetic field. The transmitted and reflected beams are projected onto respective mirrors, each orientated to return the beam directly back along its incoming path to the beamsplitter where they recombine and interfere with one another. The interference beam that is produced projects onto a screen for visual observation or an image sensor that captures the magnitude of the radiant flux incident on it. The linear displacement measurement is based on detecting and measuring the exact phase shift between the two reflected beams in terms of the number of complete and partial fringes resulting from translation of one of the reflecting mirrors and relating the result to the light-source wavelength.
Discrete photodetectors and image sensors are commonly used to detect the sinusoidal pulsing fringe pattern. The radiant flux of the fringe pattern incident on the device’s active area is mathematically derived by integrating the irradiance over the circular aperture of the light source [11]; over a circular aperture of the interferogram [12–18]; over a square/rectangular aperture of the interferogram [16,19]. Effectively, the active area of the photodetector performs the same integration function on the irradiance giving an output proportional to the radiant flux.

When using a well collimated beam and plane flat mirrors that are not perfectly aligned, i.e., they are tilted, fringe lines of equal inclination, width and spacing are produced that contract as the tilt angle is increased and expand as the tilt angle is reduced, having a significant effect on the radiant flux over the active area. This change in radiant flux can be a source of measurement error [11–17] when unconsidered in applications e.g. [20–24].

As wave front angle increases, the modulation amplitude of the dynamic fringing reduces and at a specific tilt angle the modulation amplitude of the radiant flux becomes zero [11,13–19,25]. The modulation amplitude is found to decay as a cardinal sine function with a rate of decay proportional to the area of the photodetector and inversely proportional to wavelength.

To overcome or minimise mirror tilt prevalent with flat plane mirrors, corner cube retro-reflectors [16,17], cat-eye reflectors [26] or alternate types of interferometers [19] are used.

However, the behaviour of the radiant flux with varying wave front angle considered simultaneously with photodetector area, distance from beam centre, distance from optical origin and wavelength appears not to be covered in the literature although [25] has included distance of the photodetector from the optical model origin specifically to determine the maximum offset angle for a beam-tilting spatial modulation interferometer.

Despite modulation amplitude vs. wave front angle being well understood and documented, understanding the behaviour of the radiant flux for photodetectors of varying active area, distance from the beam centre and distance from the origin for this research was paramount. This was due to the possibility that during assembly a photodetector may not be perfectly positioned on the beam centre and what affect this had on the photodetector output. Also, when designing the interferometers for the sensor, what was the optimum distance of the photodetector from the optical model origin specifically to resolve before starting to design a 6 DoF displacement sensor with the 3 Michelson interferometers. Especially so if the displacement sensor was to use image sensors where the outlying pixels are located a few or more millimetres from the beam centre.

To be specific, to the best of the author’s knowledge, the following analysis is not covered in the literature, i.e. the behaviour of the radiant flux:

- for a rectangular aperture for varying wave front angle beyond the first modulation amplitude zero
- over varying active area widths for varying distances from beam centre, e.g. row or column of pixels across an image sensor
- for varying active area widths and distances from the tilted mirror, i.e. fringe contraction/expansion
- with fringe contraction/expansion speed and mirror tilt
Literature Review

- concerning fringe transition speed with varying wave front angle, distance from optical origin, distance from beam centre and varying translation
- having a decay constant

The relevance of this theoretical analysis covering the knowledge gap was to support this research as well as to make evident how these factors may have an undesirable effect on sensor applications using homodyne interferometry. It is mostly applicable where photodetectors or image sensors are employed to sense small fractions of a fringe to achieve extremely high resolutions of measurement. It goes beyond the adverse effect of modulation amplitude reduction due to increasing wave front angle \([11,13-19,25]\) and introduces what have been termed primary nodes. The results from the mathematical analysis describes how the radiant flux behaves when the five parameters; wave front angle, wave length, photodetector width and position \((x, y)\) are varied independently and concurrently, and how this behaviour can introduce fringe counting errors. But most importantly for this research, the output from this new knowledge was essential in deciding what type of photodetector to use and the effect that distance from the beam centre and distance for the optical origin had on fringe speed. The latter govern what the minimum sampling rate can be used to accurately capture fringe count, fringe spacing and spacing slope using an image sensor.

2.3 6 DoF displacement sensor

When a torque and/or force are applied to a solid body, it undergoes angular and/or linear displacement and this is called deflection.

The predominant technology used to measure displacement within a solid body is the strain gauge (SG), which is made of metal foil, optical fibre, capacitor, piezo-resistive or piezo-electric semiconductor material all of which changes its electrical or optical properties when deformed by very small amounts.

Piezo-electric SGs are made of a crystal material that is sensitive only to shear \((x\) and \(y\) axes) or compressive/tensile \((z\) axis) forces enabling 3 SGs to be stacked in an \(x, y\) and \(z\) arrangement \([27]\). The output from each of the 3 piezo-electric crystals is a charge that is proportional to the force vector along the respective axes. A charge amplifier converts the charge to a voltage that is translated to force.

One \(x, y\) and \(z\) sensor alone as described above cannot measure deflection to 6 DoF. To do so, at least 3 such sensors must be arranged remotely from each other on the static or dynamic body to measure the linear \(x, y\) and \(z\) deflections at each location. By resolving in combination the \(x, y\) and \(z\) component outputs from the SGs, deflection to 6 DoF can be derived \([27,28]\).

Metal foil, piezo-resistive, fibre optic and capacitive SGs can also be used, but unlike the unidirectional piezo-electric SG, these SGs suffer from crosstalk, which is an erroneous measurement obtained when strain is exerted across its primary axis of sensitivity.

Any of these latter four SG types can be mounted to a solid body in an array of axial and cross-axial SG configurations to derive the \(x, y\) and \(z\) forces \([28]\). However, due to crosstalk, an additional complementary arrangement of SGs mounted to the solid body is required to cancel this effect adding complexity to the systems.

An enhancement of the single \(x, y\) and \(z\) sensor is an \(x, y\) and \(z\) sensor co-mounted with additional SG sensors that measure torque about these axes respectively to derive deflection to 6 DoF \([1,2]\). Multi-component systems such as these have accumulative
systematic and random errors therefore accuracy is undesirably reduced. Also, the size of the system is expanded by having an arrangement of at least 12 individual SG sensors to measure deflection to 6 DoF.

Deflection to 6 DoF can be measured using an elastically deformable conductive polymer placed around a solid body and contained within a casing. Electrodes, through which current is passed, are arranged axially and cross-axially around the casing and make contact with the conductive polymer. Deflection of the solid body causes a change in conductance in the polymer in 3 dimensions, which in turn causes a change in current flow between the electrodes. The measurement of deflection is resolved by an algorithm, which is a function of the electrode spacial configuration and current flow therebetween [29]. This technology is also susceptible to crosstalk as currents can flow between all electrodes making crosstalk very difficult to overcome. Although the conductive polymer is elastically deformable, maintaining its conductive properties homogeneously around the solid body is somewhat difficult, contributing to additional systematic errors in measurement to 6 DoF. It is also very bulky requiring an outer housing to not only mount but also shield the electrodes from electro-magnetic induction, which is impractical for most applications.

A further realisation of measuring deflection to 6 DoF is having an inner casing and complementary outer casing [30] that are restrained from one another by metal blocks. On each of the six sides of the two casings are mounted two conductive plates. The conductive plates on each of the opposing casing faces are arranged opposite each other and are insulated electrically from each other and from the respective casings. The distance between opposing conductive plates is of the order of tens of microns. The space between the two cases is sealed from the external environment and a dielectric liquid is pumped into the void under pressure. The opposing conductive plates consequently constitute a capacitor in an electrical circuit. Translation and rotation between the two casings causes the metal blocks to deform resulting in the distance between respective conductive plates to vary, which causes a change in the electrical circuit. The electrical changes are subsequently transformed into translation and rotation of one casing relative to the other.

Measuring deflection optically is an alternative technology to those mentioned above and there are several diverse methods in which optical technology is used to measure deflection to 6 DoF. Essentially, when detecting deflection optically, it is quantified in terms of the angular and linear displacement of one reference frame relative to another due to the non-contact nature of optical metrological technologies.

One method is having a laser aimed directly at the apex of a three-faceted triangular pyramid mirror mounted on the solid body. When in perfect alignment, the 3 beams reflected from the 3-faceted pyramid mirror are each received with equal intensity and shape by position sensitive detectors [31]. Deflection of the solid body will cause the pyramid mirror to translate and rotate causing the 3 reflected beams to change intensity and shape. The outputs of the position sensitive detectors (PSD) are resolved to determine the deflection to 6 DoF.

Variations of a non-contact sensing system for monitoring the position and orientation of a rigid body in space that have multiple distinct point light sources mounted on a rigid body are proposed by [34-36]. Complementary PSDs are arranged on a remote object to detect relative displacement between the rigid body and the remote object to 6 DoF using triangulation.
There are differing methods utilising multiple light sources incident on specifically located reflective surfaces mounted on a solid body that undergoes displacement. Displacement of the light reflected from the reflected surfaces is detected by PSDs, the outputs of which are resolved to 6 DoF [37, 38].

PSDs are only accurate to tens of microns at best, therefore the above described systems incorporating PSDs do not realise the accuracy achievable with the current research, which uses image sensors with pixel pitch just a few microns.

Another optical means to detect deflection to 6 DoF is proposed by [39-45], specifying independent methods of light sources illuminating markers such as dots, lines or patterns mounted on the object that undergoes displacement. The reflected light containing information of the dots, lines or patterns is projected onto photodetectors or image sensors, the outputs of which are resolved into 6 DoF.

An alternative system comprises patterned light sources attached to a frame of reference that are reflected by retro-reflectors mounted on the object whose position is being detected [46]. The reflected patterned beams are detected by PSDs co-mounted on the frame of reference, the outputs of which are resolved into 6 DoF.

A further means to measure deflection in solid bodies that achieves a resolution of nanometres in linear displacement and micro-radians in angular displacement is optical interferometry, which is generally more sensitive than the above mentioned optical solutions. It is based on detecting the interference pattern (interferogram) of two overlapping beams of light, usually from the same monochromatic source, and counting the number of instances of maximum constructive interference, known as fringes, which result from the displacements. This displacement measuring technology can be applied in many ways to measure deflection.

A method that uses laterally sampled white light interferometry measures solid object motion in 6 DoF by projection and measurement of interferograms [5]. This methodology requires that every pixel in each of the 3 image sensors that capture the interferogram images is analysed using digital signal processing (DSP). This requirement places a large processing load on the microprocessor and consequently influences the rate at which the object motion can be followed.

A 6 DoF displacement sensor that utilises four interferometer systems each comprises a laser and 2 pairs of diffraction gratings configured between the solid object and a reference frame [6]. This system uses a highly complex arrangement of optics and opto-electronics and requires a minimum of 40 individual components to realise 6 DoF.

Deflection of a solid body can be nano-measured to five DoF using a digital speckle pattern interferometer [7]. Digital speckle pattern interferometry is the technique of using the interferogram generated from a monochromatic light source scattered from the rough surface of an object and a reference beam from the same light source to create a speckle pattern that is captured by video recording and digital processing means to visualise displacement of the object. To calculate the magnitude and location of the applied forces in this system involves intensive DSP, which restricts the rate at which dynamic deflection can be captured to 5 DoF.

By arranging single, double and triple beam interferometers aligned with respective plane mirrors mounted on 3 orthogonal sides of a cube, 6 DoF of movement of the cube can be resolved from the 6 fringe counts [3]. A similar arrangement but with only double and triple beam interferometers and two orthogonal mirrors and additional probe can be used [4]. In the above realisations the multi-sided mirror would be attached to the solid body.
A further interferometry implementation [47] utilises a one degree of freedom deflection sensor that measures linear deflection between a fixed reference and the moving mirror. This implementation has only considered fringe count to determine deflection, whereas the interferogram generated by the Michelson interferometer contains four distinct components of information:

- mirror tilt angle – derived from fringe spacing
- mirror tilt axis angle – derived from the slope of the fringe lines
- fringe count – by counting the full and partial fringes passing a given point
- fringe direction – observing the direction in which the fringes are traversing

Moreover, none of the implementations in the literature based on the Michelson interferometer utilises all 4 data from 3 interferometers to resolve 6 DoF. As a result of this knowledge gap, when realising displacement to 6 DoF using a cube mirror, greater than 3 Michelson interferometers are required in order to resolve the 6 DoF such as [3] and [4].

This research takes advantage of the knowledge gap by utilising these four data provided by each of 3 interferometers that are arranged orthogonally about a cube mirror to find the position vectors of the cube. With the cube mirror attached to one side of a solid object and the fixed elements of the 3 interferometers attached to the other, or to a different solid object, displacement to six degrees of freedom of the one solid object relative to the other can be measured to nanoscale. As an application, the displacement can be quantified in terms of deflection by resolving the applied forces and/or moments to the solid body or bodies.

What also appears missing from the literature is the ability to determine translation direction from the 12 interferometer data. The current methods to measure translation direction using a Michelson interferometer are quadrature phase detection, heterodyning or separating the s and p polarised waves. This is because translation direction cannot be derived purely by observing the direction of fringe line transition across an image sensor without knowing the direction in which the moving mirror normal is tilted. With this research the direction that the moving mirror normal is tilted is obtained directly from the coefficients of the cube mirror position vectors. Obtaining the direction in which the fringe lines are traversing across the image sensor is quite trivial. Therefore, translation direction can be resolved simply with these data and correlation with a logic table.
3 Theoretical Analysis of Fringe Line Radiant Flux on a Rectangular Photodetector

3.1 Introduction

This section addresses specifically:

- Behaviour of the radiant flux for variable wave front angle as a function of photodetector width and position within the fringe pattern;
- Behaviour of the radiant flux on two identical photodetectors adjacent or overlapping each other;
- Magnitude of the radiant flux at wave front angle(s) of equal radiant flux;
- Behaviour of the radiant flux for variable wave front angle with variable distance of the photodetector from the tilting mirror;
- Speed of transition of the fringe lines across the photodetector for variable wave front angle;
- Speed of fringe line tilt across the photodetector for variable wave front angle;
- Damping function constant of the radiant flux for variable wave front angle.
- Fringe transition speed with variable wave front angle and translation

Much of the analysis specified above and covered in this chapter has already been published in journal paper [48].
3.2 Methodology

3.2.1 Mathematical analysis

The analysis is carried out based on a conventional Michelson interferometer that is configured as shown in Figure 1 with the following configuration constraints:

- Light source is a collimated monochromatic beam;
- Wave fronts over the area of the photodetector are approximated to be plane waves;
- Flat plane mirrors are used to reflect the transmitted and reflected beams back to the beamsplitter;
- Beamsplitter is lossless, is non-polarising and creates a transmitted and reflected beam of equal amplitude.

![Figure 1: Michelson interferometer.](image)

The two wave fronts are orientated as depicted in Figure 2 and all calculations are based on the following constraints:

- $y$ axis is taken to be normal to wave front 1;
- Origin of the Cartesian coordinate system is the point at which the centre of the incident beam is reflected by mirror M2;
- Mirror M2 tilts only about the $z$-axis;
- Mirror M2 translates only along the $y$-axis;
- Plane of the photodetector remains orthogonal to the $y$ axis;
- Shape of the active area of the photodetector is rectangular with variable side length $s$ in the $x$ direction and fixed side length $z$ (set to unity) in the $z$ direction;
Theoretical analysis of fringe line radiant flux on a rectangular photodetector

- Fringe pattern irradiates the entire active area of the photodetector;
- Output of the photodetector is assumed to be a 1:1 linear function of the incident radiant flux;
- Distance to the photodetector from mirror M2 is variable.

Figure 2: Wave fronts 1 and 2 with Mirror M2 tilted at angle $\theta/2$ about the z-axis.

The instances of maximum constructive interference are the fringe lines and when viewed in the plane of the photodetector, the fringe pattern, also known as an interferogram, will look like that shown in Figure 3.
The mathematical analysis is divided into the following subsections, the outcome of which is studied further in the Results section of this chapter:

1. Derivation of the equation for radiant flux from irradiance of the fringe pattern;
2. Identification of specific wave front angles $\theta$ with invariable radiant flux for two displaced photodetectors of equal size active areas;
3. Determination of the magnitude of the radiant flux at specific wave front angles $\theta$;
4. Determination of the linear equation defining the profile of the fringe pattern in the $x$-$y$ plane;
5. Determination of the damping function of the radiant flux with variable wave front angle $\theta$.
6. Determination of fringe count, fringe count speed and fringe transition speed with variable wave front angle $\theta$ and variable translation of moving mirror.

Note: The angle $\theta$ in this thesis refers to the angle that Wave Front 2 makes with Wave Front 1. The tilt angle of Mirror M2 is therefore $\theta/2$ relative to Mirror M1.

3.2.1.1 Derivation of the equation for radiant flux

The electric field of a plane wave is given by Equation (3.1) [49]:

$$E(r, t) = E_0 e^{i(kr - \omega t)}$$  \hspace{1cm} (3.1)

where $E$ is the time $(t)$ dependent electric field, $r$ is the unit vector of the electric field in 3 dimensional space, i.e., $r = x\hat{x} + y\hat{y} + z\hat{z}$ and $\hat{x}$, $\hat{y}$ and $\hat{z}$ are unit vectors along the $x$, $y$ and $z$ axes, $E_0$ is the vector amplitude of the wave, $k$ is the wave vector where $k = ku$, where $u$ is the unit vector defining the direction of propagation [50] and $|k| = k = 2\pi/\lambda$, $\lambda$ is the
Theoretical analysis of fringe line radiant flux on a rectangular photodetector

wavelength of the light source, \( k \) is the wave number and \( \omega \) is the angular frequency of the wave.

Figure 2 depicts the linear optical equivalent of the Michelson interferometer with the virtual source wave front approaching mirrors M1 and M2 from the top of the figure. With reference to the origin, the reflected wave fronts 1 and 2 from respective mirrors have wave vectors \( \mathbf{k}_1 \) and \( \mathbf{k}_2 \):

\[
\mathbf{k}_1 = k \hat{y} \tag{3.2}
\]

\[
\mathbf{k}_2 = k \sin \theta \hat{x} + k \cos \theta \hat{y} \tag{3.3}
\]

Also depicted in Figure 2, the source wave front travels a distance \( \Delta d_m \) further to M1 creating an optical path difference (OPD) between the wave fronts and consequently a phase lag of \( k \Delta d_m \) relative to wave front 2.

The sum of the electric fields of wave fronts 1 and 2 is therefore:

\[
E_{\text{sum}}(r, t) = E_0 e^{i(k_1r - k2\Delta d_m - \omega t)} + E_0 e^{i(k_2r - \omega t)} \tag{3.4}
\]

\[
E_{\text{sum}}(x, y, z) = E_0 e^{i(k(y - 2\Delta d_m) - \omega t)} + E_0 e^{i(k(y \cos \theta + x \sin \theta) - \omega t)} \tag{3.5}
\]

The irradiance \( I \) of an electric field is given by Equation (3.6) and is the radiant flux of the electric field delivered per area to a given surface with units \( \text{Wm}^{-2} \), i.e., radiant flux density:

\[
I = \left( \frac{n_R \epsilon_0 c}{2} \right) E_{\text{sum}} \cdot E_{\text{sum}}^* \tag{3.6}
\]

where \( n_R \) is the refractive index of the medium, \( c \) is the speed of light in vacuum, \( \epsilon_0 \) is the vacuum permittivity, and \( E_{\text{sum}}^* \) is the complex conjugate of \( E_{\text{sum}} \):

\[
I = \left( \frac{n_R \epsilon_0 c}{2} \right) E_0 e^{-i\omega t} \left( e^{i(k(y - 2\Delta d_m) + e^{i(k(y \cos \theta + x \sin \theta)}} \right) \tag{3.7}
\]

\[
= \left( \frac{n_R \epsilon_0 c}{2} \right) E_0^2 \left( 2 + 2 \cos(k(y - 2\Delta d_m - y \cos \theta - x \sin \theta)) \right)
\]

\[
= 2 \left( \frac{n_R \epsilon_0 c}{2} \right) E_0^2 \left( 1 + \cos(k(y - 2\Delta d_m - y \cos \theta - x \sin \theta)) \right)
\]

Equation (3.7) indicates that the irradiance at a point \( (x, y) \) in the fringe pattern created by wave fronts 1 and 2 is dependent on the values of \( x \) and \( y \), wave number \( k \), which is a function of wavelength, the angle \( \theta \) between the wave fronts and the optical path difference \( 2\Delta d_m \).

If Equation (3.7) is integrated along the \( x \)-axis between arbitrary points \( x_1 \) and \( x_2 \) and then multiplied by side length \( z \) in the \( z \)-direction to create an area across the photodetector, the solution is the radiant flux incident on a rectangle of side lengths \( x_2 - x_1 = s \) and \( z \). As mirror M2 is only tilted about the \( z \)-axis, variable \( z \) does not need to be included in the integration as it behaves purely as a multiplier to the solution of the integration along the \( x \)-axis. Therefore:
Chapter 3

\[ \Phi_e = z \cdot \int_1^2 \, dx \]  
\[ = 2 \left( \frac{n_{RI} e_0 c}{2} \right) E_0^2 z \int_{x_1}^{x_2} (1 + \cos(k(y - 2\Delta d_m - y \cos \theta - x \sin \theta))) \, dx \]  
\[ = 2 \left( \frac{n_{RI} e_0 c}{2} \right) E_0^2 z \int_{x_1}^{x_2} \left( \frac{1}{k \sin \theta} \left( \sin(k(x \sin \theta + y \cos \theta - y + 2\Delta d_m)) + kx \sin \theta \right) \right) \, dx \]  
\[ (3.8) \]

The radiant flux \( \Phi_e \) given by Equation (3.8) is expressed in Watts (W) and is the total radiant power of the interference beam incident on the defined rectangular active area of the photodetector.

At \( \theta = 0 \), \( \Phi_e = 0/0 \) which is indeterminate, therefore applying L'Hôpital's rule to the integral solution of Equation (3.8) for \( \theta \to 0 \) returns:

\[ \lim_{\theta \to 0} \left( \frac{\sin(k(x \sin \theta + y \cos \theta - y + 2\Delta d_m) + kx \sin \theta)}{k \sin \theta} \right) \]  
\[ = \lim_{\theta \to 0} \left( \frac{\cos(k(x \sin \theta + y \cos \theta - y + 2\Delta d_m)) \cdot k(x \cos \theta - y \sin \theta) + k \cos \theta}{k \cos \theta} \right) \]  
\[ = \lim_{\theta \to 0} \left( \frac{\cos(k2\Delta d_m)) \cdot kx + kx}{k} \right) \]  
\[ = k(1 + \cos(k2\Delta d_m)) \]  
\[ (3.9) \]

Therefore, as \( \theta \to 0 \), the radiant flux derived in Equation (3.8) tends to:

\[ \Phi_e(\theta \to 0) = 2 \left( \frac{n_{RI} e_0 c}{2} \right) E_0^2 z \cdot (1 + \cos(k2\Delta d_m)) \int_{x_1}^{x_2} \]  
\[ = 2 \left( \frac{n_{RI} e_0 c}{2} \right) E_0^2 z \cdot s(1 + \cos(k2\Delta d_m)) \]  
\[ (3.10) \]

It is worth noting that at \( \theta = 0 \), the irradiance in Equation (3.7) reduces to

\[ I_{(\theta=0)} = 2 \left( \frac{n_{RI} e_0 c}{2} \right) E_0^2 z^2(1 + \cos(k2\Delta d_m)) \]  
\[ (3.11) \]

and differs with Equation (3.10) by the area \( z \cdot s \).

Equation (3.10) indicates that as \( \theta \to 0 \) the magnitude of the radiant flux is dependent on the area \( z \cdot s \), \( k \) and \( \Delta d_m \), but is independent of \( y \).

The radiant flux and irradiance in Equations (3.10) and (3.11) are a maximum when \( \cos(k2\Delta d_m) = 1 \), i.e., when the phase lag/lead is

\[ k2\Delta d_m = 2n_f \pi \]  
\[ (3.12) \]

Therefore

\[ 2\Delta d_m = n_f \lambda \]  
\[ (3.13) \]

where \( n_f \) is an integer that expresses the optical path difference \( 2\Delta d_m \) in terms of whole wavelengths. Under this maxima condition, the two wave fronts are perfectly in phase with one another and aligned in direction creating what is termed a light fringe (as opposed to a dark fringe when the two wave fronts are in anti-phase).
Theoretical analysis of fringe line radiant flux on a rectangular photodetector

With the two wave fronts depicted in Figure 2 in phase as well as coincident in direction Equation (3.10) reduces to:

\[
\Phi_e(\theta=0, \Delta d_m=n\lambda) = 2 \left( \frac{nR\varepsilon_0 c}{2} \right) E_0 \frac{z^2}{2s} \quad (3.14)
\]

To demonstrate the behaviour of the radiant flux over differing integral boundaries, Figure 4 shows the radiant flux curves for two sets of integral boundaries that are equal in length with assigned variables defined as follows that have been substituted in Equation (3.10); \( y = 0 \), \( \lambda = 680 \times 10^{-9} \) m therefore \( k = 9,239,978 \), \( \Delta d_m = 0 \) m, integral width \( s = 0.001 \) m, red curve integral boundaries \( x_2 = 0.0005 \) m, \( x_1 = -0.0005 \) m, blue curve integral boundaries \( x_2 = 0.001 \) m, \( x_1 = 0 \) m.

It can be seen from Figure 4 that there are node points at half the normalised radiant flux that are cyclic, which have been termed primary nodes, and this phenomenon is explored further below. What is also noticeable is the two curves converge as \( \theta \to 0 \) as predicted in Equation (3.10).

![Figure 4: Normalised radiant flux vs. varying \( \theta \) for overlapping equal width active areas](image)

3.2.1.2 Identifying specific wave front angles \( \theta_n \) with invariable \( \Phi_e \) for two sets of integral boundaries

To analyse the effect of wave front tilt angle on two separate rectangular areas of equal size and determine the node points observed in Figure 4, consider only the integral solution of Equation (3.8). If we define the two intervals along the \( x \)-axis with upper and lower limits \( x_1, x_2 \) \& \( x_3, x_4 \) such that \( x_2 - x_1 = x_4 - x_3 = s \) and substitute in the Equation (3.8) we get after simplification Equations (3.15) and (3.16):

\[
\Phi_e(x_1, x_2) \propto \frac{1}{k \sin \theta} \left[ \sin(k(x_2 \sin \theta + y \cos \theta - y + 2\Delta d_m)) + kx_2 \sin \theta 
- \sin(k(x_1 \sin \theta + y \cos \theta - y + 2\Delta d_m)) - kx_1 \sin \theta \right] \quad (3.15)
\]

\[
\Phi_e(x_3, x_4) \propto \frac{1}{k \sin \theta} \left[ \sin(k\Delta d_m) + kx_4 \sin \theta 
- \sin(k(x_3 \sin \theta + y \cos \theta - y + 2\Delta d_m)) - kx_3 \sin \theta \right] \quad (3.16)
\]
To solve for $\theta$ and $x$, let Equation (3.15) = Equation (3.16):

$$
\sin(k(x_2 \sin \theta + y \cos \theta - y + 2\Delta d_m)) + kx_2 \sin \theta \\
- \sin(k(x_1 \sin \theta + y \cos \theta - y + 2\Delta d_m)) - kx_1 \sin \theta
= \sin(k(x_4 \sin \theta + y \cos \theta - y + 2\Delta d_m)) + kx_4 \sin \theta \\
- \sin(k(x_3 \sin \theta + y \cos \theta - y + 2\Delta d_m)) - kx_3 \sin \theta
$$

(3.17)

$$
\sin(k(x_2 \sin \theta + y \cos \theta - y + 2\Delta d_m)) - \sin(k(x_1 \sin \theta + y \cos \theta - y + 2\Delta d_m)) \\
= \sin(k(x_4 \sin \theta + y \cos \theta - y + 2\Delta d_m)) \\
- \sin(k(x_3 \sin \theta + y \cos \theta - y + 2\Delta d_m))
$$

(3.18)

Applying the identity in Equation (3.19)

$$
\sin A - \sin B = 2 \sin \left( \frac{A-B}{2} \right) \cos \left( \frac{A+B}{2} \right)
$$

(3.19)

and simplifying, Equation (3.18) yields:

$$
2 \sin \left( \frac{k(x_2 \sin \theta + y \cos \theta - y + 2\Delta d_m) - (x_1 \sin \theta + y \cos \theta - y + 2\Delta d_m)}{2} \right) \\
\cos \left( \frac{k(x_2 \sin \theta + y \cos \theta - y + 2\Delta d_m) + (x_1 \sin \theta + y \cos \theta - y + 2\Delta d_m)}{2} \right)
$$

(3.20)

$$
2 \sin \left( \frac{k(x_4 \sin \theta + y \cos \theta - y + 2\Delta d_m) - (x_3 \sin \theta + y \cos \theta - y + 2\Delta d_m)}{2} \right) \\
\cos \left( \frac{k(x_4 \sin \theta + y \cos \theta - y + 2\Delta d_m) + (x_3 \sin \theta + y \cos \theta - y + 2\Delta d_m)}{2} \right)
$$

(3.21)

$$
\sin \left( \frac{ks \sin \theta}{2} \right) \cos(\Delta d_m) \\
- \sin \left( \frac{ks \sin \theta}{2} \right) \cos \left( \frac{k((x_4 + x_3) \sin \theta + 2y \cos \theta - 2y + 4\Delta d_m)}{2} \right)
$$

(3.22)

To obtain the node points that satisfy Equation (3.22) for the two intervals defined above, i.e., $x_2 - x_1 = x_4 - x_3 = s$, values need to be assigned to these boundary limits. For example, let $x_1$, $x_2$, $x_3$, and $x_4$ be the two intervals depicted along the plane of the photodetector in Figure 2 with values defined as $x_1 = -s$; $x_2 = 0$; $x_3 = -s/2$; $x_4 = s/2$.

Substituting these values in Equation (3.22) and simplifying yields:

$$
\sin \left( \frac{ks \sin \theta}{2} \right) \cos \left( \frac{k((-s) \sin \theta + 2y \cos \theta - 2y + 4\Delta d_m)}{2} \right) \\
- \sin \left( \frac{ks \sin \theta}{2} \right) \cos \left( \frac{k((0) \sin \theta + 2y \cos \theta - 2y + 4\Delta d_m)}{2} \right) = 0
$$

(3.23)
The theoretical analysis of fringe line radiant flux on a rectangular photodetector

\[
\sin \left( \frac{ks \sin \theta}{2} \right) \cdot \left[ \cos \left( \frac{k((-s) \sin \theta + 2y \cos \theta - 2y + 4\Delta d_m)}{2} \right) - \cos \left( \frac{k(2y \cos \theta - 2y + 4\Delta d_m)}{2} \right) \right] = 0 \tag{3.24}
\]

The equality of Equation (3.24) is satisfied if:

a) \( \sin \left( \frac{ks \sin \theta}{2} \right) = 0 \) and/or

b) \( \cos \left( \frac{k((-s) \sin \theta + 2y \cos \theta - 2y + 4\Delta d_m)}{2} \right) - \cos \left( \frac{k(2y \cos \theta - 2y + 4\Delta d_m)}{2} \right) = 0 \)

Solving:

a) is satisfied when \( \frac{ks \sin \theta}{2} = n_p \pi \), where \( n_p \) is an integer and is the number of what is termed a primary node (see Figure 4), resulting in:

\[
\sin \theta_{n_p} = \frac{2n_p \pi}{ks} \quad \text{or} \quad \theta_{n_p} = \frac{n_p \pi}{s} \tag{3.25}
\]

If small angles are considered, implying \( \sin \theta_{n_p} = \theta_{n_p} \), where the intersection of the two radiant flux curves at \( \theta_{n_p} \) is called a primary node and \( \theta_{n_p} \) is called the primary node angle.

b) is satisfied when

\[
\cos \left( \frac{k((-s) \sin \theta + 2y \cos \theta - 2y + 4\Delta d_m)}{2} \right) = \cos \left( \frac{k(2y \cos \theta - 2y + 4\Delta d_m)}{2} \right) \tag{3.26}
\]

There are two solutions that satisfy Equation (3.26):

i. \( (-s) \sin \theta = 0 \) therefore \( \sin \theta_0 = 0 \) where \( \theta_0 = 0 \). As only small angles are of concern, \( \theta = n \pi \) is irrelevant. Note that \( \theta_0 = 0 \) is co-incident with primary node \( n_p = 0 \) from Equation (3.25).

ii. The cosines are identical for \( n_s \pi \pm \delta/2 \), where \( \delta \) is a phase shift and \( n_s \) is an integer related to secondary nodes, i.e.,:

\[
\cos \left( n_s \pi + \frac{\delta}{2} \right) = \cos \left( n_s \pi - \frac{\delta}{2} \right) \tag{3.27}
\]

\( n_s \) can be zero if the data is mirrored about \( \theta = 0 \), and \( n_s = 1 \) if the data is mirrored about \( \theta = \pi \).

From Equations (3.26) and (3.27) there are 2 identities:

\[
\cos \left( \frac{k((-s) \sin \theta + 2y \cos \theta - 2y + 4\Delta d_m)}{2} \right) = \cos \left( n_s \pi - \frac{\delta}{2} \right) \tag{3.28}
\]

\[
\cos \left( \frac{k(2y \cos \theta - 2y + 4\Delta d_m)}{2} \right) = \cos \left( n_s \pi + \frac{\delta}{2} \right) \tag{3.29}
\]

As the left- and right-hand terms refer to the same angle \( \theta \).
Chapter 3

\[ k(-s \sin \theta + 2y \cos \theta - 2y + 4\Delta d_m) = 2n_s \pi - \delta \]  \hfill (3.30)
\[ k(2y \cos \theta - 2y + 4\Delta d_m) = 2n_s \pi + \delta \]  \hfill (3.31)
\[ k(-s \sin \theta + 2y \cos \theta - 2y + 4\Delta d_m) + \delta = 2n_s \pi \]  \hfill (3.32)
\[ k(2y \cos \theta - 2y + 4\Delta d_m) - \delta = 2n_s \pi \]  \hfill (3.33)

Therefore:

\[ k(-s \sin \theta + 2y \cos \theta - 2y + 4\Delta d_m) + \delta = k(2y \cos \theta - 2y + 4\Delta d_m) - \delta \]  \hfill (3.34)

and

\[ 2\delta = k(2y \cos \theta - 2y + 4\Delta d_m) - k(-s \sin \theta + 2y \cos \theta - 2y + 4\Delta d_m) \]
\[ = k2y \cos \theta - k2y + k4\Delta d_m + ks \sin \theta - k2y \cos \theta + k2y - k4\Delta d_m \]
\[ = ks \sin \theta \]  \hfill (3.35)

The \(2n_s \pi\) term in Equations (3.32) and (3.33) lies exactly between

\[ \cos \left(\frac{k((-s) \sin \theta + 2y \cos \theta - 2y + 4\Delta d_m)}{2}\right) \]  \hfill (3.36)

and

\[ \cos \left(\frac{k(2y \cos \theta - 2y + 4\Delta d_m)}{2}\right) \]  \hfill (3.37)

as revealed from Equation (3.35), therefore:

\[ 2n_s \pi = k(2y \cos \theta - 2y + 4\Delta d_m) = \frac{ks \sin \theta}{2} \]  \hfill (3.38)

This eliminates the unknown term \(\delta\) and provides the function for secondary nodes. Simplifying Equation (3.38) renders:

\[ 2n_s \pi = k \left(2y \cos \theta - 2y + 4\Delta d_m - \frac{s \sin \theta}{2}\right) \]  \hfill (3.39)
\[ n_s \pi = k \left(y \cos \theta - y + 2\Delta d_m - \frac{s \sin \theta}{4}\right) \]  \hfill (3.40)
\[ = ky \cos \theta - ky + k2\Delta d_m - \frac{ks \sin \theta}{4} \]

Rearranging and grouping terms yields:

\[ \frac{ks \sin \theta}{4} = -n_s \pi + ky \cos \theta - ky + k2\Delta d_m \]  \hfill (3.41)
\[ \sin \theta = -\frac{n_s \pi}{ks} + \frac{4y \cos \theta}{s} - \frac{4y}{s} + \frac{8\Delta d_m}{s} \]  \hfill (3.42)
\[ \sin^2 \theta = \left(\frac{4n_s \pi}{ks}\right)^2 + \left(\frac{4y}{s}\right)^2 \cos^2 \theta + \left(\frac{8\Delta d_m}{s}\right)^2 - \frac{32n_s \pi y}{ks^2} \cos \theta \]
\[ + \frac{32n_s \pi y}{ks^2} + \frac{64n_s \pi \Delta d_m}{ks^2} - \frac{32y^2}{s^2} \cos \theta + \frac{64y \Delta d_m}{s^2} \cos \theta \]  \hfill (3.43)
Theoretical analysis of fringe line radiant flux on a rectangular photodetector

\[
\sin^2 \theta = \left( \frac{4n_s \pi}{ks} \right)^2 + \left( \frac{4y}{s} \right)^2 \cos^2 \theta + \left( \frac{4y}{s} \right)^2 + \left( \frac{8\Delta d_m}{s} \right)^2 + \frac{32n_s \pi y}{ks^2} \cos \theta - \frac{32n_s \pi y}{ks^2} \cos \theta
\]

(3.44)

\[
\left( \frac{4n_s \pi}{ks} \right)^2 + \frac{4y}{s} \cos^2 \theta + \left( \frac{4y}{s} \right)^2 + \left( \frac{8\Delta d_m}{s} \right)^2 - \frac{32n_s \pi y}{ks^2} \cos \theta + \frac{32n_s \pi y}{ks^2} \cos \theta
\]

(3.45)

\[
\left( \frac{4n_s \pi}{ks} \right)^2 + \frac{4y}{s} \cos^2 \theta + \left( \frac{4y}{s} \right)^2 + \left( \frac{8\Delta d_m}{s} \right)^2 - \frac{32n_s \pi y}{ks^2} \cos \theta + \frac{32n_s \pi y}{ks^2} \cos \theta
\]

(3.46)

\[
\left( \frac{4y}{s} \right)^2 + 1 \cos^2 \theta + \left( \frac{64\pi y \Delta d_m}{s^2} \right) - \frac{32y}{s^2} - \frac{32n_s \pi y}{ks^2} \cos \theta + \frac{32n_s \pi y}{ks^2} \cos \theta
\]

(3.47)

Let

\[
A = \left( \frac{4y}{s} \right)^2 + 1
\]

(3.48)

\[
B = \frac{64\pi y \Delta d_m}{s^2} - \frac{32y}{s^2} - \frac{32n_s \pi y}{ks^2}
\]

(3.49)

\[
C = \left( \frac{4n_s \pi}{ks} \right)^2 + \frac{4y}{s} \left( \frac{8\Delta d_m}{s} \right)^2 + \frac{32n_s \pi y}{ks^2} - \frac{64n_s \pi \Delta d_m}{ks^2} - \frac{64\pi y \Delta d_m}{s^2} - 1
\]

(3.50)

then

\[
A \cos^2 \theta + B \cos \theta + C = 0
\]

(3.51)

and

\[
\cos \theta = \cos_{s,2} \theta_{n_s} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}
\]

(3.52)

By way of demonstrating the presence of secondary nodes for the case with integral boundaries defined above as \(x_1 = -s; x_2 = 0; x_3 = -s/2; x_4 = s/2\), firstly, assume \(\Delta d_m = 0\) m.

Equations (3.48), (3.49) and (3.50) now reduce to:

\[
A = \left( \frac{4y}{s} \right)^2 + 1
\]

(3.53)

\[
B = -\frac{32y^2}{s^2} - \frac{32n_s \pi y}{ks^2}
\]

(3.54)

\[
C = \left( \frac{4n_s \pi}{ks} \right)^2 + \left( \frac{4y}{s} \right)^2 + \frac{32n_s \pi y}{ks^2} - 1
\]

(3.55)
Then, the variables in Equations (3.53), (3.54) and (3.55) are assigned values, for example those defined in Equation (3.56).

\[
y = 0.02 \text{ m} \\
\lambda = 680 \times 10^{-9} \text{ m} \text{ therefore } k = 9,239,978
\]

integral width \( s = 0.001 \text{ m} \)

Finally, Equation (3.52) is solved for different \( n_s \) (positive and negative) to find \( \theta_{n_s} \) where the radiant flux curves intersect at the secondary node points.

**Secondary nodes** for the above defined conditions are returned for

\[
\cos \theta_{n_s} = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \quad (3.57)
\]

when \( n_s \leq 0 \) and \( \theta_{n_s} \geq 0 \) and when \( 0 \leq n_s \leq 4 \) for \( \theta_{n_s} < 0 \).

Also

\[
\cos \theta_{n_s} = \frac{-B - \sqrt{B^2 - 4AC}}{2A} \quad (3.58)
\]

Consider the solution defined by Equation (3.57) and let \( n_s = -1 \). The secondary node point is calculated by this solution, i.e., \( \theta_{-1} = 0.07409^\circ \) and confirmed by substituting the above conditions into the integral part of Equation (3.8).

Mirror M2 in Figure 2 can tilt left or right of the \( y \)-axis, therefore \( \theta_{n_s} \) can be positive or negative, therefore returning secondary nodes of both polarities for \( \theta_{n_s} \) in Equations (3.57) and (3.58).

From the above derivation of the secondary nodes, a specific solution was obtained for the defined intervals specified as \( x_1 = -s; x_2 = 0; x_3 = -s/2; x_4 = s/2 \) and the variable values assigned in Equation (3.56).

The occurrence of secondary nodes is unique and specific to the defined integral boundaries \( x_1, x_2 \& x_3, x_4 \), the values of \( k, y \) and \( \Delta d_m \). Consequently, the cosine expression in Equation (3.22) has to be solved accordingly with its own set of boundary conditions. The wave front angle at secondary node angles \( \theta_{n_s} \) is incidental, unlike at primary nodes, which is cyclic and dependent only on \( k \) and \( s \) (sine expression in Equation (3.22)).

3.2.1.3 Determination of the magnitude of \( \Phi_e \) at wave front angles \( \theta_{n_p} \)

To determine the magnitude of the radiant flux of the fringe pattern at the primary nodes we solve Equations (3.15) and (3.16) independently for the intervals previously defined i.e., \( x_1 = -s; x_2 = 0; x_3 = -s/2; x_4 = s/2 \). Beginning with Equation (3.15):
Theoretical analysis of fringe line radiant flux on a rectangular photodetector

\( \Phi_{e(x_1,x_2)} \)

\[ \propto \frac{1}{k \sin \theta} \left[ \sin(k(x_2 \sin \theta + y \cos \theta - y + 2\Delta d)) + kx_2 \sin \theta 
- \sin(k(x_1 \sin \theta + y \cos \theta - y + 2\Delta d)) - kx_1 \sin \theta \right] \]  

\[ \propto \frac{1}{k \sin \theta} \left[ \sin(k(0 \sin \theta + y \cos \theta - y + 2\Delta d)) + k0 \sin \theta 
- \sin(k(-s \sin \theta + y \cos \theta - y + 2\Delta d)) + ks \sin \theta \right] \]

\[ \propto \frac{1}{k \sin \theta} \left[ \sin(k(y \cos \theta - y + 2\Delta d_m)) 
- \sin(k(-s \sin \theta + y \cos \theta - y + 2\Delta d_m)) + ks \sin \theta \right] \]

For small angles \(\sin \theta = \theta\), therefore Equation (3.59) becomes:

\[ \Phi_{e(x_1,x_2)} \propto \frac{1}{k\theta} \left[ \sin(k(y \cos \theta - y + 2\Delta d_m)) 
- \sin(k(-s \theta + y \cos \theta - y + 2\Delta d_m)) + ks \theta \right] \]  

(3.60)

At \(\theta_0\), Equation (3.60) reduces to 0/0, which is indeterminate. Therefore applying L'Hôpital's rule:

\[ \lim_{\theta \to 0} \frac{1}{k \theta} \left[ \sin(k(y \cos \theta - y + 2\Delta d_m)) - \sin(k(-s \theta + y \cos \theta - y + 2\Delta d_m)) + ks \theta \right] \]  

(3.61)

\[ = \lim_{\theta \to 0} \frac{1}{k} \left[ \cos(k(y \cos \theta - y + 2\Delta d_m)) \cdot (-ky \sin \theta) 
- \cos(k(-s \theta + y \cos \theta - y + 2\Delta d_m)) \cdot (k(-s - y \sin \theta)) + ks \right] \]

\[ = \lim_{\theta \to 0} \frac{1}{k} \left[ \cos(k(y - y + 2\Delta d_m)) \cdot (-ky\theta) - \cos(k(-s \theta + y - y + 2\Delta d_m)) 
\cdot (k(-s - y \theta)) + ks \right] \]

\[ = \lim_{\theta \to 0} \frac{ks \cos(k(2\Delta d_m)) + ks}{k} \]

\[ = \lim_{\theta \to 0} \frac{ks \cos(k(2\Delta d_m))}{k} + ks \]

\[ = s(\cos(2\Delta d_m) + 1) \]

Repeating the above for Equation (3.16) yields:

\[ \Phi_{e(x_3,x_4)} \]

\[ \propto \frac{1}{k \sin \theta} \left[ \sin \left( k \left( \frac{s}{2} \sin \theta + y \cos \theta - y + 2\Delta d_m \right) \right) + k\frac{s}{2} \sin \theta 
- \sin \left( k \left( -\frac{s}{2} \sin \theta + y \cos \theta - y + 2\Delta d_m \right) \right) + k\frac{s}{2} \sin \theta \right] \]

(3.62)

\[ \propto \frac{1}{k \sin \theta} \left[ \sin \left( k \left( \frac{s}{2} \sin \theta + y \cos \theta - y + 2\Delta d_m \right) \right) + ks \sin \theta 
- \sin \left( k \left( -\frac{s}{2} \sin \theta + y \cos \theta - y + 2\Delta d_m \right) \right) + ks \sin \theta \right] \]

For small angles \(\sin \theta = \theta\), therefore Equation (3.62) becomes:

\[ \Phi_{e(x_3,x_4)} \propto \frac{1}{k\theta} \left[ \sin \left( k \left( \frac{s}{2} \theta + y \cos \theta - y + 2\Delta d_m \right) \right) + ks \theta 
- \sin \left( k \left( -\frac{s}{2} \theta + y \cos \theta - y + 2\Delta d_m \right) \right) \right] \]  

(3.63)
Chapter 3

At $\theta_0$, Equation (3.60) reduces to $0/0$, which again is indeterminate. Applying L'Hôpital's rule

$$\lim_{\theta \to 0} \frac{1}{k} \left[ \sin \left( k \left( \frac{s}{2} \theta + y \cos \theta - y + 2\Delta d_m \right) \right) + k s \theta ight]$$

$$= \lim_{\theta \to 0} \frac{1}{k} \left[ \cos \left( k \left( \frac{s}{2} \theta + y \cos \theta - y + 2\Delta d_m \right) \right) \cdot k \left( \frac{s}{2} \theta - y \sin \theta \right) + k s \right]$$

$$= \lim_{\theta \to 0} \frac{\cos(k2\Delta d_m) \cdot k \left( \frac{s}{2} \right) + k s - \cos(k2\Delta d_m) \cdot k \left( -\frac{s}{2} \right)}{k}$$

$$= \lim_{\theta \to 0} \frac{k s \cos(k2\Delta d_m) + k s}{k}$$

$$= s \cos(k2\Delta d_m) + 1$$

Equations (3.61) and (3.64) show that as $\theta \to 0$ the radiant flux becomes equal for the integral boundaries $x_1 = -s$; $x_2 = 0$; $x_3 = -s/2$; $x_4 = s/2$. Substituting the result of Equation (3.61) or Equation (3.64) into Equation (3.8) gives Equation (3.65), which shows the magnitude of the radiant flux as $\theta \to 0$ is dependent on $k$ and $\Delta d_m$ and independent of $y$:

$$\Phi_e(\theta \to 0) = 2 \left( \frac{n_{RI} E_0 C}{2} \right) E_0 \frac{z \cdot s \cos(k2\Delta d_m) + 1}{2}$$

(3.65)

Note: Equations (3.10) and (3.65) are equal, resulting in maximum radiant flux for translations where $2\Delta d_m = n_\lambda$ (Equation (3.13)):

$$\Phi_e(\theta \to 0, 2\Delta d_m = n_\lambda) = 2 \left( \frac{n_{RI} E_0 C}{2} \right) E_0 \frac{z \cdot 2s}{2}$$

(3.66)

To work out the value of radiant flux for all other values of $\theta_{n_p}$, i.e., $\theta_{1,2,3,...}$ substitute $\theta_{n_p} = n_p \lambda/s$ from Equation (3.25) into Equations (3.60) and (3.63) independently. Beginning with Equation (3.60) and using the identity in Equation (3.19):
Theoretical analysis of fringe line radiant flux on a rectangular photodetector

\[ \Phi_{e(x_1,x_2)} \propto \frac{1}{k^2 \frac{n_0 \lambda}{s}} \left[ \sin \left( k \left( y \cos \left( \frac{n_0 \lambda}{s} \right) - y + 2\Delta d_m \right) \right) - \sin \left( k \left( -s \frac{n_0 \lambda}{s} + y \cos \left( \frac{n_0 \lambda}{s} \right) - y + 2\Delta d_m \right) \right) + ks \frac{n_0 \lambda}{s} \right] \]

\[ \propto \frac{1}{k^2 \frac{n_0 \lambda}{s}} \left[ \sin \left( k \left( y \cos \left( \frac{n_0 \lambda}{s} \right) - y + 2\Delta d_m \right) \right) - \sin \left( k \left( -n_0 \lambda + y \cos \left( \frac{n_0 \lambda}{s} \right) - y + 2\Delta d_m \right) \right) + kn_0 \lambda \right] \]

\[ \propto \frac{s}{kn_0 \lambda} \left[ 2 \sin \left( \frac{k \left( y \cos \left( \frac{n_0 \lambda}{s} \right) - y + 2\Delta d_m \right)}{2} \right) + \frac{2}{2} k \left(-n_0 \lambda + y \cos \left( \frac{n_0 \lambda}{s} \right) - y + 2\Delta d_m \right) + kn_0 \lambda \right] \cdot \cos \left( \frac{k \left( y \cos \left( \frac{n_0 \lambda}{s} \right) - y + 2\Delta d_m \right) + k \left(-n_0 \lambda + y \cos \left( \frac{n_0 \lambda}{s} \right) - y + 2\Delta d_m \right)}{2} \right) \]

\[ + kn_0 \lambda \]

\[ \propto \frac{s}{kn_0 \lambda} \left[ 2 \sin(n_0 \pi) \cdot \cos \left( k \left( y \cos \left( \frac{n_0 \lambda}{s} \right) - y + 2\Delta d_m - \frac{n_0 \lambda}{s} \right) \right) + kn_0 \lambda \right] \]

Repeating the above for Equation (3.63)
Equations (3.67) and (3.68) give the same result for the integral of the active areas at \(\theta_{np} = \theta_{1,2,3,..}\) and therefore the radiant flux at primary node angles \(\theta_{np} = \theta_{1,2,3,..}\) is:

\[
\Phi_e(\theta_{1,2,3,..}) = 2 \left(\frac{n_R l e_0 c}{2}\right) E_0 z \cdot s
\]  

Equation (3.69) also shows that the radiant flux at \(\theta_{np} = \theta_{1,2,3,..}\) is half maximum (cf. Equation (3.66)), and in contrast to the radiant flux at \(\theta_{np} = \theta_0\) given in Equation (3.65), \(\Phi_e(\theta_{1,2,3,..})\) is independent of \(k\) and \(\Delta d_m\). The reason for this is the width \(s\) of the active area is an integer multiple of the fringe line spacing at primary node angles \(\theta_{np} = \theta_{1,2,3,..}\).

When there is an exact multiple of fringe lines within the active area [18], the radiant flux across the active area is the mean of the maximum constructive and destructive interferences, i.e., 50%. This means that if the active area is moved in either direction along the x-axis, the radiant flux remains static at 0.5 normalised magnitude. If mirror M2 translates, there will be no change in the radiant flux despite the fringe pattern moving back and forth across the active area.

### 3.2.1.4 Determination of the damping function of the radiant flux curve

From Figure 4 it is evident that the normalised radiant flux curve oscillates about 0.5 maximum radiant flux and decreases its wave amplitude about this level with increasing
Theoretical analysis of fringe line radiant flux on a rectangular photodetector

wave front angle $\theta$. As this behaviour constitutes a damped function, the radiant flux decay function can therefore be derived.

For a centred photodetector of width $s$ and positioned about $x = 0$, with integral boundaries $x_2 = +s/2$ and $x_1 = -s/2$, $y = 0$ and $\Delta d_m = 0$, the variant radiant flux $\Phi_{e(x_1,x_2)}$ in Equation (3.8) reduces to:

$$\Phi_{e(x_1,x_2)} \propto \frac{1}{k \sin \theta} \left[ \sin \left( \frac{ks \sin \theta}{2} \right) + \frac{ks \sin \theta}{2} - \sin \left( \frac{-ks \sin \theta}{2} \right) - \frac{-ks \sin \theta}{2} \right]$$

(3.70)

In Equation (3.70), the radiant flux becomes maximal when the variant expression tends to $2s$ as $\theta \to 0$ (cf. Equations (3.14) and (3.66): radiant flux = constant $\times$ $z \times 2s$; $z$ is considered unity as the $z$-direction is perpendicular to fringe lines).

Dividing $\Phi_{e(x_1,x_2)}$ by $2s$ produces normalised radiant flux $\Phi_n$, i.e.:

$$\Phi_n \propto \frac{2 \sin \left( \frac{ks \sin \theta}{2} \right)}{2sk \sin \theta} + \frac{s}{2s}$$

(3.71)

As the normalised radiant flux ranges from 0 to +1, it is converted to a range from −1 to +1 in order to compare it to a standard sine function:

$$\Phi_n \propto 2 \left[ \sin \left( \frac{ks \sin \theta}{2} \right) + \frac{1}{2} \right] - 1$$

(3.72)

Considering that $k = 2\pi/\lambda$ and that a damped function (e.g., cardinal sine) corresponds to an undamped function times a decay function, the decay function $F_D$ is identical to the reciprocal of the denominator $(ks \sin \theta)/2$ in Equation (3.72). For small $\theta$, Equation (3.72) becomes:

$$\Phi_n \propto \sin \left( \frac{ks \theta}{2} \right)$$

(3.73)

This equation defines a damped sine function. The equivalent undamped sine function has the form of

$$A = \sin(2\pi f \theta)$$

(3.74)

Where $A$ is the amplitude and $f$ is the reciprocal value of the 2nd node angle ($= 2 \frac{1}{2}$), i.e., the wave length of the radiant flux function. Substituting this term into Equation (3.74) yields

31
The decay function $F_D$ is obtained when normalising the damped sine function to its undamped counterpart

$$
F_D = \frac{2 \sin \left(\frac{ks \sin \theta}{2}\right)}{sk \sin \theta \sin \left(\frac{\pi s \theta}{\lambda}\right)}
$$

(3.76)

After simplifying and considering that $\theta$ is small

$$
F_D = \frac{2 \sin \left(\frac{ks \theta}{2}\right)}{sk \sin \theta \sin \left(\frac{\pi s \theta}{\lambda}\right)}
$$

(3.77)

Considering that $k = \frac{2\pi}{\lambda}$

$$
F_D = \frac{2}{sk \cdot \frac{1}{\theta}} = \frac{\lambda}{\pi s} \cdot \frac{1}{\theta}
$$

(3.78)

Equation (3.78) is the constitutive equation of the radiant flux decay function at $x = 0$ and $y = 0$. $F_D$ is a reciprocal function of $\theta$, with a decay constant $C_D$ of $\frac{s}{sk}$ or $\frac{\lambda}{\pi s}$.

Multiplying Equation (3.75) by Equation (3.78)

$$
A_{damped} = \left(\frac{2}{sk \theta}\right) \sin \left(\frac{\pi s \theta}{\lambda}\right)
$$

(3.79)

the normalised flux results from adding 1 and dividing the sum by 2

$$
\Phi_n \propto \frac{\left(\frac{2}{sk \theta}\right) \sin \left(\frac{\pi s \theta}{\lambda}\right) + 1}{2}
$$

(3.80)

The decay constant depends on $s$ and $\lambda$. Replacing $\theta$ by the product of the first primary node angle $\theta_{n_p=1}$ of Equation (3.25) and $\theta$ normalised to $\theta_{n_p=1} \left(\frac{\theta}{\theta_{n_p=1}} = m\right)$, i.e. by $m\theta_{n_p=1}$, makes Equation (3.80) independent of any variable and converts it to a unique function applicable to any $s$ and $\lambda$.

Normalising the decay function $C_D$ to the primary node angle $\theta_{n_p}$. 

32
Theoretical analysis of fringe line radiant flux on a rectangular photodetector

\[
\Phi_n \propto \frac{2}{sk\theta_{n_p=1}m} \sin \left( \frac{n\pi \theta}{\lambda} \right) + 1
\]

\[
\propto \frac{2s\lambda}{s2\pi\lambda m} \sin \left( \frac{n\pi \theta}{\lambda} \right) + 1
\]

\[
\propto \frac{1}{\pi m} \sin \left( \frac{n\pi \theta}{\lambda} \right) + 1
\]

where \( m \) is the non-integer form of the integer node numbers \( n_p \). As the reciprocal value of 2\textsuperscript{nd} node angle \( \left( = \frac{2\lambda}{s} \right) \) is the wavelength of the flux function, the wavelength is then expressed as \( m = 2 \), and the reciprocal of the wave front angle is then \( \frac{1}{2\pi} \). From this principle, the multiple of \( \theta_{n_p=1} \) can be calculated numerically where the first minimum occurs as well as the flux value of the first minimum.

3.2.1.5 Effect of distance \( x \) and \( y \) of photodetector from origin with varying \( \theta \)

To determine the effect of distance \( y \) of the photodetector from the origin with varying \( \theta \), the position and slope of the fringe lines needs to be determined. This is done by finding the instances of maximum value of irradiance in Equation (3.7), i.e., when:

\[
\cos(k(y - 2\Delta d_m - y \cos \theta - x \sin \theta)) = 1
\]

Equation (3.82) is true when:

\[
k(y - 2\Delta d_m - y \cos \theta - x \sin \theta) = 2n_f\pi
\]

Where \( n_f \) is the \( n \)th fringe line at a point \((x, y)\) in the fringe pattern. Solving for \( y \) gives:

\[
y(1 - \cos \theta) = x \sin \theta + n_f \lambda + 2\Delta d_m
\]

\[
y = \frac{x \sin \theta}{1 - \cos \theta} + \frac{n_f \lambda + 2\Delta d_m}{1 - \cos \theta}
\]

Equation (3.85) defines the profile of the fringe pattern as illustrated in Figure 2, where \( \sin \theta/(1 - \cos \theta) \) is the slope of the fringe lines and \((n_f \lambda + 2\Delta d_m)/(1 - \cos \theta) \) is the \( y \) intercept. The equation becomes indeterminate at \( \theta = 0 \), which stands to reason as the fringe pattern is uniform across the entire \( x-y \) plane and therefore the photodetector active area also, resulting in no fringe lines being present.

By solving Equation (3.84) for \( n_f \) the number of fringes lines passing over a given point \((x, y)\) can be calculated for \( \theta \) increasing or decreasing from zero to a given wave front angle:

\[
n_f \lambda = y(1 - \cos \theta) - x \sin \theta - 2\Delta d_m
\]

\[
n_f = \frac{y(1 - \cos \theta) - x \sin \theta - 2\Delta d_m}{\lambda}
\]

The \( y \) term in the above equation is positive for \( \theta \neq 0 \) and is symmetrical in shape as a function of \( \theta \). For small angles \( \sin \theta = \theta \), therefore the coefficient of \( x \) is a linear function of angle \( \theta \). \( n_f \) can be positive or negative as it is responsive to which side of the \( y \)-axis the point \((x, y)\) is located and whether \( \theta \) and \( \Delta d_m \) is positive or negative. \( \theta \) is positive when the normal of the moving mirror lies in the \( +x - +y \) quadrant. \( \Delta d_m \) is positive when the distance \( d_1 \) to Mirror M1 is greater than the distance \( d_2 \) to Mirror M2 (Figure 2).
When $\theta = 0$, Equation (3.86) reduces to

$$-2 \Delta d_m = n_f \lambda$$  \hspace{1cm} (3.88)

$n_f$ is the number of fringes counted at a point $(x, y)$ as $\Delta d_m$ is varied.

Assuming $\Delta d_m = 0$, there is a special case in Equation (3.87) as $\theta$ is varied when:

$$\frac{y(1 - \cos \theta)}{\lambda} = \frac{x \sin \theta}{\lambda}$$  \hspace{1cm} (3.89)

and the fringe count $n_f = 0$ despite $\theta > 0$. In this case, fringe lines would have moved over point $(x, y)$ in one direction as $\theta$ is increased and then back again as $\theta$ is increased further to the angle $\theta$ that satisfies Equation (3.89).

As discussed with Equation (3.85), the slope of the fringe lines is given by:

$$\frac{y}{x} = \frac{\sin \theta}{(1 - \cos \theta)} = \cot \left( \frac{\theta}{2} \right)$$  \hspace{1cm} (3.90)

Where $\theta/2$ is the angle of the normal of mirror M2 relative to the $y$-axis (Figure 2). Therefore, there is a set of points $(x, y)$ coincident with the mirror normal that renders $n_f = 0$.

3.2.1.6 Deriving the speed of fringe movement with varying $\theta$

Consider the distances $d_1$ and $d_2$ in Figure 1 to be identical therefore $\Delta d_m = 0$. Equation (3.87) therefore reduces to:

$$n_f = \frac{y(1 - \cos \theta)}{\lambda} - \frac{x \sin \theta}{\lambda}$$  \hspace{1cm} (3.91)

Taking the derivative of the $x$-term delivers the speed of contraction/expansion of the fringe lines at the point $(x, y)$:

$$\frac{d n_f}{d \sin \theta} = -\frac{x}{\lambda}$$  \hspace{1cm} (3.92)

The speed of sideways deflection of the fringe lines at point $(x, y)$ results from calculating the derivative of the $y$-term:

$$\frac{d n_f}{d \sin \theta} = \frac{y \sin \theta}{\lambda \sqrt{1 - \sin^2 \theta}} = \frac{y}{\lambda} \tan \theta$$  \hspace{1cm} (3.93)

The overall fringe movement speed is therefore:

$$\frac{d n_f}{d \sin \theta} = \frac{x}{\lambda} + \frac{y}{\lambda} \tan \theta$$  \hspace{1cm} (3.94)

From Equation (3.94), the direction of fringe movement reverses if:

$$-\frac{x}{\lambda} + \frac{y}{\lambda} \tan \theta = 0$$  \hspace{1cm} (3.95)

and therefore the angle of movement reversal $\theta_{rev}$ is:
Theoretical analysis of fringe line radiant flux on a rectangular photodetector

\[ \theta_{rev} = \tan^{-1} \frac{x}{y} = \sin^{-1} \frac{x}{\sqrt{x^2 + y^2}} \]  
(3.96)

That is, when the perpendicular of the wave front from the origin points towards +x. The mirror tilt angle is therefore \( \theta_{rev}/2 \). If \( y \to 0 \), \( \theta_{rev} \to \pi/2 \), i.e., the smaller \( y \), the larger is \( \theta_{rev} \).

From Equation (3.91), the fringe count returns to zero if the numerical value of the \( x \)- and \( y \)-terms cancel each other out, i.e.,:

\[
\begin{align*}
y - y \cos \theta - x \sin \theta &= 0 \\
1 - \frac{x}{y} \sin \theta &= \cos \theta \\
1 - \frac{x}{y} \sin \theta &= \sqrt{1 - \sin^2 \theta} \\
1 + \frac{x^2}{y^2} \sin^2 \theta - 2 \frac{x}{y} \sin \theta &= 1 - \sin^2 \theta \\
\frac{x^2}{y^2} \sin^2 \theta - 2 \frac{x}{y} \sin \theta &= -\sin^2 \theta \\
\frac{x^2}{y^2} \sin \theta - 2 \frac{x}{y} &= -\sin \theta \\
\frac{x^2}{y^2} \sin \theta + \sin \theta &= 2 \frac{x}{y} \\
\sin \theta \left(\frac{x^2}{y^2} + 1\right) &= 2 \frac{x}{y}
\end{align*}
\]

which has solution:

\[
\sin \theta = \frac{2 \frac{x}{y}}{\left(\frac{x^2}{y^2} + 1\right)} = \frac{2 \frac{x}{y}}{\frac{x^2 + y^2}{y^2}} = \frac{2xy^2}{y(x^2 + y^2)} = \frac{2xy}{x^2 + y^2}
\]

(3.105)

This occurs at the angle \( \theta_{n=0} \):

\[
\theta_{n=0} = \sin^{-1} \frac{2xy}{(x^2 + y^2)}
\]

(3.106)

If \( x = y \), then \( \theta_{n=0} = \pi/2 \), if \( x > y \), then \( \theta_{n=0} > \pi/2 \) and conversely, if \( y > x \), then \( \theta_{n=0} < \pi/2 \). The relationship between \( \theta_{n=0} \) and \( \theta_{rev} \) results in the identity of \( \theta_{n=0} \equiv 2\theta_{rev} \).

From Equations (3.96) and (3.106):

\[
\theta_{rev} = \frac{1}{2} \sin^{-1} \frac{2xy}{(x^2 + y^2)} = \sin^{-1} \frac{x}{\sqrt{x^2 + y^2}}
\]

(3.107)

Resulting in:

\[
\sin 2\theta_{rev} = 2 \sin \theta_{rev} \cos \theta_{rev} = \frac{2xy}{(x^2 + y^2)}
\]

(3.108)

and proving the identity of \( \theta_{n=0} \equiv 2\theta_{rev} \) being correct.
3.2.1.7 Fringe count speed

The fringe count speed is dependent on the dynamic motion of moving mirror M2 in terms of both mirror tilt and translation. If the motion is defined by a differentiable function then the fringe count speed can be determined. For example, let the following time dependent functions be applied to $\theta(t)$ and $\Delta d_m(t)$ where the frequency of oscillation for both is identical:

\[ \theta(t) = \theta_{\text{max}} \sin(\omega t) \]  \hspace{1cm} (3.109)

\[ \Delta d_m(t) = \Delta d_{\text{max}} \sin(\omega t) \]  \hspace{1cm} (3.110)

Substituting in Equation (3.87) produces:

\[ n_f(t) = \frac{y(1 - \sqrt{1 - \sin^2(\theta_{\text{max}} \sin(\omega t))}) - x \sin(\theta_{\text{max}} \sin(\omega t))}{\lambda} - \frac{2\Delta d_{\text{max}} \sin(\omega t)}{\lambda} \]  \hspace{1cm} (3.111)

The fringe count speed can be derived by taking the derivative of $n_f(t)$ with respect to time.

\[ \frac{dn_f(t)}{dt} = \frac{y \omega \theta_{\text{max}} \cos(\omega t) \cos(\theta_{\text{max}} \sin(\omega t)) \sin(\theta_{\text{max}} \sin(\omega t))}{\sqrt{1 - \sin^2(\theta_{\text{max}} \sin(\omega t))}} \]

\[ - \frac{x}{\lambda} \omega \theta_{\text{max}} \cos(\omega t) \cos(\theta_{\text{max}} \sin(\omega t)) - \frac{2}{\lambda} \omega \Delta d_{\text{max}} \cos(\omega t) \]

\[ = \frac{y \omega \theta_{\text{max}} \cos(\omega t) \sin(\theta_{\text{max}} \sin(\omega t))}{\lambda} \cos(\theta_{\text{max}} \sin(\omega t)) \]

\[ - \frac{x}{\lambda} \omega \theta_{\text{max}} \cos(\omega t) \cos(\theta_{\text{max}} \sin(\omega t)) - \frac{2}{\lambda} \omega \Delta d_{\text{max}} \cos(\omega t) \]

\[ = \frac{y}{\lambda} \omega \theta_{\text{max}} \cos(\omega t) \sin(\theta_{\text{max}} \sin(\omega t)) \]

\[ - \frac{x}{\lambda} \omega \theta_{\text{max}} \cos(\omega t) \cos(\theta_{\text{max}} \sin(\omega t)) - \frac{2}{\lambda} \omega \Delta d_{\text{max}} \cos(\omega t) \]  \hspace{1cm} (3.112)

To find the instances when the fringe count is minimum and maximum let Equation (3.112) = 0

\[ \frac{y}{\lambda} \omega \theta_{\text{max}} \cos(\omega t) \sin(\theta_{\text{max}} \sin(\omega t)) - \frac{x}{\lambda} \omega \theta_{\text{max}} \cos(\omega t) \cos(\theta_{\text{max}} \sin(\omega t)) - \frac{2}{\lambda} \omega \Delta d_{\text{max}} \cos(\omega t) = 0 \]  \hspace{1cm} (3.113)

The extreme values of Equation (3.113) are when:

\[ \cos(\omega t) = 0 \Rightarrow \omega t = \pm \frac{(2n_0 + 1)\pi}{2} \]  \hspace{1cm} (3.114)

where $n_0$ is an integer, therefore:

\[ t = \pm \frac{(2n_0 + 1)}{4f} \]  \hspace{1cm} (3.115)

Equation (3.115) indicates at what instances in time the fringe count is a maxima or minima, where $f$ is the frequency of oscillation of the mirror tilting and translating.
To find the instances of maxima and minima fringe count speed, differentiate Equation (3.112) and equate the result to zero:

\[
\frac{d}{dt} \left( \frac{y}{\lambda} \omega \theta_{\text{max}} \cos(\omega t) \sin(\theta_{\text{max}} \sin(\omega t)) - \frac{x}{\lambda} \omega \theta_{\text{max}} \cos(\omega t) \cos(\theta_{\text{max}} \sin(\omega t)) - \frac{2}{\lambda} \omega \Delta d_{\text{max}} \cos(\omega t) \right) = \frac{y}{\lambda} \omega^2 \theta_{\text{max}}^2 (1 - \sin^2(\omega t)) \cos(\theta_{\text{max}} \sin(\omega t)) - \frac{y}{\lambda} \theta_{\text{max}} \omega^2 \sin(\omega t) \sin(\theta_{\text{max}} \sin(\omega t)) - \frac{y}{\lambda} \omega^2 \theta_{\text{max}}^2 (1 - \sin^2(\omega t)) \sin(\theta_{\text{max}} \sin(\omega t)) - \frac{y}{\lambda} \theta_{\text{max}} \omega^2 \sin(\omega t) \cos(\theta_{\text{max}} \sin(\omega t)) + \frac{2}{\lambda} \omega^2 \Delta d_{\text{max}} \sin(\omega t)
\]

Simplifying and equating to zero:

\[
y \theta_{\text{max}}^2 \cos^2(\omega t) \left( \cos(\theta_{\text{max}} \sin(\omega t)) - \sin(\theta_{\text{max}} \sin(\omega t)) \right) - \frac{y}{\lambda} \theta_{\text{max}} \omega^2 \sin(\omega t) \left( \sin(\theta_{\text{max}} \sin(\omega t)) + \cos(\theta_{\text{max}} \sin(\omega t)) \right) + 2 \Delta d_{\text{max}} \sin(\omega t) = 0
\]

For small \( \theta_{\text{max}} \) angles, \( \sin(\theta_{\text{max}} \sin(\omega t)) = \theta_{\text{max}} \sin(\omega t) \) and \( \cos(\theta_{\text{max}} \sin(\omega t)) = 1 \), therefore:

\[
y \theta_{\text{max}}^2 \cos^2(\omega t) \left( 1 - (\theta_{\text{max}} \sin(\omega t)) \right) - y \theta_{\text{max}} \omega^2 \sin(\omega t) (\theta_{\text{max}} \sin(\omega t) + 1) + 2 \Delta d_{\text{max}} \sin(\omega t) = 0
\]

Substituting \( \sin(\omega t) = A \) and \( \cos(\omega t) = B \) and simplifying,

\[
y \theta_{\text{max}}^2 B^2 - y \theta_{\text{max}}^3 B A - y \theta_{\text{max}}^2 A^2 - y \theta_{\text{max}} A + 2 \Delta d_{\text{max}} A = 0 \quad (3.119)
\]  
\[
y \theta_{\text{max}}^2 (1 - A^2) - y \theta_{\text{max}}^3 (1 - A^2) A - y \theta_{\text{max}}^2 A^2 - y \theta_{\text{max}} A + 2 \Delta d_{\text{max}} A = 0 \quad (3.120)
\]  
\[
y \theta_{\text{max}}^2 - y \theta_{\text{max}}^3 A^2 - y \theta_{\text{max}} A^3 + y \theta_{\text{max}}^2 A^2 - y \theta_{\text{max}} A + 2 \Delta d_{\text{max}} A = 0 \quad (3.121)
\]  
\[
y \theta_{\text{max}}^2 - 2 y \theta_{\text{max}}^2 A^2 - y \theta_{\text{max}} A + y \theta_{\text{max}}^3 A^3 - y \theta_{\text{max}} A + 2 \Delta d_{\text{max}} A = 0 \quad (3.122)
\]

Substituting \( y \theta_{\text{max}} = C, y \theta_{\text{max}}^2 = D, y \theta_{\text{max}}^3 = E \) and \( 2 \Delta d_{\text{max}} = F \) renders

\[
EA^3 - 2DA^2 + A(F - E - C) + D = 0 \quad (3.123)
\]

As this a polynomial to the third power it has real and complex roots, of which the latter can be ignored. The real solution will reveal the instances in time the maxima and minima fringe count speed occur.

The dynamic motion of the interferometer mirror in the above example was defined by Equations (3.109) and (3.110). To find the fringe count speed for any other motion defined by a differentiable function, the same procedure must be applied.

### 3.2.1.8 Fringe transition speed

Fringe transition speed is the speed in ms\(^{-1}\) at which the fringe lines cross a point in the fringe pattern due to the dynamic moving mirror tilt and translation. It is the product of the distance between the fringe lines and the fringe count speed.
To work out the distance between fringe lines use is made of Equation (3.87) for consecutive fringe lines.

\[
\eta_f(n+1) = \frac{y(1 - \cos \theta)}{\lambda} \frac{x_{n+1} \sin \theta}{\lambda} - \frac{2\Delta d_m}{\lambda} \\
\eta_f(n) = \frac{y(1 - \cos \theta)}{\lambda} \frac{x_n \sin \theta}{\lambda} - \frac{2\Delta d_m}{\lambda}
\]  

The values \( y \) (distance to the plane of the photodetector) and \( \Delta d_m \) (distance between fixed and moving mirrors) will be identical for both the above equations, therefore:

\[
\eta_f(n+1) - \eta_f(n) = \frac{x_{n+1} \sin \theta}{\lambda} - \frac{x_n \sin \theta}{\lambda} \\
1 = \frac{\Delta d_f \sin \theta}{\lambda}
\]  

where \( \Delta d_f \) is the distance between fringe lines in metres, \( \lambda \) is the wavelength of the laser in metres and \( \theta \) is the angle of the tilted wave front in radians. For small angles \( \sin \theta = \theta \), resulting in:

\[
\Delta d_f = \frac{\lambda}{\theta}
\]

Fringe count speed is obtained by taking the derivative of Equation (3.87) wrt time, therefore, fringe transition speed is given by:

\[
\frac{\lambda}{\theta} \frac{d}{dt} \left( \frac{y(1 - \cos \theta(t))}{\lambda} \frac{x \sin \theta(t)}{\lambda} - \frac{2\Delta d_m(t)}{\lambda} \right)
\]  

3.3 Results

The radiant flux and the fringe count are influenced by a range of parameters, namely the wave front angle \( \theta \), the position \( x \) of the photodetector with respect to the beam centre line, the laser wave length \( \lambda \), the distance \( y \) between the mirror and the photodetector, and the side length \( s \), i.e., the size of the photodetector, and the translation \( \Delta d_m \) of the moving mirror M2, all of which are variable. The influence of these parameters is explained in a systematic way based on the equations derived in the Mathematical Analysis section above.

3.3.1 Influence of \( \theta \) on the radiant flux \( \Phi_e \)

The effect of \( \theta \) on the magnitude of the radiant flux \( \Phi_e \) also depends on the values of the other abovementioned variable parameters. In order to demonstrate this, the radiant flux is examined with four different conditions, in which some parameters are kept constant whereas others are variable.

3.3.1.1 \( \theta \) = variable, \( x = 0, y = 0, s = \) variable

Figure 5(a) shows the radiant flux curves of two different photodetector areas. With \( \Delta d_m = 0 \) and \( \theta = 0 \), the radiant flux is 100%. With \( \Delta d_m = 0 \) and increasing \( \theta \), the radiant flux decreases first, reaches the first primary node at a radiant flux magnitude of 50% and oscillates about the 50% level with decreasing radiant flux amplitude. The normalised radiant flux curve corresponds to a damped sine function with a damping function of \( 2/(sk\theta) \) or \( \lambda/(n\theta) \). Multiplying a sine function of the form \( \sin (n\theta/\lambda) \), where \( \lambda/s \)
represents the reciprocal value of the first node angle $\theta_{n_p}$, delivers a damped sine wave, which, after adding 1 and dividing the sum by 2, results in the normalised radiant flux curve. In contrast to a standard sine wave, where the first minimum is at $(3/2)\pi$, i.e., at $1.5\theta_{n_p}$, the non-linear decay rate causes the first minimum to be located at an angle of $1.4304\theta_{n_p}$, with a normalised radiant flux magnitude of 0.3913832. This magnitude is independent of $s$ and $\lambda$.

According to Equation (3.25), the smaller $s$, the larger is the wave front angle at the first node point. If $s = 100\, \mu m$ and $\lambda = 680\, nm$, the first and tenth primary nodes are at $\theta = 0.3896^\circ$ and $3.896^\circ$, respectively, and if $s = 10\, \mu m$, the first primary node is at $3.896^\circ$ (Figure 5(a)). The smaller is $s$, the slower the radiant flux decreases with increasing wave front angle. If $s = 10\, \mu m$, $100\, \mu m$, and $1\, mm$, the angle at 99% radiant flux is located at $\theta = 0.43^\circ$, $0.043^\circ$, and $0.0043^\circ$ (Figure 5(b)), respectively. According to Equation (3.25), $s$ and $\lambda$ have opposite effects: reducing $s$ by a factor of two results in the same angles of primary node points and 99% radiant flux as does a two-fold increase of $\lambda$. Figure 6 exemplifies this principle in a contour plot of equal angles of 99% radiant flux as a function of $s$ and $\lambda$. 

![Figure 5: Normalised radiant flux against wave front angle; a) node points; b) wave front angle at 99% radiant flux; $s = 10\, \mu m$, $100\, \mu m$, and $1\, mm$; $\lambda = 680\, nm.$](image)
At this point it has to be mentioned that the same radiant flux curves are obtained if $y > 0$ and $x = y(1/\sin \theta - 1/\tan \theta)$ according to Equation (3.90). Equation (3.90) provides the solution for $n_I = 0$, which deviates in $x$-direction if $y > 0$. This is explained in more detail below in the section dealing with influence on fringe count.

3.3.1.2 $\theta =$ variable, $x =$ variable, $y = 0$, $s =$ variable

Photodetector positions $x \neq 0$ changes the curves shown in Figure 5(a) insofar as the number of 50% radiant flux transitions is larger than the number of primary node points. The larger $x$, the more the radiant flux curve oscillates between the node points. In Figure 7(a), the radiant flux curve at $x = s/2$ intersects the 0.5 radiant flux level once between each pair of node points, the curve at $x = s$ does so twice, at $x = 2s$ four times and at $x = 10s$ twenty times. The larger $x$, the higher is the density of the radiant flux curve filling up the area under the radiant flux curve at $x = 0$ (Figure 7(b)), which acts like an envelope curve for radiant flux oscillations at larger $x$. This is insofar important to note as it shows that the modulation amplitude of the radiant flux across $x$ (Figure 8), is unaffected by $x$. 

![Figure 6: Contour plot of equal wave front angles of 99% radiant flux as a function of the side length $s$ of a square photodetector area and laser wave length $\lambda$.](image-url)
Figure 7: Normalised radiant flux vs. wave front angle a); at $x = 0, s/2, s, 2s$ and b); at $x = 0$ and $10s$. In both examples $s = 0.001 \text{ m}$ and $\lambda = 680 \text{ nm}$; secondary nodes are marked with green dots in the first cycle up to the 2nd node point in a).

Figure 8: Normalised radiant flux across an $x$-range of 4 mm at different wave front angles $\theta$ (in degrees); the dashed green lines indicate the $x$-position of the blue radiant flux curve shown in Figure 7(a) ($s = 1 \text{ mm}, x = s/2, s, 2s$); the dashed purple line indicates the radiant flux level (intersections of green dashed lines and purple radiant flux curve) at the first triple secondary node (at $\theta = 0.02597^\circ$); the 1st minimum refers to the radiant flux curve at $x = 0$ (Figure 7(a)).

Figure 7(a) shows secondary node points, i.e., intersections of the two curves at radiant flux magnitudes other than 50%. Independent of the position $x$, all curves intersect at the primary node points. The radiant flux at the primary node points is constant (50%), whereas the radiant flux at the secondary nodes is variable and a function of $x$. For example, at multiples of $0.02597^\circ$ (Figure 7(a)), the three radiant flux curves of $x = s/2, s$
and $2s$, with $s = 0.001$, intersect; at the 3rd and 6th intersection, the secondary nodes are identical to the primary nodes (2nd and 4th).

As the angle $\theta$ increases, so does the number of fringe lines per unit $x$ (Figure 8). At $\theta = 0$, the radiant flux is constant at 100%. After a slight increase in $\theta$, the radiant flux oscillates between 100% and 0%, i.e., the maximal radiant flux is still very close to 100% (Figure 5(b)). Further increase in $\theta$ reduces the radiant flux amplitude, which fluctuates about 50% until the modulation amplitude converges to 0 at the 1st node point and remains constant at 50% radiant flux. Further increase in $\theta$ expands the modulation amplitude, however, the sign of the radiant flux curve changes, i.e., peaks at $x = 0$ before the node point are converted to troughs after the node point.

3.3.1.3 $\theta = \text{variable}, x = 0, y = \text{variable}, s = \text{variable}$

When introducing the distance $y$ between the plane of the photodetector and the tilting mirror, the radiant flux curve can be entirely below or above the 50% radiant flux level, touching it only at the primary node points (Figure 9). The radiant flux curve is then superimposed by a further oscillation of a longer wave length. At the 6th primary node point of $\theta = 0.23377^\circ$, the radiant flux curve does not cross the 50% radiant flux level; nevertheless, the sign changes in the same way as shown in Figure 8. The distance $y$ does not affect the primary node points according to Equation (3.25), whereas the secondary nodes are a function of $y$ (as well as $s$ and $\theta$).

**Figure 9: Normalised radiant flux against wave front angle at two different $x$; $\lambda = 680$ nm, $y = 0.02$ m.**

Increasing $y$ (Figure 10) has the same effect as increasing $x$ (Figure 7): the radiant flux curve oscillates more frequently under the envelope of the radiant flux at $y = 0$ (Figure 9(a)). This does not affect the modulation amplitude of the radiant flux (Figure 10(b)), which remains the same across $x$ at a specific angle $\theta$, however, the centre fringe line is more deflected off-centre with $\theta$, the larger is $y$ (Figure 10(b)).
Figure 10: a) Normalised radiant flux vs. wave front angle at examples of $y$ (m) and b) normalised radiant flux vs. $x$; $s = 0.00001$ m, $y = 0, 0.0002, 0.002$, and 0.02 m; b) shows the position of the centre fringe (i.e., fringe number 0) radiant flux at angle $\theta = +1^\circ$ and movement of the fringe pattern with increasing $y$ (note that amplitude range and fringe density are independent of $y$).

3.3.1.4 $\theta =$ variable, $x =$ variable, $y =$ variable, $s =$ variable

Figure 11 summarises the influence of $\theta$, $s$, $x$, $y$ and $\lambda$ on the normalised radiant flux. The difference between Figure 11(a)–(d) is that the radiant flux before the first primary node point decays slower the smaller $s$ is. Figure 11(d) shows for small angles that the modulation amplitude remains constant across $x$ and $\theta$.

Figure 11(e), (f) shows with greater $y$, the more the centre fringe line deflects towards larger positive $x$ and that fringe lines from the negative $x$-side cross over to the positive-side.

The dotted lines in Figure 11(a), (b), (e), (f) shows the primary nodes for $s = 1$ mm, $s = 0.5$ mm and that the primary nodes are dependent on $s$ and independent of $y$ for constant $\lambda$.

Figure 11(g), (h) shows with greater $\lambda$ for equivalent $s$ (cf. Figure 11(a), (b)) that the interval of primary node angles is greater. Also, the radiant flux decays slower before the 1st primary node for greater $\lambda$.

Figure 11(g), (h) shows with greater $\lambda$ for equivalent $s$ (cf. Figure 11(a), (b)) that the radiant flux drops off slower before the 1st primary node.

Figure 11 shows at $\theta = 0^\circ$, the normalised radiant flux = 1 and is independent of $x$, $s$, $y$ and $\lambda$. 

Theoretical analysis of fringe line radiant flux on a rectangular photodetector
Theoretical analysis of fringe line radiant flux on a rectangular photodetector

**Figure 11:** Normalised radiant flux vs. wave front angle (θ = 0° to 0.2°) and distance from the beam centre (x = 0 mm to 5 mm) at y = 20 mm, and λ = 680 nm; a): s = 1 mm & y = 20 mm, λ = 680 nm; b): s = 0.5 mm & y = 20 mm, λ = 680 nm; c): s = 0.1 mm & y = 20 mm, λ = 680 nm; d): s = 0.01 mm & y = 20 mm, λ = 680 nm; e): s = 1 mm & y = 1 m, λ = 680 nm; f): s = 0.5 mm & y = 1 m, λ = 680 nm; g): s = 1 mm & y = 1 m, λ = 1,550 nm; h): s = 0.5 mm & y = 1 m, λ = 1,550 nm.

### 3.3.2 Influence of \( x \) and \( y \) on the fringe count

The number of fringe lines \( n_f \) passing over a point \((x, y)\) within the fringe pattern is given by Equation (3.87), which is a function of the tilting fringe lines (\( y \) term), the contracting/expanding fringe lines (\( x \) term) and moving mirror displacement (\( \Delta d_m \) term).

Assuming \( \Delta d_m = 0 \), as the moving mirror tilts from orthogonality, fringe lines are produced with a slope (in cross-section) that is parallel to the normal of the mirror (Equation (3.90) and Figure 2). As the wave front angle increases then so does the slope of the fringe lines therefore increasing the number of fringe lines \( n_f \) passing over point \((x, y)\).

Although the angle \( \theta \) between the two wave fronts can be positive or negative, the coefficient of the \( y \)-term is always positive and therefore only has an additive effect on \( n_f \) as seen by the centered \( \cup \) shaped curve in Figure 12. With the \( x \) coordinate and \( \theta \) kept constant, the greater the magnitude of \( y \), the greater the number of fringe lines that tilt across the point (compare Figure 12a) with b).

For small angles \( \sin \theta = \theta \), therefore the coefficient of \( x \) in Equation (3.87) is a linear function of angle \( \theta \) with the negative slope for positive \( x \) seen in Figure 12.

The example shown in Figure 12b) clearly demonstrates that as positive \( \theta \) is increased, the coefficient of the \( x \) term is initially greater in magnitude but opposite in sign to the \( y \) coefficient. Therefore, the fringe count is initially negative and increases negatively until the tilt angle is approximately 0.1 degrees at which point the negative value ceases increasing. The angle that this occurs at is termed \( \theta_{rev} \) in Equation (3.96) and is indicated in the figure. Thereafter, the \( y \) coefficient dominates and as \( \theta \) is increased further the fringe count returns to zero and then becomes positive.

![Figure 12: The effect of variable \( y \) on the fringe count; a) \( x = 0.001 \) m, \( y = 0.02 \) m, \( \lambda = 680 \) nm, \( \Delta d_m = 0 \); b) \( x = 0.001 \) m, \( y = 0.573 \) m, \( \lambda = 680 \) nm, \( \Delta d_m = 0 \).](image)

The \( x \)-term of Equation (3.87) is linked to contraction/expansion of the fringe lines with varying wave front angle. The focal point of the fringe contraction/expansion is the mirror normal that is coincident with the mirror tilt axis. With \( \Delta d_m = 0 \), as \( \theta \) is increased a fringe...
line is generated that is centred on this tilt axis mirror normal. As $\theta$ is further increased, fringe lines develop to the left and right of this normal and contract toward it. For a given point $(x, y)$ in the fringe pattern, as $\theta$ is increased, more and more fringe lines will develop and cross over the point. The central fringe line is essentially static, therefore the fringe lines to the far left and right move substantially quicker than those closer to the central fringe line.

Equation (3.92) and Figure 13 confirm this behaviour showing that the speed of contraction/expansion (from the $x$-term of Equation (3.91)) of the fringe pattern is a constant. Therefore an active area located further from the centre of the beam will experience more fringe lines passing over it than one located closer to the centre when mirror M2 is tilted. Parameters $x$ and $\lambda$ have opposing effects on the contraction/expansion speed.

In cross section, the tilt speed of the fringe lines (from the $y$-term of Equation (3.91)) is derived in Equation (3.93). This fringe tilt speed is a tangent function of the wave front angle, independent of $x$, i.e., of the lateral position of the photodetector, but dependent on $y$ and $\lambda$. The tilt speed of the fringe lines is initially smaller than the speed of contraction, as the former is zero if $\theta = 0$ ($\tan \theta = 0$).

Figure 13 shows the effect of increasing and then decreasing fringe numbers with progressive wave front angle. The fringe count on the positive $x$-side are acutely curved initially, the speed of the centre fringe line deflection lags behind the speed of contraction. Subsequently, the former speed term catches up and eventually overtakes the latter term. This results in the fringe lines initially moving over an off-centre photodetector in one direction and then moving over the same detector again but in opposite direction, thereby first increasing the fringe count and subsequently decreasing it. Figure 13 also shows that the larger $x$, the larger is $\theta_{rev}$.

![Figure 13](image_url)

_Figure 13: Fringe counting as a function of $x$ and $\theta$; $y = 0.30$ m, $\lambda = 680$ nm, $\Delta d = 0$; the fringe count is shown on the right side, for $x = 0$, 0.001, and 0.0015 m; “0” = peak of centre fringe; positive and negative fringe numbers refer to negative and positive $x$, respectively, i.e., to left and right sides of the centre fringe line “0”._

If $\Delta d_m \neq 0$, then from Equation (3.87) $\Delta d_m$ introduces a shift of the fringe numbers $n_f$ in Figure 13 above, one curve line for every $\lambda/2$ translation of Mirror M2 (Figure 2).

When an interferometer is used to measure translation, the number of fringes $n_f$ sensed by the photodetector are counted and the translated distance is calculated using Equation
Theoretical analysis of fringe line radiant flux on a rectangular photodetector

(3.88). However, the above theory shows that despite \( \Delta d_m \) being kept constant, i.e. translation = 0, the \( x \) and \( y \) position of the photodetector has a profound effect on \( n_r \) and small changes in mirror tilt angle can be mis-interpreted as translation.

To illustrate by way of one example, let \( s = 0.01 \) mm, \( y = 30 \) mm, \( \lambda = 680 \) nm, \( \Delta d_m = 0 \) and the photodetector is centred on the \( y \) axis (i.e. \( x = 0 \) mm). From Equation (3.8) the normalised radiant flux is calculated to be 0.997 for a wave front angle \( \theta \) of 0.05°. Relocating the photodetector at \( y = 200 \) mm returns a normalized flux of 0.881, which is reduced from the first location as a result of the tilted fringe. To work out what this change in radiant flux represents in terms of change in fringe position, substitute each flux value into the equation \( \Phi = \Phi_0 \sin(kx') \), where \( \Phi_0 = 1 \) is the maximum flux amplitude, \( k = 2\pi/\Delta d_f \), \( \Delta d_f = \lambda/\theta \) is the fringe width and \( x' \) represents the first and second position respectively of the fringe lines. Solving for \( x' \) in each case and subtracting the two equations gives a change in fringe position of 51.5 μm. Fringe width is 779.2 μm, therefore the change in fringe position due to the photodetector being located further away can be interpreted as a 6.6% shift in translation, i.e. 6.6% of 340 nm = 22.5 nm.

Changing the position of the photodetector in the \( x \)-direction with \( \Delta d_m = 0 \) and varying \( \theta \) has an even greater effect on the radiant flux values, creating an even greater mis-interpretation of translation.

3.3.3 Fringe count speed

The wavelength of visible light ranges from approximately 400 nm to 700 nm, which enables optical interferometers to measure translations with high precision. With such a small measurement resolution, the ability to accurately count the number of fringes as translation is a function of the position and size of the photodetector within the fringe pattern as well as the magnitude and rate of mirror translation and tilt angle being applied.

By applying to the moving mirror a sinusoidal oscillation of translation and tilt angle such as the functions in Equations (3.109) and (3.110), and substituting differing variables in Equations (3.111) and (3.112), the effect on fringe count and fringe count speed becomes apparent as depicted in Figure 14 and Figure 15 respectively.

Using the variables assigned to Figure 14 a) as a basis for comparison of fringe count, i.e. \( x = 0.001 \) m, \( y = 0.1 \) m, \( \theta_{\text{max}} = 0.5^\circ \), \( \Delta d_{\text{max}} = 50 \) μm, \( f = 5 \) Hz and \( \lambda = 680 \) nm, the maxima and minima values of total fringe count are seen to be different. As mentioned in the previous section, this is due to the effect of fringe tilt and fringe contraction/expansion having an additive or subtractive effect on the translation fringe count (= 147 when unaffected by mirror tilt Equation (3.88)).

In Figure 14b), the photodetector is moved 1 mm across in the \( x \)-direction from (0.001, 0.1) to (0.002, 0.1) and the maxima and minima have increased quite significantly from the initial location. Compare that to Figure 14 c) where the photodetector has been moved up the \( y \)-axis by 200 mm from (0.001, 0.1) to (0.001, 0.3). The change to total fringe count by moving up the \( y \)-axis is not as significant despite it being 200 times greater in magnitude than in the \( x \)-direction.

In Figure 14 d) the maximum fringe tilt angle has been increase from 0.5° to 0.8° and it is noteworthy even though this is a 60% increase in tilt angle, it has not resulted in a significant change in total fringe count.
From the above observation, despite the fringe count due to translation being dominant, tilting and contraction/expansion of the fringe lines must be factored for when wanting to achieve accurate translation measurement.

Remember that angle $\theta$ is the wave front angle therefore the tilt angle of the moving mirror is half this angle.
Figure 14: Fringe count with concurrent sinusoidal varying $\theta$ and $\Delta dm$, with $\lambda = 680$ nm, for: a) $x = 0.001$ m, $y = 0.1$ m, $\theta_{\text{max}} = 0.5^\circ$, $\Delta dm = 50 \, \mu$m, $f = 5$ Hz; b) $x = 0.002$ m, $y = 0.1$ m, $\theta_{\text{max}} = 0.5^\circ$, $\Delta dm = 50 \, \mu$m, $f = 5$ Hz; c) $x = 0.001$ m, $y = 0.3$ m, $\theta_{\text{max}} = 0.5^\circ$, $\Delta dm = 50 \, \mu$m, $f = 5$ Hz; d) $x = 0.001$ m, $y = 0.1$ m, $\theta_{\text{max}} = 0.8^\circ$, $\Delta dm = 50 \, \mu$m, $f = 5$ Hz.

Figure 15 illustrates the fringe count speed curves for the same parameters as the respective fringe count curves above. The instances of fringe count speed equal to zero coincides whenever there is a maxima or minima fringe count, i.e. when the mirror has reached the extent of translation in each direction and reverses direction (0.05 s, 0.15 s, 0.25 s, etc.) and is confirmed by Equation (3.115).

What is noticeable from Figure 15 is that the magnitude of the maxima and minima total fringe count speed is identical for each set of parameters. The instance that this occurs is when the $x$, $y$ and translation fringe count speeds all lie above or below the zero line and Equation (3.123) can be used to find the instances that this occurs.

Similarly to the analysis of fringe count, moving the photodetector along the $x$-axis has several orders of magnitude more effect on fringe count speed than moving the photodetector along the $y$-axis or increasing the wave front tilt angle $\theta$. 

![Fringe Count Speed Graph](image-url)
3.3.4 Fringe transition speed

When a Michelson interferometer is in perfect alignment and the light is approximated to be plane waves, the two wave fronts are parallel to each other. Therefore, when applying a tilt angle to the wave fronts, the fringe lines are generated from infinity and contract at high speed towards the mirror normal coincident with the tilt axis.

Compounded with mirror translation, the fringe lines at times cross the photodetector at speeds in excess of 1000 ms$^{-1}$ as seen in Figure 16 for an oscillation of 5 Hz, a maximum translation of 50 μm and the photodetector positioned 1 mm from the central axis of the interferogram. These speeds occur when fringe count speed is near maximum and the mirror tilt is approximately zero.

As the wave front angle increases and the fringe lines contract closer together, the fringe transition speed drops quickly and in the example depicted in Figure 16, for the majority of the time it is 0.1 ms$^{-1}$ to 10 ms$^{-1}$. To freeze the fringe motion when the lines are transitioning at such high speeds it is essential that the characteristic of the photodetector is suitable and if sampling is applied, the sampling frequency is greater than or equal to the Nyquist frequency.

If mirror translation is zero, fringe transition speed is dictated by the position the photodetector is from the central axis of the interferogram as it is a linear function for...
small angles. Therefore, to reduce the impact on the characteristics of photodetector, the photodetector should be positioned on or as close as possible to the central axis.

3.3.5 Radiant flux vs. sinusoidal wave front angle and translation

For a photodetector width = 0.0001 m and wavelength = 680 nm, the first primary node occurs at 0.39° (Equation (3.25)). Figure 17 depicts the radiant flux across the photodetector for a sinusoidally varying wave front angle and mirror translation and the occurrence of these nodes for every complete sinusoidal oscillation of wave front angle, four per cycle. The radiant flux oscillates primarily due to the sinusoidal translation under an envelope curve of variant modulation amplitude brought about by the sinusoidal wave front angle.

3.4 Summary

The focus of this chapter has been to establish the behaviour of the radiant flux of the interferogram over a photodetector of rectangular aperture that is variable; in size; in displacement across the interferogram; and, in axial distance from the source of interference, for variable angle between the two wave fronts and variable wavelength.
The most apparent observation from the mathematical analysis in this study is that the radiant flux decays rapidly with increasing wave front angle with the recurrence of primary nodes where the radiant flux decays to 50% maximum and the modulation amplitude reduces to zero. This observation is also confirmed by [3,5–10,17] where radiant flux is calculated over a disc and also by [8,11] where the radiant flux is calculated over a square area. The modulation amplitude in this study and the literature is found to decay as a cardinal sine function. However, this study has gone further to determine that the radiant flux decays with a decay function that is a reciprocal function of the wave front angle with decay constant that is proportional to the wavelength and is inversely proportional to the photodetector active area width.

In the literature [3,5–11,17], the boundary of the radiant flux calculation is centred on the interferogram, giving just a single radiant flux curve decaying as a cardinal sine function. Whereas, in this study, the radiant flux boundary is derived to be variable across the interferogram. This variant shows that as the boundary is moved off centre, the radiant flux oscillates increasingly about the 50% normalised maximum as an integer multiple of the distance from centre. Additionally, the amplitude of the oscillation is bounded by the centred radiant flux curve.

A further finding from this study, which to the best of the author's knowledge, is not mentioned in the literature is how the radiant flux is affected when the fringe lines tilt and contract/expand with varying wave front angle. Having included as an integral parameter the axial distance of the photodetector from the interferometer, it is found for a centred photodetector that fringe tilt initially lags fringe contraction, but then fringe tilt becomes increasingly dominant on the radiant flux with increasing distance of the photodetector from the beamsplitter.

Interferometry applications that use a plane flat mirror with translation stage, and discrete photodetectors [12–15] or position sensitive device [16] will suffer erroneous measurements if the wave front angle is not limited to give acceptable modulation amplitude. This can be done by choosing an appropriate photodetector aperture width that is an order of magnitude less than the fringe line spacing. Increasing the wavelength improves modulation amplitude for equivalent photodetector aperture widths.

Alignment of the photodetector with the centre of the interferogram reduces the susceptibility of the contracting fringe lines crossing over the photodetector as a linear function of distance from centre with increasing wave front angle.

With increasing wave front angle the tilt angle of the fringe lines increases and more cross over the central axis of the interference beam with increasing distance from the beamsplitter. Irrespective of which side of the central axis the wave front is tilted, the sign of the fringe count due to this tilting is always positive. Therefore fringe counting when the mirror is translating in the positive direction will be greater than when the mirror is translating in the negative direction. Therefore to limit the error in fringe count as a result, the photodetector should be located as close as possible to the beamsplitter.

By far the most dominant variable producing fringe count is translation. Superimposed on the translation fringe count is fringe count due to fringe tilt and fringe contraction/expansion. In much the same way fringe count is affected by varying the axial (y) and lateral (x) position of the photodetector within the interferogram for a periodic tilt angle and translation, the same can be said for fringe count speed, which is the number of fringe lines crossing the photodetector in a given time.

Fringe transition speed is the product of fringe count speed and the distance between fringe lines. Theoretically, if the interferometer is perfectly aligned and the light is
approximated as plane waves then the distance between fringe lines is infinity and so is fringe transition speed. As wave front tilt angle is applied, fringe lines are generated from infinity and contract rapidly towards the centre of the interferogram. For a photodetector positioned a millimetre from this central axis and without any mirror translation, the fringe transition speed can be in excess of 100 ms\(^{-1}\) in that location as the angle approaches 0.0001 degrees. For small angles, the transition speed doubles for a doubling of distance from the central axis. Despite these speeds rapidly decreasing as tilt angle is increased further, to captured fringe movement at the very small tilt angles, the response time and sampling rate of the photodetector has to be given careful consideration.
Chapter 4

4 Utilising Image Sensors to Capture Mirror Tilt Angle and Tilt Axis Angle

4.1 Introduction

The previous chapter dealt with how the radiant flux on a rectangular photodetector of varying width changes when varying the wave front tilt angle, the x- and y-position of the photodetector relative to the beamsplitter, the mirror translation and the light wavelength. An image sensor comprises an array of photodetectors commonly known as pixels arranged in rows and columns. Each pixel captures the magnitude of the radiant flux incident on it, which is digitally quantised in terms of a proportional voltage and is stored in a logically addressable table. The proportional voltage of each pixel can be digitally processed in real time to analyse the movement of the fringe lines across the sensor as well as to derive the distance between the fringe lines. The proportional voltage can also be used to reproduce the fringe pattern on a monitor.

Image sensor technology is advancing rapidly with the physical distance between pixels (pixel pitch) currently achievable being in the order of microns and getting smaller. This enables image sensors to be manufactured with side lengths of less than 1 mm for applications that require such miniaturisation.

For an image sensor that has a pixel pitch of less than 10 μm, it was shown in Section 3.3.1.1 that the radiant flux modulation amplitude drops to 99% at a wave front angle of 0.43°. For a pixel pitch of 5 μm, the 99% level is reached at 0.86°. Therefore, if the image sensor is required to capture fringe lines for wave front angles of this magnitude and modulation amplitude no less than 99%, then the pixel pitch must be in the order of a few microns.

From the mathematical analysis, the further a pixel is from the centre of the image sensor the faster is the fringe transition speed. Consequently, to capture an acceptable fringe image, the image sensor scanning rate must be calculated for those pixels furthest from the centre.
This chapter presents one basic method of capturing the orientation and distance between fringe lines but starts with a simple explanation of the operation of the Michelson interferometer.

4.2 Basic operation of the Michelson Interferometer

With reference to Figure 1, to use the interferometer to measure displacement, the laser, beamsplitter and the fixed mirror (M1) are kept fixed relative to one another and mounted to a reference. The moving mirror (M2) is mounted to the object that undergoes displacement relative to the reference. When displacement occurs there is a change in the phase relationship of the recombining beams causing the interference beam to vary in intensity, which is captured and measured using the image sensor. Whenever the recombining beams are in phase with one another maximum constructive interference occurs and the intensity of the interference beam will be at its greatest, which is called a fringe.

The difference in distance between the return paths of the transmitted and reflected beams is called the optical path difference and when the two wave fronts are aligned in direction, the OPD is given by Equation (3.13) at a resolution of wavelength as \( n_f \) is an integer. However, when the two wave fronts are not perfectly aligned then solving for the OPD in Equation (3.83) returns:

\[
\text{OPD} = 2d_1 - 2d_2 = 2\Delta d_m = y(1 - \cos \theta) - x \sin \theta - n_f \lambda
\]  

(4.1)

where \( d_1 \) and \( d_2 \) are the respective distances from fixed and moving mirror to the beamsplitter (Figure 1) and \( n_f \) is the number of fringes counted traversing the point \( (x, y) \). The sign of \( \Delta d_m \) is dependent on: the designation of \( d_1 \) and \( d_2 \) in Figure 1 producing a phase lag or lead; the magnitude and sign of \( \theta \); the position of the point along the \( x \)-axis and the sign of \( n_f \) given to the direction that the fringe lines traverse. Note that \( y \) is always positive as the point \( (x, y) \) is always along the positive \( y \)-axis.

4.3 Determining fringe line orientation and spacing

In Figure 1, the interference beam is shown projecting onto the plane of a photodetector, in this case an image sensor, arranged in \( X \) columns and \( Z \) rows.

By way of example, consider Figure 18 below which depicts an image sensor with an array of 18 x 18 pixels upon which a fringe pattern is superimposed. The coordinate system is shown with the positive \( x \) axis pointing towards the reader, the \( y \) axis is horizontal and the \( z \) axis vertical. The array has been divided into 4 quadrants as shown in the figure. Digital signal processing (DSP) can be used to determine the pixels with the highest radiant flux incident on them and these are indicated in the figure with the black dot. The width and height of each pixel is known from the image sensor specification sheet, therefore trigonometry can be applied to work out the angle of inclination of the fringe lines as well as the fringe line spacing.

Having derived the highest flux pixels, a simple function using DSP would be to determine, for example, pixels \( P1 (P_{1x}, P_{1z}) \), \( P2 (P_{2x}, P_{2z}) \) and \( P3 (P_{3x}, P_{3z}) \) that are the apexes of a right angled triangle, therefore

\[
\varphi_x = \tan^{-1} \left( \frac{P_{2z} - P_{3z}}{P_{3y} - P_{1y}} \right)
\]  

(4.2)
and

\[ \Delta d_f = (P_{3y} - P_{3y}) \sin \varphi_x \]  \hspace{1cm} (4.3)

where \( \varphi_x \) is the angle of inclination of the fringe lines with respect to the \( y \) axis, and \( \Delta d_f \) is the orthogonal distance between the fringe lines.

From Equation (3.128), the angle between the two wave fronts can be calculated, i.e.:

\[ \theta_x = \frac{\lambda}{\Delta d_f} \]  \hspace{1cm} (4.4)

where \( \theta_x \) is the angle of the tilted wave front with respect to the \( x \) axis.

The angle of inclination \( \varphi_x \) of the fringe lines is parallel to the tilt axis of the mirror.

![Fringe line inclinations and fringe spacing obtained from an image sensor provides information of two degrees of freedom of the moving mirror M2 (Figure 1), namely, the orientation of the axis about which the mirror is tilted and the magnitude of tilt (Equation (3.128)). A third degree of freedom can be obtained if the fringe lines are counted as they move across the image sensor.

Several methodologies exist that use Digital Signal Processing (DSP), for example Durango [57], for interferogram analysis that can determine tilt axis orientation and fringe spacing. What cannot be gleaned from this information is in which direction the mirror is tilted. In order to do so requires 3 interferometers orthogonally arranged about 3 sides of a cube mirror, as explained in the following chapter.

\[ \alpha_y \]
5 DERIVING CUBE MIRROR POSITION VECTORS

5.1 Introduction

The 3 degrees of freedom that can be determined from an image sensor, i.e. mirror tilt axis angle, fringe line spacing (tilt angle) and fringe count (translation), lack critical information whether the tilt angle is positive or negative and in which direction is the translation.

Methods exist in the literature [3-5] using 5 or 6 interferometers to resolve this uncertainty, however, it can be done in a more novel way using 3 interferometers as described below.

This chapter covers the realisation to the following Research Questions:

i. How can fringe spacing and fringe slope be used to determine the tilt and rotation of the cube mirror about each Cartesian axis? (Section 1.2i)

ii. How can translation direction be determined solely from fringe spacing, fringe slope, fringe count & fringe direction? (Section 1.2ii)

iii. Using a cube mirror inherently generates translation error, how can this be compensated for? (Section 1.2iii)

5.2 Methodology

5.2.1 Arranging 3 Michelson Interferometers orthogonally to derive the components of the position vectors of a cube mirror

Three interferometers as depicted in Figure 1 are arranged orthogonally as shown in Figure 19, which define the x, y and z Cartesian axes in 3 dimensional space with the origin being the point at which the laser beams would intersect one another were it not for them being reflected by the cube mirror.
The interferometers measure the cube mirror’s position or change in position in terms of the cube mirror’s 3 position vectors by deriving:

- the 3 tilt angles and tilt axis angles, which is the subject of the next sub-chapter 5.2.2, and
- translation of the cube mirror along each of the axes in terms of the number of full and partial fringes produced by the mirror’s change in position, which is discussed in detail in a sub-chapter 5.2.3

The position vectors of each side of the cube mirror are the superimposition of that side’s tilt/rotation vector derived from mirror tilt axis angle and tilt angle and translation vector. The mirror’s tilt/rotation vector is derived separately from deriving the mirror translation vector. As a consequence, the movement of the cube mirror is resolved to 6 degrees of freedom.

Although the wave front from a collimated laser beam is not perfectly planar, it can be assumed to be approximately so within a few centimetres from the collimating lens. The fringe lines created by two plane wave fronts can therefore be considered of equal inclination, width and spacing. This makes it possible to accurately derive the mirror tilt axis angles and tilt angles.

Figure 19: 3 Michelson Interferometers arranged orthogonally about a cube mirror.
5.2.2 Mathematical analysis for deriving the cube mirror position vectors for tilt axis angle and tilt angle

For derivation of the 3 position vectors of the cube mirror due to angular displacement, the centre of the cube mirror is the origin of the reference frame. This is because:

1. the 3 orthogonal interferometer beams always define the direction of the 3 Cartesian axes within the space they intersect

2. the pitch, roll and yaw parameters of the cube mirror measured by the image sensors (fringe spacing and fringe tilt axis) is identical despite the actual position of the cube within the space

3. the centre of the cube is a theoretical origin with its orthogonal axes aligned with those define by the interferometers, therefore the instantaneous position of the cube centre within the space is irrelevant

This simplifies the vector analysis as the position vectors always have magnitude unity and the vector components reduce to trigonometric functions. When measuring change in pitch, roll and yaw, orthogonal alignment of cube with the interferometers is not necessary as the initial cube rotational position can be taken as an offset to any subsequent change.

Figure 20a) defines the first position of the cube mirror with origin (0,0,0) and three known points.

Positive x-axis, position vector \( \mathbf{A} \), \((A_x, A_y, A_z) = (1,0,0)\)

Positive y-axis, position vector \( \mathbf{B} \), \((B_x, B_y, B_z) = (0,1,0)\)

Positive z-axis, position vector \( \mathbf{C} \), \((C_x, C_y, C_z) = (0,0,1)\)

Thus, the non-zero components are: \(A_x, B_y, C_z = (1,1,1)\); i.e., the resultant of the three positive axes.

The fixed mirrors M1 (Figure 1) of each of the 3 interferometers can be considered as the 3 orthogonal sides of the cube mirror in the first position giving rise to vectors \( \mathbf{A}, \mathbf{B} \) and \( \mathbf{C} \).

Figure 20b) defines the second position of the cube mirror with origin (0,0,0) and three unknown points.

Vector \( \mathbf{X} \), \((X_x, X_y, X_z)\)

Vector \( \mathbf{Y} \), \((Y_x, Y_y, Y_z)\)

Vector \( \mathbf{Z} \), \((Z_x, Z_y, Z_z)\)

The first and second position vectors are depicted on a single vector diagram in Figure 21 where \( \mathbf{i}, \mathbf{j} \) and \( \mathbf{k} \) are unit vectors in the respective \(x, y\) and \(z\) directions such that \( \mathbf{i} = \mathbf{A}, \mathbf{j} = \mathbf{B} \) and \( \mathbf{k} = \mathbf{C} \).

With the first and second positions of the cube mirror superimposed on one another as shown in Figure 22,
the tilt angles $\alpha_x$, $\beta_y$ and $\gamma_z$ are defined as the angles the mirror normals (i.e. the perpendicular to each facet) make with the respective $x$, $y$ and $z$ axes. They are determined by measuring the distance between fringe lines (Equation (4.4)) on the image sensor to derive the wave front angles $\theta_x$, $\theta_y$ and $\theta_z$ and then dividing these angles by 2 to obtain $\alpha_x$, $\beta_y$ and $\gamma_z$ respectively

$$
\alpha_x = \frac{\theta_x}{2}, \beta_y = \frac{\theta_y}{2}, \gamma_z = \frac{\theta_z}{2}
$$

(5.1)

the tilt axis angles $\varphi_x$, $\varphi_y$ and $\varphi_z$ are defined as the angles the respective facets make with one another in the respective $y$-$z$, $x$-$z$ and $x$-$y$ planes and are determined from measuring the slope of the fringe lines on the image sensor (Equation (4.2))

Figure 20: Defining a) first position and b) second position vectors.

Figure 21: First and second position vectors depicted together.
Figure 22: First and second cube mirror positions defining tilt angles and tilt axis angles.

Superimposing the interferograms generated from each of the 3 interferometer image sensors onto the cube mirror would appear as illustrated in Figure 23. Each image sensor has captured the fringe tilt angle, which coincides with the mirror tilt axis angle, and the fringe line spacing. From these 3 interferograms the mirror tilt angles $\alpha_x, \beta_y$ and tilt axis angles $\varphi_x, \varphi_y$ and $\varphi_z$ can be resolved to derive the 3 position vectors of the cube mirror.

Two methods are described below how the 3 position vectors can be resolved.

Figure 23: $x$-, $y$- and $z$- axis interferograms superimposed on cube mirror.

5.2.2.1 Method 1

Vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are orthogonal to one another, therefore vector $\mathbf{C}$ can be derived from the cross product of vectors $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = \mathbf{i} \times \mathbf{j} = \hat{\mathbf{k}}$$ (5.2)
Similarly, vectors $X$, $Y$ and $Z$ in the second position are orthogonal to one another therefore vector $Z$ can be derived from the cross product of $X$ and $Y$:

$$X \times Y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ Xx & Xy & Xz \\ Yx & Yy & Yz \end{vmatrix} = Z$$  \hspace{1cm} (5.3)

$$Z = (XyYz - YyXz)\mathbf{i} + (XzYx - YzXx)\mathbf{j} + (XxYy - YxXy)\mathbf{k}$$  \hspace{1cm} (5.4)

The tilt angles $\alpha_x$, $\beta_y$ and $\gamma_z$ are derived from the fringe spacing of the three interferograms, therefore their cosine values produce the position vector components $Xx$, $Yy$ and $Zz$ from the dot product of the second position vectors and the respective unit vectors as follows:

$$\mathbf{i} \cdot X = |X| \cos \alpha_x \hspace{1cm} \therefore Xx = Gx$$  \hspace{1cm} (5.5)

$$\mathbf{j} \cdot Y = |Y| \cos \beta_y \hspace{1cm} \therefore Yy = Gy$$  \hspace{1cm} (5.6)

$$\mathbf{k} \cdot Z = |Z| \cos \gamma_z \hspace{1cm} \therefore XxYy - YxXy = Zz = Gz$$  \hspace{1cm} (5.7)

Note:

$$|X| = |Y| = |Z| = 1$$  \hspace{1cm} (5.8)

The slopes of the fringe lines on the interferograms are parallel to the tilt axes of each respective side of the cube mirror. The tilt axes vectors can be determined by taking the cross products of the first and second cube mirror positions as follows:

$$T_x = \mathbf{i} \times X = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ Xx & Xy & Xz \end{vmatrix} = -Xz\mathbf{j} + Xy\mathbf{k}$$  \hspace{1cm} (5.9)

$$T_y = \mathbf{j} \times Y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ Yx & Yy & Yz \end{vmatrix} = Yz\mathbf{i} - Yx\mathbf{k}$$  \hspace{1cm} (5.10)

$$T_z = \mathbf{k} \times Z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ (XyYz - YyXz) & (XzYx - YzXx) & (XxYy - YxXy) \end{vmatrix} = -(XzYx - YzXx)\mathbf{i} + (XyYz - YyXz)\mathbf{j}$$  \hspace{1cm} (5.11)

where vectors $T_x$, $T_y$ and $T_z$ are the tilt axis vectors in the respective $y$-$z$, $x$-$z$ and $x$-$y$ planes.

The slope of the fringe lines are the tangent values of the tilt axis angles $\phi_x$, $\phi_y$ and $\phi_z$ captured from the 3 image sensors. Therefore:

$$\tan \phi_x = \frac{Xy}{-Xz} = Fx$$  \hspace{1cm} (5.12)

$$\tan \phi_y = \frac{Yz}{-Yx} = Fy$$  \hspace{1cm} (5.13)
Deriving cube mirror position vectors

\[
\tan \varphi_z = \frac{(XyYz - YyXz)}{-(XzYx - YzXx)} = \frac{XyYz - YyXz}{YzXx - XzYx} = F_z \tag{5.14}
\]

Solving Equations (5.5), (5.6), (5.7), (5.12), (5.13) and (5.14) for the unknowns \(X_y, X_z, Y_x, \) and \(Y_z\) gives:

\[
X_y = \frac{-(J + H)}{2Gy} \tag{5.15}
\]
\[
X_z = \frac{J + H}{2GyF_x} \tag{5.16}
\]
\[
Y_x = \frac{J - H}{2GxF_xF_yF_z} \tag{5.17}
\]
\[
Y_z = \frac{-(J - H)}{2GxF_xF_z} \tag{5.18}
\]

where

\[
H = GzF_z - GxGyFz + GzFxFy - GxGyFxFy \tag{5.19}
\]

and

\[
J = \pm(Gx^2Gy^2Fx^2Fy^2 - 2Gx^2Gy^2FxFyFz + Gx^2Gy^2Fz^2 - 2GxGyGzFx^2Fy^2 - 2GxGyGzFz^2 + Gz^2Fx^2Fy^2 + 2Gz^2FxFyFz + Gz^2Fz^2) \frac{1}{2} \tag{5.20}
\]

Referring back to Equation (5.4), having now calculated components \(X_y, X_z, Y_x, \) and \(Y_z,\) the components of \(Z\) can be calculated, i.e.:

\[
Z_x = XyYz - YyXz \tag{5.21}
\]
\[
Z_y = XzYx - YzXx \tag{5.22}
\]
\[
Z_z = XxYy - YxXy = Gz \tag{5.23}
\]

5.2.2.1.1 Method to determine the correct position vectors

From Equation (5.20), 2 solutions result for each of the components \(X_y, X_z, Y_x, \) and \(Y_z,\) which gives rise to 16 possible permutations of \(Xx, Xy, Xz, Yx, Yy, \) and \(Yz.\)

5.2.2.1.1.1 Use vector dot product to find the permutation of vectors \(X\) and \(Y\) that are orthogonal to one another

To find the correct permutation, each permutation in turn has to be tested for orthogonality using the vector dot product to find which return a result of zero.

\[
\mathbf{X} \cdot \mathbf{Y} = 0 = \cos 90^\circ \tag{5.24}
\]

5.2.2.1.1.2 Test orthogonal permutations for \(X\) and \(Y\) vectors of unit length

There may be more than one permutation of \(X\) and \(Y\) vectors that prove to be orthogonal to one another. When this occurs the \(X\) and \(Y\) vectors have to be tested to determine which permutation has vectors of unit length, e.g.

\[
\|\mathbf{X}\| = \sqrt{X^2_x + X^2_y + X^2_z} = 1 \tag{5.25}
\]
5.2.2.1.3 Filtering method to verify the correct position vector permutation

The dot product of each permutation of vectors $\mathbf{X}$ and $\mathbf{Y}$ produce an angle very close to $90^\circ$, if not exactly $90^\circ$. Also, the length of each $\mathbf{X}$ and $\mathbf{Y}$ vector is very close to unity, if not exactly 1. To unambiguously select the correct $\mathbf{X}$ and $\mathbf{Y}$ position vectors each permutation is passed through the following mathematical filter to find the permutation with the highest score.

1. Take the absolute value of the dot product of vectors $\mathbf{X}$ and $\mathbf{Y}$
2. Take the common logarithm of the dot product
3. Take the absolute value of 1 minus length of vectors $\mathbf{X}$ and $\mathbf{Y}$
4. Take the common logarithm of the result of bullet 3 above
5. Take the absolute value of the product of all 3 values
6. Select the permutation with the highest score

5.2.2.1.4 Use cross product of vectors $\mathbf{X}$ and $\mathbf{Y}$ to find vector $\mathbf{Z}$

Having found the permutation of vectors $\mathbf{X}$ and $\mathbf{Y}$ that returns the highest score, they are used to derive position vector $\mathbf{Z}$ using the vector cross product.

In conclusion, there will be instances when the cube mirror will be in an orientation where the slope of the fringe lines (Equations (5.12) - (5.14)) will be zero or indeterminate, i.e. interferometer aligned = homogeneous fringing. Therefore Equations (5.16) - (5.18) will be indeterminate rendering it impossible to resolve the cube mirror position vectors in these instances. Methods to overcome or avoid these instances are treated in section 5.2.5.

5.2.2.2 Method 2

Vectors $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ can be expressed in terms of direction angles as defined in Figure 24, Figure 25 and Figure 26.

\[
\mathbf{X} = |\mathbf{X}|(\cos \alpha_x \hat{i} + \cos \beta_x \hat{j} + \cos \gamma_x \hat{k}) = \cos \alpha_x \hat{i} + \cos \beta_x \hat{j} + \cos \gamma_x \hat{k} \\
\mathbf{Y} = |\mathbf{Y}|(\cos \alpha_y \hat{i} + \cos \beta_y \hat{j} + \cos \gamma_y \hat{k}) = \cos \alpha_y \hat{i} + \cos \beta_y \hat{j} + \cos \gamma_y \hat{k} \\
\mathbf{Z} = |\mathbf{Z}|(\cos \alpha_z \hat{i} + \cos \beta_z \hat{j} + \cos \gamma_z \hat{k}) = \cos \alpha_z \hat{i} + \cos \beta_z \hat{j} + \cos \gamma_z \hat{k}
\]

(5.26)
(5.27)
(5.28)

Referring to Figure 24, the rejection vector of vector $\mathbf{X}$ onto the y-z plane is vector $\mathbf{R}_x$, which lies at angle $\epsilon_x$ relative to the y-axis. By convention, $\epsilon_x$ is measured counter clockwise with respect to the y-axis and will determine the sign of the components of $\mathbf{R}_x$, which is given by:

\[
\mathbf{R}_x = \sin \alpha_x \cos \epsilon_x \hat{j} + \sin \alpha_x \sin \epsilon_x \hat{k}
\]

(5.29)
Figure 24: Vector X direction angles, rejection vector $R_x$ and tilt axis vector $T_x$.

Substituting the components of Equation (5.29) into Equation (5.26) gives

$$X = \cos \alpha_x \hat{i} + \sin \alpha_x \cos \varepsilon_x \hat{j} + \sin \alpha_x \sin \varepsilon_x \hat{k}$$  \hspace{1cm} (5.30)

The tilt axis vector $T_x$ is the cross product of the unit vector $\hat{i}$ and second position cube vector $X$ and consequently also lies on the $y$-$z$ plane at angle $\varphi_x$ relative to the $y$-axis. Figure 24 shows the slope of the fringe lines, which are aligned with the tilt axis vector $T_x$, therefore;

$$\varepsilon_x = \varphi_x - \frac{\pi}{2}$$  \hspace{1cm} (5.31)

Substituting for $\varepsilon_x$ into Equation (5.30) gives

$$X = \cos \alpha_x \hat{i} + \sin \alpha_x \cos \left(\varphi_x - \frac{\pi}{2}\right) \hat{j} + \sin \alpha_x \sin \left(\varphi_x - \frac{\pi}{2}\right) \hat{k}$$  \hspace{1cm} (5.32)

Referring to Figure 25, the rejection vector of vector $Y$ onto the $x$-$z$ plane is vector $R_y$, which lies at angle $\varepsilon_y$ relative to the $z$-axis. Similarly to rejection vector $R_x$ above, the sign of $\varepsilon_y$ will determine of the sign of the components of $R_y$.

$$R_y = \sin \beta_y \sin \varepsilon_y \hat{i} + \sin \beta_y \cos \varepsilon_y \hat{k}$$  \hspace{1cm} (5.33)
Substituting the components of Equation (5.33) into Equation (5.27) gives

\[
Y = \sin \beta_y \sin \epsilon_y \hat{i} + \cos \beta_y \hat{j} + \sin \beta_y \cos \epsilon_y \hat{k}
\]  
(5.34)

The tilt vector \(T_y\) is the cross product of the unit vector \(\hat{i}\) and second position cube vector \(Y\) and consequently also lies on the \(x-z\) plane at angle \(\varphi_y\) relative to the \(z\)-axis. Therefore,

\[
\epsilon_y = \varphi_y - \frac{\pi}{2}
\]  
(5.35)

Substituting for \(\epsilon_y\) into Equation (5.34) gives

\[
Y = \sin \beta_y \sin \left(\varphi_y - \frac{\pi}{2}\right) \hat{i} + \cos \beta_y \hat{j} + \sin \beta_y \cos \left(\varphi_y - \frac{\pi}{2}\right) \hat{k}
\]  
(5.36)

Referring to Figure 26, the rejection vector of vector \(Z\) onto the \(x-y\) plane is vector \(R_z\), which lies at angle \(\epsilon_x\) relative to the \(x\)-axis. As mentioned previously, the sign of \(\epsilon_x\) will determine the sign of the components of \(R_z\).

\[
R_z = \sin \gamma_z \cos \epsilon_x \hat{i} + \sin \gamma_z \sin \epsilon_x \hat{j}
\]  
(5.37)
Deriving cube mirror position vectors

Substituting the components of Equation (5.37) into Equation (5.28) gives

\[ \mathbf{Z} = \sin \gamma_z \cos \varepsilon_z \mathbf{i} + \sin \gamma_z \sin \varepsilon_z \mathbf{j} + \cos \gamma_z \mathbf{k} \]  

(5.38)

The tilt vector \( \mathbf{T}_z \) is the cross product of the unit vector \( \mathbf{k} \) and second position cube vector \( \mathbf{Z} \) and consequently also lies on the \( x-y \) plane at angle \( \varphi_z \) relative to the \( x \)-axis. Therefore,

\[ \varepsilon_z = \varphi_z - \frac{\pi}{2} \]  

(5.39)

Substituting for \( \varepsilon_z \) into Equation (5.38) gives

\[ \mathbf{Z} = \sin \gamma_z \cos \left( \varphi_z - \frac{\pi}{2} \right) \mathbf{i} + \sin \gamma_z \sin \left( \varphi_z - \frac{\pi}{2} \right) \mathbf{j} + \cos \gamma_z \mathbf{k} \]  

(5.40)

In the same way explained with Method 1, there will be instances when the cube mirror will be in an orientation where the slope of the fringe lines (Equations (5.12) - (5.14)) will be indeterminate, i.e. interferometer aligned = homogeneous fringing. This occurs when one or more tilt angles \( \alpha_x, \beta_y, \text{and} \gamma_z \) are zero.

Taking Equation (5.40) as an example, when \( \gamma_z = 0 \) then \( \varphi_z \) is indeterminate. However, when \( \gamma_z = 0 \), position vector \( \mathbf{Z} \) has only one non-zero component anyway, i.e. \( \mathbf{Z} = \mathbf{k} \) and is of unit length. Therefore, despite the first two components of Equation (5.40) being mathematically indeterminate they in this instance can be considered zero and ignored.

5.2.2.2.1 Method to determine the correct position vectors

Theoretically, the mirror tilt angles \( \alpha_x, \beta_y, \text{and} \gamma_z \) can range from 0 to \( \pi \) rad, however, the maximum angle that the cube mirror surfaces will tilt relative to the respective \( x, y, \text{and} z \) axes is less than 1 degree. Therefore the sine and cosines of these angles will always be positive.
Chapter 5

The tilt axis angles \( \varphi_x \), \( \varphi_y \) and \( \varphi_z \) are all measured from the inclination of the fringe lines with respect to their respective primary axes. The tilt vectors \( \mathbf{T}_x \), \( \mathbf{T}_y \) and \( \mathbf{T}_z \) lie along a line parallel to their fringe lines but their direction is unknown, therefore until it is known which direction is correct, \( \varphi_x \), \( \varphi_y \) and \( \varphi_z \) must initially be treated as having two values:

\[
\begin{align*}
\varphi_x &= a \text{ or } \varphi_x = a + \pi \\
\varphi_y &= b \text{ or } \varphi_y = b + \pi \\
\varphi_z &= c \text{ or } \varphi_z = c + \pi
\end{align*}
\]

This means there are eight possible solutions to the components of second position cube vectors \( \mathbf{X} \), \( \mathbf{Y} \) and \( \mathbf{Z} \). A method is therefore required to determine which one of the eight possibilities is correct.

5.2.2.2.1.1 Create all \( \mathbf{X} \), \( \mathbf{Y} \) and \( \mathbf{Z} \) vector permutations by substituting both values of \( \varphi_x \), \( \varphi_y \) and \( \varphi_z \)

Each of the eight permutations of tilt axis angles \( \varphi_x \), \( \varphi_y \) and \( \varphi_z \) (Equations (5.41) to (5.43)) in turn is substituted into the \( \mathbf{X} \), \( \mathbf{Y} \) and \( \mathbf{Z} \) vector components derived in Equations (5.32), (5.36) and (5.40);

5.2.2.2.1.2 Calculate the dot product of respective \( \mathbf{X} \), \( \mathbf{Y} \) and \( \mathbf{Z} \) vector permutations to test for orthogonality

If vectors \( \mathbf{X} \), \( \mathbf{Y} \) and \( \mathbf{Z} \) are orthogonal to one another, then the dot product of each pair of vectors should equal 0, i.e.:

\[
\begin{align*}
\mathbf{X} \cdot \mathbf{Y} &= 0 = \cos 90^\circ \\
\mathbf{Y} \cdot \mathbf{Z} &= 0 = \cos 90^\circ \\
\mathbf{Z} \cdot \mathbf{X} &= 0 = \cos 90^\circ
\end{align*}
\]

Find the permutation that satisfies Equations (5.44) - (5.46) or is closest to orthogonality.

5.2.2.2.1.3 Filtering method to verify the correct position vector permutation

A more robust method to unambiguously select the correct \( \mathbf{X} \), \( \mathbf{Y} \) and \( \mathbf{Z} \) position vectors is to pass each permutation through the following numerical filter to find the permutation with the highest score.

1. Take the absolute value of the dot products \( \mathbf{X} \cdot \mathbf{Y} \), \( \mathbf{Y} \cdot \mathbf{Z} \) and \( \mathbf{Z} \cdot \mathbf{X} \)
2. Take the common logarithm of the dot products
3. Take the absolute value of the product of all 3 values
4. Select the permutation with the highest score

The outcome of the above procedure may result in more than one permutation with the highest score, however, on close examination it will be found that the components of vectors \( \mathbf{X} \), \( \mathbf{Y} \) and \( \mathbf{Z} \) producing the highest score will all be identical.

Note: the components of vectors \( \mathbf{X} \), \( \mathbf{Y} \) and \( \mathbf{Z} \) are defined by their cosine angles, therefore by definition the vectors are of unit length.
5.2.3 Deriving translation direction using only image sensors

To count the number of fringes $n_f$ passing across an image sensor one pixel is selected to step a counter every time the highest value of radiant flux is detected. Preferably, the selected pixel should be located at or closed the centre of the image sensor, for example pixel $P3 (P_{3y}, P_{3z})$ in Figure 27. With the wavelength $\lambda$ of the laser being known, the displacement $\Delta d_m$ of the moving mirror can be calculated using Equation (4.1).

To capture the direction of fringe movement using the image sensor, two pixels [51] are selected in close proximity to one another and the voltage proportional to the radiant flux of each pixel is measured. Figure 27 shows two such pixels $P3 (P_{3y}, P_{3z})$ and $P4 (P_{4y}, P_{4z})$ whose outputs are fed into a quadrature phase decoder (QPD) [51-54], which controls an up/down counter. With the fringe movement traversing in one direction the QPD will, for example, control the counter to count up and if fringe movement reverses the QPD will control the counter to count down.

When the tilted moving mirror translates, the fringe lines will move along the plane of the image sensor. Figure 27 depicts the fringe lines projected onto the image sensor parallel with the z axis, hence the fringe lines will move in the positive or negative x direction.

If the orientation of the moving mirror is CCW about the z axis and the direction of translation is in the positive y direction as shown in Figure 28, the fringe lines will move in the negative x direction. Conversely, the fringe lines will move in the positive x direction if the mirror orientation is CW for translation in the positive y direction as shown in Figure 29. Hence, the direction of the moving mirror has been in the positive y direction for both cases but the direction of the fringe movement has been in the negative or positive x direction as the case may be. It is therefore imperative to concurrently know the position vectors of the mirror and the direction in which the fringe lines are moving to derive the direction of translation of the moving mirror.

In reality, the tilt axis angles of each side of the cube mirror could range from 0 to $2\pi$ radians dependent on the orientation of the cube. Therefore the position vectors of the cube – which correspond to the normal of each of the 3 mirrors – will point to one of the 4 quadrants of the image sensor shown in Figure 27. Having already determined the 3 position vectors of the cube mirror in the preceding sections above, knowledge of which quadrant it happens to be is already known. If it happens to be towards quadrants 2 or 3 and the mirror is translating in the positive y direction, then the conditions of Figure 28 apply. Conversely, if the position vector points towards quadrants 1 or 4 and the mirror is translating in the positive y direction, then the conditions of Figure 29 apply.

Thus Research Question 1.2v has been resolved.

Generally, when using a Michelson interferometer for measuring translation, the beamsplitter, moving and fixed mirrors are aligned to maximise the fringe intensity. In this case, the fringe intensity will be homogeneous across the entire image sensor. If perfect alignment is maintained during translation then no fringe lines will be observed, just pulsing of the interferogram. In this situation, pixels $P3 (P_{3y}, P_{3z})$ and $P4 (P_{4y}, P_{4z})$ outputs will be identical and fringe direction across the image sensor will be zero making it impossible to determine mirror translation direction despite pixel $P3 (P_{3y}, P_{3z})$ continuing to count fringes. Ways of overcoming this indeterminate are dealt with in section 5.2.5.
Figure 27: Fringe lines aligned with the z axis.

Figure 28: $\alpha_y$ inclined towards quadrants 2 or 3.
5.2.4 Eliminating translation error when using a cube mirror

When using a Michelson interferometer with plain flat mirrors to measure translation, tilting of the moving mirror is unwanted because tilting induces a reduction in modulation amplitude. The reduction in modulation amplitude can cause translation errors, which can be corrected by means of an algorithm [55] or avoided by utilising corner cube retroreflectors [16,17], cat-eye reflectors [26] or alternate types of interferometers [19].

When using a single Michelson interferometer, translation is only measured in 1 DOF. Using a cube mirror and 3 orthogonally arranged Michelson interferometers, translation can be measured simultaneously along the 3 Cartesian axes. However, using this arrangement translation along one Cartesian axis induces a translation measurement error in the other two when the cube mirror has undergone a tilt and rotation. Fortunately, because the components of the position vectors due tilt and rotation are known, the translation error can be resolved as follows.

Figure 30 shows the cube mirror in the x-y perspective having translated $L_y$ distance from the first position to the second position. In this case, the X interferometer detected translation $L_{xy}$ in the x-direction when in fact there should not have been any. As the components of the position vector $X$ are known, the erroneous translation $L_{xy}$ can be calculated.

\[
\xi_{xy} = \tan^{-1}\frac{Y_y}{X_x} \tag{5.47}
\]

\[
L_{xy} = L_y \tan \xi_{xy} = L_y \tan\left(\tan^{-1}\frac{Y_y}{X_x}\right) = L_y \frac{Y_y}{X_x} \tag{5.48}
\]

Similarly, the tilt in the x-z perspective results in erroneous translation $L_{xz}$.
Translation along the $x$-axis purely due to displacement of the cube mirror in that direction is $L_x$ therefore, the total translation $L_{Tx}$ along the $x$-axis is the sum of the translations along each of the 3 Cartesian axes, i.e.

$$L_{Tx} = L_x + L_{xy} + L_{xz}$$

(5.50)

When the interferometer is in perfect alignment translation can be derived from Equation (3.13), i.e. $2 \Delta d_m = n_x \lambda$. However, if the wave front is tilted by an angle $\theta_x$ such as depicted in Figure 2, then the total translation becomes

$$L_{Tx} = L_x + L_{xy} + L_{xz} = \frac{n_x \lambda}{2 \cos \theta_x} = \frac{n_x \lambda}{2 \cos 2 \alpha_x}$$

(5.51)

where $n_x$ is the $x$-interferometer fringe count and $\theta_x = 2 \alpha_x$ (Equation (5.1)).

Similarly,

$$L_{Ty} = L_{yx} + L_y + L_{yz} = \frac{n_y \lambda}{2 \cos 2 \beta_y}$$

(5.52)

and

$$L_{Tz} = L_{xz} + L_{yz} + L_z = \frac{n_z \lambda}{2 \cos 2 \gamma_z}$$

(5.53)

Expanding

$$L_x + L_y + L_z = \frac{n_x \lambda}{2 \cos 2 \alpha_x}$$

(5.54)

$$L_x \frac{Y_y}{Y_y} + L_y + L_z \frac{Y_y}{Y_y} = \frac{n_y \lambda}{2 \cos 2 \beta_y}$$

(5.55)

$$L_x \frac{Z_x}{Z_x} + L_y \frac{Z_y}{Z_y} + L_z = \frac{n_z \lambda}{2 \cos 2 \gamma_z}$$

(5.56)

Let Equation (5.58) represent Equation (5.57)

$$\begin{vmatrix}
1 & X_x & X_x \\
Y_x & Y_x & L_y \\
Z_x & Z_x & L_z
\end{vmatrix} = \begin{vmatrix}
\frac{n_x \lambda}{2 \cos 2 \alpha_x} \\
\frac{n_y \lambda}{2 \cos 2 \beta_y} \\
\frac{n_z \lambda}{2 \cos 2 \gamma_z}
\end{vmatrix}$$

(5.57)

$$\begin{bmatrix}
1 & A & B \\
C & 1 & D \\
E & F & 1
\end{bmatrix} \begin{bmatrix}
L_x \\
L_y \\
L_z
\end{bmatrix} = \begin{bmatrix}
L_x \\
M \\
N
\end{bmatrix}$$

(5.58)

Working out the cofactor of matrix $[R]$. 

72
Deriving cube mirror position vectors

\[
[R_{\text{cofactor}}] = \begin{bmatrix}
1 - FD & -(C - ED) & CF - E \\
-(A - FB) & 1 - EB & -(F - EA) \\
AD - B & -(D - CB) & 1 - CA
\end{bmatrix}
\] (5.59)

Transposing \(R_{\text{cofactor}}\)

\[
[R_{\text{cofactor}}]^T = \begin{bmatrix}
1 - FD & -(A - FB) & AD - B \\
-(C - ED) & 1 - EB & -(D - CB) \\
CF - E & -(F - EA) & 1 - CA
\end{bmatrix}
\] (5.60)

The inverse of the determinant of \([R]\) is

\[
\frac{1}{\det[R]} = \frac{1}{1 + ADE + BCF - EB - FD - CA}
\] (5.61)

\[
\frac{1}{\det[R]} = \frac{1}{1 + \frac{X_y Y_z Z_x}{X_x Y_y Z_z} + \frac{X_z Y_y Z_x}{X_x Y_y Z_z} - \frac{Z_x X_z}{Z_z X_x} - \frac{Z_y Y_z}{Z_z Y_y} - \frac{Y_z X_z}{Y_y X_x}}
\] (5.62)

Therefore

\[
\begin{bmatrix}
L_x \\
L_y \\
L_z
\end{bmatrix} = \frac{1}{\det[R]} \begin{bmatrix}
1 - FD & FB - A & AD - B \\
ED - C & 1 - EB & CB - D \\
CF - E & EA - F & 1 - CA
\end{bmatrix} \begin{bmatrix}
L \\
M \\
N
\end{bmatrix}
\] (5.63)

\[
L_x = \frac{1}{\det[R]} [(1 - FD)L + (FB - A)M + (AD - B)N]
\] (5.64)

\[
L_x = \frac{\lambda}{2\det[R]} \left[ (1 - \frac{Z_y Y_z}{Z_z Y_y}) \frac{n_x}{\cos 2\alpha_x} + \left( \frac{Z_y X_z}{Z_z X_x} - \frac{X_y}{X_x} \right) \frac{n_y}{\cos 2\beta_y} \right]
\] (5.65)

\[
L_x = \frac{\lambda}{2\det[R]} \left[ (1 - \frac{Z_y Y_z}{Z_z Y_y}) \frac{n_x}{\cos 2\alpha_x} + \left( \frac{Z_y X_z}{Z_z X_x} - \frac{X_y}{X_x} \right) \frac{n_y}{\cos 2\beta_y} \right]
\] (5.66)

Similarly

\[
L_y = \frac{\lambda}{2\det[R]} \left[ \frac{Z_y X_z}{Z_z Y_y} - \frac{Y_z}{Y_y} \frac{n_x}{\cos 2\alpha_x} + \left( 1 - \frac{Z_y X_z}{Z_z X_x} \right) \frac{n_y}{\cos 2\beta_y} \right]
\] (5.67)

\[
L_z = \frac{\lambda}{2\det[R]} \left[ \frac{Y_z X_z}{Y_z Y_y} - \frac{Z_y}{Z_y} \frac{n_x}{\cos 2\alpha_x} + \left( 1 - \frac{Y_z X_z}{Y_z X_x} \right) \frac{n_y}{\cos 2\beta_y} \right]
\] (5.68)

All the variables in Equations (5.66), (5.67) and (5.68) are known which therefore enables the actual translation of the cube mirror along each axis to be calculated.

Thus Research Question 1.2iii has been resolved.
5.2.5 Overcoming position vector indeterminates

With 3 Michelson interferometers perfectly aligned with the cube mirror, all 3 interferograms will be homogeneous across their respective image sensors. If the cube mirror is translated purely along one or more of axes then the associated interferograms will pulse but no fringe lines would be present. If the cube mirror is rotated about one axis then the image sensor along that axis will have a homogenous interferogram, the other two will have fringe lines across them.

A homogeneous interferogram implies the pertinent tilt axis angle \( (\varphi_x, \varphi_y, \varphi_z) \) is indeterminate and any equations implicated to resolve the position vectors of the cube mirror are meaningless. Therefore, all three position vectors of the cube will be unknown rendering it impossible to derive the displacement of the cube mirror to six degrees of freedom without the application of additional aids.

Such aids can be:

- modification(s) to the design of the orthogonal interferometer system
- intelligence built into the DSP
- a combination of both of the above

5.2.5.1 Modification to the design of the orthogonal interferometer system

The essence of this thesis is to design a sensor that measures displacement to 6 DoF using optical interferometry. When applied to a solid body under a 3 dimensional load, the displacement of one part of the solid body relative to another can be translated to the applied forces and moments, and in turn, deflection. The proposed solution is an orthogonal arrangement of 3 Michelson interferometers about a cube mirror. The cube mirror is the element of the system that undergoes displacement in concert with the deflection imparted on a solid body. Installation of the sensor within or on the solid body can be done in many different ways. One way, which may seem obvious, is to align the cube mirror to the solid body such that their respective coordinate systems align.
If the deflection imparted on the solid body is primarily along or about one or more of the axes, then the situation will predominate where the tilt axis angles are indeterminate. To overcome this unwanted occurrence the design of the system can be modified in any of the following ways:

5.2.5.1.1 Cube mirror imparted with a known tilt/rotation

In the solid body rest position the cube mirror can be given a known misalignment with the sensor coordinate system therefore applying a known offset in the rest position. As a consequence, all 3 interferograms will have fringe lines present. Any translation, tilt and/or rotation of the cube mirror from the rest position can then be calculated as the tilt angles \( \alpha \), \( \beta \), \( \gamma \) and tilt axis angles \( \varphi_x \), \( \varphi_y \) and \( \varphi_z \) can all be derived from the spacing, inclination and transition of the fringe lines.

5.2.5.1.2 Deflection sensor imparted with a known tilt/rotation

With the solid body in the rest position, the 3 aligned interferometers can be installed in or on the solid body so that the coordinate systems of the sensor and solid body misalign. In the rest position all three interferograms will be homogeneous.

This implementation is only suitable when the initial deflection to the solid body will be known to commence along and/or about one of the coordinate axes, therefore, immediately inducing fringing across all three interferograms.

5.2.5.1.3 One or more interferometers deliberately misaligned

Prior to installing the sensor on or in the solid body, one or more of the interferometers can be aligned to impart a known misalignment to the fixed mirror. This will provide an initial offset of tilt angle and tilt axis angle for the relevant interferometer. In doing so, there will never be the situation where all three interferograms will have no fringe lines present.

To correct for the offset, the digital signal processing can incorporate a transition matrix to change from the interferometer frame of reference to one that is orthogonal when deriving the cube mirror position vectors.

5.2.5.1.4 Known misalignment of cube mirror, deflection sensor and interferometer

In order to make the deflection sensor application independent, a combination of the above three imparted misalignments can be implemented to eliminate the possibility of an indeterminate occurring in the cube mirror position vector calculation.

5.2.5.2 Intelligence built into the digital signal processing

The data captured from the image sensors are fed into a digital signal processor that determines the fringe count, fringe transition direction, tilt angle and tilt axis angle for each interferometer. This data are applied to derive the cube mirror position vectors. By programming the software accordingly, the DSP can implement functionality that anticipates and overcomes the occurrence of indeterminates and/or corrects for applied misalignment of reference frames as mentioned above.

5.2.5.2.1 Track the position vector trajectory

The 3 sides of the cube mirror are orthogonal therefore the position vectors will be orthogonal. As the cube mirror moves with the solid body deflection the DSP can track its trajectory, which is especially useful when the cube mirror passes through perfect alignment of one or more interferometers producing mathematical indeterminates. When
the deflection is dynamic and the cube mirror is frequently passing through an instance of perfect alignment, by tracking the cube mirror trajectory, the instances of indeterminacy can be predicted and mitigated.

In addition, tracking of the position vector trajectories can be used to smooth the trajectory path by means of statistical averaging techniques.

5.2.5.2.2 Correction of misaligned coordinate systems

When the interferometer system as a whole has been misaligned with the solid body coordinate system or the cube mirror alone has been misaligned, the misalignment can be corrected for by the DSP by applying a change of basis transition matrix to an orthogonal reference frame.

5.2.5.2.3 Correction of misaligned interferometers

If one or more interferometers have been purposefully misaligned to overcome instances the tilt axis angles are indeterminate, then once again, this offset can be programmed into the DSP as change of basis transition matrix to an orthogonal reference frame.

5.2.5.3 Combination of system design and intelligent DSP

The design of the deflection sensor together with the solid body will be on a per application basis. Therefore, if it is required to impart a mechanical misalignment to the sensor installation as described above to suit the application, the programming of the DSP will tailored accordingly to overcome the applied misalignments.

This capability is a powerful tool enabling the anticipation of indeterminates in the mathematical resolution of the cube mirror position vectors to be managed and overcome.

By being able overcome or mitigate for indeterminates in the design of the sensor as well as applying intelligence to the DSP, Research Question 1.2\textsuperscript{v} has been theoretically satisfied.

5.3 Results

To test the two methods of deriving the position vectors of a cube mirror, a model cube was defined with the following initial orthogonal position vectors aligned with its respective \(x\), \(y\) and \(z\) coordinate axes.

\[
\mathbf{A} = (A_x, A_y, A_z) = (1, 0, 0) = \mathbf{\hat{x}}
\]

\[
\mathbf{B} = (B_x, B_y, B_z) = (0, 1, 0) = \mathbf{\hat{y}}
\]

\[
\mathbf{C} = (C_x, C_y, C_z) = (0, 0, 1) = \mathbf{\hat{z}}
\]

where \(\mathbf{\hat{x}}, \mathbf{\hat{y}}\) and \(\mathbf{\hat{z}}\) are unit vectors along the respective \(x\)-, \(y\)- and \(z\)-axes.

The cube mirror was located along a virtual line of a second reference frame at +45° to the respective \(x\)'-\(y\)'- and \(z\)'- axes, as illustrated in Figure 31a). The \(x\) and \(x\)'-axes, \(y\) and \(y\)'-axes, and \(z\) and \(z\)'-axes are parallel to one another respectively. Yaw, roll and pitch were applied in 5 steps to the modelled cube mirror as follows to derive the final position vectors:

Step 1 – 45° negative yaw about the \(z\)-axis so that the virtual line is aligned with the \(x\) axis.
Step 2 – 35.264° negative pitch about the y-axis so that the virtual line is aligned with the x-axis

Step 3 - 1° positive roll about the x-axis thereby applying rotation to the model cube

Step 4 - 35.264° positive pitch about the y-axis therefore returning the virtual line to the same position as in Step 1

Step 5 - 45° positive yaw about the z-axis therefore returning the virtual line back to the Initial Position. The cube is now in the Final Position.

Each step is illustrated in Figure 31 and the components of vectors A, B and C after each step are tabulated in Table 1.

The final position of vectors A, B and C at the end of the 5 steps were:

\[
A^v = (A_x^v, A_y^v, A_z^v) = (0.999897529, 0.010173626, -0.010071156) \quad (5.72)
\]

\[
B^v = (B_x^v, B_y^v, B_z^v) = (-0.010071156, 0.999897529, 0.010173626) \quad (5.73)
\]

\[
C^v = (C_x^v, C_y^v, C_z^v) = (0.010173626, -0.010071156, 0.999897529) \quad (5.74)
\]

Alternatively in 1 step, the model cube could have been rotated counter-clockwise by 1° about the axis of the virtual line.

The orientation of the cube mirror having these position vectors would in reality generate respective interferograms from each interferometer.

To derive the tilt angles \(\alpha_x, \beta_y, \) and \(\gamma_z\) of vectors A, B and C, between the initial position and the final position, use is made of the vector dot product of the initial and final position vectors, e.g.

\[
A \cdot A^v = A_x A_x^v + A_y A_y^v + A_z A_z^v = \cos \alpha_x \quad (5.75)
\]

\[
\alpha_x = 0.014315895 \quad (5.76)
\]

Similarly for \(B \cdot B^v\) and \(C \cdot C^v,\)

\[
\beta_y = 0.014315895 \quad (5.77)
\]

\[
\gamma_z = 0.014315895 \quad (5.78)
\]

Using Equation (3.128) and assuming the wavelength of the monochromatic light is 680 nm

\[
\Delta d_f = \frac{\lambda}{\theta} = \frac{680 \times 10^{-9}}{2 \times 0.014315895} = 23.74982 \times 10^{-6} \text{ m} \quad (5.79)
\]

This is the fringe spacing that would be captured by each image sensor as the tilt angles are identical for the 3 orthogonal sides of the cube mirror.

The tilt axis angle of each side of the model cube mirror can be derived by first of all taking the vector cross product of the initial and final position vectors (see Equation (5.9)), e.g.
We now need to know what angle tilt vector $\mathbf{T}_x$ makes with the $y$-axis as this is the tilt axis angle of the fringe lines. This is accomplished by taking the vector dot product of the tilt vector $\mathbf{T}_x$ and the $y$-axis unit vector $\hat{y}$ to find the cosine angle.

\[
\mathbf{T}_x \cdot \hat{y} = (A_y A'_x - A_z A'_y) \cdot 0 + (A_z A'_x - A_x A'_y) \cdot 1 + (A_x A'_y - A_y A'_x) \cdot 0
\]

\[
= \|\mathbf{T}_x\| \|\hat{y}\| \cos \varphi_x
\]

\[
= -0.010071156
\]

\[
\therefore \varphi_x = 45.2900495^\circ
\]

Similarly, the tilt axis angles $\varphi_y$ and $\varphi_z$ are obtained by taking the respective dot products $\mathbf{T}_y \cdot \hat{z}$ and $\mathbf{T}_x \cdot \hat{x}$ to produce:

\[
\varphi_y = 45.29000495^\circ \quad \text{and} \quad \varphi_z = 45.29000495^\circ
\]
Deriving cube mirror position vectors

Figure 31: Five step orientation of model cube mirror: a) Initial position, b) Step 1, c) Step 2, d) Step 3, e) Step 4, f) Final position
Chapter 5

Position vectors \( \mathbf{A} \), \( \mathbf{B} \) and \( \mathbf{C} \) of unit length of a model cube mirror with initial position \( \mathbf{A} = (Ax, Ay, Az) = (1, 0, 0) \), \( \mathbf{B} = (Bx, By, Bz) = (0, 1, 0) \), \( \mathbf{C} = (Cx, Cy, Cz) = (0, 0, 1) \) and with origin = \( (0, 0, 0) \)

Step 1 - 45° negative yaw about the \( z \)-axis so that the virtual line is aligned with the \( x \) axis

Step 2 - 35.264° negative pitch about the \( y \)-axis so that the virtual line is aligned with the \( x \) axis

Step 3 - 1° positive roll about the \( x \)-axis thereby rotation the model cube

Step 4 - 35.264° positive pitch about the \( y \)-axis therefore returning the virtual line to the same position as in Step 1

Step 5 - 45° positive yaw about the \( z \)-axis therefore returning the virtual line back to the Initial Position. The model cube is now in the Final Position.

Table 1: Position vectors of a model cube mirror from initial position to final position via 5 steps of yaw, roll and pitch, and used for subsequent analysis.

<table>
<thead>
<tr>
<th>Model Cube Vectors</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coeffs</td>
<td>( Ax )</td>
<td>( Ay )</td>
<td>( Az )</td>
</tr>
<tr>
<td>Initial Position</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Step 1(^1)</td>
<td>0.707106781</td>
<td>-0.707106781</td>
<td>0</td>
</tr>
<tr>
<td>Step 2(^{\text{II}})</td>
<td>0.577350269</td>
<td>-0.707106781</td>
<td>-0.40824829</td>
</tr>
<tr>
<td>Step 3(^{\text{III}})</td>
<td>0.577350269</td>
<td>-0.699840483</td>
<td>-0.420582887</td>
</tr>
<tr>
<td>Step 4(^{\text{IV}})</td>
<td>0.714228164</td>
<td>-0.699840483</td>
<td>-0.010071156</td>
</tr>
<tr>
<td>Step 5(^{\text{V}}) (Final Position)</td>
<td>0.999897529</td>
<td>0.010173626</td>
<td>-0.010071156</td>
</tr>
</tbody>
</table>
5.3.1 Method 1

Following the methodology given in Section 5.2.2.1, to find $F_x$, $F_y$ and $F_z$ we use Equations (5.12) to (5.14) and Equations (5.82) and (5.83) to give:

$$F_x = 1.010174669; \quad F_y = 1.010174669; \quad F_z = 1.010174669$$

To find $G_x$ and $G_y$ we use Equations (5.5), (5.6), (5.76) and (5.77) to give:

$$G_x = 0.999897529; \quad G_y = 0.999897529$$

Substituting the above variables into Equations (5.19) and (5.20) to find $H$ and $J$ we get:

$$H = 0.000208058; \quad J = \pm 0.020553226$$

The variable $J$ has two solutions, therefore there are two possible values for components $X_y$, $X_z$, $Y_x$ and $Y_z$. The two possible values for each of these components for each sign of $J$ is given along each row of Table 2.

Table 2: Method 1 components for $X_y$, $X_z$, $Y_x$ and $Y_z$ due to $\pm J$ expression.

<table>
<thead>
<tr>
<th>$J$</th>
<th>$X_y$</th>
<th>$X_z$</th>
<th>$Y_x$</th>
<th>$Y_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.020553226</td>
<td>0.010173626</td>
<td>0.010277140</td>
<td>0.009869300</td>
<td>0.010173626</td>
</tr>
<tr>
<td>-0.020553226</td>
<td>-0.010381706</td>
<td>-0.010071156</td>
<td>-0.010071156</td>
<td>-0.009969717</td>
</tr>
</tbody>
</table>

There is only one value for components $X_x$ and $Y_y$, which were derived in Equations (5.5) and (5.6) respectively. They are the cosine values of the tilt angles $\alpha_x$ and $\beta_y$ calculated in Equations (5.76) and (5.77), i.e. from the final position of the model cube mirror position vectors (i.e. step 5 in Table 1). With a single value for $X_x$ and $Y_y$ and two values for each of the four components $X_y$, $X_z$, $Y_x$ and $Y_z$ implies there are 16 permutations of the components for position vectors $X$ and $Y$, which are listed in Table 4.

From Section 5.2.2.1.1.1, the next step was to take the dot product of each permutation of position vectors $X$ and $Y$ to find those that are orthogonal to one another. The result of the dot product calculation is given in the second last column of Table 4. The outcome of this exercise returned two permutations (highlighted in Table 4 in green) that were at right angle to each other, i.e. permutations 4 and 7.

As stated in Section 5.2.2.1.1.2, when more than one permutation of vectors $X$ and $Y$ prove to be orthogonal to one another, they have to be tested for unit length using Equation (5.25). In this case, it is the permutation 7 in Table 4 that has both vectors $X$ and $Y$ of unit length (highlighted in yellow).

However, to unambiguously select the correct $X$ and $Y$ vectors from the 16 permutations, each permutation needs to be applied to the mathematical filter defined in Section 5.2.2.1.1.3. Table 5 lists the results of the calculations and Figure 32 shows graphically the product of values $A$, $B$ and $C$ against the respective permutations. Once again, permutation 7 stands out clearly from the other permutations with its vectors $X$ and $Y$ the closest to being orthogonal to one another and their respective lengths being closest to unity.

Moving on to the next step of the process (Section 5.2.2.1.4), having completed the above filtering process, Equations (5.21) - (5.23) are used to determine the components of
position vector $Z$ by taking the vector cross product of the above highlighted permutation of vectors $X$ and $Y$.

Table 3 collates the only permutation of 16 that satisfies the above dot product and unit length tests as well as the permutation filter test.

**Table 3: Position vectors X, Y and Z derived from model cube mirror tilt angle and tilt axis angles.**

<table>
<thead>
<tr>
<th></th>
<th>$X_x$</th>
<th>$X_y$</th>
<th>$X_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>0.999897529</td>
<td>0.010173626</td>
<td>-0.010071156</td>
</tr>
<tr>
<td>Y</td>
<td>0.010071156</td>
<td>0.999897529</td>
<td>0.010173626</td>
</tr>
<tr>
<td>Z</td>
<td>0.010173626</td>
<td>-0.010071156</td>
<td>0.999897529</td>
</tr>
</tbody>
</table>

Comparing the components of position vectors $X$, $Y$ and $Z$ in Table 3 with the components of position vectors of $A^\gamma$, $B^\gamma$ and $C^\gamma$ in Table 1, it can be seen that they have are identical, therefore validating Method 1.

The above procedures proves that Method 1 is a solution to finding the correct position vector components of the cube mirror from the tilt angles and tilt axis angles captured from the interferograms projected onto the respective image sensors.

**Therefore, Method 1 theoretically satisfies Research Question 1.2i.**
Table 4: Components of position vectors X and Y creating 16 permutations. Permutation 7 being the only one that is orthogonal and of unit length.

| Xx  | Xy     | Xz     | ||X|| | Yx     | Yy     | ||Y|| | X · Y | Permutation |
|-----|--------|--------|-------|--------|--------|-------|-------|-------------|
| 0.01017363 | 0.01027714 | 1.00000209 | 0.00986930 | 0.01017363 | 0.0099997987 | 0.020145428 | 1 |
| 0.01017363 | 0.01027714 | 1.00000209 | -0.01007116 | -0.00996972 | 0.999997946 | 2.637415E-11 | 4 |
| 0.01017363 | -0.01007116 | 1 | 0.00986930 | 0.01017363 | 0.999997987 | 0.019938413 | 5 |
| 0.01017363 | -0.01007116 | 1 | -0.01007116 | -0.00996972 | 0.999995934 | 0.000207016 | 6 |
| 0.01017363 | -0.01007116 | 1 | 0.00986930 | 0.01017363 | 0.999997946 | 2.627209E-11 | 7 |
| 0.01017363 | -0.01007116 | 1 | -0.01007116 | -0.00996972 | 0.999995934 | 0.000207016 | 8 |
| -0.01038171 | 0.01027714 | 1.00000423 | 0.00986930 | 0.01017363 | 0.999997987 | -0.000407798 | 9 |
| -0.01038171 | 0.01027714 | 1.00000423 | -0.01007116 | -0.00996972 | 0.999995934 | -0.000614814 | 10 |
| -0.01038171 | 0.01027714 | 1.00000423 | -0.01007116 | 0.01017363 | 0.999997946 | -0.002034621 | 11 |
| -0.01038171 | 0.01027714 | 1.00000423 | -0.01007116 | -0.00996972 | 0.999995934 | -0.020553226 | 12 |
| -0.01038171 | -0.01007116 | 1.00000213 | 0.00986930 | 0.01017363 | 0.999997987 | -0.000614814 | 13 |
| -0.01038171 | -0.01007116 | 1.00000213 | 0.00986930 | -0.00996972 | 0.999995934 | -0.000411947 | 14 |
| -0.01038171 | -0.01007116 | 1.00000213 | -0.01007116 | 0.01017363 | 0.999997946 | -0.020553226 | 15 |
| -0.01038171 | -0.01007116 | 1.00000213 | -0.01007116 | -0.00996972 | 0.999995934 | -0.020350359 | 16 |
### Table 5: Permutation filter calculations.

<table>
<thead>
<tr>
<th>Permutation</th>
<th>$|X \cdot Y|$</th>
<th>$\log_{10}(|X \cdot Y|) = A$</th>
<th>$\text{ABS}(1 - |X|)$</th>
<th>$\log_{10}\left(\text{ABS}(1 - |X|)\right) = B$</th>
<th>$\text{ABS}(1 - |Y|)$</th>
<th>$\log_{10}(\text{ABS}(1 - |Y|)) = C$</th>
<th>$\text{ABS}(A \times B \times C)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.020145428</td>
<td>-1.695823491</td>
<td>2.09568E-06</td>
<td>-5.678674378</td>
<td>2.01257E-06</td>
<td>-5.696248436</td>
<td>54.85503997</td>
</tr>
<tr>
<td>2</td>
<td>0.019938413</td>
<td>-1.700309423</td>
<td>2.09568E-06</td>
<td>-5.678674378</td>
<td>4.06629E-06</td>
<td>-5.390802036</td>
<td>52.05090823</td>
</tr>
<tr>
<td>3</td>
<td>0.000207016</td>
<td>-3.683996134</td>
<td>2.09568E-06</td>
<td>-5.678674378</td>
<td>2.62721E-11</td>
<td>-10.58050528</td>
<td>221.3464396</td>
</tr>
<tr>
<td>4</td>
<td>2.63741E-11</td>
<td>-10.5782153</td>
<td>2.09568E-06</td>
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<td>-5.687455406</td>
<td>54.54381005</td>
</tr>
</tbody>
</table>
Deriving cube mirror position vectors

Figure 32: Graph of results from permutation filter showing permutation 7 with the highest score.
5.3.2 Method 2

The tilt angles $\alpha_x$, $\beta_y$, and $\gamma_z$ and the tilt axis angles $\varphi_x$, $\varphi_y$ and $\varphi_z$ of the model cube mirror were obtained using Equations (5.76) - (5.78) and (5.82) - (5.83) respectively from the resulting position vectors of the model cube mirror. This simulation of angular displacement of the cube mirror generated only one set of tilt axis angles $\varphi_x$, $\varphi_y$ and $\varphi_z$ as the 5 angular displacement steps of the model cube mirror were done with known increments as tabled in Table 1. However, in reality, when trying to determine $\varphi_x$, $\varphi_y$ and $\varphi_z$ from their respective interferograms these angles can take on two values as specified by Equations (5.41) - (5.43). This therefore presents 8 possible solutions for the cube mirror position vectors.

To find the correct position vector of the cube mirror the method detailed in section 5.2.2.2.1 must be followed.

To begin with the position vectors $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ of the cube mirror are calculated for every permutation of tilt axis angles $\varphi_x$, $\varphi_y$ and $\varphi_z$ as stated in Section 5.2.2.2.1.1. Table 6 lists all eight permutations of tilt angles $\alpha_x$, $\beta_y$, and $\gamma_z$ and tilt axis angles $\varphi_x$, $\varphi_y$ and $\varphi_z$ and the position vectors $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ that they generate.

The next step of the procedure (Section 5.2.2.1.1.2) was to test whether each respective pair of vectors were orthogonal to one another by finding their dot products (Equations (5.44) - (5.46)).

Table 7 lists the result of each dot product calculation and it can be seen that vectors $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ of permutation 1 are all almost exactly orthogonal to one another, i.e. their dot products are very close to zero.

However, the robust way of determining the correct permutation is by applying the numerical filter defined in Section 5.2.2.2.1.3 and the permutation with the highest score will reveal the vectors with the correct components.

Table 7 lists the outcome of each step of the procedure and Figure 33 illustrates the result graphically.

Comparing the components of position vectors $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ in permutation 1 in Table 6 with the components of position vectors of $\mathbf{A}^r$, $\mathbf{B}^r$ and $\mathbf{C}^r$ in Table 1, it can be seen that they are identical, therefore validating Method 2.

The above procedure proves that Method 2 is a solution to finding the correct position vector components of the cube mirror from the tilt angles and tilt axis angles captured from the interferograms projected onto the respective image sensors.

Therefore, Method 2 theoretically satisfies Research Question 1.2i.
Table 6: Eight possible solutions to the cube mirror position vectors.

<table>
<thead>
<tr>
<th>Tilt angles from Equations (5.76) - (5.78)</th>
<th>( \alpha_x = 0.014315895 )</th>
<th>( \beta_y = 0.014315895 )</th>
<th>( \gamma_z = 0.014315895 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 8 ) Permutations of tilt axis angles Equations (5.82) - (5.83) and Equations (5.41) - (5.43)</td>
<td>( \varphi_x )</td>
<td>( \varphi_y )</td>
<td>( \varphi_z )</td>
</tr>
<tr>
<td>(-0.78033662)</td>
<td>2.36125603</td>
<td>-0.78033662</td>
<td>2.36125603</td>
</tr>
<tr>
<td>0.78033662</td>
<td>0.78033662</td>
<td>2.36125603</td>
<td>0.78033662</td>
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<tr>
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<td>0.78033662</td>
<td>0.78033662</td>
<td>2.36125603</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>8 Permutations of vectors ( X, Y ) and ( Z )</th>
<th>Permutation 1</th>
<th>Permutation 2</th>
<th>Permutation 3</th>
<th>Permutation 4</th>
<th>Permutation 5</th>
<th>Permutation 6</th>
<th>Permutation 7</th>
<th>Permutation 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_x )</td>
<td>0.99989753</td>
<td>0.999897529</td>
<td>0.999897529</td>
<td>0.999897529</td>
<td>0.999897529</td>
<td>0.999897529</td>
<td>0.999897529</td>
<td>0.999897529</td>
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<tr>
<td>( X_y )</td>
<td>0.01017363</td>
<td>-0.010173626</td>
<td>0.010173626</td>
<td>-0.010173626</td>
<td>0.010173626</td>
<td>-0.010173626</td>
<td>-0.010173626</td>
<td>-0.010173626</td>
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<tr>
<td>( X_z )</td>
<td>-0.01007116</td>
<td>-0.010071156</td>
<td>-0.010071156</td>
<td>-0.010071156</td>
<td>-0.010071156</td>
<td>-0.010071156</td>
<td>-0.010071156</td>
<td>-0.010071156</td>
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<tr>
<td>( Y_x )</td>
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<td>0.010071156</td>
<td>0.010071156</td>
<td>-0.010071156</td>
<td>-0.010071156</td>
<td>-0.010071156</td>
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<td>( Y_y )</td>
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<td>0.999897529</td>
<td>0.999897529</td>
<td>0.999897529</td>
<td>0.999897529</td>
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<td>0.999897529</td>
<td>0.999897529</td>
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<tr>
<td>( Y_z )</td>
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<td>0.010071156</td>
<td>0.010071156</td>
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<tr>
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<td>0.999897529</td>
<td>0.999897529</td>
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</tr>
</tbody>
</table>
Table 7: The dot product of permutations of vectors X, Y and Z, their common logarithm and the product of all their values.

<table>
<thead>
<tr>
<th>Permutation</th>
<th>X·Y</th>
<th>(\log_{10}|X \cdot Y| = A)</th>
<th>Y·Z</th>
<th>(\log_{10}|Y \cdot Z| = B)</th>
<th>Z·X</th>
<th>(\log_{10}|Z \cdot X| = C)</th>
<th>ABS(A * B * C)</th>
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<tr>
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<td>-1.691516473</td>
<td>-5.48346E-15</td>
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<td>0.00010246</td>
<td>-3.989444908</td>
<td>96.23587874</td>
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<td>0.00010246</td>
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<td>0.00010246</td>
<td>-3.989444908</td>
<td>0.00010246</td>
<td>-3.989444908</td>
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<tr>
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</table>

Figure 33: Graph of results from permutation filter showing permutation 1 with the highest score.
5.4 Summary

The interferogram from a single Michelson interferometer provides information on the tilt angle, tilt axis angle and translation of the moving mirror. However, what is lacking from the information is in which direction the mirror is tilted and in which direction the mirror is translating – despite having captured the direction of fringe line transition.

To overcome this, a novel method using 3 Michelson interferometers orthogonally arranged about a cube mirror not only resolves the 3 degrees of freedom of the single interferometer but resolves the 6 DoF of the cube mirror.

With reference to Research Question 1.2i, two methods were described to resolve the components of the position vectors of the cube mirror by utilising the tilt angle and tilt axis angles captured from the interferograms projected onto respective image sensors. Both methods gave rise to several possible position vector solutions, therefore a procedure for each was set out how the correct solution could be found. Both procedures were validated by creating a model cube mirror that had been orientated from an initial position to a final position. The tilt angles and tilt axis angles of the 3 orthogonal sides of the cube mirror were derived from the position vectors of the model cube mirror in the final position. These angles were a simulation of the data that would have been obtained from the interferograms that the orientated cube mirror would have generated. By substituting these angles into the equations for each Methods 1 and 2, the procedures correctly determined the simulated cube mirror position vectors.

Common practice for determining the direction of mirror translation with the Michelson interferometer is by modifying the basic arrangement and introducing additional optics, opto-electronics and quadrature phase decoding. With an orthogonal arrangement of 3 Michelson interferometers, the direction of translation is determined by capturing the direction of fringe line movement and correlating it with the mirror normal tilt angle. This methodology avoids the need for additional optics, opto-electronics and quadrature phase decoding and satisfies Research Question 1.2ii.

When the cube mirror is tilted/rotated about one or more axes, translation of the cube mirror along one axis will introduce a translation error in one or both the other two axes. To fulfil Research Question 1.2iii, a method was derived to mathematically correct for this induced translation utilising the fringe count of all three interferometers and the components of the 3 position vectors of the cube mirror.

Whenever one or more interferometers are perfectly aligned with the cube mirror the associated interferogram(s) will be devoid of fringe lines therefore there will be no tilt axis angle to measure. This introduces indeterminates into the mathematics used to derive the cube mirror position vectors. The occurrence of these indeterminates can be managed or overcome by incorporating changes to the design of the orthogonal system such as imparting a known misalignment of the cube mirror or a known misalignment of one or more of the interferometers. The sensor itself could be imparted with a known misalignment with the solid body it is mounted on or within. These changes to the mechanical design can be corrected for by the digital signal processing used to derive the position vectors from the interferogram data. By being able to introduce a misalignment to overcome the indeterminates resolves Research Question 1.2v.

To test the accuracy of the vector analysis in deriving the cube mirror position vectors an apparatus was required capable of inducing known amounts of tilt to the cube mirror about each of the 3 Cartesian axes. To avoid the high cost of purchasing or hiring such an instrument that was accurate to an order of magnitude of thousandths of a degree and that
was totally immune from undesired angular displacement about the first and second axes when adjusting the tilt about the third axis, it was necessary to design and build such a sensor as part of this thesis.
6 DESIGN OF MIRROR TILT AND TILT AXIS MEASUREMENT SYSTEM

6.1 Introduction

To test the accuracy of the methods described in Chapter 5 to derive the position vectors of the cube mirror, a calibrated 3 axis tilt and rotation stage was required. By mounting the cube mirror on the tilt/rotation stage and applying measured changes to the orientation of the cube mirror, these changes could be correlated with the derived position vectors of the cube mirror.

The primary requirements for a 3 axis tilt and rotation stage were twofold; a) to have a tilt measurement accuracy of one thousandth of a degree or better about each axis and; b) for the instrument to give a read out of any induced angular displacement to an axis when adjusting angular displacement about another axis. It was found that not only did the cost of obtaining a precision commercial or laboratory 3 axis tilt and rotation stage far exceed the budget of this thesis but none of the products researched gave an actual measurement of induced crosstalk.

It was therefore decided to design and manufacture a device - mounted on an inexpensive tilt/rotation stage - that could measure tilt and rotation about the 3 axes to an accuracy of one thousandth of a degree. Because the device was independent of the adjustment controls of the tilt/rotation stage and could simultaneously measure tilt of all 3 axes, it could measure any inter-axis crosstalk.

Consequently, a Tilt/Rotation Stage (model 860-0110) was purchased from Eksma Optics [56] onto which the cube mirror would be mounted. The stage provided two independent tilt adjustments with ±5° range (x- and y-axis) and ±4° in-plane rotation adjustment (z-axis).

The tilt/rotation stage, like with any mechanical micrometric device machined to the highest quality, was susceptible to typical errors of a linear nature such as inaccuracy of the screw thread pitch, drunken thread and backlash. In addition, errors of a rotational nature arising from imperfections in machining the two tilt and the single rotation mechanisms would affect precision of the independent tilt/rotation movements. Therefore a change in the attitude of one of the three axes would have some effect on the other two. Therefore, having a separate instrument to measure the tilt and rotation of the
stage would also capture the magnitude of any crosstalk that were to occur when making a tilt adjustment about any one of the axes.

The basis for designing and manufacturing a 3 axis tilt/rotation measuring device was to use the linear properties of light transmission and the optical lever principle. A narrow collimated beam of light can be projected tens or even hundreds of metres onto a screen. Any small change in direction of the beam can be converted into an angle using trigonometry. To amplify the change in angle to shorten the need for long projection distances, the light beam can be channelled between two mirrors serving as a waveguide. If one of the mirrors moves in concert with tilt or rotation, the change in angle of the mirror can be multiplied by the number of reflections within the waveguide. The following sections explain the principle in detail.

6.2 Methodology

6.2.1 Optical lever

Figure 34 shows a diagram of a mirror in two positions reflecting an incident light ray. With the mirror in the first position, the incident ray makes angle $\theta_i$ with the normal therefore the reflected ray departs the mirror at angle $\theta_i$ to the normal. With the mirror in the second position tilted at angle $\delta$ from the first position, the incident and reflected rays makes angle $\theta_i + \delta$ with the normal. As a consequence, a change in mirror tilt of $\delta$ has produced a change of $2\delta$ between the reflected ray in first and second positions.

![Optical lever principle](image)

Figure 34: Optical level principle.

If the optical level principle is used between two mirrors, one static and the other tilting, a change in tilt angle of the tilting mirror produces a change in angle of the output light beam that is the product of the number of reflections and the tilt angle.
Using a collimated laser beam of narrow waist as input to the optical lever, the output can be projected a great distance from the two mirrors. The tilt angle can be calculated as follows:

\[
\Delta d_i = l \theta_{ai} = ln \delta
\]  (6.1)

\[
\delta = \frac{\Delta d_i}{ln}
\]  (6.2)

Where \( l \) is the distance from the optical lever to the projection surface, \( \Delta d_i \) is the change in distance in metres of the laser spot at the projection surface, \( \theta_{ai} \) is the angle subtended by lengths \( l \) and \( \Delta d_i \) (\( l \gg \Delta d_i \)), \( n \) is the number of mirror reflections and \( \delta \) is the mirror tilt angle.

The principle for the optical lever designed for calibration of each tilt/rotation axis is illustrated in Figure 35. It is shown with a separation distance between the static and tilting mirrors of 2.8 mm to produce 12 reflections. However, the number of reflections can be increased to increase the sensitivity by reducing this distance, as will be encountered later in the experimentation chapter.

The narrow collimated beam was derived from an Egismos H436351D/R 1 milliwatt laser module [58] and was projected at the tilting mirror at 45°. The tilting mirror in Figure 35 is shown in 3 positions:

- Black – tilting mirror parallel to the static mirror
- Green – tilting mirror rotated counter clockwise
- Red – tilting mirror rotated clockwise

Based on the designed mirror lengths and indicated distance they are apart, there are 11 reflections before the laser beam departs the optical lever. The departing beam is then projected onto a steering mirror, which reflects the beam in a desired direction to suit the space available for the experiment and to maximise the distance between the optical lever and the projection surface.

**Figure 35: Tilt calibration optical lever.**

### 6.2.2 Orthogonal arrangement

Three such optical levers in the experiment rig are arranged orthogonally with the tilting mirrors mounted on the tilt/rotation stage. The steering mirrors are angled so that all three departing beams project in the same direction so that they can be observed together. Figure 36 depicts the arrangement without showing the mounting of the components. The
red, green and blue beams identify the optical lever about the \(x\)-, \(y\)- and \(z\)-axes respectively. As the static and tilting mirrors of the X and Y optical lever are in the same \(x\)-\(y\) plane as one another, to save space they share a common round tilting mirror and a common round static mirror as seen in Figure 36.

![Diagram of optical lever setup](image)


**Figure 36: Three orthogonally arranged optical levers.**

From a starting configuration (Figure 37a) where the two sets of static and tilting mirrors are parallel to one another, if there is a rotation of the Tilt/Rotation Stage purely about the \(z\)-axis (Figure 37b) then only the blue beam will move horizontally on the projection screen in concert with the rotation.

If the Tilt/Rotation Stage tilts purely about the \(x\)-axis (Figure 37c), then the blue beam will remain static, the red beam will move vertically and the green beam will move equally horizontally and vertically as shown.

However, if the tilt is purely about the \(y\)-axis (Figure 37d) then on the projection screen the red beam will move horizontally, the green beam will move equally in the horizontal and vertical direction, and the blue beam will move vertically.

If there is tilt/rotation about two or more axes, then the movement of all three beams will be a combination of those tilts/rotations.

Vector analysis was used to determine the magnitude of the tilt/rotation of the Tilt/Rotation Stage.
6.2.3 Vector analysis to obtain the reflection vector

Consider the vector diagram illustrated in Figure 38, which represents the reflection of a light ray \( \mathbf{R} \) about a mirror normal \( \mathbf{N} \). The incident ray represented by vector \( \mathbf{I} \) has been reversed in direction to simplify calculating the components of ray \( \mathbf{R} \). By definition, the incident ray, the mirror normal and the reflected ray are all in the same plane and the angle the incident ray and reflected ray make with the normal are identical. The reflected ray is essentially a reflection isometry of the incident ray about the plane containing the mirror normal, which is perpendicular to the incident ray. Vector \( \mathbf{I}' \) is the vector projection of vector \( \mathbf{I} \) on normal vector \( \mathbf{N} \) and vector \( \mathbf{I''} \) is the vector rejection of vector \( \mathbf{I} \) and is perpendicular to vector \( \mathbf{I}' \).

Let

\[ \|\mathbf{I}\| = \|\mathbf{N}\| = 1 \]  \hspace{1cm} (6.3)

Therefore

\[ \mathbf{I} \cdot \mathbf{N} = \|\mathbf{I}\| \|\mathbf{N}\| \cos \theta_l \]  \hspace{1cm} (6.4)

\[ \cos \theta_l = \frac{\mathbf{I} \cdot \mathbf{N}}{\|\mathbf{I}\| \|\mathbf{N}\|} \]  \hspace{1cm} (6.5)

where \( n_{ix}, n_{iy} \) and \( n_{iz} \) are the \( x, y \) and \( z \) components of normal vector \( \mathbf{N} \) and \( \theta_l \) is the angle between vectors \( \mathbf{I} \) and \( \mathbf{N} \). Hence, the norm of projection vector \( \mathbf{I}' \) is
Chapter 6

\[ \|I'\| = \|I\| \cos \theta_1 = \cos \theta_1 = c \]  
(6.6)

which is the scalar multiple of vector \( \mathbf{N} \) used to derive the components of \( \mathbf{I}' \), i.e.:

\[ \mathbf{I}' = \cos \theta_1 \mathbf{N} = \left( \cos \theta_1 n_{i_x}, \cos \theta_1 n_{i_y}, \cos \theta_1 n_{i_z} \right) \]  
(6.7)

Rejection vector \( \mathbf{I}'' \) is derived as follows

\[ \mathbf{I}'' = \mathbf{I} - \mathbf{I}' \]
\[ = \left( i_x - \cos \theta_1 n_{i_x}, i_y - \cos \theta_1 n_{i_y}, i_z - \cos \theta_1 n_{i_z} \right) \]  
(6.8)

and because \( \mathbf{R} \) is the isometric reflection of \( \mathbf{I} \), its components are simply

\[ \mathbf{R} = \mathbf{I}' - \mathbf{I}'' \]
\[ = \left( \cos \theta_1 n_{i_x} - i_x + \cos \theta_1 n_{i_x}, \cos \theta_1 n_{i_y} - i_y + \cos \theta_1 n_{i_y}, \cos \theta_1 n_{i_z} - i_z + \cos \theta_1 n_{i_z} \right) \]
\[ = \left( 2 \cos \theta_1 n_{i_x} - i_x, 2 \cos \theta_1 n_{i_y} - i_y, 2 \cos \theta_1 n_{i_z} - i_z \right) \]
(6.9)

If the reflection isometry is about the \( z \)-axis then \( n_{i_x} = n_{i_y} = 0 \), and \( n_z = 1 \). Substituting into Equation (6.4) then

\[ \cos \theta_1 = i_z \]  
(6.10)

Continuing, the norm of projection vector \( \mathbf{I}' \) is

\[ \|\mathbf{I}'\| = \|\mathbf{I}\| \cos \theta_1 = \cos \theta_1 = i_z = c \]  
(6.11)

which is the scalar multiple of vector \( \mathbf{N} \) used to derive the components of \( \mathbf{I}' \), i.e.:

\[ \mathbf{I}' = \cos \theta_1 \mathbf{N} = \left( 0, 0, i_z \right) = (0, 0, i_z) \]  
(6.12)

Rejection vector \( \mathbf{I}'' \) is

\[ \mathbf{I}'' = \mathbf{I} - \mathbf{I}' \]
\[ = \left( i_x - 0, i_y - 0, i_z - i_z \right) \]
\[ = \left( i_x, i_y, 0 \right) \]  
(6.13)

and because \( \mathbf{R} \) is the isometric reflection of \( \mathbf{I} \), its components are simply

\[ \mathbf{R} = \mathbf{I}' - \mathbf{I}'' \]
\[ = \left( -i_x, -i_y, i_z \right) \]  
(6.14)

Proving \( \mathbf{R} \) also has norm of unity
Design of mirror tilt and tilt axis measurement system

\[ \| \mathbf{R} \|^2 = (2 \cos \theta_t n_{1x} - i_x)^2 + (2 \cos \theta_t n_{1y} - i_y)^2 + (2 \cos \theta_t n_{1z} - i_z)^2 \]  
\[ = (2 \cos \theta_t n_{1x})^2 - 4 \cos \theta_t n_{1x}i_x + i_x^2 + (2 \cos \theta_t n_{1y})^2 - 4 \cos \theta_t n_{1y}i_y + i_y^2 + (2 \cos \theta_t n_{1z})^2 - 4 \cos \theta_t n_{1z}i_z + i_z^2 \]  
\[ = (2 \cos \theta_t)^2 (n_{1x}^2 + n_{1y}^2 + n_{1z}^2) - 4 \cos \theta_t (n_{1x}i_x + n_{1y}i_y + n_{1z}i_z) + 1 \]  
\[ = (2 \cos \theta_t)^2 - 4 \cos \theta_t \cos \theta_t + 1 \]  
\[ \therefore \| \mathbf{R} \| = 1 \]  

(6.15)

(6.16)

Figure 38: Finding the components of the reflection vector.

6.2.4 Deriving the components of the tilting mirror normal of an optical lever with 12 reflections

The methodologies described in this Section 6.2.4 are based on an optical lever system having 12 reflections. Even though the optical lever can be adjusted to have fewer or more than 12 reflection, the same methodologies will apply.

Referring to Figure 39, which shows the path of a 12 reflection optical lever about the x-axis, vector I is the incident ray from the laser (not shown), R_1 to R_11 are the vectors of the reflection beams through the optical lever, R_12 is the vector of the reflection that is projected onto a distant screen (not shown), normal vector to the static mirror is designated N_s, normal vector to the tilting mirror is designated N_T and the normal vector to the steering mirror is designated N_SM. All vectors are reduced to norm = 1.

The tilting mirror has 3 degrees of freedom for which the components of its normal vector N_T are unknown. What is known by measurement are the components of incident vector I, static mirror normal vector N_s, steering mirror normal vector N_SM and normalised reflection vector R_12. The components of vectors I, N_s and N_SM remain static as the laser, static and steering mirrors are fixed to a mounting, whereas, vector R_12 is dynamic and its components have to be measured for every displacement of the beam on the projection.
surface. To find the components of tilting mirror normal $N_T$, first the components of reflection $R_{11}$ must be found. Two methods are shown how this can be done:

1. From the direction of the incident vector $I$, i.e. from the laser. The path of reflections from the tilting and static mirrors are calculated as the light beam progresses between the two mirrors. This method will produce components of $R_{11}$ that are a function of known components of $I$ and $N_S$, and the unknown components of $N_T$.

2. From the direction of the projection surface, i.e. from the opposite direction to 1 above. As $R_{11}$ is a reflection isometry of normalised vector $R_{12}$ about $N_{SM}$, and as the components of $R_{12}$ and $N_{SM}$ are known, the components of $R_{11}$ can be calculated.

To find the components of tilting mirror normal $N_T$, the two sets of components of $R_{11}$ deduced from opposite directions in the optical lever are resolved as a system of polynomials demonstrated in the following sections below.

![Figure 39: Vector diagram of a 12 reflection optical lever about the x-axis.](image)

6.2.4.1 Deriving the components of $R_{11}$ from the direction of the incident vector $I$

Two methods to derive the components of $R_{11}$ from the direction of the incident vector $I$ are presented below, the first by using reflection isometry step by step through the optical lever and the second by simulating a virtual mirror.

6.2.4.1.1 Deriving the components of $R_{11}$ by step-wise reflection isometry

Starting with the components of incident beam $I$ and static mirror $N_S$, which are known, reflection vector $R_{11}$ is derived by calculating the components of the reflections $R_1$ to $R_{11}$ as they progress through the optical lever. The reflection components are represented as functions of the known components of vectors $I$ and $N_S$ and the unknown components of tilting mirror normal $N_T$. The procedure is as follows.

Calculating $R_1$ by following the procedure in the previous section

$$-I \cdot N_T = \cos \theta_1 = -i_xn_{TX} - i_yn_{TY} - i_zn_{TZ}$$  \hspace{1cm} (6.17)

$$I' = \cos \theta_1 N_T = (\cos \theta_1 n_{TX}, \cos \theta_1 n_{TY}, \cos \theta_1 n_{TZ})$$  \hspace{1cm} (6.18)

$$I'' = -I - I'$$

$$= (-i_x - \cos \theta_1 n_{TX}, -i_y - \cos \theta_1 n_{TY}, -i_z - \cos \theta_1 n_{TZ})$$ \hspace{1cm} (6.19)
Design of mirror tilt and tilt axis measurement system

\[ \mathbf{R}_1 = 1' - 1'' \]
\[ = (\cos \theta_1 n_{TX} + i_x + \cos \theta_1 n_{TY} + i_y + \cos \theta_1 n_{TZ} + i_z + \cos \theta_1 n_{TX}) \]
\[ = (2 \cos \theta_1 n_{TX} + i_x, 2 \cos \theta_1 n_{TY} + i_y, 2 \cos \theta_1 n_{TZ} + i_z) \]

Reflection \( \mathbf{R}_1 \) is the incident ray of reflection \( \mathbf{R}_2 \) about static mirror normal \( \mathbf{N}_S \). To simplify the calculation of the components of \( \mathbf{R}_2 \) the direction of \( \mathbf{R}_1 \) is reversed to that depicted in Figure 39. Calculating the components of \( \mathbf{R}_2 \)

\[ -\mathbf{R}_1 \cdot \mathbf{N}_S = \cos \theta_2 \]
\[ = -((2 \cos \theta_1 n_{TX} + i_x)n_{SX} - (2 \cos \theta_1 n_{TY} + i_y)n_{SY} - (2 \cos \theta_1 n_{TZ} + i_z)n_{SZ}) \]
\[ \mathbf{R}_1' = \cos \theta_2 \mathbf{N}_S = (\cos \theta_2 n_{SX}, \cos \theta_2 n_{SY}, \cos \theta_2 n_{SZ}) \]
\[ \mathbf{R}_1'' = -\mathbf{R}_1 - \mathbf{R}_1' \]
\[ = (-((2 \cos \theta_1 n_{TX} + i_x) - \cos \theta_2 n_{SX}, -(2 \cos \theta_1 n_{TY} + i_y) - \cos \theta_2 n_{SY}, -(2 \cos \theta_1 n_{TZ} + i_z) - \cos \theta_2 n_{SZ}) \]
\[ \mathbf{R}_2 = \mathbf{R}_1' - \mathbf{R}_1'' \]
\[ = (\cos \theta_2 n_{SX} + (2 \cos \theta_1 n_{TX} + i_x) + \cos \theta_2 n_{SY} + \cos \theta_2 n_{SY} + \cos \theta_2 n_{SZ} + (2 \cos \theta_1 n_{TY} + i_y) + \cos \theta_2 n_{SZ} + 2 \cos \theta_1 n_{TZ} + i_z + 2 \cos \theta_1 n_{TZ} + i_z + \cos \theta_2 n_{SX} + 2 \cos \theta_1 n_{TY} + i_y, 2 \cos \theta_2 n_{SZ} + 2 \cos \theta_1 n_{TZ} + i_z) \]

Similarly, calculating the components of \( \mathbf{R}_3 \)

\[ -\mathbf{R}_2 \cdot \mathbf{N}_T = \cos \theta_3 \]
\[ = -((2 \cos \theta_2 n_{SX} + (2 \cos \theta_1 n_{TX} + i_x)n_{TX} - (2 \cos \theta_2 n_{SY} + 2 \cos \theta_1 n_{TY} + i_y)n_{TY} - (2 \cos \theta_2 n_{SZ} + 2 \cos \theta_1 n_{TZ} + i_z)n_{TZ}) \]
\[ \mathbf{R}_2' = \cos \theta_3 \mathbf{N}_T = (\cos \theta_3 n_{TX}, \cos \theta_3 n_{TY}, \cos \theta_3 n_{TZ}) \]
\[ \mathbf{R}_2'' = -\mathbf{R}_2 - \mathbf{R}_2' \]
\[ = (-((2 \cos \theta_2 n_{SX} + (2 \cos \theta_1 n_{TX} + i_x) - \cos \theta_3 n_{TX}, -(2 \cos \theta_2 n_{SY} + 2 \cos \theta_1 n_{TY} + i_y) - \cos \theta_3 n_{TY}, -(2 \cos \theta_2 n_{SZ} + 2 \cos \theta_1 n_{TZ} + i_z) - \cos \theta_3 n_{TZ}) \]
\[ \mathbf{R}_3 = \mathbf{R}_2' - \mathbf{R}_2'' \]
\[ = (\cos \theta_3 n_{TX} + (2 \cos \theta_2 n_{SX} + (2 \cos \theta_1 n_{TX} + i_x) + \cos \theta_3 n_{TX}, \cos \theta_3 n_{TY} + (2 \cos \theta_2 n_{SY} + 2 \cos \theta_1 n_{TY} + i_y) + \cos \theta_3 n_{TY}, \cos \theta_3 n_{TZ} + (2 \cos \theta_2 n_{SZ} + 2 \cos \theta_1 n_{TZ} + i_z) + \cos \theta_3 n_{TZ}) \]

Building the pattern for the polynomial expressions for the reflection vector components, the reflection vector \( \mathbf{R}_{11} \) has components
Similarly, there is a pattern to the cosine values of the reflection angles, i.e.:

\[
\cos \theta_1 = -i_x n_{TX} - i_y n_{TY} - i_z n_{TZ}
\]

\[
\cos \theta_2 = -(2 \cos \theta_1 n_{TX} + i_x) n_{SX} - (2 \cos \theta_1 n_{TY} + i_y) n_{SY} - (2 \cos \theta_1 n_{TZ} + i_z) n_{SZ}
\]

\[
\cos \theta_3 = -(2 \cos \theta_2 n_{SX} + 2 \cos \theta_1 n_{TX} + i_x) n_{SY} - (2 \cos \theta_2 n_{SY} + 2 \cos \theta_1 n_{TY} + i_y) n_{ST} - (2 \cos \theta_2 n_{SZ} + 2 \cos \theta_1 n_{TZ} + i_z) n_{SZ}
\]

\[
\cos \theta_4 = -(2 \cos \theta_3 n_{TX} + 2 \cos \theta_2 n_{SX} + 2 \cos \theta_1 n_{TX} + i_x) n_{SY} - (2 \cos \theta_3 n_{TY} + 2 \cos \theta_2 n_{SY} + 2 \cos \theta_1 n_{TY} + i_y) n_{ST} - (2 \cos \theta_3 n_{TZ} + 2 \cos \theta_2 n_{SZ} + 2 \cos \theta_1 n_{TZ} + i_z) n_{SZ}
\]

\[
\cos \theta_5 = -(2 \cos \theta_4 n_{SX} + 2 \cos \theta_3 n_{TX} + 2 \cos \theta_2 n_{SX} + 2 \cos \theta_1 n_{TX} + i_x) n_{SY} - (2 \cos \theta_4 n_{SY} + 2 \cos \theta_3 n_{TY} + 2 \cos \theta_2 n_{SY} + 2 \cos \theta_1 n_{TY} + i_y) n_{ST} - (2 \cos \theta_4 n_{SZ} + 2 \cos \theta_3 n_{TZ} + 2 \cos \theta_2 n_{SZ} + 2 \cos \theta_1 n_{TZ} + i_z) n_{SZ}
\]

\[
\cos \theta_6 = -(2 \cos \theta_5 n_{TX} + 2 \cos \theta_4 n_{SX} + 2 \cos \theta_3 n_{TX} + 2 \cos \theta_2 n_{SX} + 2 \cos \theta_1 n_{TX} + i_x) n_{SY} + 2 \cos \theta_5 n_{TY} + 2 \cos \theta_4 n_{SY} + 2 \cos \theta_3 n_{TY} + 2 \cos \theta_2 n_{SY} + 2 \cos \theta_1 n_{TY} + i_y) n_{ST} + 2 \cos \theta_5 n_{TZ} + 2 \cos \theta_4 n_{SZ} + 2 \cos \theta_3 n_{TZ} + 2 \cos \theta_2 n_{SZ} + 2 \cos \theta_1 n_{TZ} + i_z) n_{SZ}
\]

...
Expanding the components of reflection vector $R_{11}$ (Equations (6.29) - (6.31)) with the $\cos(\theta_n)$ terms given in Equations (6.32) - (6.38) produces components as a function of large powers of $n_{Tx}$, $n_{Ty}$, and $n_{Tz}$, which is extremely unwieldy to resolve when it comes to correlating $R_{11}$ with $R_{12}$ using reflection isometry. If the optical lever is adjusted to have greater than 12 reflections then the equations become even more cumbersome. This method will not be pursued any further.

6.2.4.1.2 Deriving the components of $R_{11}$ by virtual mirror simulation

6.2.4.1.2.1 Creating a virtual mirror

Figure 40 illustrates a vector diagram of a virtual tilted mirror with normal $N'_T$ of unit length with components:

$$N'_T = (n'_{Tx}, n'_{Ty}, n'_{Tz})$$

(6.39)

\[ \text{Virtual tilted mirror} \]

Figure 40: Vector diagram of tilted mirror normal and angles made with axes.

Referring back to the 12 reflection $x$-axis optical lever in Figure 39, there are 6 reflections from the tilting mirror and 5 reflections from the static mirror. The normal of the static mirror is coincident with the $z$-axis, therefore the $x$ and $y$ components of all 5 reflection vectors from the static mirror are the inverse of the respective incident vector whilst the $z$ component remains unchanged (Equation (6.14)). By contrast, as the tilting mirror is not fixed in position its normal is variant, and when not aligned with the $z$-axis, all three components of each of the 6 reflection vectors differ from their respective incident vectors. Consequently, the optical lever can be represented as a single reflection of the incident beam about the normal of a virtual mirror that summates the 11 reflections relative to the $x$- and $y$-axes and summates the 6 reflections relative to the $z$-axis.

To illustrate how to determine the magnitude of the components of the normal for the virtual mirror, consider the generic optical lever depicted in Figure 41 having $n$ reflections. The tilting mirror is tilted by angle $\delta$ to the $z$-axis and incident beam $I$ strikes the mirror at angle $\rho + \delta$ to the mirror normal as depicted in the enlargement (top left). The beam progresses through the optical lever until it emerges as reflection $R_n$ at an angle $\rho + n\delta$.
with the mirror normal (enlargement top right), where \( n \) is the number of reflections. The angle that \( R_n \) makes with the \( z \)-axis is \( \tau \).

\[
\tau = (\rho + n\delta) + \delta = \rho + (n + 1)\delta
\]

(6.40)

The normal of the virtual mirror would be located at half \( \zeta \), i.e.

\[
\frac{\zeta}{2} = \rho + \frac{(n + 1)\delta}{2}
\]

(6.42)

Having found \( \zeta \), the tilt angle \( \delta' \) of the virtual mirror with the \( z \)-axis is therefore

\[
\delta' = \frac{\zeta}{2} - \rho = \frac{(n + 1)}{2}\delta
\]

(6.43)

For an optical lever that has 11 reflections, the relationship between the normal angles of the tilting mirror and the virtual tilting mirror is:

\[
\delta' = 6\delta
\]

(6.44)

---

**Figure 41: Close up of the incident beam and reflection \( R_{11} \).**

To create a virtual mirror that produces reflection \( R_n \) with angle \( \tau \) to the \( z \)-axis from incident beam \( I \) having angle \( \rho \) to the \( z \)-axis, the first step is to find the whole angle \( \zeta \) between \( I \) and \( R_n \) in Figure 42, i.e.

\[
\zeta = \rho + \tau = 2\rho + (n + 1)\delta
\]

(6.41)
6.2.4.1.2.2 Deriving the components of $R_{11}$ as a function of the virtual mirror simulating 11 reflections

Assuming a 12 reflection optical lever, then reflection vector $R_n = R_{11}$ is a reflection isometry of the incident beam $-I$ about the virtual tilted mirror normal $N_T'$ and its components are derived using the procedure associated with Figure 38 as follows

\[ \| -I \| = \| N_T' \| = 1 \quad (6.45) \]

Therefore

\[ -I \cdot N_T' = \| -I \| \| N_T' \| \cos \theta_i' \quad (6.46) \]

\[ \cos \theta_i' = \frac{-I \cdot N_T'}{\| -I \| \| N_T' \|} \]

\[ = - (i_x n_{T_x}' + i_y n_{T_y}' + i_z n_{T_z}') \quad (6.47) \]

Hence, the norm of projection vector $I'$ is

\[ \| I' \| = \| -I \| \cos \theta_i' = \cos \theta_i \quad (6.48) \]

which is the scalar multiple of vector $N_T'$ used to derive the components of $I'$, i.e.:

\[ I' = \cos \theta_i' N_T' = (\cos \theta_i' n_{T_x}', \cos \theta_i' n_{T_y}', \cos \theta_i' n_{T_z}') \quad (6.49) \]

Deriving the rejection vector

\[ I'' = -I - I' \]

\[ = (-i_x - \cos \theta_i' n_{T_x}', -i_y - \cos \theta_i' n_{T_y}', -i_z - \cos \theta_i' n_{T_z}') \quad (6.50) \]

and because $R_{11}$ is the isometric reflection of $I$, its components are
\[ R_{11} = I' - I'' \]
\[ = \left( \cos \theta'_n n'_{Tx} - (-i_x - \cos \theta'_n n'_{Ty}) \right) \cos \theta'_n n'_{Ty} \]
\[ - \left( -i_y - \cos \theta'_n n'_{Ty} \right) \cos \theta'_n n'_{Tz} - (-i_z - \cos \theta'_n n'_{Tz}) \]
\[ = \left( 2 \cos \theta'_n n'_{Tx} + i_x, 2 \cos \theta'_n n'_{Ty} + i_y, 2 \cos \theta'_n n'_{Tz} + i_z \right) \]
\[ = \left( -2 \left( i_x n'_{Tx}^2 + i_y n'_{Ty} n'_{Tx} + i_z n'_{Tz} n'_{Tx} \right) + i_x \right) \]
\[ = \left( -2 \left( i_x n'_{Ty} n'_{Ty} + i_y n'_{Ty}^2 + i_z n'_{Tz} n'_{Tz} \right) + i_y \right) \]
\[ = \left( -2 \left( i_x n'_{Tz} n'_{Tz}^2 + i_y n'_{Tz} n'_{Tz} + i_z n'_{Tz}^2 \right) + i_z \right) \]
\[ = \left( r_{11x}, r_{11y}, r_{11z} \right) \]

Note: The components of reflection vector \( R_{11} \) in Equation (6.51) have been derived in the direction from the laser towards the projection surface.

### 6.2.4.2 Deriving the components of \( R_{11} \) in the direction from the projection surface towards the laser

The components of reflection vector \( R_{11} \) can also be derived in the reverse direction, i.e. from the projection surface going towards the laser.

To differentiate \( R_{11} \) and its components \((r_{11x}, r_{11y}, r_{11z})\) obtained in Equation (6.51) from \( R_{11} \) and its components that will be derived from the direction of the projection surface, the latter will be designated \( R'_{11} \) and its components \((r'_{11x}, r'_{11y}, r'_{11z})\) respectively. Therefore,

\[ R'_{11} = -R_{11} \]  
(6.52)

### 6.2.4.2.1 Normalising reflection vector \( R_{12} \)

\( R_{12} \) is projected onto the projection surface, which is graduated to a scale of 1 millimetre. The origin of the 12\(^{th}\) reflection is at the base of the normal to the steering mirror. Beam \( R_{12} \) is represented as a vector

\[ R_{12} = \left( r_{12x}, r_{12y}, r_{12z} \right) \]  
(6.53)

where \( r_{12x}, r_{12y}, \) and \( r_{12z} \) are its measured components.

By calculating the norm of \( R_{12} \)

\[ \| R_{12} \| = \sqrt{r_{12x}^2 + r_{12y}^2 + r_{12z}^2} \]  
(6.54)

the magnitude of \( R_{12} \) can be normalised

\[ R'_{12n} = \left( \frac{r_{12x}}{\sqrt{r_{12x}^2 + r_{12y}^2 + r_{12z}^2}}, \frac{r_{12y}}{\sqrt{r_{12x}^2 + r_{12y}^2 + r_{12z}^2}}, \frac{r_{12z}}{\sqrt{r_{12x}^2 + r_{12y}^2 + r_{12z}^2}} \right) \]  
(6.55)

where \( r_{12xn}, r_{12yn}, \) and \( r_{12zn} \) are its normalised components.
6.2.4.2 Deriving the components of $R'_{11}$ using reflection isometry

From Equation (6.51), the components of reflection vector $R_{11}$ were derived as functions of incident vector $I$ and normalised virtual tilting mirror normal $N'_T$.

As mentioned previously, $R'_{11}$ has a reflection isometry about steering mirror normal $N_{SM}$ with $R_{12n}$. By once again using the procedure associated with Figure 38, the components of $R'_{11}$ can be found.

\[
R_{12n} \cdot N_{SM} = \|R_{12n}\|\|N_{SM}\| \cos \theta_l \\
\cos \theta_l = \frac{R_{12n} \cdot N_{SM}}{\|R_{12n}\|\|N_{SM}\|} \\
= r_{12xn}n_{SMx} + r_{12yn}n_{SMy} + r_{12zn}n_{SMz}
\]

Hence the norm of projection vector of $R_{12n}$ on vector $N_{SM}$ is

\[
R'_{12n} = \cos \theta_l N_{SM} = \left( \cos \theta_l n_{SMx}, \cos \theta_l n_{SMy}, \cos \theta_l n_{SMz} \right)
\]

The rejection vector is derived as follows

\[
R''_{12n} = R_{12n} - R'_{12n} \\
= \left( r_{12xn} - \cos \theta_l n_{SMx}, r_{12yn} - \cos \theta_l n_{SMy}, r_{12zn} - \cos \theta_l n_{SMz} \right)
\]

Therefore

\[
R'_{11} = \frac{R'_{12n} - R''_{12n}}{\left( \cos \theta_l n_{SMx} - (r_{12xn} - \cos \theta_l n_{SMx}), \cos \theta_l n_{SMy} - (r_{12yn} - \cos \theta_l n_{SMy}), \cos \theta_l n_{SMz} - (r_{12zn} - \cos \theta_l n_{SMz}) \right)} \\
= \left( 2 \cos \theta_l n_{SMx} - r_{12xn}, 2 \cos \theta_l n_{SMy} - r_{12yn}, 2 \cos \theta_l n_{SMz} - r_{12zn} \right)
\]

Expanding $R'_{11}$

\[
R'_{11} = \left( \begin{array}{c} 2(r_{12xn}n_{SMx} + r_{12yn}n_{SMy} + r_{12zn}n_{SMz})n_{SMx} - r_{12xn} \\ 2(r_{12xn}n_{SMx} + r_{12yn}n_{SMy} + r_{12zn}n_{SMz})n_{SMy} - r_{12yn} \\ 2(r_{12xn}n_{SMx} + r_{12yn}n_{SMy} + r_{12zn}n_{SMz})n_{SMz} - r_{12zn} \\ \left(2n_{SMx}^2 - 1\right)r_{12xn} + 2n_{SMx}n_{SMy}r_{12yn} + 2n_{SMx}n_{SMz}r_{12zn} \\ \left(2n_{SMy}^2 - 1\right)r_{12yn} + 2n_{SMy}n_{SMx}r_{12xn} + 2n_{SMy}n_{SMz}r_{12zn} \\ \left(2n_{SMz}^2 - 1\right)r_{12zn} + 2n_{SMz}n_{SMx}r_{12xn} + 2n_{SMz}n_{SMy}r_{12yn} \end{array} \right)
\]

All the terms in Equation (6.61) are known as the steering mirror is mounted on the experiment rig in a specifically designed location and the components of normalised reflection vector $R_{12}$ have been calculated.

6.2.4.3 Resolving $-R_{11} = R'_{11}$ to derive the components of the optical lever tilting mirror normal

Reflection vectors $-R_{11}$ and $R'_{11}$ are the same vector (see Equation (6.52)), except their respective components were derived from opposite ends of the optical level system. Reflection vector $R'_{11}$ components are all known whereas reflection vector $-R_{11}$ are not as they are functions of the virtual tilting mirror normal $N'_T$ components ($n'_Tx$, $n'_Ty$, $n'_Tz$), which are what need to be found. Equating the respective components renders:
Chapter 6

\[
\begin{pmatrix}
-r_{11x} \\
r_{11y} \\
r_{11z}
\end{pmatrix}
= \begin{pmatrix}
 r'_{11x} \\
r'_{11y} \\
r'_{11z}
\end{pmatrix}
\] (6.62)

\[
\begin{pmatrix}
 2(i_x n'_T x^2 + i_y n'_T y^2 + i_z n'_T z^2) - i_x \\
 2(i_x n'_T x y + i_y n'_T y z + i_z n'_T z x) - i_y \\
 2(i_x n'_T x z + i_y n'_T y z + i_z n'_T z y) - i_z
\end{pmatrix}
= \begin{pmatrix}
 r'_{11x} \\
r'_{11y} \\
r'_{11z}
\end{pmatrix}
\] (6.63)

Annotating the terms

\[
2(ax^2 + bxy + cxz) - a - d = 0
\] (6.65)

\[
2(axy + by^2 + cyz) - b - e = 0
\] (6.66)

\[
2(azx + byz + cz^2) - c - f = 0
\] (6.67)

Equations (6.65)- (6.67) are a system of polynomial equations that has the follow solution for \(x, y\) and \(z\):

\[
x = \frac{(a + d)}{\sqrt{2}} \sqrt[4]{\frac{1}{(a^2 + ad + b^2 + be + c^2 + cf)}}
\] (6.68)

or

\[
x = - \frac{(a + d)}{\sqrt{2}} \sqrt[4]{\frac{1}{(a^2 + ad + b^2 + be + c^2 + cf)}}
\]

\[
y = \frac{(b + e)}{\sqrt{2}} \sqrt[4]{\frac{1}{(a^2 + ad + b^2 + be + c^2 + cf)}}
\] (6.69)

or

\[
y = - \frac{(b + e)}{\sqrt{2}} \sqrt[4]{\frac{1}{(a^2 + ad + b^2 + be + c^2 + cf)}}
\]

\[
z = \frac{(c + f)}{\sqrt{2}} \sqrt[4]{\frac{1}{(a^2 + ad + b^2 + be + c^2 + cf)}}
\] (6.70)

or

\[
z = - \frac{(c + f)}{\sqrt{2}} \sqrt[4]{\frac{1}{(a^2 + ad + b^2 + be + c^2 + cf)}}
\]

106
Design of mirror tilt and tilt axis measurement system

Expanding $a$, $b$, $c$, $d$, $x$, $y$ and $z$ produces:

$$n'_{Tx} = \frac{(i_x + r'_{11x})}{\sqrt{2}} \sqrt{\frac{1}{(i_x^2 + i_x r'_{11x} + i_y^2 + i_y r'_{11y} + i_z^2 + i_z r'_{11z})}}$$  

(6.71)

or

$$n'_{Ty} = \frac{(i_y + r'_{11y})}{\sqrt{2}} \sqrt{\frac{1}{(1 + i_x r'_{11x} + i_y r'_{11y} + i_z r'_{11z})}}$$  

or

$$n'_{Tz} = \frac{(i_z + r'_{11z})}{\sqrt{2}} \sqrt{\frac{1}{(1 + i_x r'_{11x} + i_y r'_{11y} + i_z r'_{11z})}}$$  

(6.73)

The above derivation was based on the orientation of the example virtual tilted mirror shown in Figure 40. Because only very small angular displacements are being considered, and because the mirror normal is closely aligned with the $z$-axis, the $z$-axis component will always be positive. Therefore the choice of $n'_{Tz}$ in Equation (6.73) will be the one that is positive. The sign of $n'_{Ty}$ and $n'_{Tx}$ is unknown. Similarly for the X and Y optical levers, the normal component that is most closely aligned with the axis in which the normal vector is pointed will always be positive. The sign of the other two tilting mirror normal components will be unknown.

6.2.4.3.1 Determining the sign of the unknown components of the optical lever tilting mirror normal

Figure 37 shows how the transition of the beams projecting on the screen can be observed as they move from their initial position to their next or final position. The directions in which the beams move across the screen indicates whether the tilt/rotation of the optical lever tilting mirrors are increasing or decreasing. The instantaneous position of each beam relative to its initial position indicates in which direction the mirror normal is tilted, therefore enabling the sign of the unknown components above to be determined. In this way, the sign of all three components in Equations (6.71) - (6.73) have been resolved.

With reference to Figure 37b), the blue beam signifies the $z$-axis optical lever and the normal vector to its tilting mirror is designated $N'_{Tx}$:
If the blue beam is located along the negative $x$-axis from its initial position, then component $n'_{Tx,y}$ is positive.

With reference to Figure 37c), the red beam signifies the $x$-axis optical lever and the normal vector to its tilting mirror is designated $N'_{T_x}$. The green beam signifies the $y$-axis optical lever and the normal vector to its tilting mirror is designated $N'_{T_y}$.

If the red beam is located along the positive $z$-axis from its initial position, then components $n'_{Tx,y}$ and $n'_{Ty,y}$ are positive.

With reference to Figure 37d), because the $y$-axis optical lever steering mirror reflects the beam moving under the influence of rotation about the $y$-axis is a little more difficult than the other two optical levers.

If the green beam is located above a line with a negative 45° slope through the $y$-axis initial position, then $n'_{Tx}$, $n'_{Ty}$ and $n'_{Tz}$ are negative.

### 6.2.4.4 Deriving the components of the tilting mirror normal $N_T$

The purpose of the optical lever system was to measure to approximately $1/1000°$ accuracy, the 3 DoF tilt and rotation of the tilt/rotation stage with a measurement range of $±0.4°$.

The reason for this defined accuracy was because it was felt that a minimum of 2 fringe lines across the image sensor would provide an accurate measure of fringe spacing and fringe slope. With a wavelength of 680 nm, this represented a tilt angle measurement of 0.01° for the image sensor being used. The resolution of the optical lever was therefore 10x greater than the 6 DoF displacement sensor.

The reason for the range was due to the maximum tilt and rotation angles of adjustment built safely into the experiment rig without the optics touching one another.

The method of measuring the tilt and rotation angles was by projecting a laser beam through the optical lever onto a projection screen a substantial distance away. With such small angles, the magnitude of the component along the length of the beam will therefore be very much greater than those components that are orthogonal. The virtual mirror depicted in Figure 40 shows component $n'_{Tz}$ very much greater than both components $n'_{Tx}$ and $n'_{Ty}$. Therefore the large magnitude component of the mirror normal vector will be a several orders of magnitude greater than the other two components. This simplifies translating the virtual tilting mirror components $n'_{Tx}$, $n'_{Ty}$ and $n'_{Tz}$ into the tilting mirror components $n_{Tx}$, $n_{Ty}$ and $n_{Tz}$ by using approximation.

Referring to Figure 43, assume the tilting mirror is orientated with its normal projecting in the positive $z$-axis direction. The figure portrays the relative magnitude of the components. Equation (6.44) illustrates the 6:1 relationship between the angles of the tilting mirror and the virtual tilting mirror normal for an optical lever with 11 reflections. With components $n'_{Tx}$ and $n'_{Ty}$ being orders of magnitude less than $n'_{Tz}$ then:

$$n_{Tx} \approx \frac{n'_{Tx}}{6} \quad (6.74)$$

and
Design of mirror tilt and tilt axis measurement system

\[ n_{Ty} \approx \frac{n'_{Ty}}{6} \quad (6.75) \]

As the tilting mirror normal is defined as a unit vector the \( n_{Tz} \) component is derived as follows:

\[ n_{Tz} \approx \sqrt{1 - n_{Ty}^2 + n_{Ty}^2} \quad (6.76) \]

Equations (6.74) to (6.76) are the approximated components of the tilted mirror normal depicted in Figure 39. As Figure 36 shows, there are three such optical lever systems orthogonal to one another to simultaneously measure tilt and rotation of the tilt/rotation stage. Therefore, each optical lever has an associated set of equations as those mentioned above that define its associated tilting mirror normal vectors.

![Figure 43: 1/6 approximation of tilting mirror normal components.](image)

6.2.4.5 Aligning the optical lever normal vectors and cube mirror position vectors

The X, Y and Z interferometers are aligned with the Cartesian coordinate system as depicted in Figure 23. Conversely, the X and Y optical lever round tilting mirror normal is aligned with the negative z-axis and the Z optical lever normal is aligned with the positive x-axis as depicted in Figure 36. Therefore, to eventually correlate the components of the X and Y optical lever normals with those of the Z interferometer position vectors, the direction of these optical lever normal vectors must be reversed. The Z optical lever is already aligned with the X interferometer, therefore its normal vector direction remains unchanged.

6.2.4.6 Deriving the third normal from the optical lever system

From Figure 36 it can be seen that the z-axis optical lever has its own static and tilting mirrors, therefore its tilting mirror normal vector \( \mathbf{N}_{Tz} \) is unique. Whereas, the x- and y-
axis optical levers share common static and tilting mirror. Therefore, the X and Y optical lever tilting mirror normal vectors $N_{Tx}$ and $N_{Ty}$ are identical.

Essentially, the optical lever system produces a pair of orthogonal vectors, $N_{Tx} \& N_{Ty}$ and $N_{Ty} \& N_{Tx}$, which theoretically are identical to one another. However, the components of $N_{Tx}$ and $N_{Ty}$ are calculated independently using the derived equations above and they may differ slightly due to different measured variables being used in the calculations. To find which of normal vectors $N_{Tx}$ and $N_{Ty}$ are more closely orthogonal to normal vector $N_{Tz}$ the dot product of the two vector pairs must be determined. The vector pair whose value is closest to zero should be used as the two optical lever tilting mirror normals, i.e. the lesser of:

$$N_{Tx} \cdot N_{Tz} \quad \text{or} \quad N_{Ty} \cdot N_{Tz} \quad (6.77)$$

Also, from Figure 36 it can be seen that the normal of the Z tilting mirror $N_{Tz}$ is aligned with the x-axis, whereas normals $N_{Tx}$ and $N_{Ty}$ of the X & Y tilting mirror are aligned with the negative z-axis. To provide a third optical lever vector that is orthogonal to $N_{Tx}/N_{Ty}$ and $N_{Tz}$, the cross product of the two vectors must be calculated, i.e. the normal vector aligned with the z-axis crossed with the normal vector aligned with the x-axis as follows:

$$N_{Tz} \times N_{Tx} = N_{y} \quad \text{or} \quad N_{Tz} \times N_{Ty} = N_{y} \quad (6.78)$$

where $N_{y}$ is the third position vector from the optical lever aligned with the y-axis.

The components of the optical lever system vectors have now all been calculated and can therefore be directly correlated with the respective position vectors of the interferometers.

### 6.3 Results

The results for this chapter on the ability of the orthogonal optical lever system to function as an accurate measure of tilt and rotation of the cube mirror are covered in the experimentation results discussed in Chapter 8.

### 6.4 Summary

With the use of a laser with a beam width of approximately 1 mm, a good tape measure and adequate space to project the laser beam onto a sheet of graph paper, the slightest change in the inclination of the laser can be measured very accurately using elementary trigonometry.

This is the principle on which the optical lever tilt/rotation angle measurement system was based. The system was needed because a 3 DoF tilt/rotation stage could not be obtained that actually measured crosstalk induced in the second and third axes when adjusting the tilt about the first axis.

The orthogonal optical lever system was mounted on an inexpensive tilt/rotation stage and each of the 3 beams was projected in the same direction to simplify measurement of their vector components.

On the ingress side of the optical lever the components of the incident vector (laser beam) were known. On the egress side, the components of the reflection vector projecting onto the distant screen were measured. What needed to be determined in between were the
components of the tilted mirror normal. Using reflection isometry, two methods were described how to obtain the unknown components. The more simple method simulated the optical lever as a single virtual mirror. To simulate the virtual mirror with a single reflection with an optical lever with multiple reflections, a simulation factor was formulated to reduce the magnitude of the virtual mirror components to approximately those of the actual tilted mirror normal components.

The sign of the dominant component of the tilted mirror normal is trivial to determine due to knowledge in which direction the mirror is facing. To determine the sign of the other two components, the position of the projected laser beam relative to the initial or “zero” position of the optical lever can be observed on the screen. This identified the orientation of the mirror normal therefore enabling the sign of the other two components to be deduced.

Having designed the orthogonal interferometer displacement sensor as well as the optical lever system to evaluate the accuracy of the displacement sensor, the two systems needed to be built into an experiment rig to perform the necessary experimentation and extract the data. This is the subject of the next chapter.
7 EXPERIMENT RIG

7.1 Introduction

The experiment rig was designed to apply linear and angular displacement to the 6 DoF displacement sensor cube mirror and to the optical lever system. The 6 DoF displacement sensor image sensors were to capture fringe count, fringe line spacing and fringe line inclination so that the applied linear and angular displacement could be derived in terms of the 3 cube mirror position vectors. The 3 beams from the optical lever systems were to be read off the graph paper on the projection surface to derive the applied angular displacement in terms of the 3 optical lever vectors.

With regards to an application, the applied linear and angular displacement of the cube mirror could be considered as (or simulate) deflection on a solid body being subjected to compression, tension, torsion, shear and bending, or it could considered as nano-positioning of an object such as the stage of a photo-lithographical apparatus.

An XYZ translation stage (XYZ Stage) enabled translation to be induced to the cube mirror along the respective axes and a tilt/rotation stage (T/R Stage) induced pitch, tilt and yaw. The two stages came as separate devices, therefore the T/R Stage was mounted upon the XYZ Stage so that the whole T/R Stage moved whenever translation was applied but only the T/R Stage moved whenever tilt or rotation was applied. In this way, the cube mirror, which was indirectly attached to the T/R Stage, could be moved to 6DoF.

The XYZ Stage also enabled the separation distance between the static and tilting mirrors of the optical lever to be adjusted to increase or decrease the number of reflections passing through the optical lever.

The XYZ Stage was bolted to a Base plate, which in turn was mounted to a solid structure to reduce the occurrence of vibration.

The rig also comprised the optical lever system detailed in Chapter 6 that was designed to measure the angular displacement of the tilt/rotation stage. The sensor was an orthogonal arrangement of optical levers with the beams exiting the sensor projected towards a screen that was placed a distance away from the experiment rig. The change of position of the beams on the screen was proportional to the magnitude of pitch, tilt and yaw applied to the tilt/rotation stage. The direction of projection of the beams was designed so that they project away from where one sat to operate the experiment.
Above the translation and tilt/rotation stages the experiment rig was built up of a number of layers comprising mounting hardware for the orthogonal optical levers and the orthogonal interferometers.

At the first level, the tilt/rotation stage provided a mounting platform to which a bracket (T/R Stage Bracket 1) was attached whose purpose was to mount the tilting mirror of the z axis optical levers and provide an attachment point for a further bracket (T/R Stage Bracket 2) onto which the x- and y-axis optical lever tilting mirror and the cube mirror were mounted.

The experiment rig also comprised two vertical uprights (Upright 1 and Upright 2) at right angles to each other that were secured to the base as well as to one another. The uprights served as a structure for attaching mounting brackets for the fixed components of the optical levers and interferometers.

The first of these was the Optical Lever Fixed Component Mounting Bracket (OLFCM Bracket) located at the second level above the T/R Stage platform onto which all the fixed components of the 3 orthogonal optical levers were mounted, i.e. the lasers, static and steering mirrors.

At the third level was the aforementioned T/R Stage Bracket 2, to which the tilting mirror of the x- and y-axis optical levers was mounted as well as the cube mirror for the three orthogonal interferometers.

The penultimate level of the equipment rig was the Interferometer Fixed Component Mounting Bracket (IFCM Bracket), the fixed components of the interferometers being the lasers, beamsplitters, fixed mirrors and the webcams, which were used as image sensors. The webcams were designed to be easily installed or removed so that each interferogram could be projected at a distant surface to facilitate aligning the three interferometers after having first zeroed the T/R Stage.

Finally, the top level of the equipment rig was the IFCM Bracket Support to which the IFCM Bracket was bolted. The IFCM Bracket Support was then aligned with and bolted down to the two orthogonal uprights.

A detailed explanation of the design and assembly of the experiment rig follows.

7.2 Methodology

7.2.1 XYZ Translation Stage

Translation was applied using a Fiber Coupling Stage (model 860-0210), which had a travel range of 2 mm in each of the x, y and z directions with advertised 0.2 μm sensitivity and reading accuracy of 1.25 μm (Figure 44a). The Fiber Coupling Stage (XYZ Stage) came with a platform with a number of threaded holes onto which apparatus could be bolted.

7.2.2 Tilt/Rotation Stage

The Tilt/Rotation Stage (model 860-0110) provided two independent tilt adjustments with an advertised ±5° range (x- and y-axis) and an in-plane rotation adjustment (z-axis) with an advertised ±4° range (Figure 44b). The base of the T/R Stage was provided with four slots that enabled it to be bolted down to the surface below. The platform of the stage had 4 threaded holes for apparatus to be fastened to it.
Both Fiber Coupling Stage and Tilt/Rotation Stage were purchased from Eksma Optics [56].

![Fiber Coupling Stage](image1.png)

![Tilt/Rotation Stage](image2.png)

Figure 44: a) Fiber Coupling Stage (model 860-0210), Note: the picture (obtained from the manufacturer’s specification) is a mirror image of the actual device, b) Tilt/Rotation Stage (model 860-0110). Copyright - Optolita UAB, Mokslininkų Street 11, LT-08412 Vilnius, Lithuania.

7.2.3 Base

The Base of the experiment rig was an aluminium plate (Figure 45) into which two slots were milled orthogonal to one another with associated countersunk screw holes, which together were used to accurately locate two uprights to support the mountings for the immovable optical components of the experiment. The dimensions of the Base were 400 mm (L) x 300 mm (W) x 10 mm (H).

Four holes, one adjacent each corner, enabled the Base to be bolted down to a solid external structure to minimise vibration.

In addition, there were three apertures drilled in the Base into which close sliding fit locating pins were inserted to accurately locate the XYZ Stage. Two threaded bolt holes in the Base enabled the XYZ Stage to be bolted to it (Figure 46).

![Base](image3.png)

![XYZ Stage](image4.png)

Figure 45: Base. Figure 46: XYZ Stage.
7.2.4 Adapter Plate

Despite the XYZ Stage platform having a number of threaded holes, none of them aligned well with the four slots of the base of the T/R Stage. To overcome this, an Adapter Plate was designed (Figure 47) as an interface between the two stages so that they could be securely fastened to one another (Figure 48).

The Adapter Plate had two countersunk holes that aligned with two holes in the XYZ Stage and enable these two parts to be screwed together. On the upper side of the Adapter Plate there were two lugs that engaged with two slots in the base of the T/R Stage and provided alignment with the XYZ Stage. Two threaded holes were also provided in the Adapter Plate that aligned with the other two slots in the base of the T/R Stage for the T/R Stage to be bolted to it.

7.2.5 Tilt/Rotation Stage Bracket 1

The platform of the T/R Stage could impart 6 DoF under the control of the 3 adjustment knobs of the XYZ Stage and the 3 adjustment knobs of the T/R Stage. To transfer this movement to the cube mirror of the 6 DoF sensor and the tilting mirrors of the 3 optical levers, these components were mounted to the platform of the T/R Stage by a series of brackets. The first of these was the T/R Stage Bracket 1 (Figure 49a).

Figure 49b shows the bracket itself having a recess to mount the tilting mirror of the z-axis optical lever as well as a boss to mount a further bracket (T/R Stage Bracket 2) to which the x- and y-axis optical lever tilting mirror and the cube mirror were mounted. The boss had a close tolerance locating lug that made a close sliding fit with a recess on the further bracket.
7.2.6 Uprights

7.2.6.1 Upright 1

Upright 1 (Figure 50a) had a boss along its lower surface that mated with a close sliding fit into the shorter of the two orthogonal slots in the Base plate and was screwed down with two screws from the bottom of the Base.

The upright had a hole through it though which to project the interferogram of the x-axis interferometer when initially aligning the cube mirror.

The far side of the upright had a vertical slot that served to accurately align it with Upright 2. There were two counter sunk holes associated with the slot to screw Uprights 1 & 2 together.

There was also a horizontal slot milled in Upright 1 to locate the mounting for the optical lever fixed components.

7.2.6.2 Upright 2

Similarly to Upright 1, Upright 2 (Figure 50b) mated with a close sliding fit with the longer of the two orthogonal slots in the Base. It had a rectangular hole through which the 3 optical lever beams were projected towards a distant screen.

It also had a horizontal slot located close to the rectangular hole that provided a mounting, together with Upright 1, for the optical lever fixed components.

The sturdy interconnection of the two uprights with the Base and with themselves made it a rigid arrangement to attach the other mountings of the experiment rig.

7.2.7 Optical Lever Fixed Component Mounting Bracket

The fixed components of each optical lever were the laser, static mirror and steering mirror. Not only did these components need to be fixed in space relative to one another, but each optical lever needed to be fixed orthogonal to one another. Despite the sheer number of components and the various angles they were required to be mounted relative to one another (45°, 90° and 22.5°), a single Optical Lever Fixed Component Mounting Bracket (OLFCM Bracket) (Figure 51 a – d)) was designed for the purpose and as a consequence, was the most difficult of the equipment rig parts to manufacture.
The OLFCM Bracket had complementary bosses that interfaced with a close sliding fit with respective horizontal slots in Uprights 1 and 2. The bracket was secured to Uprights 1 and 2 by means of countersink screws fastened from the outer side of the uprights.

Sufficient clearance between the OLFCM Bracket and the T/R Stage Bracket ensured the two parts did not foul one another when the full extent of tilt and/or rotation was applied to the T/R Stage.

Prior to mounting the OLFCM Bracket to the rig, the lasers and mirrors were mounted to the OLFCM Bracket by epoxy glue.

The fixed mirrors were mounted in their respective recesses as follows:

- $x$-axis optical lever – parallel to the $x$-$y$ plane
- $y$-axis optical lever – parallel to the $x$-$y$ plane
- $z$-axis optical lever – parallel to the $y$-$z$ plane

Note: As the fixed mirror of the $X$ and $Y$ optical lever are in the same plane as one another, they shared a round mirror to save space as seen in Figure 51c.

The beam of each laser was projected at $45^\circ$ to the normal of their respective fixed mirrors and along the following planes:

- $x$-axis optical lever – parallel to the $y$-$z$ plane
- $y$-axis optical lever – parallel to the $x$-$z$ plane
- $z$-axis optical lever – parallel to the $x$-$y$ plane

The steering mirror of each optical lever was orientated to reflect the beam exiting the respective optical lever through the square shaped projection hole in Upright 2 (Figure 50b) and on towards the distant projection surface.
7.2.8 Tilt/Rotation Stage Bracket 2

The Tilt/Rotation Stage Bracket 2 (T/R Stage Bracket 2) provided a mounting for the cube mirror of the 3 orthogonal interferometers on its upper surface (Figure 52c) and for the tilting mirrors of the $x$ & $y$ axis optical levers on its lower surface (Figure 52d).

As illustrated in Figure 52a) and b), T/R Stage Bracket 2 was fastened to the boss of T/R Stage Bracket 1 by means of countersink screws. The complementary slot and boss of the two brackets respectively mated with a close sliding fit to accurately align them with one another. T/R Stage Bracket 2 was also designed with a further locating slot seen in Figure 52d) with the intention of reusing the bracket and cube mirror for a prototype application that required displacement to be measured to 6 DoF. The bracket could then be mounted to a solid body (part of the prototype application) so that cube mirror could be aligned with the fixed components of the 3 orthogonal interferometers.

Similar to the previously mentioned fixed mirror of the X and Y optical lever having a single round mirror (Figure 51c), a complementary round mirror was used for the tilting mirror (Figure 52d)) for the X and Y optical levers.
7.2.9 Interferometer Fixed Component Mounting Bracket

The Interferometer Fixed Component Mounting Bracket (IFCM Bracket) provided a mounting for the fixed components of the 3 orthogonal interferometers about the cube mirror, i.e., the lasers, beamsplitters, fixed mirrors and image sensors, (Figure 53a – d).

For each interferometer an aperture was machined in the bracket into which the laser was inserted. The axis of the aperture aligned with the centre of the respective beamsplitter and fixed mirror and was perpendicular to their incident surfaces.

The upper and lower surfaces of the IFCM Bracket had recesses and bosses that were designed to accurately position and align the beamsplitters and fixed mirrors. These can be seen in the close up in Figure 53c) and d). The depth/height of the beamsplitter recesses/bosses were designed so that they did not block the edges of the laser beams passing through. The beamsplitters and fixed mirrors were carefully pressed into position and glued in place using epoxy glue.

The lugs for mounting the x-axis interferometer webcam are indicated in the figures. The lugs for the y-axis interferometer webcam are obscured from view and the z-axis interferometer webcam required a separate bracket, which is discussed in section 7.2.10.

At each of the four corners of the IFCM Bracket were threaded screw holes that were used, together with the locating boss along the upper surface of the bracket, to screw it to a further bracket (IFCM Bracket Support) with a close sliding fit. The IFCM Bracket was also designed with it being capable of being reused in a prototype application that required displacement to be measured to 6 DoF. The threaded screw holes and locating boss were positioned on the bracket so that it could be mounted to a solid body (part of the prototype application) so that the interferometers could be aligned with the cube mirror.

To save space on the IFCM Bracket, two mirrored surfaces of a surplus cube mirror was used instead of individual fixed mirrors for the X and Y interferometers.
7.2.10 Z-Axis Interferometer Webcam Bracket and Webcams

Due to the complexity of the IFCM Bracket, it was not possible to include an integrated webcam mounting for the z-axis interferometer. Instead, a separate bracket was manufactured and screwed to the IFCM Bracket as shown in Figure 54a straddling the z-axis beamsplitter.

Three Logitech C600 webcams [59] were purchased and stripped down to their circuit boards to expose their image sensors.

Figure 54b) shows the three webcams screwed to their respective mountings. Each webcam was orientated with respect to the other two axes using the right hand rule so that all three interferograms could be viewed on monitors alongside one another with the correct orientation.

The centre of the image sensor on each webcam circuit board was aligned with the axis of the respective laser beam. The 1600 x 1200 pixels image sensors had a pixel pitch of 2.835 microns in both the horizontal and vertical directions.
Included in the design of the experiment rig was the need for easy access to fasten or unfasten each webcam from its mounting. The reason for this was so that the interferograms could be projected onto a distant surface to perfectly align each interferometer when the tilt/rotation stage was zeroed.

The wiring of the lasers and webcams is not shown in the figures.

![z interferometer webcam bracket](image1)

**Figure 54:** a) z axis interferometer webcam bracket, b) x, y and z-axis interferometer webcams mounted to the IFCM Bracket.

The axes of the interferograms captured by the X, Y and Z webcams are orientated as illustrated in Figure 55.

![IFCM Bracket Support](image2)

**7.2.11 IFCM Bracket Support**

The IFCM Bracket Support has an aperture through which the z-axis interferometer fixed components mounted upon the IFCM Bracket can project. It also has four screw holes that align with the threaded holes of the IFCM Bracket so the two elements can be screwed together. This task was done before fastening them to the rest of the experiment rig.

The underside of the IFCM Bracket Support has two orthogonal slots that have a close sliding fit with the bosses projecting above Uprights 1 and 2 seen in Figure 50b). The two holes either side of the orthogonal slots allow the IFCM Bracket Support to be bolted down to the two uprights.

The assembled experiment rig was now ready to connect the wiring for the lasers to the driver circuitry and the webcams to a computer.
7.2.12 Aligning the Optical Levers

The narrow collimated beams for the optical levers were produced by Egismos H436351D/R 1 milliwatt laser modules \[58\]. It was found that the definition of the laser spot cast on the distant projection screen was enhanced by reducing the optical power of the laser using a simple constant current source.

Although the laser module was cylindrical in shape of 10 mm length and 4 mm diameter, when tested by rotating the cylinder about its axis, it was found that the beam was not perfectly centred on the axis. The apertures for the lasers machined into the Optical Lever Fixed Component Mounting Bracket (section 7.2.7) were at 45° to their respective axes, therefore to maintain this angle as accurately as possible, adapters were designed to correct for this axial misalignment.

The adapters are shown in Figure 57a) with the recess for the laser modules not concentric with the adapter axis. With an adapter at each end of the laser module, each adapter could be rotated to adjust the direction of the laser beam. Figure 57b) shows the laser inserted between the two adapters.

To perform the adjustment, an alignment tube Figure 57c) was fabricated into which at one end the laser/adapter assembly was inserted. The tube was in turn inserted into the race of a needle bearing of complementary diameter and the needle bearing clamped in a vice, as shown in Figure 58. This enabled the tube to be smoothly rotated to observe the laser beam describing a circle on the distant projection screen. The front and rear adapters with their non-concentricity were rotated with a specially designed tool (not shown) until the smallest diameter circle was achieved. The angle of deviation from the virtual axis of the laser module was derived from the distance of the laser to the projection surface and the radius of the circle. The deviation angle from being truly axial for each laser beam was measured to be:

- x-axis laser 0.1°
- y-axis laser 0.2°
- z-axis laser 0.15°
Having aligned the optical lever laser modules for directionality, the laser and adapters were epoxied together and each assembly inserted into their respective apertures in the OLFCM Bracket.

The distance between the static and tilting mirrors of the optical levers determined the number reflections passing through and in turn the amplification of the induced angular displacements to the T/R stage.

For the X and Y optical levers the mirror spacing was approximately 1.75 mm creating 18 reflections.

For the Z optical lever the mirror spacing was approximately 2.8 mm creating 12 reflections.

**Figure 57: Optical lever laser alignment, a) alignment adapters, b) laser inserted into adapters, c) laser and adapters inserted into alignment tube.**
7.2.13 Displacement sensor interferometers

An image sensor was used to capture fringe line spacing, fringe tilt angle and fringe count from the interferogram. Image sensors are manufactured with an active area varying in aspect ratio and pixel pitch dependent on the purpose for which they are to be used. The image sensor chosen for the experiment was the Logitech C600 2 MP Webcam that had an active area of 4.536 x 3.402 mm [59].

To maximise the effectiveness of the image sensor a Thorlabs C560TME mounted aspheric lens [61] with a clear aperture of 5.1 mm and effective focal length of 13.86 mm was used to collimate the laser beam.

7.2.13.1 Interferometer laser driver circuit

Vixar 680-0000-B001 Red Vertical Cavity Surface Emitting Lasers (VCSEL) [60] were used as the light source to the interferometers. For experimentation, a constant current source of \( \approx 2 \text{mA} \) was designed to drive each laser.

Referring to Figure 59, a 2.9 V stable reference voltage was generated using an LM336 reference diode in series with a signalling diode 1N4148. The 2.9V reference was buffered by an LM741 operational amplifier used as a voltage follower. The output of the opamp was connected to the base of a PNP BC327 transistor, therefore each emitter was fixed at 3.5 V. This provided a 2.5 V drop across the 1K2 resistor providing a constant current of approx. 2 mA through the emitter-collector junction of the transistor. The VCSEL was connected between the collector and ground.

**Figure 58: Aligning the optical laser beam for concentricity with laser module.**
7.2.13.2 Laser Assembly

An exploded view of the laser assembly in shown in Figure 60, which had to provide the following characteristics:

- Red Vertical Cavity Surface Emitting Laser (VCSEL) in TO-46 of approx. 1 mW optical power
- A beam waist that just overlapped the active area of the image sensor, i.e., approx. 5 mm
- Focal length approaching infinity
- Beam irradiance attenuation (Neutral Density (ND) filter) and/or Bull’s Eye apodisation filter
- Laser beam direction adjustment

The left part of the laser housing (shown in section) was designed to allow the aspheric lens to be focused and accommodate an ND filter and/or Bull’s Eye apodisation filter to be installed, if required, between two fibre washers. The spring served two purposes, one to hold the filter(s) firmly in place and two, to apply force to the threaded lens mounting to take up play in the focusing adjustment threads and hold it firmly in place.

To the right of the laser housing is shown the laser adapter into which the laser was firmly inserted. To hold the laser firmly in place a fibre washer was used to press onto the TO-46 can by means of the laser retainer and 3 threaded screws.

The right part of the laser housing had a hemispherical recess machined to accommodate the complementary hemispherical boss of the laser adapter. The hemispherical interface enabled the laser adapter to be adjusted in tilt and rotation to align the axis of the laser with the aspheric lens. The 3 threaded screws were screwed into the laser housing and adjusted in combination to axially align the laser.
7.2.13.2.1 Focusing the Laser

To get the beam collimated to the specified 5.1 mm clear aperture the laser housing was clamped in a vice so that the laser beam projected with a horizontal trajectory onto a surface approximately 8 metres away. The threaded aspheric collimating lens was adjusted to give a consistent beam diameter over that distance. This adjustment was done by eye.

7.2.13.2.2 Adjusting the beam axis

As mentioned above, the axis of the laser was axially adjusted by means of the hemispherical interface between the laser housing and the laser adapter shown in section in Figure 61. Rotational adjustment was obtained by rotating the laser relative to the laser adapter. This adjustment helped to maximise the intensity in the centre of the beam.

7.2.13.2.3 Beam Waist

The spacing between fringe lines is inversely proportional to the tilt angle of the moving mirror, i.e., the greater the fringe spacing the smaller the tilt angle. For the experiment, a Logitech C600 2 MP Webcam was select that has a 4.536 x 3.402 mm image sensor.
Although it was possible to visually see and measure fringe spacing and slope of half a fringe across the shorter of the two sides of the IS, there was a level of uncertainty in determining an accurate measurement. However, with 2 fringe lines visible across the interferogram the level of uncertainty was dramatically reduced, therefore this was taken as the basis for deciding the measurement resolution. Using Equation (3.128), with a laser wavelength of 680 nm, the fringe spacing equates to a tilt angle $\alpha$ of:

$$\alpha_{\text{mirror}} = \frac{1}{2} \frac{680 \times 10^{-9}}{1.701 \times 10^{-3}} \approx 0.01 \text{ degrees} \quad (7.1)$$

7.2.13.3 Optics

7.2.13.3.1 Cube mirror

The collimating lens clear aperture was 5.1 mm, therefore to ensure the entire beam never went beyond the edges of the reflective surface with a side to side translation of 0.4 mm, the 3 reflective surfaces were made with a side length of 6 mm. However, as the high reflectance coating tool blocked each edge by 0.5 mm, each side of the cube mirror was made 7 mm in length.

The cube mirrors were manufactured by Deln Industrial Development Co. [62] to the following specification:

<table>
<thead>
<tr>
<th>Material:</th>
<th>BK7 grade A optical glass</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dimensions/tolerance:</td>
<td>7 x 7 x 7 mm ±0.2mm</td>
</tr>
<tr>
<td>Parallism:</td>
<td>10°</td>
</tr>
<tr>
<td>90° angles:</td>
<td>10°</td>
</tr>
<tr>
<td>Bevel:</td>
<td>&lt;0.25 mm x 45°</td>
</tr>
<tr>
<td>Surfaces S1, S2 and S3:</td>
<td>Optical</td>
</tr>
<tr>
<td>Angle of Incidence:</td>
<td>0°</td>
</tr>
<tr>
<td>Wavelength:</td>
<td>680 nm</td>
</tr>
<tr>
<td>Coating:</td>
<td>HR Dielectric, R &gt;99.5%</td>
</tr>
<tr>
<td>Surface Quality:</td>
<td>40/20 scratch/dig</td>
</tr>
<tr>
<td>Wavefront Distortion:</td>
<td>$\lambda/10$ per 25mm</td>
</tr>
<tr>
<td>All other surfaces:</td>
<td>Non-optical</td>
</tr>
</tbody>
</table>

The cube mirror was mounted on Tilt/Rotation Stage Bracket 2 as depicted in Figure 52c) and glued in place using cyanoacrylate "super-glue".

7.2.13.3.2 Beamsplitters

The beamsplitters (part no. CBS201, 650 – 900 nm) used for the interferometers were obtained from Deln Industrial Development Co. [62] and had the following specification:

<table>
<thead>
<tr>
<th>Material:</th>
<th>BK7 grade A optical glass</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dimensions/tolerance:</td>
<td>10 x 10 x 10 mm ± 0.2mm</td>
</tr>
<tr>
<td>Wavelength:</td>
<td>650 – 900 nm</td>
</tr>
<tr>
<td>Flatness:</td>
<td>$l/4 @ 632.8 \text{ nm}$</td>
</tr>
<tr>
<td>Surface Quality:</td>
<td>60/40 scratch and dig</td>
</tr>
<tr>
<td>Transmission/Reflection:</td>
<td>50/50 ± 5%, $T=(Ts+Tp)/2$, $R=(Rs+Rp)/2$</td>
</tr>
<tr>
<td>Beam Deviation:</td>
<td>&lt;3 arc minutes</td>
</tr>
<tr>
<td>Coatings:</td>
<td>Broadband partial reflectance: on hypotenuse face</td>
</tr>
<tr>
<td>Anti reflection coating:</td>
<td>on all input and output face</td>
</tr>
</tbody>
</table>

© Deln Industrial Development Co.
The beamsplitters were mounted on the Interferometer Fixed Component Mounting Bracket as shown in Figure 53c) and d), and initially glued using cyanoacrylate glue, which was found to be a bad choice as it gave off a vapour that clouded the optical surface. After discovering the consequence of using cyanoacrylate glue, a fast drying epoxy adhesive was subsequently used to mount all the optical components as it did not compromise the quality of the optical surfaces.

7.2.13.3.3 Fixed mirrors

From Figure 53d), it can be seen that two sides of a cube mirror were used as the fixed mirrors of the respective X and Y interferometers instead of using separate fixed mirrors for each.

However, the z-axis interferometer required its own fixed mirror as illustrated in Figure 53c) and fixed in place with epoxy adhesive. It was also procured from Deln Industrial Development Co. [62] and had the following optical specification:

Material: BK7 grade A, optical glass
Dimensions: 10 x 10 x 6.35 mm
Width Tolerance: +0, -0.2 mm
Thickness Tolerance: ±0.2 mm
Wavelength: 650 nm
Clear Aperture: >80%
Flatness: λ/10@632.8 nm
Parallelism: < 1 arc minute
Surface Quality: 20/10 scratch/dig
Coating: HR coating on S1, R>99.5%, Uncoated on S2

7.2.13.4 Image sensors

The image sensor used for the experiment was the Logitech C600 2 MP Webcam that has an active area of 4.536 x 3.402 mm, 1600 x 1200 pixels and a pixel pitch of 2.835 microns [59].

To expose its image sensor, the webcam was stripped of its plastic housing and lens. With the collimated beam having a clear aperture of 5.1 mm, the interferogram covered all but the very corners of the active area. Also, having a pixel pitch of 2.835 microns meant that wave front angles of ± 0.8° could be applied to the cube mirror without perceptible drop in modulation amplitude (refer to Figure 6).

However, with the webcam having a maximum frame rate of 30 frames per second, a translation speed of less than 15 fringes per second should not be exceeded without occurring fringe count errors. This limitation was not deemed an issue for this thesis as using interferometry for accurately measuring translation is a mature technology and therefore nothing was to be gained by demonstrating something that is already proven and commercially available. Utilising image sensors with far higher frame rates or designing a bespoke image sensor for the purpose of capturing ultra-high fringe count speed, which is covered in section 11.1.3, can be used to overcome this limitation.

7.2.14 Experiment set up

The experiment rig was placed on a stable level surface in such a way that the beams from the optical level calibration system could project onto a vertical surface a reasonable distance away as illustrated in Figure 62. For example, at 2 metres a change in beam
position of 4 mm represents a change in angle of 0.01 degrees when the optical lever is adjusted to produce twelve reflections, i.e. an angle magnification of 12.

In the foreground of the figure the three x, y and z axis translation knobs of the XYZ Translation Stage and the respective tilt/rotation knobs of the T/R Stage can be seen. Onto the distant wall, the 3 optical lever beams can be seen projecting onto graph paper that was used to measure the movement of the beams in the x-z plane when the tilt/rotation knobs were adjusted.

The 3 webcams were connected to a laptop from which the interferograms could be captured and saved as a JPEG file for a still image or as a WMV file for a video clip. The Durango Interferometry Software package was used to derive the mirror tilt angles $\alpha_x$, $\beta_y$, and $\gamma_z$ and mirror tilt axis angles $\varphi_x$, $\varphi_y$ and $\varphi_z$ from the distance between fringe lines and the fringe line slope respectively.

By adjusting the knobs on the tilt/rotation stage, rotation about each of the 3 Cartesian axes could be induced and the magnitude of the rotation could be accurately determined from the change in position of the 3 optical lever beams. These measurements were inserted into an Excel spreadsheet that contained the equations developed in Chapter 6 to find the position vector of the optical lever tilting mirror normals.

The same spreadsheet included the equations to determine the position vectors of the cube mirror that were developed in Chapter 5. The outcome of the data acquisition and analysis in covered in detail in Chapter 8.

![Figure 62: Photograph of experiment set up.](image)
7.3 Results

7.3.1 Aligning the interferometers with the cube mirror

After having assembled the experiment rig, the optical lever system and the 3 orthogonal interferometers were switched on. The orientation of the cube mirror was adjusted to try and obtain alignment of all 3 interferometers concurrently.

Aligning the fixed and moving mirrors of a Michelson interferometer was best done with the respective beams projecting onto a distant surface. Therefore, the webcams were not installed at this stage of the assembling the experiment rig so that the interferograms could be projected through the apertures provided in Upright 1 (X interferometer), Upright 2 (Y interferometer and Z interferometer webcam bracket).

It had been assumed during the experiment rig design that machining the IFCM Bracket to a tolerance of ISO 2768 – fH would provide an almost perfect alignment of the 3 orthogonal interferometers after having pressed and glued the optical components into their machined recesses. It was thought that after having mounted the fixed optical components in place, all that was then require was to orientate the cube mirror into perfect alignment, which unfortunately proved incorrect. What could have been microscopic imperfections in the machining process e.g. burrs or ridges quite possibly caused slight misalignment of the interferometers with the cube mirror.

Because of the above assumption, no interferometer alignment adjustment had been built into the design of the experiment rig and therefore it had to be done by ungluing the parts, aligning them by hand and re-gluing them.

With the experiment rig completely assembled except for the webcams, it can be seen from Figure 54b) and Figure 56b) that the beamsplitter and fixed mirrors of the X and Y interferometers were impossible to access to make these alignment adjustments. The Z interferometer could be accessed from above therefore, it was decided to unbol the IFCM Bracket Support (section 7.2.11) from Uprights 1 & 2 with the IFCM Bracket (section 7.2.9) still attached, turn this sub-assembly upside down so that the X and Y interferometers were now exposed. With the cube mirror still mounted to the experiment rig, a spare cube mirror was required as a substitute to perform the x and y alignment.

The substitute cube mirror was mounted on the IFCM Bracket and fixed in position as close to where the experiment rig mirror would have been with the sub-assembly the right way up. The cyanoacrylate glue to the x-interferometer beamsplitter was scraped away freeing the beamsplitter, which could then be manoeuvred so that the transmitted and reflected beams perfectly overlapped in space and direction. The beamsplitter was re-glued and left to dry. This operation took several attempts to get right as the drying glue would microscopically re-position the beamsplitter as it set and pull the beams slightly out of alignment.

Having aligned the X interferometer the same procedure was followed with aligning the Y interferometer. The sub-assembly was then remounted to the experiment rig and the cube mirror orientated to align the X and Y interferometers as best as possible. Maintaining the cube mirror in this position the Z interferometer was aligned to the same procedure as the other two.

Following this rudimentary method of aligning the 3 interferometers with the cube mirror, it resulted in the X and Z interferometers and the Y and Z interferometers in perfect alignment, but for an unknown reason, the X and Y interferometers had become slightly
misaligned with one another by $0.0286^\circ$. The interferograms from the 3 interferometers in the “zero position” are captured in Figure 64. It can be seen that the interferograms of the X and Z interferometers are homogeneous, whereas the Y interferograms shows the fringing causes by the slight misalignment.

Essentially, the interferometer misalignment can be thought of as a new reference frame rather than the intended orthogonal Cartesian reference frame. Therefore, by using a change of basis transition matrix, the misaligned reference frame was transformed into an orthogonal reference frame. Application of the transition matrix to compensate for this misalignment is covered in Chapter 8 on Experimentation.
7.3.1.1 Beam displacement relationship between the X, Y and Z optical levers

From Section 6.2.2 and Figure 37 it is shown that if there is a tilt about the x-axis then there is a change in beam position of both the x- and y-axis optical lever beams. Likewise, if there is a tilt about the y-axis then there is a change in position of the x-, y- and z-axis optical lever beams. Because the beams are orthogonal to one another there is a fixed relationship between them whenever there is a tilt/rotation of any one of their tilting mirrors. With the optical levers all having the same number of reflections, this relationship is defined as follows:

1. If there is purely a positive rotation about the z-axis (Figure 37b) then,
   - the z-axis beam moves horizontally on the projection screen in concert with the rotation in the positive x-direction
   - the x- and y-axis beams remain static

2. If there is purely positive rotation about the x-axis (Figure 37c) then,
   - the x-axis beam will move vertically on the projection screen in the positive z-direction in concert with the tilting
   - the y-axis beam moves at 45° to the positive x- and z-axes a distance \( \frac{1}{\sqrt{2}} \) the distance traversed by the x-axis beam
   - the z-axis beam remains static

3. If there is purely positive rotation about the y-axis (Figure 37d) then,
   - the y-axis beam will move at 45° to the negative x-axis and positive z-axis
   - the x-axis beam will move horizontally along the negative x-axis a distance \( \frac{1}{\sqrt{2}} \) the magnitude traversed by the y-axis beam
   - the z-axis beam will move vertically along the negative z-axis a distance \( \frac{1}{\sqrt{2}} \) the magnitude traversed by the y-axis beam

If the rotations about the x-, y- and z-axes are opposite to those specified above, then the directions traversed by the beams will also be opposite to those specified.

With the optical levers having 12 reflections, the fixed relationship can be defined by the following equations:

\[
\begin{align*}
    r_{12xx} &\equiv r_{12yy} - \frac{r_{12xz}}{2} \equiv r_{12y_x} - r_{12y_z} - r_{12z_x} \\
    r_{12xz} &\equiv 2r_{12yx} - 2r_{12zx} \equiv 2r_{12zx} + 2r_{12y_z} \\
    r_{12yx} &\equiv r_{12zx} + \frac{r_{12xz}}{2} \\
    r_{12yz} &\equiv -r_{12zx} + \frac{r_{12xz}}{2} \\
    r_{12yz} &\equiv r_{12xz} - r_{12yz} \equiv r_{12y_x} - r_{12z_x} - r_{12y_z}
\end{align*}
\]
where \( r_{12x_x} \) is the \( x \)-component of the \( x \)-axis optical lever reflection vector \( \mathbf{R}_{12} \). Similarly, \( r_{12x_z} \) is the \( z \)-component of the \( x \)-axis optical lever reflection vector \( \mathbf{R}_{12} \). All the other components are designated accordingly.

This relationship between the displacements of the optical lever beams was used to determine that the system was working perfectly by ensuring:

- the beam displacements complied with Equations (7.2) - (7.6)
- the beams moved horizontally/vertically/45° (as the case may be) when tilt/rotation was induced one axis at a time - to confirm the experiment rig was level

For the experimentation, the X and Y optical lever adjustment was set at 18 reflections.

7.3.1.2 Adjusting the Interferometers to Zero Tilt/Rotation

If the machined recesses in the experiment rig had provided perfect alignment of the interferometers with the cube mirror then in the “zero position” the 3 optical lever beams would have projected onto the graph paper screen in positions X, Y and Z as shown in Figure 65. However, due to the alignment procedure that had to be adopted for the 3 interferometers, which was elaborated above, the actual “zero position” of the aligned interferometers resulted in the optical lever beams being positioned as marked X’, Y’ and Z’ in Figure 65.

This displacement of the "zero positions" of the optical lever beams from the ideal positions to the actual aligned interferometer positions was irrelevant. This was because the optical lever system measured change in tilt/rotation of the T/R Stage between one position and another. The important aspect was to initially capture the position of the 3 optical lever beams when the cube mirror was in alignment with all 3 interferometers. Thereafter, whenever the orientation of the T/R Stage was adjusted, the tilt and rotation of the cube mirror was derived using the methodology detailed in Chapter 6 by measuring the change in position of the 3 beams of the optical lever system relative to the X’, Y’ and Z’ "zero positions".

The experiment rig was now ready to start applying adjustment to the T/R Stage and capturing the fringe line spacing and fringe slope from the 3 interferograms as well as the change in position of the 3 optical lever beams. The outcome of the experimentation and data analysis is covered in detail in Chapter 8.
Figure 65: Zero position of the projected optical lever beams $X'$, $Y'$ and $Z'$ compared with $X$, $Y$ and $Z$ where they were designed to be with best alignment of the $X$, $Y$ and $Z$ interferometers with the cube mirror.

7.4 Summary

The experiment rig provided the ability to apply displacement to 6 DoF to the cube mirror by applying adjustments to the XYZ Stage and the T/R Stage, and to measure the magnitude of the applied tilt and rotations by means of the specially designed optical lever system.

The moving mirrors of the optical lever system were mounted with the cube mirror so any applied tilt and rotation adjustment moved all the mirrors equally. The webcam mountings were designed so that the webcams could be removed, which enabled the interferogram to be projected onto a distant screen to align each interferometer by eye as best as possible. The experiment rig was designed so that the 3 optical lever beams all projected in the same direction towards a distant screen making it easy for the operator to observe changes to all 3 beams together.

As the beam of a laser diode is not always perfectly concentric with the package or module in which it is mounted, the mounting arrangement of the interferometer and optical lever lasers in the experiment rig enabled the lasers to be adjusted so that their respective beams could be aligned with the designed optical axes.

It had been expected that machining the mounting points for the beamsplitters and fixed mirrors to a high tolerance and gluing these optical components in position would have automatically aligned the interferometers. After switching on the experiment rig it was found that the cube mirror could not be orientated to align all 3 interferometers together.
as there was slight misalignment of each. As no manual alignment of the interferometers had been designed into the experiment rig a procedure was adopted where the beamsplitters were in turn unglued and manually repositioned and re-glued. In this way, orthogonality was almost achieved except for a slight misalignment of 0.0286° for the Y interferometer. This misalignment was compensated for in the calculations of the cube mirror position vectors by applying a transition matrix to transform the cube mirror position vectors to an orthogonal Cartesian coordinate system.

Because the alignment of the interferometers were not quite intended with the design of experiment rig, the position of the optical lever beams on the distant screen were consequently not aligned with the expected “zero position”. However, this was irrelevant as the optical lever system worked on the change in position of the 3 beams with respect to their respective “zero position”, wherever they happen to be.

The data from the experiment rig and the associated calculations to determine the cube mirror position vectors and the optical lever system vectors resulting from applied tilt and rotations is covered in Chapter 8 on experimentation.
8 EXPERIMENTATION

8.1 Introduction

After having assembled and performed initial testing of the experimentation rig it was found that there was a slight misalignment of the Y interferometer. The misalignment of the Y interferometer was corrected for by multiplying a change of basis transition matrix to the X, Y and Z interferometer position vectors.

In Sections 5.2.2.1 and 5.2.2.2, two methods were described how the cube mirror position vectors could be found using different approaches. With Method 1, it was found that there were 16 permutations that needed to be filtered to find the correct permutation. Also, whenever any one of the orthogonal interferometers was in perfect alignment, the inability to define a mirror tilt axis angle for that interferometer meant that some of the solution equations became indeterminate. Method 2, however, only had 8 permutations to filter to find the correct cube mirror position vectors and did not suffer from indeterminate equations. Therefore, the experimentation was done using only Method 2.

Using a Michelson interferometer for measuring translation is a well-known and well proven technology, therefore it was not deemed necessary to test the sensor for translation measurement. What is novel in this thesis is using three Michelson interferometers arranged orthogonally about a cube mirror to derive the 3 position vectors of the cube mirror due to angular displacement. This is done by measuring the fringe spacing and fringe line tilt angle from each of the 3 interferograms, which are used to calculate the position vectors. Due to the nature of light, angular displacement of the cube mirror will produce exactly the same components for the 3 position vectors irrespective of linear displacement of the cube mirror. Angular displacement and the magnitude of linear displacement of the cube mirror are derived independently* from one another using different methodologies. Therefore, to find the cube mirror position vectors resulting from angular and linear displacement, the respective components of the position vectors from each type of displacement are simply added to derive displacement to 6 degrees of freedom.

*Note: Although the method of measuring the magnitude of linear displacement is independent of that measuring angular displacement, to determine the direction of linear displacement of the cube mirror requires knowledge of the cube mirror as discussed in Section 5.2.3.
Having completed preparation of the experiment rig, it was ready to use for experimentation.

8.2 Methodology

With reference to Figure 62, the experiment rig was set up on a sturdy work bench perpendicular to a wall approximately 2.2 metres away and checked to be level.

An A3 sheet of graph paper was stuck to the wall and served to measure the displacement of the 3 optical lever beams to an accuracy of 0.5 mm. The horizontal and vertical alignment of the graph paper was checked to match that of the experiment rig. This procedure made use of the linear relationship between the displacement of the optical lever beams discussed in Section 7.3.1.1.

After adjusting the 3 interferometers to give the best “zero position” alignment as illustrated in Figure 64, the spot of each optical lever beam was marked with a cross on the graph paper, illustrated as X’, Y’ and Z’ in Figure 65.

The 3 webcams [59] were connected to a laptop computer to capture the interferograms from each interferometer.

Installed on the laptop was the Durango Interferometer Software [57] analysis program, which accepted JPEG files from the webcams and had the tools and functionality, amongst other capability, to find pixels of the highest and lowest radiant flux on the interferogram. Each interferogram could be marked to record the exact location of the selected pixels used determine the fringe line spacing and slope.

An Excel workbook was constructed that included all the relevant equations from Chapters 4, 5 and 6. The workbook provided the following functionality:

- It accepted the row and column positions of the selected pixels as inputs and worked out the mirror tilt angles and tilt axis angles, i.e. \( \alpha_x, \beta_y, \) and \( \gamma_z \) and \( \varphi_x, \varphi_y \) and \( \varphi_z \), from the fringe line spacing and fringe slope respectively.

- From the tilt angle and tilt axis angles it calculated the position vectors of the cube mirror.

- It applied the transition matrix to the cube mirror position vectors to compensate for the misalignment of the Y interferometer relative to the X and Z interferometers.

- As there were two possible values for tilt axis angles \( \varphi_x, \varphi_y \) and \( \varphi_z \), all eight permutations of cube mirror position vectors were tabulated and the dot product of each set of adjacent position vectors was calculated.

- The mathematical filter was applied to each permutation of position vectors to determine the correct permutation.

- It accepted the measurements of displacement of the optical lever beams as inputs to working out the X, Y and Z optical lever position vectors.

- As there were two solutions for each optical level position vector, all permutations were tabulated.
• The correct permutation was selected from the position of the 3 optical lever beams relative to their “zero position” on the graph paper and compared directly with the cube mirror position vectors.

• It was used to perform a gradient and intercept analysis of the components and the residual standard deviation in terms of Euler angles, pitch and yaw.

In all, 16 tests were run for which data was captured, the first test being done with the 3 interferometers in the “zero position.” The tests were conducted using the following procedure.

8.2.1 Procedure
Activities designated as “Step nn” that are listed below coincide with the tabulation of values in Table 17 - Table 31 in the Result section 8.3.

8.2.1.1 Data capture
Sixteen tests of tilt and rotation were applied to the T/R Stage.

For each test the position of the optical lever beams on the graph paper was photographed (see Figure 66d) and Figure 78d - Figure 92d). The position of the optical lever beams were measured off the graph paper and entered into the Excel spreadsheet.

For each test the 3 webcam interferogram images were saved as individual JPEG files, imported into the Durango Interferometry Software program [57] and displayed on the laptop screen (see Figure 66a),b), c) and Figure 78a),b), c) - Figure 92 a),b), c)).

8.2.1.2 Deriving fringe slope and mirror tilt angle
Lines coinciding with peaks or troughs of the fringe lines were marked on the interferograms using one of the software masking tools.

**Step 1.** Referring to Figure 18, to determine the fringe line slope, i.e. tilt axis angles \( \varphi_x, \varphi_y \) and \( \varphi_z \), one of the marked lines was chosen and the horizontal and vertical coordinates of a pixel was recorded at each end of the chosen line. Excel calculated the tilt axis angle from the pixel coordinates using the tangent function (Equation (4.2)).

**Step 2.** Referring again to Figure 18, to derive the mirror tilt angle, i.e. \( \alpha_x, \beta_y \) and \( \gamma_z \), the fringe line spacing had to first be determined:

- the number of complete fringe lines visible across the interferogram were counted
- the number of pixels along the horizontal or vertical axis coinciding with the number of counted fringe lines were measured off the interferogram and entered into Excel
- by multiplying the number of pixels by the pixel pitch (the width of each pixel) the length of the hypotenuse of a right angle triangle was obtained
- the included angle was deduced from the fringe slope derived above, therefore the perpendicular distance between the fringes could be derived using the sine or cosine function as the case may be
Experimentation

- by dividing this distance by the number of fringe lines, the fringe line spacing was obtained (see Equation (4.3))
- using Equations (4.4) and (5.1), the mirror tilt angles were calculated

The calculations from all the tests are tabulated in Table 13 and Table 17 - Table 31.

8.2.1.3 Deriving the cube mirror position vectors

As discussed in Section 5.2.2.2.1, there are two possible values for the mirror tilt angles found in Step 2 above, therefore both angles have to be used to determine the correct cube mirror position vectors. The cube mirror position vectors are calculated using Equations (5.32), (5.36) and (5.40) by substituting the mirror tilt angles and tilt axis angles into the equations.

**Step 3.** the components of the cube mirror X, Y and Z position vectors are calculated for the first value of $\phi_x$, $\phi_y$ and $\phi_z$

**Step 4.** the components of the cube mirror X, Y and Z position vectors are calculated for the second value of $\phi_x$, $\phi_y$ and $\phi_z$, i.e. $\phi_x + \pi$, $\phi_y + \pi$ and $\phi_z + \pi$

8.2.1.4 Correcting the cube mirror position vectors for the misalignment of Y interferometer

With reference to the interferograms in Figure 64 and Section 7.3.1, when the cube mirror was orientated to the “zero position”, there was a slight misalignment of the Y interferometer and this needed to be compensated for. From the fringe spacing of the Y interferogram the mirror tilt angle was derived, however, the direction of its tilt was unknown. Having followed the procedure in the above section it can be seen in Table 8 that there are two possible position vectors for the Y interferometer:

**Table 8: Possible X, Y and Z cube mirror position vectors with the Y interferometer slightly misaligned.**

<table>
<thead>
<tr>
<th>Cube mirror X, Y and Z components with $\phi_x$, $\phi_y$, $\phi_z$</th>
<th>Cube mirror X, Y and Z components with $\phi_x + \pi$, $\phi_y + \pi$, $\phi_z + \pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$-axis</td>
<td>$y$-axis</td>
</tr>
<tr>
<td>X</td>
<td>Y</td>
</tr>
<tr>
<td>1</td>
<td>-0.000496945</td>
</tr>
<tr>
<td>0</td>
<td>0.999999876</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

However, the correct Y interferometer position vector was determined by rotating the cube mirror about the $z$-axis to find the position at which the Y interferometer became aligned. By observing the direction of movement of the Z optical lever beam on the graph paper whilst rotating the cube mirror, the orientation of the misalignment of the cube mirror relative to the $y$-axis was determined. The components of Y interferometer position vector was found to be those on the left of Table 8 with fringe slope $\phi_y$.

Therefore, the reference frame for the cube mirror with the misalignment is the set of X, Y and Z position vector components of the left half of Table 8.

The reference frame for the experiment rig and the X, Y and Z optical lever position vectors are given in Table 9.
Table 9: Cube mirror components when in perfect alignment.

<table>
<thead>
<tr>
<th>Cube mirror X, Y and Z components in</th>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>perfect alignment</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>X</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Y</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Z</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

To convert the cube mirror position vectors from the misaligned reference frame to the Cartesian reference frame a change of basis transition matrix was created. Consequently, all cube mirror position vectors derived from the interferograms were transformed by the transition matrix in Table 10 to produce the cube mirror position vectors relative the orthogonal optical lever reference frame. In this way the corresponding cube mirror and optical lever position vector component could be directly compared.

Table 10: Transition matrix to convert from misaligned reference frame to Cartesian coordinate system.

<table>
<thead>
<tr>
<th>Transition Matrix</th>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>X</td>
<td>Y</td>
<td>Z</td>
</tr>
<tr>
<td>x-axis</td>
<td>1</td>
<td>0.000496945</td>
<td>0</td>
</tr>
<tr>
<td>y-axis</td>
<td>0</td>
<td>0.999999876</td>
<td>0</td>
</tr>
<tr>
<td>z-axis</td>
<td>0</td>
<td>-3.82906E-05</td>
<td>1</td>
</tr>
</tbody>
</table>

Step 5. The cube mirror position vector components obtained in Step 3 and Step 4 were transformed by the transition matrix defined in Table 10.

8.2.1.5 Determining the correct cube mirror position vector permutation

Having produced the two possible sets of components for each of the X, Y and Z cube mirror position vectors in the orthogonal coordinate system, the eight permutations needed to be filtered to find the correct permutation. To do this the methodology described in Sections 5.2.2.2.1.2 and 5.2.2.2.1.3 was followed.

Step 6. The dot product of each adjacent position vector was calculated, the common logarithm of each was found and the 3 values were multiplied together. The permutation with the highest score was the set of position vectors most orthogonal to one another and was therefore the correct permutation.

Having found the X, Y and Z position vectors of the cube mirror it needed to be determined whether they truly represented the displacement of the cube mirror due to the applied tilt and rotation of the T/R Stage. In order to do so, the position vectors of the optical levers were required to compare with the respective cube mirror position vector components.

The next steps are to do with finding the position vectors of the optical levers remembering that the X and Y optical levers had 18 reflections and the Z optical lever had 12.
Experimentation

8.2.1.6 Calculating the normal of the optical lever reflection vectors

The X, Y and Z optical lever beams projecting from the experiment rig onto the graph paper are represented by reflection vectors $R_{18x}$, $R_{18y}$ for the X and Y optical levers and $R_{12z}$ for the Z optical lever. Having already entered the measured components of these vectors into Excel:

**Step 7.** Normalise the X, Y and Z optical lever components to produce vectors $R_{18xn}$, $R_{18yn}$ and $R_{12zn}$.

8.2.1.7 Deriving the components of $R'_{17x}$, $R'_{17y}$ and $R'_{11z}$ using the reflection isometry method

From Step 7, the reflection vectors $R_{18xn}$, $R_{18yn}$ and $R_{12zn}$ are the normalised vectors projecting from the experiment rig onto the graph paper. Referring back to Section 6.2.4.2.2, they are produced by vectors $R'_{17x}$, $R'_{17y}$ and $R'_{11z}$ that exit the optical levers, reflect off the respective steering mirrors, which then become $R_{18x}$, $R_{18y}$ and $R_{12z}$. Reflection vectors $R'_{17x}$, $R'_{17y}$ and $R'_{11z}$ and normalised vectors $R_{18xn}$, $R_{18yn}$ and $R_{12zn}$ are therefore reflection isometries about the steering mirror normals $N_{SMx}$, $N_{SMy}$ and $N_{SMz}$.

The components of the steering mirror normals $N_{SMx}$, $N_{SMy}$ and $N_{SMz}$ are known from the design of the experiment rig, and are given in Table 11:

<table>
<thead>
<tr>
<th>Table 11: Components of the steering mirror normal vectors.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{SMx}$</td>
</tr>
<tr>
<td>-----------</td>
</tr>
<tr>
<td>x-axis</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

**Step 8.** To find the components of reflection vectors $R'_{17x}$, $R'_{17y}$ and $R'_{11z}$, the components of the static mirror normals $N_{SMx}$, $N_{SMy}$ and $N_{SMz}$ and the normalised components of vectors $R_{18xn}$, $R_{18yn}$ and $R_{12zn}$ are substituted into Equation (6.61).

8.2.1.8 Deriving the components of the virtual tilting mirror normal vectors

To find the magnitude of the tilts and rotations of the T/R Stage, the normal vectors $N_{Tx}$, $N_{Ty}$ and $N_{Tz}$ of the X, Y and Z optical lever tilting mirrors had to be found as it was these mirrors that were mounted to the T/R Stage causing displacement of the beams on the distant graph paper. Referring to Section 6.2.4.1.2, the simplest method of finding the components of these vectors was by means of simulating each tilting mirror with a virtual mirror.

By doing so, the path of the laser beam through each optical lever was reduced from multiple reflections to just 1 reflection vector, i.e. vectors $R_{17x}$, $R_{17y}$ and $R_{11z}$ for the X, Y and Z optical levers respectively.

The single reflection vectors $R_{17x}$, $R_{17y}$ and $R_{11z}$ were derived by Equation (6.51) in terms of the laser beam vectors and the virtual tilting mirror normal vectors. The components of the laser beam vectors $I_x$, $I_y$ and $I_z$ are known from the design of the experiment rig, and are given in Table 12, the virtual tilting mirror normal vectors are what needed to be found.
Table 12: Components of the laser beam vectors.

<table>
<thead>
<tr>
<th></th>
<th>I_x</th>
<th></th>
<th>I_y</th>
<th></th>
<th>I_z</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>-0.70710</td>
<td>0.70710</td>
<td>-0.70710</td>
<td>0</td>
<td>0.70710</td>
</tr>
</tbody>
</table>

The objective of the next step was to find the unknown virtual tilting mirror normal vectors.

Referring to Section 6.2.4.2, reflection vectors \(-\mathbf{R}_{17x}\), \(-\mathbf{R}_{17y}\) and \(-\mathbf{R}_{11z}\) and reflection vectors \(\mathbf{R'}_{17x}\), \(\mathbf{R'}_{17y}\) and \(\mathbf{R'}_{11z}\) are the same vector respectively, they have just been designated differently because the former vectors have been derived from the direction of the laser beam and the latter from the direction of the graph paper.

**Step 9.** To find the virtual tilting mirror normal \(\mathbf{N}'_T\) components for the X, Y and Z optical levers, the components of the laser beam vectors \(\mathbf{I}_x\), \(\mathbf{I}_y\) and \(\mathbf{I}_z\) and vectors \(\mathbf{R'}_{17x}\), \(\mathbf{R'}_{17y}\) and \(\mathbf{R'}_{11z}\) derived in Step 8 are substituted into Equations (6.71), (6.72) and (6.73). There is a positive and negative value to each of the components therefore both values are determined.

**8.2.1.9 Using the displacement of the optical lever beams to determine the correct \(\mathbf{N}'_T\) components for the X, Y and Z optical levers**

**Step 10.** Section 6.2.4.3.1 described the method of determining the angular displacement of the optical levers by observing the position of the optical lever beams relative to their “zero position.” Using this methodology, the correct sign for each \(\mathbf{N}'_T\) component could be selected.

**8.2.1.10 Converting the components of the virtual mirrors to the actual optical lever tilting mirrors and finding the most orthogonal X, Y and Z optical lever position vectors**

The virtual mirror reduced the 18 and 12 reflections of the respective X/Y and Z optical levers to a single reflection to simplify the vector analysis. Having derived the virtual mirror normals, the normals of the actual optical lever tilting mirrors could now be determined.

Following Section 6.2.4.4, which provided the explanation and equations to convert from the virtual tilting mirror normals \(\mathbf{N'}_{Tx}\), \(\mathbf{N'}_{Ty}\) and \(\mathbf{N'}_{Tz}\) to the actual optical lever tilting mirror normals \(\mathbf{N}_{Tx}\), \(\mathbf{N}_{Ty}\) and \(\mathbf{N}_{Tz}\), the minor axis components were divided by 9 and 6 respectively and the major axis component was obtained using Equation (6.76).

Recall from Section 6.2.4.5 that the \(z\)-axis optical lever had its own static and tilting mirrors whereas the \(x\)- and \(y\)-axis optical levers shared common static and tilting mirrors. Therefore, the X and Y optical lever tilting mirror normal vectors \(\mathbf{N}_{Tx}\) and \(\mathbf{N}_{Ty}\) should in theory be identical. However, in practice each had been calculated from different measured variables, i.e. from the displacement of the X and Y optical lever laser beams on the graph paper, therefore they may not prove to be identical. As there was only one Z optical lever normal vector \(\mathbf{N}_{Tz}\) it was best to select the most orthogonal of the vector pairs, \(\mathbf{N}_{Tx} & \mathbf{N}_{Tz}\) and \(\mathbf{N}_{Ty} & \mathbf{N}_{Tz}\).

**Step 11.** Following Section 6.2.4.4, calculate the actual tilting mirror normal vectors from the virtual tilting mirror normal vectors and then take the dot products
Experimentation

\[ \mathbf{N}_{x} \cdot \mathbf{N}_{z} \text{ and } \mathbf{N}_{y} \cdot \mathbf{N}_{z} \] to find the pair that is closest to being orthogonal. Use the most orthogonal pair to progress to the next step of the procedure.

8.2.1.11 Reversing the direction of the X or Y optical lever normal vectors and taking the cross product with the Z normal vector to create the 3rd optical lever vector

The direction of the X and Y optical lever normal vectors by virtue of the design of the optical lever system had their direction aligned with the negative z-axis of the cube mirror. Therefore, in order to directly correlate the components of the Z interferometer position vector with the components of the X or Y optical lever normal vectors, the direction of the optical lever vector needed to be reversed (see Section 6.2.4.5).

As the direction of the Z optical lever normal vector and the direction of the X interferometer position vectors were the same, no action was required.

**Step 12.** Reverse the direction of the selected X or Y optical lever normal vectors and referring to Equation (6.78) take the cross product with the Z optical lever normal vector to create the 3rd optical lever position vector \( \mathbf{N}_y \)

An example of the above 12 step procedure for deriving the position vectors of the cube mirror and the position vectors of the optical levers is giving below for Test 4 whilst all the tests are documented in Appendix A.
Table 13a): Test 4, Steps 1 – 4, determining the magnitude of Y interferometer misalignment.

<table>
<thead>
<tr>
<th>Interferometer</th>
<th>Step 1</th>
<th>Step 2</th>
<th>Step 3</th>
<th>Step 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Fringe slope</td>
<td>Fringe spacing</td>
<td>Mirror tilt angle</td>
<td>Fringe slope</td>
</tr>
<tr>
<td>X</td>
<td>( \varphi_x )</td>
<td>-3.737</td>
<td>0.180</td>
<td>( \alpha_x )</td>
</tr>
<tr>
<td>Y</td>
<td>( \varphi_y )</td>
<td>-20.18</td>
<td>0.952</td>
<td>( \beta_y )</td>
</tr>
<tr>
<td>Z</td>
<td>( \varphi_z )</td>
<td>-87.69</td>
<td>0.174</td>
<td>( \gamma_z )</td>
</tr>
</tbody>
</table>

Table 13b): Test 4, Steps 5 & 6.

<table>
<thead>
<tr>
<th>Step 5</th>
<th>Step 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cube mirror ( \mathbf{X}, \mathbf{Y}, \mathbf{Z} ) components with ( \varphi_x, \varphi_y, \varphi_z ) after re-aligning ( \mathbf{Y} )</td>
<td>The most orthogonal permutation of cube mirror ( \mathbf{X}, \mathbf{Y}, \mathbf{Z} ) components</td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td>0.999995604</td>
<td>-0.00012381</td>
</tr>
<tr>
<td>0.000163949</td>
<td>0.999999957</td>
</tr>
<tr>
<td>-0.00309449</td>
<td>-0.00012261</td>
</tr>
</tbody>
</table>

Table 13c): Test 4, Step 7.

<table>
<thead>
<tr>
<th>Step 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured components of projected optical lever reflection vectors</td>
</tr>
<tr>
<td>x-axis</td>
</tr>
<tr>
<td>( R_{18x} )</td>
</tr>
<tr>
<td>( R_{18y} )</td>
</tr>
<tr>
<td>( R_{12z} )</td>
</tr>
</tbody>
</table>
### Table 13d): Test 4, Step 8.

**Step 8**
Normalised optical lever components and steering mirror components substituted into Equation (6.61) to produce $R'_{17}/R'_{11}$ reflection vectors

<table>
<thead>
<tr>
<th></th>
<th>$R'_{17x}$</th>
<th></th>
<th>$R'_{17y}$</th>
<th></th>
<th>$R'_{11z}$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>0.02495678</td>
<td>y-axis</td>
<td>0.70904749</td>
<td>z-axis</td>
<td>0.70471896</td>
<td></td>
</tr>
<tr>
<td>x-axis</td>
<td>0.73102989</td>
<td>y-axis</td>
<td>0.00247093</td>
<td>z-axis</td>
<td>0.68234097</td>
<td></td>
</tr>
<tr>
<td>x-axis</td>
<td>-0.70534285</td>
<td>y-axis</td>
<td>0.70867938</td>
<td>z-axis</td>
<td>0.01627906</td>
<td></td>
</tr>
</tbody>
</table>

### Table 13e): Test 4, Step 9.

**Step 9**
Use Equations (6.71) – (6.73) find both solutions of virtual mirror normal vectors

<table>
<thead>
<tr>
<th></th>
<th>$N'_{T_x}$ positive</th>
<th></th>
<th>$N'_{T_y}$ positive</th>
<th></th>
<th>$N'_{T_z}$ positive</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>0.017674177</td>
<td>y-axis</td>
<td>0.001374396</td>
<td>z-axis</td>
<td>0.999842855</td>
<td></td>
</tr>
<tr>
<td>x-axis</td>
<td>0.01721513</td>
<td>y-axis</td>
<td>0.001778086</td>
<td>z-axis</td>
<td>0.999850228</td>
<td></td>
</tr>
<tr>
<td>x-axis</td>
<td>-0.99993297</td>
<td>y-axis</td>
<td>0.00111331</td>
<td>z-axis</td>
<td>0.01152464</td>
<td></td>
</tr>
<tr>
<td>$N'_{T_x}$ negative</td>
<td></td>
<td>$N'_{T_y}$ negative</td>
<td></td>
<td></td>
<td>$N'_{T_z}$ negative</td>
<td></td>
</tr>
<tr>
<td>x-axis</td>
<td>-0.01767417</td>
<td>y-axis</td>
<td>-0.00137439</td>
<td>z-axis</td>
<td>-0.99984285</td>
<td></td>
</tr>
<tr>
<td>x-axis</td>
<td>-0.01721513</td>
<td>y-axis</td>
<td>-0.00177808</td>
<td>z-axis</td>
<td>-0.99985022</td>
<td></td>
</tr>
<tr>
<td>x-axis</td>
<td>0.99993297</td>
<td>y-axis</td>
<td>-0.00111331</td>
<td>z-axis</td>
<td>-0.01152464</td>
<td></td>
</tr>
</tbody>
</table>

### Table 13f): Test 4, Step 10.

**Step 10**
Use displacement of the optical lever beams relative to "zero position" to determine the correct solution from Step 9

<table>
<thead>
<tr>
<th></th>
<th>$N'_{T_x}$</th>
<th></th>
<th>$N'_{T_y}$</th>
<th></th>
<th>$N'_{T_z}$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>-0.01767417</td>
<td>y-axis</td>
<td>-0.99984285</td>
<td>z-axis</td>
<td>-0.01721513</td>
<td></td>
</tr>
<tr>
<td>x-axis</td>
<td>-0.00137439</td>
<td>y-axis</td>
<td>-0.99984285</td>
<td>z-axis</td>
<td>-0.00177808</td>
<td></td>
</tr>
<tr>
<td>x-axis</td>
<td>0.99993297</td>
<td>y-axis</td>
<td>-0.00111331</td>
<td>z-axis</td>
<td>-0.01152464</td>
<td></td>
</tr>
</tbody>
</table>
### Chapter 8

#### Table 13g): Test 4, Step 11.

<table>
<thead>
<tr>
<th>Dot product of virtual mirror vectors</th>
<th>Convert the components of the virtual mirrors to the actual optical lever tilting mirrors and use the dot product to select the most orthogonal pair of vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{Tx} \cdot N_{Tz}$</td>
<td>$N_{Ty} \cdot N_{Tz}$</td>
</tr>
<tr>
<td>-0.006148635</td>
<td>-0.005689084</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-0.00191279</td>
</tr>
<tr>
<td></td>
<td>-0.999991838</td>
</tr>
<tr>
<td></td>
<td>0.00018555</td>
</tr>
</tbody>
</table>

#### Table 13h): Test 4, Step 12.

<table>
<thead>
<tr>
<th>Reverse the direction of the selected X or Y optical lever normal vectors and take the cross product with the Z optical lever normal</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{Tz}$ (aligned with X cube mirror components)</td>
</tr>
<tr>
<td>x-axis</td>
</tr>
<tr>
<td>y-axis</td>
</tr>
<tr>
<td>z-axis</td>
</tr>
<tr>
<td>0.999998138</td>
</tr>
<tr>
<td>-0.00191279</td>
</tr>
<tr>
<td>-0.00192077</td>
</tr>
</tbody>
</table>
Experimentation

Figure 66: Test 4, a) X interferometer interferogram, b) Y interferometer interferogram, c) Z interferometer interferogram, d) optical lever beam positions.
Chapter 8

8.3 Results

The cube mirror position vectors from Step 6 of each test have been collated in Table 14. Similarly, the optical lever position vectors from Step 12 of each test have been collated in Table 15. Test 4 is documented in the previous section and all other tests have been documented in Appendix A.

In Table 14 and Table 15, the tests that returned the optical levers back to the “zero position” are indicated highlighted, i.e. tests 9 and 15.

The vectors \(X, Y\) and \(Z\) for the cube mirror and optical levers have components \(X_y, X_z, Y_x, Y_z, Z_x\) and \(Z_y\) that follow a linear function, however, components \(X_x, Y_y\) and \(Z_z\) do not as they are derived from the equation, e.g.:

\[
X_x = \sqrt{1 - \left(\text{linear function of } X_y\right)^2 - \left(\text{linear function of } X_z\right)^2} \tag{8.1}
\]

Vectors \(X, Y\) and \(Z\) are therefore viewed in terms of Euler angles yaw and pitch as these do follow a linear function enabling linear regression to be performed on their values.

These vectors are considered orthogonal to one another and are analysed in each of the three yaw and pitch reference frames depicted in Figure 67. This was done to get a complete understanding of the behaviour of the cube mirror position vectors relative to those of the optical lever.

![Figure 67: Yaw and pitch reference frames, a) XYZ, b) YZX, c) ZXY.](image)

8.3.1 Standard linear regression

8.3.1.1 Gradient analysis

For the gradient analysis, the gradient of the linear regression function should be 1 for components \(X_y, X_z, Y_x, Y_z, Z_x\) and \(Z_y\), as well as yaw and pitch angles.

From Table 16 it can be seen from the components that the worst gradients off the expect gradient of 1 is 6.7% for \(Z_y\), the others being less than 3.4%.

For the Euler yaw and pitch angles:

XYZ frame – 6.1% is worst case for pitch \(Z\), as pitch for \(Z\) vector is close to 90° in this reference frame.
YZX frame – 8.9% is worst case for yaw X, as pitch for X vector is close to 90° in this reference frame.

ZXY frame – 13.3% is worst case for yaw Y, as pitch for Y vector is close to 90° in this reference frame.

From the cases referred to above for gradient of yaw and pitch, the inaccuracy can be attributed to the relevant vector being close to 90° in pitch in that reference frame, which causes large change in yaw for small errors in pitch.

8.3.1.2 Intercept analysis

Components $X_y$, $X_z$, $Y_x$, $Y_z$, $Z_x$ and $Z_y$ are all close to zero. If the unit = metre, then the worst case is 66.6 micrometres for $Z_y$, however, the cube mirror had a unit vector length of 3.5 mm, which represents an intercept of 233 nm.

XYZ frame – For yaw Y, the intercept is high because yaw Y is close to 90° and the magnitude of vector Y being close to 1. For yaw Z, the intercept is high caused by small errors of high pitch, i.e. with pitch close to 90° causing large errors of yaw. Similarly for pitch Z, the intercept is high because pitch Z is close to 90°.

YZX frame – For yaw X, the intercept is high caused by small errors of high pitch. For yaw Y, the intercept is high because of yaw Y being close to 90° and the magnitude of vector Y being close to 1.

ZXY frame – For yaw Y, the intercept is high caused by small errors of high pitch. For pitch Y, the intercept is high because of pitch Y being close to 90°. For yaw Z, the intercept is high because of yaw Z being close to 90° and the magnitude of vector Z being close 1.

For each frame of reference, the closeness to the origin of the intercept of the regression function is affected greatest whenever the relevant vector is close to 90° in either yaw or pitch. Small errors in pitch cause large errors in yaw.

Instead of a standard linear regression ($y = mx + c$), a through-origin regression could have been used ($y = mx$), as if e.g. $X_{yinterf} = 0$ then $X_{yoptlev} = 0$. However, the intercept of a standard linear regression adds another dimension to the accuracy analysis of the "interferometer system."

8.3.1.3 R-squared

Except for yaw Y in the ZXY reference frame, which had $r^2 = 0.9281$, all other parameters had $r^2 > 0.9850$, which indicates a high closeness of the data to the regression line. This is confirmed by having analysed the residual plots (not depicted) and provides a strong indication that the position vectors of the cube mirror system are accurate - on the assumption that the optical lever system itself is accurate by virtue of its design and the linear propagation of light.

8.3.2 Residual standard deviation

The intention of the displacement sensor is to measure small angles, therefore components $X_y$, $X_z$, $Y_x$, $Y_z$, $Z_x$ and $Z_y$ are extremely small. Because of this the residual standard deviations are difficult to put into perspective, therefore no residual standard deviation values have been listed for these components in Table 16.
However, the residual standard deviation for cube mirror position vectors $X$, $Y$ and $Z$ are given in terms of Euler angles yaw and pitch being easier to understand and to interpret, and accuracy is also more evident.

In the same way that gradient and intercept were most affected when the relevant vectors were close to 90° in either yaw or pitch, similar behaviour is evident for residual standard deviation for each of the reference frames.

**XYZ frame – vector $Z$ pitch is close to 90° therefore small errors in pitch has resulted in yaw $Z$ residual standard deviation being 10.1814 degree. Although this may appear high, it only equates to 2.828% of full circle.**

**YZX frame - vector $X$ pitch is close to 90°, which can be attributed to yaw $X$ residual standard deviation being 14.6297 degree, 4.0638% of full circle.**

**ZXY frame - vector $Y$ pitch is close to 90°, which can be attributed to yaw $Y$ residual standard deviation being 25.3556 degree, 7.04321% of full circle.**

The yaw of the above $X$, $Y$ and $Z$ vectors viewed in the stated reference frame are analogous with the slope of the fringe lines in their interferogram, i.e. the $\phi$ angles in Figure 23.

All other yaw components for each reference frame exhibit negligible residual standard deviation with respect to full circle.

The pitch of the above vectors in the stated reference frames is analogous to the respective mirror tilt angle, i.e. angles $\alpha$, $\beta$ and $\gamma$ in Figure 23. The worst case residual standard deviation for pitch of these vectors is 0.004915° for vector $Z$ in XYZ frame. As mentioned in Section 7.2.13.2.3, the measurement resolution for deriving the mirror tilt angle from the fringe line spacing was 0.01°, therefore the residual standard deviation is approximately half of this for very small angles.

From the above gradient, intercept, r-squared and residual standard deviation analysis applied to the test results, the cube mirror position vectors can be considered to closely correlate with those produced by the optical lever system. The objective of this thesis was to research a sensor using optical interferometry to measure displacement to 6 DoF. This Chapter has proven, using 3 Michelson interferometers arranged orthogonally about a cube mirror, that when angular displacement is applied to the cube mirror the position vectors resulting from the angular displacement can be derived from the fringe line spacing and slope. As mentioned previously, measuring linear displacement using optical interferometry is a mature technology and therefore no value would be added to the knowledge by including linear displacement in the experimentation.

**Therefore, having established the accuracy and sensitivity of the interferometry sensor, Research Question 1.2iv has been satisfied.**
### Table 14: Cube mirror position vectors from Step 6 of each test.

<table>
<thead>
<tr>
<th></th>
<th>X_CUBE MIRROR</th>
<th>Y_CUBE MIRROR</th>
<th>Z_CUBE MIRROR</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Xx</td>
<td>Yx</td>
<td>Zx</td>
</tr>
<tr>
<td>Zero Position</td>
<td>1   0</td>
<td>-6.1725E-11</td>
<td>1   4.75609E-12</td>
</tr>
<tr>
<td>Test 1</td>
<td>0.99999988</td>
<td>-0.00049018</td>
<td>2.63369E-05</td>
</tr>
<tr>
<td>Test 2</td>
<td>0.999999814</td>
<td>-0.00048884</td>
<td>-0.00036431</td>
</tr>
<tr>
<td>Test 3</td>
<td>1   0</td>
<td>0</td>
<td>1   0.999999876</td>
</tr>
<tr>
<td>Test 4</td>
<td>0.999998224</td>
<td>-0.0012287</td>
<td>-0.00188081</td>
</tr>
<tr>
<td>Test 5</td>
<td>0.999997647</td>
<td>-0.00216927</td>
<td>1.09475E-05</td>
</tr>
<tr>
<td>Test 6</td>
<td>0.999996531</td>
<td>0.002634103</td>
<td>-1.0086E-07</td>
</tr>
<tr>
<td>Test 7</td>
<td>0.999983951</td>
<td>-0.00506743</td>
<td>-0.00253352</td>
</tr>
<tr>
<td>Test 8</td>
<td>0.999999896</td>
<td>-0.00045256</td>
<td>4.89432E-05</td>
</tr>
<tr>
<td>Test 9</td>
<td>0.999999996</td>
<td>7.97403E-05</td>
<td>-4.1378-05</td>
</tr>
<tr>
<td>Test 10</td>
<td>0.999973004</td>
<td>-0.00533532</td>
<td>-0.00505237</td>
</tr>
<tr>
<td>Test 11</td>
<td>0.999999868</td>
<td>3.40515E-05</td>
<td>0.000512518</td>
</tr>
<tr>
<td>Test 12</td>
<td>0.999999975</td>
<td>0.000142343</td>
<td>0.000174819</td>
</tr>
<tr>
<td>Test 13</td>
<td>0.999999996</td>
<td>8.69825E-05</td>
<td>-0.00026890</td>
</tr>
<tr>
<td>Test 14</td>
<td>0.999999862</td>
<td>4.55932E-05</td>
<td>0.000522567</td>
</tr>
<tr>
<td>Test 15</td>
<td>0.999999998</td>
<td>5.84508E-05</td>
<td>-3.1161-05</td>
</tr>
</tbody>
</table>
Table 15: Optical lever position vectors from Step 12 of each test.

<table>
<thead>
<tr>
<th>Zero Position</th>
<th>$N_{Tz}$ (aligned with X cube mirror components)</th>
<th>$N_{Tz} \times N_{Tx}/N_{Ty} = N_y$ (aligned with Y cube mirror components)</th>
<th>$N_{Tx}/N_{Ty}$ (aligned with Z cube mirror components)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$Xx$</td>
<td>$Xy$</td>
<td>$Xz$</td>
</tr>
<tr>
<td>Test 1</td>
<td>0.999999879</td>
<td>-0.000491578</td>
<td>0</td>
</tr>
<tr>
<td>Test 2</td>
<td>0.999999793</td>
<td>-0.000510781</td>
<td>-0.00039051</td>
</tr>
<tr>
<td>Test 3</td>
<td>0.999999997</td>
<td>2.0879E-08</td>
<td>-8.3425E-05</td>
</tr>
<tr>
<td>Test 4</td>
<td>0.999998138</td>
<td>-0.000185551</td>
<td>-0.00192077</td>
</tr>
<tr>
<td>Test 5</td>
<td>0.999997576</td>
<td>-0.00220172</td>
<td>0</td>
</tr>
<tr>
<td>Test 6</td>
<td>0.999996373</td>
<td>0.002692835</td>
<td>-5.4711E-05</td>
</tr>
<tr>
<td>Test 7</td>
<td>0.99998376</td>
<td>-0.005125268</td>
<td>-0.00249230</td>
</tr>
<tr>
<td>Test 8</td>
<td>0.999999998</td>
<td>3.9336E-05</td>
<td>-5.5603E-05</td>
</tr>
<tr>
<td>Test 9</td>
<td>0.99997239</td>
<td>-0.005477459</td>
<td>-0.00502153</td>
</tr>
<tr>
<td>Test 10</td>
<td>0.999999888</td>
<td>6.70439E-07</td>
<td>0.00472736</td>
</tr>
<tr>
<td>Test 11</td>
<td>0.999999983</td>
<td>0.000118038</td>
<td>0.000138942</td>
</tr>
<tr>
<td>Test 12</td>
<td>0.999999952</td>
<td>3.96072E-05</td>
<td>-0.0030582</td>
</tr>
<tr>
<td>Test 13</td>
<td>0.999999888</td>
<td>3.99676E-05</td>
<td>0.000472624</td>
</tr>
<tr>
<td>Test 14</td>
<td>0.999999999</td>
<td>7.42364E-09</td>
<td>-4.4493E-05</td>
</tr>
</tbody>
</table>

Chapter 8
Table 16: Linear regression, r-squared and residual standard deviation of cube mirror and optical lever components (Euler angles pitch and yaw).

<table>
<thead>
<tr>
<th>Component</th>
<th>Linear regression function</th>
<th>r-squared</th>
<th>Residual Standard Deviation</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Xy</td>
<td>( Y = 1.016074043 * X - 2.067296672E-005 )</td>
<td>0.999808</td>
<td></td>
<td>Components too small</td>
</tr>
<tr>
<td>Xz</td>
<td>( Y = 0.9866187973 * X - 3.015915341E-005 )</td>
<td>0.999693</td>
<td></td>
<td>Components too small</td>
</tr>
<tr>
<td>Yx</td>
<td>( Y = 1.027762848 * X - 6.394413823E-006 )</td>
<td>0.9999</td>
<td></td>
<td>Components too small</td>
</tr>
<tr>
<td>Yz</td>
<td>( Y = 1.033930359 * X - 3.898079975E-005 )</td>
<td>0.998738</td>
<td></td>
<td>Components too small</td>
</tr>
<tr>
<td>Zx</td>
<td>( Y = 1.033552915 * X - 2.573077366E-005 )</td>
<td>0.996596</td>
<td></td>
<td>Components too small</td>
</tr>
<tr>
<td>Zy</td>
<td>( Y = 1.066911173 * X + 6.655367415E-005 )</td>
<td>0.99467</td>
<td></td>
<td>Components too small</td>
</tr>
</tbody>
</table>

**XYZ Reference frame**

<table>
<thead>
<tr>
<th>Component</th>
<th>Linear regression function</th>
<th>r-squared</th>
<th>Residual Standard Deviation</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>yawX</td>
<td>( Y = 1.016604414 * X - 0.001014130313 \text{deg} )</td>
<td>0.999833</td>
<td>0.001598 \text{deg} \Rightarrow 0.00044 % (with respect to a full circle = 360 degrees)</td>
<td></td>
</tr>
<tr>
<td>pitchX</td>
<td>( Y = 0.9865236656 * X - 0.00175139947 \text{deg} )</td>
<td>0.999691</td>
<td>0.001451 \text{deg} \Rightarrow 0.0004030% (with respect to a full circle = 360 degrees)</td>
<td></td>
</tr>
<tr>
<td>yawY</td>
<td>( Y = 1.027579768 * X - 2.481875153 \text{deg} )</td>
<td>0.999902</td>
<td>0.001154 \text{deg} \Rightarrow 0.0003206% (with respect to a full circle = 360 degrees)</td>
<td>intercept high because of magnitude Y and yaw Y being close to 90 deg</td>
</tr>
<tr>
<td>pitchY</td>
<td>( Y = 1.034440942 * X - 0.002142862387 \text{deg} )</td>
<td>0.998743</td>
<td>0.0025126 \text{deg} \Rightarrow 0.0006979% (with respect to a full circle = 360 degrees)</td>
<td></td>
</tr>
<tr>
<td>yawZ</td>
<td>( Y = 1.057748216 * X + 4.049508245 \text{deg} )</td>
<td>0.989469</td>
<td>10.181443 \text{deg} \Rightarrow 2.8281785% (with respect to a full circle = 360 degrees)</td>
<td>intercept and residual stdev high because small errors of high pitch, i.e. close to 90 deg, cause large errors of yaw</td>
</tr>
<tr>
<td>pitchZ</td>
<td>( Y = 1.061368953 * X - 5.521299867 \text{deg} )</td>
<td>0.997674</td>
<td>0.004915 \text{deg} \Rightarrow 0.0013653% (with respect to a full circle = 360 degrees)</td>
<td>intercept high because of the pitch Z being close to 90 deg</td>
</tr>
</tbody>
</table>
## YZX Reference frame

<table>
<thead>
<tr>
<th>Reference</th>
<th>Equation</th>
<th>Intercept</th>
<th>Residual Stdev</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{yawX}$</td>
<td>$Y = 1.088824249 \times X - 8.749790705 \text{ deg}$</td>
<td>0.985481</td>
<td>14.6297276 deg $\Rightarrow$ 4.0638% (with respect to a full circle = 360 degrees)</td>
<td>Intercept and residual stdev high because small errors of high pitch, i.e. close to 90 deg, cause large errors of yaw</td>
</tr>
<tr>
<td>$\text{pitchX}$</td>
<td>$Y = 1.010402007 \times X + 0.9362496569 \text{ deg}$</td>
<td>0.999717</td>
<td>0.00217519 deg $\Rightarrow$ 0.00060% (with respect to a full circle = 360 degrees)</td>
<td></td>
</tr>
<tr>
<td>$\text{yawY}$</td>
<td>$Y = 1.033931558 \times X + 3.051606769 \text{ deg}$</td>
<td>0.998739</td>
<td>0.00252671 deg $\Rightarrow$ 0.00070% (with respect to a full circle = 360 degrees)</td>
<td>Intercept high because of magnitude of $Y$ and yaw $Y$ being close to 90 deg</td>
</tr>
<tr>
<td>$\text{pitchY}$</td>
<td>$Y = 1.027763067 \times X + 0.00366365302 \text{ deg}$</td>
<td>0.9999</td>
<td>0.00119028 deg $\Rightarrow$ 0.00033% (with respect to a full circle = 360 degrees)</td>
<td></td>
</tr>
<tr>
<td>$\text{yawZ}$</td>
<td>$Y = 1.066911866 \times X - 0.00381256279 \text{ deg}$</td>
<td>0.99467</td>
<td>0.00521711 deg $\Rightarrow$ 0.00144% (with respect to a full circle = 360 degrees)</td>
<td></td>
</tr>
<tr>
<td>$\text{pitchZ}$</td>
<td>$Y = 1.033553378 \times X - 0.001474271763 \text{ deg}$</td>
<td>0.996596</td>
<td>0.00497856 deg $\Rightarrow$ 0.00138% (with respect to a full circle = 360 degrees)</td>
<td></td>
</tr>
</tbody>
</table>

## ZXY Reference frame

<table>
<thead>
<tr>
<th>Reference</th>
<th>Equation</th>
<th>Intercept</th>
<th>Residual Stdev</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{yawX}$</td>
<td>$Y = 0.9868052923 \times X + 0.00168225037 \text{ deg}$</td>
<td>0.999685</td>
<td>0.00142783 deg $\Rightarrow$ 0.00040% (with respect to a full circle = 360 degrees)</td>
<td></td>
</tr>
<tr>
<td>$\text{pitchX}$</td>
<td>$Y = 1.016074191 \times X - 0.001184481573 \text{ deg}$</td>
<td>0.999808</td>
<td>0.00162027 deg $\Rightarrow$ 0.00045% (with respect to a full circle = 360 degrees)</td>
<td></td>
</tr>
<tr>
<td>$\text{yawY}$</td>
<td>$Y = 1.132643621 \times X - 7.17121445 \text{ deg}$</td>
<td>0.928106</td>
<td>25.3555509 deg $\Rightarrow$ 7.04321% (with respect to a full circle = 360 degrees)</td>
<td>Intercept and residual stdev high because small errors of high pitch, i.e. close to 90 deg, cause large errors of yaw</td>
</tr>
<tr>
<td>$\text{pitchY}$</td>
<td>$Y = 1.040159844 \times X - 3.61237817 \text{ deg}$</td>
<td>0.999774</td>
<td>0.00185898 deg $\Rightarrow$ 0.00052% (with respect to a full circle = 360 degrees)</td>
<td>Intercept high because of pitch Y being close to 90 deg</td>
</tr>
<tr>
<td>$\text{yawZ}$</td>
<td>$Y = 1.034098243 \times X + 3.067250267 \text{ deg}$</td>
<td>0.996582</td>
<td>0.00497839 deg $\Rightarrow$ 0.00138% (with respect to a full circle = 360 degrees)</td>
<td>Intercept high because of</td>
</tr>
</tbody>
</table>
Experimentation

| pitchZ | $Y = 1.06544822 \times X + 0.004103647979$ deg | 0.99482 | 0.00521608 deg $\Rightarrow$ 0.00144% (with respect to a full circle = 360 degrees) | magnitude $Y$ and yaw $Y$ being close to 90 deg |
8.4 Summary

The experiment rig was set up approximately 2.2 metres from a wall onto which the 3 optical lever beams projected onto graph paper. Displacement of the optical lever beams was measured to an accuracy of 0.5 mm.

Sixteen separate tests were conducted applying varying amounts of tilt and rotation to the tilt/rotation stage. Three of the tests were re-zeroing of the optical lever beam system to see whether there was any perceived drift of the interferometer sensor. Each test followed a 12 step procedure to capture the fringe spacing and slope from the interferograms, the displacement of the optical lever beams and deriving their respective position vectors.

Data from all the tests was tabulated in the Chapter as well as Appendix A including snapshots of the 3 interferograms and displacement of the optical lever beams on the graph paper.

The data was entered into a spreadsheet, which was also used to calculate the standard linear regression of the position vector components using Euler angles. As the 3 orthogonal position vectors can be viewed in terms of Euler angles in 3 difference reference frames, the data was analysed and present with respect to all 3 frames.

It was found that for an expected gradient of 1, the worst case gradient was 13.3% and the worst case r-squared was 0.9281. From the residual standard deviation analysis, the worst case deviation was a yaw of 7.0432% of full circle of the cube mirror position vector due to the vector being close to being 90° in pitch.

It was clearly evident from the data analysis that the worst case residual standard deviation of components was obtained when the vector in question was either close to 90° in pitch or yaw. This is due to the asymptotic behaviour of the tangent function close to ±90°.

The experimentation showed an extremely close correlation of the cube mirror and optical lever position vectors.
9 DISCUSSION

The objective of this thesis was to research, develop and test a novel sensor using optical interferometry to measure displacement to 6 degrees of freedom.

Outlined below are highlights of the research, methodologies, experiment design, data capture and analysis that are of note.

This research focused on the novel aspect of the sensor, which was primarily its ability to accurately measure angular displacement of a cube mirror using just 3 interferometers as opposed to 6 interferometers described in [3,4]. It achieves this by utilising mirror tilt angle and fringe slope information provided by the interferograms compared with the latter, which only measures linear translation of the cube mirror to derive the angular displacement.

To determine the accuracy and sensitivity of the sensor to angular displacement, a method of measuring angular displacement was required, which resulted in an optical lever system being designed and built for this purpose. The interferometer displacement sensor was integrated with the optical lever system onto an experiment rig as a common platform for verification of one system using the other. This methodology contributed to the robustness of the data analysis.

Since no adjustment had been designed on the rig for aligning the interferometers with the cube mirror in its zero position, alignment was done by manoeuvring each beamsplitter by hand and then gluing it in position. This resulted in a slight orthogonal misalignment of the y-axis interferometer relative to the X and Z interferometers, but was solved by way of a change of basis transition matrix to correct the cube mirror position vectors. Although adjustment of the rig would ideally have been allowed for, what this demonstrated was that in the event of interferometer misalignment, orthogonal alignment can be achieved mathematically when the data is processed. Therefore the methodology adopted did not adversely affect the data obtained, but showed that absolutely perfect orthogonal alignment of the 3 interferometers is not essential. In fact, it could be used to advantage to overcome indeterminates in the cube mirror position vector calculations by always having fringe lines present on at least 2 of the 3 interferograms.

The design of the optical lever tilt/rotation measurement system enabled not only accurate measurement of tilt and rotation about the 3 axes, but also overcame cross-axis crosstalk. Crosstalk occurs in mechanical tilt/rotation stages when reading tilt and
rotation from dials on its adjustment knobs. For example, making an adjustment with one knob may induce a slight tilt/rotation about one or both of the other axes due to imperfections in the mechanism. The induced crosstalk would not be registered because the other two knobs would not move with the crosstalk, thus giving the same reading as before, resulting in measurement error. Therefore the optical lever system was adopted as it was considered to provide greater accuracy than a commercially affordable mechanical tilt/rotation stage with graduated adjustment knobs.

The optical lever system generated 2 orthogonal position vectors of unit length. The third position vector, orthogonal to both, was obtained using the vector cross product. This would mean that any slight orthogonal misalignment of the 2 measured position vectors would be transferred to the cross product vector, resulting in an error of tilt and rotation measurement. However, the worst-case orthogonal misalignment of the 2 measured vectors in the experiments was 0.0025 degrees, which was considered minimal with little adverse impact on the data.

Despite the optical levers being able to project their laser beams a distance of 20 metres or more, for convenience the experiment rig was set up in a room at a distance of 2.2 metres from the projection wall. The degree of resolution of laser beam displacement measurement on the graph paper was 0.5mm. For the 12 reflection optical lever this represented an error of 0.001 degrees. For the 18 reflection optical lever this represented an error of 0.0007 degrees. Therefore, using a projection distance greater than the adopted distance was proved to be unnecessary.

No instrumentation was available to accurately collimate the laser beams for the interferometers. Instead, this was done by eye by small adjustments of the focal length of the aspheric lens and observing the beam over a distance of 8 metres until the collimation appeared uniform over that distance. This had the probable result that the fringe-line formation was not totally perfect and therefore would have resulted in some error in determining both fringe spacing and fringe slope. Despite this, the cube mirror position vectors calculated from the measured fringe spacing and slope had a very close correlation with the optical lever system with worse case r-squared value = 0.9281 and all other values > 0.9850.

To simplify determining the optical lever normal vectors, a virtual tilting mirror was used to simulate the actual tilting mirror. The virtual mirror reduced the number of reflections (12 and 18) to a single reflection. To convert from the virtual mirror to the actual tilting mirror, a scaling factor derived by equation was applied to the two small vector components. In doing so, a close approximation to the actual tilting mirror components from the virtual mirror was obtained. The third vector component was derived by normalising the vector to unity. As the third component was always 2-3 orders of magnitude greater than the other two, the errors induced by this methodology were considered minimal. However, from the intercept and residual standard deviation analysis it can be seen for each reference frame that the vertical vector had a significantly higher intercept and yaw error than those vectors in the horizontal plane. The approximation methodology of converting from the virtual to actual tilting mirror normal could have contributed to creating these intercept and yaw errors.

The 2 methods identified in the literature review that use a cube mirror are [3,4], which require at least 5 interferometers to derive 6 DoF. With [3], the arrangement involves single, double and triple interferometers arranged orthogonally to one another. For each interferometer a discrete photodetector is used to purely measure linear displacement from which angular displacement is derived. In contrast, this research adopts 3 interferometers and derives both linear and angular displacement from image sensors.
Because each realisation uses a different method to detect and measure linear and angular displacement, the applications of [3,4] versus this research are different, hence this research reduces the knowledge gap.

In terms of angular displacement, when using a photodetector as in [3,4], as discussed in the fringe analysis theory in Chapter 3 the modulation amplitude of radiant flux on the photodetectors drops off rapidly with mirror tilt. For example, for a 1mm width photodetector, the modulation amplitude reduces to zero at approx. 0.04°. This explains the angular measurement range of ±2 arc minutes (0.033°) quoted by [3]. By way of comparison with this research, the 3 interferometers using image sensors to capture the interferogram, achieve a measurement range of greater than 0.5 degrees (i.e. the range in this research is at least 15× greater than [3]).

In terms of linear displacement, [3] has a 2m measurement range and moving mirror translation rate of 600mm/second, which currently is only possible using discrete photodetectors to obtain the fringe count. In this research the intended linear displacement measurement range is 1mm and the translation rate is dependent on the maximum sampling rate of the image sensor used. As [4] is a theoretical paper it therefore gives no detail on measurement capabilities with which to compare with this research.

This research therefore fills a gap in the knowledge for possible use of the technology in measuring displacement where the linear displacement is expected to be in the order of fractions of a millimetre and angular displacement is expected to be \( \leq 0.5° \), and great accuracy is required.

With the knowledge from this research, and with the use of optical/photonic instrumentation for collimating the laser beams and aligning the interferometers accurately, far higher levels of accuracy in determining the cube mirror position vectors could be achieved. Including a simple mechanical means of helping to achieve alignment of the interferometers would be a great advantage.

Nevertheless in this research:

- the worst case gradient from the standard linear regression was 13.3% from the expect gradient of 1.
- the worst case r-squared was value 0.9281 with all other values > 0.9850.
- from the residual standard deviation analysis the worst case yaw due to the cube mirror position vectors being close to 90° in pitch was 7.0432% of full circle.

Using 3 interferometers to measure linear displacement along the x-, y- and z axes has not been included in the experimentation as it is common knowledge and offers nothing new. It is a straight forward exercise to add the linear displacement measured by each of the X, Y and Z interferometers to the respective components that are derived for angular displacement in order to obtain displacement to 6 DoF.

In conclusion, the research has successfully demonstrated that it is possible to accurately determine the displacement of a cube mirror to 6 DoF using only 3 interferometers arranged orthogonally about a cube mirror. This is a significant outcome when compared with existing knowledge.
10 CONCLUSION

The radiant flux $\Phi$ across the active area of a photodetector is a damped sine function (i.e., a cardinal sine function) of the wave front angle $\theta$, with a reciprocal decay of the ratio of wavelength $\lambda$ to detector width $s$. The larger $s$ and the smaller $\lambda$, the faster is the decay of the radiant flux.

If the radiant flux magnitude at wavefront angle $\theta = 0$ is normalised to 100%, then the radiant flux magnitude at the primary node points is 50% and at the first minimum is 39.14%. The sign of the radiant flux changes exclusively at every primary node point. The radiant flux magnitude at specific $\theta$ is independent of any parameter if the centre of the fringe beam coincides with the centre of the photodetector.

The larger the distance $x$ of the photodetector from the centre of the fringe beam and/or the distance $y$ between mirror and photodetector, the more the radiant flux oscillates under the envelope radiant flux curve generated if $x$ and $y = 0$. These two parameters do not affect the modulation amplitude of the radiant flux.

The movement of fringe lines with increasing $\theta$ is a combined effect of fringe contraction ($x$-dependent; the faster the more the detector is off centre) and fringe tilt ($y$-dependent; the faster the larger $y$). Fringe contraction and tilt movement can have opposite effects, with fringe contraction lagging behind fringe tilt, such that the fringe count first increases, then decreases and then returns to zero.

Consequently, significant fringe count errors occur if the photodetector is operated near or beyond primary nodes where radiant flux modulation reduces to zero or if the $x$ and $y$ distances of the photodetector are large therefore increasing the fringe contraction and fringe tilt influence.

In addition, fringe transition speed significantly increases the greater the $x$- and $y$-distances of the photodetector are from the central axis of the interferometer.

In all, these findings for this research pointed to utilisation of photodetector widths that are less than 10 µm, to reduce the impact of modulation amplitude reduction with increasing wave-front angle. In addition, the $x$- and $y$-distances need to be kept to the minimum possible. This indicates that to maximise the capture of the radiant flux across the interferogram, an image sensor was required that was located as close as possible to the optical origin, and with a pixel pitch of less than 10 µm. Deciding upon an image
The information given by an interferogram consists of fringe spacing, fringe slope, fringe count and fringe transition direction. From this information, both moving mirror tilt angle and moving mirror tilt axis angle can be calculated, but the direction of tilt is unknown. The magnitude of linear translation can be derived from the fringe count, however fringe transition direction does not provide indication of translation direction.

To resolve these unknowns, three orthogonally-arranged interferometers about a cube mirror are required.

It has been demonstrated using two methodologies that when applying angular displacement to a cube mirror, the cube mirror position vectors can be theoretically derived from mirror tilt angles and tilt axis angles. To verify this theory, known magnitudes of yaw, pitch and roll were applied to a mathematically modelled cube to find its resultant position vectors.

Had each of the three orthogonal sides of the cube mirror been the moving mirror of a Michelson interferometer, they would have each generated an interferogram from which the resulting fringe spacing and fringe slope could have been derived. By substituting this data into the two methodologies for deriving the cube mirror position vectors, the resulting position vectors were identical to the modelled cube, thereby successfully verifying the methodologies.

In doing so, this has filled a knowledge gap and resolved Research Question 1.2i.

The direction of fringe transition is dependent upon the direction of translation of the moving mirror, as well as its direction of tilt. Having determined the position vectors of each of the three sides of the cube mirrors, as well as determining the direction of fringe transition across the interferogram, the direction of translation of the cube mirror along each of the Cartesian axes can be determined.

Hereby, Research Question 1.2ii has been resolved.

When utilising orthogonally arranged interferometers about a cube mirror to measure linear displacement, a translation error occurs along a first axis due to translation about the second and/or third axis. In the literature, this error does not appear to have been taken into consideration, therefore resulting in a translation error. In the interest of maximising linear displacement accuracy when using a cube mirror, a method of correcting for this error was derived.

In doing so, this has filled a knowledge gap and resolved Research Question 1.2iii.

To test the cube mirror and the optical lever system theory in practice, the displacement sensor needed to be built. To validate the accuracy of the sensor, a method of calibrating tilt and rotation of the cube mirror about the Cartesian axes was required. To induce tilt and rotation to the cube mirror, a 3DoF tilt and rotation stage was purchased. However, this device provided no measurement of the tilt and rotations about the x, y and z axes. Therefore, a device to specifically measure the angular displacement needed to be designed and integrated with the tilt rotation stage. The tilt and rotation measuring device was based on three orthogonal optical levers that had a variable angle.
magnification – a magnification of 12 and 18 being used in the experimentation. A method was developed to derive the position vectors of the optical lever system.

Angular displacements were applied to the tilt and rotation stage of the experiment rig, and for each test the interferogram data and optical lever system data were captured. For each test, the position vectors for the cube system and optical lever system were derived. On analysis it was found that one of the interferometers was slightly misaligned, however by applying a change of basis transition matrix, this misalignment was corrected for mathematically.

The interferograms were analysed visually with the aid of fringe analysis software to determine fringe spacing and fringe slope. The resolution of measurement of the displacement measuring system was limited by visual determination of the fringe spacing across the image sensor. As the minimum number of fringes that can be accurately determined visually across the image sensor = 2 fringes this represents 1/100 degree angular displacement. From a standard linear regression of the experimentation results it was found that for an expected gradient of 1, the worst case gradient was 13.3% and the worst case r-squared was 0.9281. From the residual standard deviation analysis, the worst case deviation was a yaw of 7.0432% of full circle of the cube mirror position vector due to the vector being close to being 90° in pitch. The experimentation showed an extremely close correlation of the cube mirror and optical lever position vectors.

Hereby, Research Question 1.2iv has been resolved.

The two methods of deriving the position vectors of the cube mirror were dependent upon the determination of the slope angle of the fringe lines, i.e. the mirror tilt axis angle. When an interferometer is perfectly aligned, the radiant flux across the interferogram is homogeneous. Therefore, the fringe slope is indeterminate, which renders it impossible to theoretically resolve the position vectors of the cube mirror. However, whenever the tilt angle is zero, the position vector of the cube mirror has only one non-zero component. Therefore, to overcome the indeterminate in deriving the cube mirror position vectors, in data, if a tilt angle is found to be zero, then the off-axis components of the vector shall be set to zero. Applying this rule in method 2 simplifies the vector analysis and overcomes the indeterminate.

This resolves Research Question 1.2v.

This thesis has shown that it is possible to accurately measure six degrees of freedom using three interferometers orthogonally arranged about a cube mirror. Using a mathematically modelled cube mirror subjected to angular displacement, and deriving the position vectors of the cube mirror using the methods developed, has shown direct equivalence with the model. Building and testing the displacement sensor, by way of an experiment rig, demonstrated an extremely close correlation of the cube mirror position vectors and those of the optical lever system. In doing so, all research questions have been resolved, a knowledge gap in measuring displacement to 6DoF has been filled and the therefore the objective of this thesis has been fulfilled.
11 FOR FURTHER STUDY

11.1 Optimising the resolution of the interferometer output

To determine fringe line orientation and spacing, digital signal processing (DSP) is required to analyse the radiant flux pattern of the interferogram that is captured by the image sensor. DSP will use statistical methods to accurately resolve the pixel position of the maximum and minimum radiant flux instances as well as to reduce or eliminate unwanted noise from the signal. The process is simplified with the knowledge that the flux profile is sinusoidal, so undesirable disturbances are easily identifiable and filtered from the actual signal. Thereafter, using trigonometry, the fringe pattern can be mathematically modelled to match the maximum and minimum radiant flux instances from which fringe line orientation and spacing can be determined.

The components that come together to make up an interferometer are not perfect in their function. For example,

- the wave front from the collimated laser beam is not perfectly planar
- the irradiance profile of a laser beam is Gaussian and also includes manufacturing imperfections
- beamsplitters and mirrors contain optical imperfections
- the conversion gain and responsivity of the pixel array is not perfectly linear
- the image sensor ADC is not perfectly linear over the conversion range

To overcome these imperfections to a large extent, the radiant flux profile of the laser beam and the voltage output from the image sensor can be normalised before the DSP further processes the signal.

In addition, other DSP techniques can be used to improve the accuracy of the fringe line profile captured by the image sensor, one of which is covered in this section.

11.1.1 Radiant flux normalisation using an apodising filter

A collimated laser beam has a profile with maximum irradiance centred on the beam's axis and diminishes radially as a Gaussian function as depicted in Figure 68. One of the definitions to describe the width of a laser beam is the distance between two diametrically opposite points where the irradiance has diminished to $1/e^2$ of the maximum. Therefore,
if the interferometer is in perfect alignment, the irradiance will always be greater at the centre of the interferogram, except at the instance of maximum destructive interference.

If the interferometer is not perfectly aligned then fringe lines will be formed across the interferogram and they will differ in irradiance dependent on whether they describe a diameter or chord across the beam.

A Bull’s Eye™ apodising neutral density filter is a filter that has highest optical density at the centre and reduces radially as a Gaussian function or as close to it as practicable (see Figure 69). Using such a filter in the path of the laser beam will tend to flatten the irradiance profile across it and therefore the full sensitivity range of the image sensor can be better utilised over the whole beam width.

11.1.2 Radiant flux normalisation by pixel output normalisation

If a Michelson interferometer is perfectly aligned and in phase, and assuming a Bull’s Eye™ filter is not inserted into the beam path, the interferogram will be at its maximum irradiance with a profile illustrated in Figure 68. With the interferometer in this state, the output of each pixel will be at its maximum (assuming that no pixels at the centre have reached saturation). To normalise the radiant flux, the voltage value of each pixel must be saved and every subsequent voltage measurement taken of the interferogram divided by this value. Every pixel output will therefore be a ratiometric value of its maximum and consequently every pixel output can be directly compared with one another.

To carry out this normalisation method requires a precise translation/rotation/tilt stage and it may be difficult to perfectly align the interferometer and adjust it to be perfectly in phase. To overcome this difficulty, and assuming that the beamsplitter has a 50:50 transmission/reflection transfer ratio, the normalisation can be done with just one beam of the interferometer. To do this, the beam incident on the moving mirror must be blocked so that only the radiant flux from the fixed mirror is measured by the image sensor. Each
pixel output voltage is then stored in memory. As the stored data is representative of the radiant flux from one beam of the interferometer, each value must be multiplied by two. Thereafter, every subsequent voltage measurement taken of the interferogram is divided by this value to give a ratiometric value of each pixel maximum.

As a result, the Gaussian profile of the interferogram has been digitally flattened. However, the full sensitivity of the pixels away from the beam centre is not being utilised as the full scale measurement range of the image sensor must be set for the highest irradiance without reaching saturation, i.e., the centre pixels.

When used in conjunction with a Bull’s Eye™ filter, the voltage normalisation will overcome the likelihood that the optical density profile of the filter is not perfectly Gaussian.

In reality, a collimated single mode laser diode is not perfectly Gaussian and will have other imperfections that affect the quality of the beam, e.g. asymmetry and transverse electromagnetic modes. Also, the beamsplitter and mirrors will have optical imperfections, the pixel output is not perfectly linear, nor is the ADC. Altogether, by normalising the pixel voltage output all these imperfections can be overcome to a great extent.

11.1.3 Optimising the required number of pixel rows and columns

11.1.3.1 Fringe line orientation when one or more fringe lines span the image sensor

Having implemented the normalisation methods above, all pixels after DSP will produce an output that is equally scaled over the full range of radiant flux that they individually capture. This is not only useful when one or more fringe lines are within the width of the image sensor, but it is essential when determining the fringe orientation and spacing when it exceeds the width of the image sensor.

The slope of the fringe lines on an interferogram are aligned with the tilt axis angle of the moving mirror. When the moving mirror is part of a dynamic system that has 6 degrees of freedom, the slope of the fringe lines will follow the mirror tilt axis angle throughout its 360º transition.

The horizontal and vertical pixel array of an image sensor is capable of capturing the fringe pattern in real time and with the aid of DSP, the mirror tilt axis angle can be derived.

Assume the image sensor shown in Figure 70 has only one row and one column of pixels and it is illuminated with the interferogram depicted in Figure 71. The DSP will process the row and column data and identify the positions of the maximum and minimum radiant flux instances as shown in Figure 72.

The DSP then determines the distance between the same number of maximum and minimum instances in the row and column and derives the tilt axis angle $\theta$ from the formula,

$$\theta = \tan^{-1}\frac{\sum y}{\sum x} = \tan^{-1}\frac{\Delta y}{\Delta x}$$

(11.1)

where $\Delta x$ and $\Delta y$ are one or several horizontal and vertical distances between the fringe lines and $n$ is the number of fringe lines used in the calculation. Once $\theta$ has been
calculated, the DSP then takes each maximum and minimum instance in turn along the horizontal array and using trigonometry, correlates it with a respective max/min instance up the vertical array to find the correct fringe line orientation.

Whenever the fringe lines are not centred with a maximum or minimum instance on the centre pixel, there is only one way they can be orientated. Figure 73 shows an interferogram where the fringe lines have the same spacing as Figure 72 but are orientated at angle minus $\theta$. It can be clearly seen from the figure that there is a misalignment of the fringe lines with the maximum and minimum instances that were captured by the vertical pixel array. Therefore, whenever the fringe lines are not centred about the centre pixel, their orientation can be derived using only one horizontal and one vertical line of pixels.

As the moving mirror translates and/or its tilt angle varies, there will be times when the fringe lines are perfectly centred on the centre pixel as pictured in Figure 74. With the fringe lines in this position, the captured locations of the maximum and minimum radiant flux are perfectly symmetrical about the centre of the image sensor as depicted in the figure and therefore the tilt axis angle $\theta$ could be the other way as seen in Figure 75. To overcome this anomaly, a further horizontal and vertical linear pixel array is required seen detailed in Figure 76. The alternative tilt axis angle that is presented in Figure 77 shows
misalignment of the fringe lines with the location of some max/min flux. Therefore this alternative can be eliminated from the two possibilities.

The same outcome of eliminating one of the two tilt axis angle alternatives that is realised using two horizontal and two vertical pixel arrays can also be achieve using 3 horizontal pixels arrays and one vertical, or vice versa. Depiction of this option is not illustrated.

The example above is an ideal case. In reality, purity of fringe formation is subject to a number of parameters such as imperfections in beam collimation, intensity profile of the laser beam, quality of the apodising neutral density filter, quality of the beamsplitter and mirror optics, stability of the laser, coherent length, etc. Therefore, the number of horizontal and vertical pixel array would be greater than 2 each as shown in the example. However, the number of linear pixel arrays required to accurately resolve the fringe spacing and fringe slope would be 3 or 4 orders of magnitude less than commonly available image sensors for capturing video. Therefore, a bespoke image sensor with a minimum number of horizontal and vertical pixel arrays being sampled simultaneously would allow extremely high frame rates to be achieved.
12 REFERENCES


References


Chapter 12


References


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58 Egismos Technology Corporation, 6025 Sussex Ave, STE#81140, Burnaby B.C., Canada V5H 3C2, from

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61  Thorlabs Inc., 56 Sparta Avenue, Newton, New Jersey 07860, USA, Model C560TME-B - f = 13.86 mm, NA = 0.18, Mounted Geltech Aspheric Lens, from https://www.thorlabs.com/thorproduct.cfm?partnumber=C560TME-B

APPENDIX A

Appendix A is a tabulation of all the experimentation test results (except for Test 4 tabulated in Section 8.2.1 above), the images of the interferograms and displacement of the optical lever beams projected onto graph paper.

The steps indicated in the tables refer to the steps in the methodology of capturing the data and calculating the components of the cube mirror position vectors and optical lever vectors given in Section 8.2.1 for the 16 tests.

### Table 17a) Zero position test, Y interferometer slightly misaligned, Steps 1 to 4.

<table>
<thead>
<tr>
<th>Interferometer</th>
<th>Step 1</th>
<th>Step 2</th>
<th>Step 3</th>
<th>Step 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Fringe slope</td>
<td>Fringe spacing mm</td>
<td>Mirror tilt angle</td>
<td>Fringe slope</td>
</tr>
<tr>
<td>X</td>
<td>$\varphi_x$</td>
<td>indet</td>
<td>$\infty$</td>
<td>$\alpha_x$</td>
</tr>
<tr>
<td>Y</td>
<td>$\varphi_y$</td>
<td>4.40</td>
<td>0.682</td>
<td>$\beta_y$</td>
</tr>
<tr>
<td>Z</td>
<td>$\varphi_z$</td>
<td>indet</td>
<td>$\infty$</td>
<td>$\gamma_z$</td>
</tr>
</tbody>
</table>

### Table 17b) Zero position test, Steps 5 & 6.

<table>
<thead>
<tr>
<th>Step 5</th>
<th>Step 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cube mirror $X, Y$ and $Z$ components with $\varphi_x, \varphi_y, \varphi_z$ after re-aligning $Y$</td>
<td>The most orthogonal permutation of cube mirror $X, Y$ and $Z$ components</td>
</tr>
<tr>
<td>$x$-axis</td>
<td>$x$-axis</td>
</tr>
<tr>
<td>$y$-axis</td>
<td>$y$-axis</td>
</tr>
<tr>
<td>$z$-axis</td>
<td>$z$-axis</td>
</tr>
<tr>
<td>X</td>
<td>-6.17258E-11</td>
</tr>
<tr>
<td>Y</td>
<td>0</td>
</tr>
<tr>
<td>Z</td>
<td>0</td>
</tr>
</tbody>
</table>
Experimentation Results

Table 17c): Zero position test, Step 7.

<table>
<thead>
<tr>
<th></th>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
<th></th>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{18x}$</td>
<td>0</td>
<td>-2123</td>
<td>0</td>
<td>$R_{18x_n}$</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$R_{18y}$</td>
<td>0</td>
<td>-2145</td>
<td>0</td>
<td>$R_{18y_n}$</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$R_{12z}$</td>
<td>0</td>
<td>-2119</td>
<td>0</td>
<td>$R_{12z_n}$</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 17d): Zero position test, Step 8.

<table>
<thead>
<tr>
<th></th>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
<th></th>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R'_{17x}$</td>
<td>0</td>
<td>0.70710678</td>
<td>0.70710678</td>
<td></td>
<td>0.70710678</td>
<td>-2.22045E-16</td>
<td>-2.22045E-16</td>
</tr>
<tr>
<td>$R'_{17y}$</td>
<td>0.70710678</td>
<td>0.70710678</td>
<td>-2.22045E-16</td>
<td></td>
<td>0.70710678</td>
<td>-2.22045E-16</td>
<td>0.70710678</td>
</tr>
<tr>
<td>$R'_{11z}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 17e): Zero position test, Step 9.

<table>
<thead>
<tr>
<th></th>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
<th></th>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N'_{T_x}$ positive</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td></td>
<td>1.57009E-16</td>
<td>-1.57009E-16</td>
<td>1</td>
</tr>
<tr>
<td>$N'_{T_y}$ positive</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td></td>
<td>1.57009E-16</td>
<td>-1.57009E-16</td>
<td>1</td>
</tr>
<tr>
<td>$N'_{T_z}$ positive</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td></td>
<td>1.57009E-16</td>
<td>-1.57009E-16</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
<th></th>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
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</thead>
<tbody>
<tr>
<td>$N'_{T_x}$ negative</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td></td>
<td>-1.57009E-16</td>
<td>1.57009E-16</td>
<td>-1</td>
</tr>
<tr>
<td>$N'_{T_y}$ negative</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td></td>
<td>1.57009E-16</td>
<td>-1.57009E-16</td>
<td>1</td>
</tr>
<tr>
<td>$N'_{T_z}$ negative</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td></td>
<td>1.57009E-16</td>
<td>-1.57009E-16</td>
<td>1</td>
</tr>
</tbody>
</table>
Table 17f): Zero position test, Step 10.

<table>
<thead>
<tr>
<th>Step 10</th>
<th>Use displacement of the optical lever beams relative to “zero position” to determine the correct solution from Step 9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N'_{T_x}$</td>
<td>$N'_{T_y}$</td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 17g): Zero position test, Step 11.

<table>
<thead>
<tr>
<th>Step 11</th>
<th>Dot product of virtual mirror vectors</th>
<th>Convert the components of the virtual mirrors to the actual optical lever tilting mirrors and use the dot product to select the most orthogonal pair of vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{T_x} \cdot N_{T_z}$</td>
<td>$N_{T_y} \cdot N_{T_z}$</td>
<td>$N_{T_x}/N_{T_y}$</td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>0</td>
<td>-1.57009E-16</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 17h): Zero position test, Step 12.

<table>
<thead>
<tr>
<th>Step 12</th>
<th>Reverse the direction of the selected X or Y optical lever normal vectors and take the cross product with the Z optical lever normal</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{T_z}$ (aligned with X cube mirror components)</td>
<td>$N_{T_x} \times N_{T_z}/N_{T_y}$ (aligned with Y cube mirror components)</td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
Figure 78: Zero position test, a) X interferometer interferogram, b) Y interferometer interferogram, c) Z interferometer interferogram, d) optical lever beam positions.
Table 18a): Test 1, Steps 1 – 4, determining the magnitude of Y interferometer misalignment.

<table>
<thead>
<tr>
<th>Interferometer</th>
<th>Step 1</th>
<th>Step 2</th>
<th>Step 3</th>
<th>Step 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Fringe slope</td>
<td>deg</td>
<td>Fringe spacing mm</td>
<td>Mirror tilt angle</td>
</tr>
<tr>
<td>X</td>
<td>$\varphi_x$</td>
<td>86.92</td>
<td>0.692</td>
<td>$\alpha_x$</td>
</tr>
<tr>
<td>Y</td>
<td>$\varphi_y$</td>
<td>indet</td>
<td>$\infty$</td>
<td>$\beta_y$</td>
</tr>
<tr>
<td>Z</td>
<td>$\varphi_z$</td>
<td>indet</td>
<td>$\infty$</td>
<td>$\gamma_z$</td>
</tr>
</tbody>
</table>

Table 18b): Test 1, Steps 5 & 6.

<table>
<thead>
<tr>
<th>Step 5</th>
<th>Cube mirror X, Y and Z components with $\varphi_x, \varphi_y, \varphi_z$ after re-aligning Y</th>
<th>Step 6</th>
<th>The most orthogonal permutation of cube mirror X, Y and Z components</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>X</td>
<td>0.99999988</td>
<td>0.000490175</td>
<td>-2.63E-05</td>
</tr>
<tr>
<td>Y</td>
<td>0.00049695</td>
<td>0.999999876</td>
<td>-3.82E-05</td>
</tr>
<tr>
<td>Z</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 18c): Test 1, Step 7.

<table>
<thead>
<tr>
<th>Step 7</th>
<th>Measured components of projected optical lever reflection vectors</th>
<th>Normalised components of optical lever reflection vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>x-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td>$R_{18x}$</td>
<td>-0.5</td>
<td>-2123</td>
</tr>
<tr>
<td>$R_{18y}$</td>
<td>-0.5</td>
<td>-2145</td>
</tr>
<tr>
<td>$R_{12z}$</td>
<td>12.5</td>
<td>-2119</td>
</tr>
</tbody>
</table>
Experimentation Results

Table 18d): Test 1, Step 8.

<table>
<thead>
<tr>
<th></th>
<th>$R'_{17x}$</th>
<th></th>
<th>$R'_{17y}$</th>
<th></th>
<th>$R'_{11z}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>0.000235516</td>
<td>y-axis</td>
<td>0.707273277</td>
<td>z-axis</td>
<td>0.706940207</td>
</tr>
<tr>
<td></td>
<td>0.707223312</td>
<td>x-axis</td>
<td>0.000164827</td>
<td>y-axis</td>
<td>0.70690212</td>
</tr>
<tr>
<td></td>
<td></td>
<td>z-axis</td>
<td>-0.70292332</td>
<td>x-axis</td>
<td>0.711265635</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>y-axis</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 18e): Test 1, Step 9.

<table>
<thead>
<tr>
<th></th>
<th>$N'_{Tx}$ positive</th>
<th></th>
<th>$N'_{Ty}$ positive</th>
<th></th>
<th>$N'_{Tz}$ positive</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>0.000166554</td>
<td>y-axis</td>
<td>0.000117744</td>
<td>z-axis</td>
<td>0.999999979</td>
</tr>
<tr>
<td></td>
<td>0.999999997</td>
<td>x-axis</td>
<td>8.24065E-05</td>
<td>y-axis</td>
<td>0.999999997</td>
</tr>
<tr>
<td></td>
<td>0.999999997</td>
<td>z-axis</td>
<td>-0.99999564</td>
<td>x-axis</td>
<td>0.002949466</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>y-axis</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>z-axis</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 18f): Test 1, Step 10.

<table>
<thead>
<tr>
<th></th>
<th>$N'_{Tx}$</th>
<th></th>
<th>$N'_{Ty}$</th>
<th></th>
<th>$N'_{Tz}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>-0.00016655</td>
<td>y-axis</td>
<td>-0.00011774</td>
<td>z-axis</td>
<td>-0.99999977</td>
</tr>
<tr>
<td></td>
<td>-0.99999997</td>
<td>x-axis</td>
<td>-8.24066E-05</td>
<td>y-axis</td>
<td>-0.99999997</td>
</tr>
<tr>
<td></td>
<td></td>
<td>z-axis</td>
<td>0.99995647</td>
<td>x-axis</td>
<td>-0.00294946</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>y-axis</td>
<td>0</td>
</tr>
</tbody>
</table>

Step 10

Use displacement of the optical lever beams relative to "zero position" to determine the correct solution from Step 9
Appendix A

Table 18g): Test 1, Step 11.

<table>
<thead>
<tr>
<th>Step 11</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Dot product of virtual mirror vectors</td>
<td>Convert the components of the virtual mirrors to the actual optical lever tilting mirrors and use the dot product to select the most orthogonal pair of vectors</td>
</tr>
<tr>
<td>$\mathbf{N}<em>{Tx} \cdot \mathbf{N}</em>{Tz}$</td>
<td>$\mathbf{N}<em>{Ty} \cdot \mathbf{N}</em>{Tz}$</td>
</tr>
<tr>
<td>-1.84996E-05</td>
<td>-9.14992E-06</td>
</tr>
<tr>
<td>-9.15629E-06</td>
<td>-1.29511E-05</td>
</tr>
</tbody>
</table>

Table 18h): Test 1, Step 12.

<table>
<thead>
<tr>
<th>Step 12</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Reverse the direction of the selected X or Y optical lever normal vectors and take the cross product with the Z optical lever normal</td>
<td></td>
</tr>
<tr>
<td>$\mathbf{N}_{Tz}$ (aligned with X cube mirror components)</td>
<td>$\mathbf{N}<em>{Tz} \times \mathbf{N}</em>{Tx}/\mathbf{N}<em>{Ty} = \mathbf{N}</em>{y}$ (aligned with Y cube mirror components)</td>
</tr>
<tr>
<td>$\mathbf{N}_{Tx}$ (aligned with X cube mirror components)</td>
<td>x-axis</td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td>0.999999879</td>
<td>-0.00049157</td>
</tr>
</tbody>
</table>
Figure 79: Test 1, a) X interferometer interferogram, b) Y interferometer interferogram, c) Z interferometer interferogram, d) optical lever beam positions.
Table 19a): Test 2, Steps 1 – 4, determining the magnitude of Y interferometer misalignment.

<table>
<thead>
<tr>
<th>Interferometer</th>
<th>Step 1</th>
<th>Step 2</th>
<th>Step 3</th>
<th>Step 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Fringe slope</td>
<td>Fringe spacing mm</td>
<td>Mirror tilt angle</td>
<td>deg</td>
</tr>
<tr>
<td>X</td>
<td>( \varphi_x )</td>
<td>-53.30</td>
<td>0.557</td>
<td>( \alpha_x )</td>
</tr>
<tr>
<td>Y</td>
<td>( \varphi_y )</td>
<td>indet</td>
<td>( \infty )</td>
<td>( \beta_y )</td>
</tr>
<tr>
<td>Z</td>
<td>( \varphi_z )</td>
<td>-87.55</td>
<td>0.815</td>
<td>( \gamma_z )</td>
</tr>
</tbody>
</table>

Table 19b): Test 2, Steps 5 & 6.

<table>
<thead>
<tr>
<th>Step 5</th>
<th>Step 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cube mirror X, Y and Z components with ( \varphi_x, \varphi_y, \varphi_z ) after re-aligning Y</td>
<td>Cube mirror X, Y and Z components with ( \varphi_x + \pi, \varphi_y + \pi, \varphi_z + \pi ) after re-aligning Y</td>
</tr>
<tr>
<td>The most orthogonal permutation of cube mirror X, Y and Z components</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Step 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured components of projected optical lever reflection vectors</td>
</tr>
<tr>
<td>Normalised components of optical lever reflection vectors</td>
</tr>
</tbody>
</table>

181
Experimentation Results

**Table 19d): Test 2, Step 8.**

<table>
<thead>
<tr>
<th></th>
<th>$R'_{17x}$</th>
<th></th>
<th>$R'_{17y}$</th>
<th></th>
<th>$R'_{11z}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>0.005181277</td>
<td>y-axis</td>
<td>0.707263803</td>
<td>z-axis</td>
<td>0.706930738</td>
</tr>
<tr>
<td>y-axis</td>
<td>0.71210046</td>
<td>y-axis</td>
<td>0.00049447</td>
<td>z-axis</td>
<td>0.07027741</td>
</tr>
<tr>
<td>z-axis</td>
<td>-0.70275164</td>
<td>y-axis</td>
<td>0.71142759</td>
<td>z-axis</td>
<td>0.00330336</td>
</tr>
</tbody>
</table>

**Table 19e): Test 2, Step 9.**

<table>
<thead>
<tr>
<th></th>
<th>$N'_{Tx}$ positive</th>
<th></th>
<th>$N'_{Ty}$ positive</th>
<th></th>
<th>$N'_{Tz}$ positive</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>0.003664148</td>
<td>y-axis</td>
<td>0.000111044</td>
<td>z-axis</td>
<td>0.999993281</td>
</tr>
<tr>
<td>y-axis</td>
<td>0.00354365</td>
<td>y-axis</td>
<td>0.00035089</td>
<td>z-axis</td>
<td>0.99999366</td>
</tr>
<tr>
<td>z-axis</td>
<td>-0.00306469</td>
<td>y-axis</td>
<td>0.00234303</td>
<td>z-axis</td>
<td>0.99999366</td>
</tr>
</tbody>
</table>

**Table 19f): Test 2, Step 10.**

<table>
<thead>
<tr>
<th></th>
<th>$N'_{Tx}$</th>
<th></th>
<th>$N'_{Ty}$</th>
<th></th>
<th>$N'_{Tz}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>-0.00366414</td>
<td>y-axis</td>
<td>-0.00011104</td>
<td>z-axis</td>
<td>-0.99999328</td>
</tr>
<tr>
<td>y-axis</td>
<td>-0.00354365</td>
<td>y-axis</td>
<td>-0.00035089</td>
<td>z-axis</td>
<td>-0.99999366</td>
</tr>
<tr>
<td>z-axis</td>
<td>0.99999256</td>
<td>y-axis</td>
<td>0.00306469</td>
<td>z-axis</td>
<td>0.99999366</td>
</tr>
</tbody>
</table>
Table 19g): Test 2, Step 11.

<table>
<thead>
<tr>
<th>Dot product of virtual mirror vectors</th>
<th>Convert the components of the virtual mirrors to the actual optical lever tilting mirrors and use the dot product to select the most orthogonal pair of vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_{T_x} \cdot N_{T_z} )</td>
<td>( N_{T_y} \cdot N_{T_z} )</td>
</tr>
<tr>
<td>-0.001320766</td>
<td>-0.00119953</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 19h): Test 2, Step 12.

<table>
<thead>
<tr>
<th>( N_{T_z} ) (aligned with X cube mirror components)</th>
<th>( N_{T_x} \times N_{T_x}/N_{T_y} = N_{y} ) (aligned with Y cube mirror components)</th>
<th>( N_{T_x}/N_{T_y} ) (aligned with Z cube mirror components)</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>0.999999793</td>
<td>-0.00051078</td>
<td>-0.00039050</td>
</tr>
</tbody>
</table>
Figure 80: Test 2, a) X interferometer interferogram, b) Y interferometer interferogram, c) Z interferometer interferogram, d) optical lever beam positions.
Table 20a): Test 3, Steps 1 – 4, determining the magnitude of Y interferometer misalignment.

<table>
<thead>
<tr>
<th>Interferometer</th>
<th>Step 1</th>
<th>Step 2</th>
<th>Step 3</th>
<th>Step 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Fringe slope</td>
<td>Fringe spacing mm</td>
<td>Mirror tilt angle</td>
<td>deg°</td>
</tr>
<tr>
<td>X</td>
<td>φ_x</td>
<td>indet</td>
<td>α_x</td>
<td>0</td>
</tr>
<tr>
<td>Y</td>
<td>φ_y</td>
<td>-70.86</td>
<td>β_y</td>
<td>0.084</td>
</tr>
<tr>
<td>Z</td>
<td>φ_z</td>
<td>-3.750</td>
<td>γ_z</td>
<td>0.078</td>
</tr>
</tbody>
</table>

Cube mirror X, Y and Z components with φ_x, φ_y, φ_z before re-aligning Y

<table>
<thead>
<tr>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00000000</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Cube mirror X, Y and Z components with φ_x + π, φ_y + π, φ_z + π before re-aligning Y

<table>
<thead>
<tr>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.009977517</td>
<td>0.999998615</td>
<td>0.001347032</td>
</tr>
</tbody>
</table>

Table 20b): Test 3, Steps 5 & 6.

<table>
<thead>
<tr>
<th>Step 5</th>
<th>Step 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cube mirror X, Y and Z components with φ_x, φ_y, φ_z after re-aligning Y</td>
<td>The most orthogonal permutation of cube mirror X, Y and Z components</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00000000</td>
<td>0</td>
<td>0</td>
<td>1.00000000</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-0.00014244</td>
<td>-0.00212780</td>
<td>-0.99999726</td>
<td>-8.98357E-05</td>
<td>-0.00136025</td>
<td>0.999999071</td>
</tr>
</tbody>
</table>

Table 20c): Test 3, Step 7.

<table>
<thead>
<tr>
<th>Step 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured components of projected optical lever reflection vectors</td>
</tr>
<tr>
<td>Normalised components of optical lever reflection vectors</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>R_{18x}</td>
<td>-1.5</td>
<td>-2123</td>
<td>-56</td>
<td>R_{18x}</td>
<td>-0.0007063</td>
</tr>
<tr>
<td>R_{18y}</td>
<td>-29.5</td>
<td>-2145</td>
<td>-28</td>
<td>R_{18y}</td>
<td>-0.0137504</td>
</tr>
<tr>
<td>R_{12z}</td>
<td>0</td>
<td>-2119</td>
<td>-1.5</td>
<td>R_{12z}</td>
<td>0</td>
</tr>
</tbody>
</table>
Experimentation Results

Table 20d): Test 3, Step 8.

<table>
<thead>
<tr>
<th>R'_{17x}</th>
<th>R'_{17y}</th>
<th>R'_{11z}</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>0.0007063</td>
<td>0.7255061</td>
<td>0.6882153</td>
</tr>
<tr>
<td>0.7073293</td>
<td>0.0189517</td>
<td>0.7071066</td>
</tr>
<tr>
<td>0.7066301</td>
<td>-0.7071066</td>
<td>0.0007079</td>
</tr>
</tbody>
</table>

Table 20e): Test 3, Step 9.

<table>
<thead>
<tr>
<th>N'_T_x positive</th>
<th>N'_T_y positive</th>
<th>N'_T_z positive</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>0.00050615</td>
<td>0.01318531</td>
<td>0.99991294</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>N'_T_x negative</th>
<th>N'_T_y negative</th>
<th>N'_T_z negative</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>-0.00050614</td>
<td>-0.01318531</td>
<td>-0.99991294</td>
</tr>
</tbody>
</table>

Table 20f): Test 3, Step 10.

<table>
<thead>
<tr>
<th>N'_T_x</th>
<th>N'_T_y</th>
<th>N'_T_z</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>-0.00050614</td>
<td>-0.01318531</td>
<td>-0.99991294</td>
</tr>
</tbody>
</table>
### Appendix A

#### Table 20g): Test 3, Step 11.

<table>
<thead>
<tr>
<th>Dot product of virtual mirror vectors</th>
<th>Convert the components of the virtual mirrors to the actual optical lever tilting mirrors and use the dot product to select the most orthogonal pair of vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{Tx} \cdot N_{Tz}$</td>
<td>$N_{Ty} \cdot N_{Tz}$</td>
</tr>
<tr>
<td>-5.64614E-06</td>
<td>0.000343127</td>
</tr>
<tr>
<td>$N_{Tx}/N_{Ty}$</td>
<td>$N_{Tz}$</td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td>-5.6239E-05</td>
<td>-1.4650E-03</td>
</tr>
<tr>
<td>1.00000000</td>
<td>2.0879E-08</td>
</tr>
</tbody>
</table>

#### Table 20h): Test 3, Step 12.

<table>
<thead>
<tr>
<th>$N_{Tz}$ (aligned with X cube mirror components)</th>
<th>$N_{Tz} \times N_{Tx}/N_{Ty}$ (aligned with Y cube mirror components)</th>
<th>$N_{Tz}/N_{Ty}$ (aligned with Z cube mirror components)</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>0.999999997</td>
<td>2.0879E-08</td>
<td>-8.34246E-05</td>
</tr>
</tbody>
</table>
Figure 81: Test 3, a) X interferometer interferogram, b) Y interferometer interferogram, c) Z interferometer interferogram, d) optical lever beam positions.
### Appendix A

#### Table 21a: Test 5, Steps 1 – 4, determining the magnitude of Y interferometer misalignment.

<table>
<thead>
<tr>
<th>Interferometer</th>
<th>Step 1</th>
<th>Step 2</th>
<th>Step 3</th>
<th>Step 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Fringe slope</td>
<td>Fringe spacing</td>
<td>Mirror tilt angle</td>
<td>Cube mirror X, Y and Z components with $\varphi_x$, $\varphi_y$, $\varphi_z$ before re-aligning Y</td>
</tr>
<tr>
<td>X</td>
<td>$\varphi_x$ = 89.71</td>
<td>0.156</td>
<td>$\alpha_x$ = 0.1242</td>
<td>0.999997647</td>
</tr>
<tr>
<td>Y</td>
<td>$\varphi_y$ = -1.530</td>
<td>0.199</td>
<td>$\beta_y$ = 0.0974</td>
<td>-0.00120337</td>
</tr>
<tr>
<td>Z</td>
<td>$\varphi_z$ = 30.12</td>
<td>2.510</td>
<td>$\gamma_z$ = 0.0077</td>
<td>6.79023E-05</td>
</tr>
</tbody>
</table>

#### Table 21b: Test 5, Steps 5 & 6.

<table>
<thead>
<tr>
<th>Step 5</th>
<th>Step 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cube mirror X, Y and Z components with $\varphi_x$, $\varphi_y$, $\varphi_z$ after re-aligning Y</td>
<td>The most orthogonal permutation of cube mirror X, Y and Z components</td>
</tr>
<tr>
<td>Cube mirror X, Y and Z components with $\varphi_x + \pi$, $\varphi_y + \pi$, $\varphi_z + \pi$ after re-aligning Y</td>
<td></td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td>--------</td>
<td>--------</td>
</tr>
<tr>
<td>X</td>
<td>0.999997611</td>
</tr>
<tr>
<td>Y</td>
<td>0.000121826</td>
</tr>
<tr>
<td>Z</td>
<td>0.00010767</td>
</tr>
</tbody>
</table>

#### Table 21c: Test 5, Step 7.

<table>
<thead>
<tr>
<th>Step 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured components of projected optical lever reflection vectors</td>
</tr>
<tr>
<td>x-axis</td>
</tr>
<tr>
<td>--------</td>
</tr>
<tr>
<td>$R_{18x}$</td>
</tr>
<tr>
<td>$R_{18y}$</td>
</tr>
<tr>
<td>$R_{12z}$</td>
</tr>
</tbody>
</table>
Experimentation Results

### Table 21d): Test 5, Step 8.

**Step 8**
Normalised optical lever components and steering mirror components substituted into Equation (6.61) to produce $\mathbf{R'}_{17}/\mathbf{R'}_{11}$ reflection vectors

<table>
<thead>
<tr>
<th>R'_{17x}</th>
<th>R'_{17y}</th>
<th>R'_{11z}</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>0.000235516</td>
<td>0.707439753</td>
<td>0.706773614</td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>0.706990212</td>
<td>0.000164827</td>
<td>0.707223312</td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>-0.68817939</td>
<td>0.725540568</td>
<td>0</td>
</tr>
</tbody>
</table>

### Table 21e): Test 5, Step 9.

**Step 9**
Use Equations (6.71) – (6.73) find both solutions of virtual mirror normal vectors

<table>
<thead>
<tr>
<th>N'_{Tx} positive</th>
<th>N'_{Ty} positive</th>
<th>N'_{Tz} positive</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>0.000166574</td>
<td>0.000235502</td>
<td>0.999999958</td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>-8.24202E-05</td>
<td>0.000116541</td>
<td>0.999999999</td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>-0.99991274</td>
<td>0.013210321</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>N'_{Tx} negative</th>
<th>N'_{Ty} negative</th>
<th>N'_{Tz} negative</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>-0.00016657</td>
<td>-0.00023550</td>
<td>-0.999999955</td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>8.24202E-05</td>
<td>-0.000116545</td>
<td>-0.999999999</td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>0.99991274</td>
<td>-0.013210321</td>
<td>0</td>
</tr>
</tbody>
</table>

### Table 21f): Test 5, Step 10.

**Step 10**
Use displacement of the optical lever beams relative to "zero position" to determine the correct solution from Step 9

<table>
<thead>
<tr>
<th>N'_{Tx}</th>
<th>N'_{Ty}</th>
<th>N'_{Tz}</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>-0.00016657</td>
<td>-0.00023550</td>
<td>-0.99999995</td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>8.24202E-05</td>
<td>-0.000116545</td>
<td>-0.999999999</td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>0.99991274</td>
<td>-0.013210321</td>
<td>0</td>
</tr>
</tbody>
</table>
Appendix A

Table 21g): Test 5, Step 11.

<table>
<thead>
<tr>
<th>Dot product of virtual mirror vectors</th>
<th>Convert the components of the virtual mirrors to the actual optical lever tilting mirrors and use the dot product to select the most orthogonal pair of vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{N}<em>{T_x} \cdot \mathbf{N}</em>{T_z}$</td>
<td>$\mathbf{N}<em>{T_y} \cdot \mathbf{N}</em>{T_z}$</td>
</tr>
<tr>
<td>-0.000163448</td>
<td>8.39525E-05</td>
</tr>
</tbody>
</table>

Table 21h): Test 5, Step 12.

<table>
<thead>
<tr>
<th>$\mathbf{N}_{T_z}$ (aligned with X cube mirror components)</th>
<th>$\mathbf{N}<em>{T_x}/\mathbf{N}</em>{T_y}$ (aligned with Y cube mirror components)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{N}_{T_z}$ (aligned with X cube mirror components)</td>
<td>$\mathbf{N}<em>{T_x}/\mathbf{N}</em>{T_y}$ (aligned with Z cube mirror components)</td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td>0.999997576</td>
<td>-0.00220172</td>
</tr>
</tbody>
</table>
Figure 82: Test 5, a) X interferometer interferogram, b) Y interferometer interferogram, c) Z interferometer interferogram, d) optical lever beam positions.
Table 22a): Test 6, Steps 1 – 4, determining the magnitude of Y interferometer misalignment.

<table>
<thead>
<tr>
<th>Interferometer</th>
<th>Step 1</th>
<th>Step 2</th>
<th>Step 3</th>
<th>Step 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Fringe slope</td>
<td>deg°</td>
<td>Fringe spacing mm</td>
<td>Mirror tilt angle</td>
</tr>
<tr>
<td>X</td>
<td>$\varphi_x$</td>
<td>90</td>
<td>0.129</td>
<td>$\alpha_x$</td>
</tr>
<tr>
<td>Y</td>
<td>$\varphi_y$</td>
<td>1.91</td>
<td>0.108</td>
<td>$\beta_y$</td>
</tr>
<tr>
<td>Z</td>
<td>$\varphi_z$</td>
<td>indet</td>
<td>$\infty$</td>
<td>$\gamma_z$</td>
</tr>
</tbody>
</table>

Table 22b): Test 6, Steps 5 & 6.

<table>
<thead>
<tr>
<th>Step 5</th>
<th>Step 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cube mirror X, Y and Z components with $\varphi_x, \varphi_y, \varphi_z$ after re-aligning Y</td>
<td>The most orthogonal permutation of cube mirror X, Y and Z components</td>
</tr>
<tr>
<td>---------------</td>
<td>---------------</td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td>X</td>
<td>0.999996477</td>
</tr>
<tr>
<td>Y</td>
<td>-0.00266813</td>
</tr>
<tr>
<td>Z</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 22c): Test 6, Step 7.

<table>
<thead>
<tr>
<th>Step 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured components of projected optical lever reflection vectors</td>
</tr>
<tr>
<td>x-axis</td>
</tr>
<tr>
<td>$R_{18x}$</td>
</tr>
<tr>
<td>$R_{18y}$</td>
</tr>
<tr>
<td>$R_{12z}$</td>
</tr>
</tbody>
</table>
Experimentation Results

Table 22d): Test 6, Step 8.

<table>
<thead>
<tr>
<th>Step 8</th>
<th>Normalised optical lever components and steering mirror components substituted into Equation (6.61) to produce $\mathbf{R}'<em>{17}/\mathbf{R}'</em>{11}$ reflection vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\mathbf{R}'_{17x}$</td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td>0.000235516</td>
<td>0.706940207</td>
</tr>
</tbody>
</table>

Table 22e): Test 6, Step 9.

<table>
<thead>
<tr>
<th>Step 9</th>
<th>Use Equations (6.71) − (6.73) find both solutions of virtual mirror normal vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\mathbf{N}'_{Tx}$ positive</td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td>0.000166515</td>
<td>-0.000117772</td>
</tr>
<tr>
<td>$\mathbf{N}'_{Tx}$ negative</td>
<td>$\mathbf{N}'_{Ty}$ negative</td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td>-0.00016651</td>
<td>0.000117772</td>
</tr>
</tbody>
</table>

Table 22f): Test 6, Step 10.

<table>
<thead>
<tr>
<th>Step 10</th>
<th>Use displacement of the optical lever beams relative to &quot;zero position&quot; to determine the correct solution from Step 9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{N}'_{Tx}$</td>
<td>$\mathbf{N}'_{Ty}$</td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td>-0.00016651</td>
<td>0.000117772</td>
</tr>
</tbody>
</table>
Appendix A

Table 22g): Test 6, Step 11.

<table>
<thead>
<tr>
<th>Dot product of virtual mirror vectors</th>
<th>Convert the components of the virtual mirrors to the actual optical lever tilting mirrors and use the dot product to select the most orthogonal pair of vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{T_x} \cdot N_{T_z}$</td>
<td>$N_{T_y} \cdot N_{T_z}$</td>
</tr>
<tr>
<td>3.62441E-05</td>
<td>6.39032E-05</td>
</tr>
<tr>
<td>-1.85017E-05</td>
<td>1.30857E-05</td>
</tr>
</tbody>
</table>

Table 22h): Test 6, Step 12.

<table>
<thead>
<tr>
<th>$N_{T_z}$ (aligned with X cube mirror components)</th>
<th>$N_{T_z} \times N_{T_x}/N_{T_y} = N_y$ (aligned with Y cube mirror components)</th>
<th>$N_{T_x}/N_{T_y}$ (aligned with Z cube mirror components)</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>0.999996373</td>
<td>0.002692835</td>
<td>-5.47105E-05</td>
</tr>
</tbody>
</table>
Figure 83: Test 6, a) X interferometer interferogram, b) Y interferometer interferogram, c) Z interferometer interferogram, d) optical lever beam positions.
Appendix A

Table 23a): Test 7, Steps 1 – 4, determining the magnitude of Y interferometer misalignment.

<table>
<thead>
<tr>
<th>Interferometer 0–meter</th>
<th>Step 1</th>
<th>Step 2</th>
<th>Step 3</th>
<th>Step 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Fringe slope</td>
<td>Fringe spacing mm</td>
<td>Mirror tilt angle</td>
<td>deg°</td>
</tr>
<tr>
<td>X</td>
<td>φ_x</td>
<td>-63.43</td>
<td>0.060</td>
<td>α_x</td>
</tr>
<tr>
<td>Y</td>
<td>φ_y</td>
<td>42.96</td>
<td>0.055</td>
<td>β_y</td>
</tr>
<tr>
<td>Z</td>
<td>φ_z</td>
<td>-33.47</td>
<td>0.069</td>
<td>γ_z</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Step 3</th>
<th>Cube mirror X, Y and Z components with φ_x, φ_y, φ_z before re-aligning Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>z-axis</td>
<td>x-axis</td>
</tr>
</tbody>
</table>

Table 23b): Test 7, Steps 5 & 6.

<table>
<thead>
<tr>
<th>Step 5</th>
<th>Cube mirror X, Y and Z components with φ_x, φ_y, φ_z after re-aligning Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td>0.9999790</td>
<td>-0.00510630</td>
</tr>
<tr>
<td>0.999972735</td>
<td>0.00619842</td>
</tr>
<tr>
<td>0.999970654</td>
<td>0.004268088</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Step 6</th>
<th>The most orthogonal permutation of cube mirror X, Y and Z components</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td>0.999983951</td>
<td>-0.00506743</td>
</tr>
<tr>
<td>0.999978911</td>
<td>0.004958454</td>
</tr>
<tr>
<td>0.999988106</td>
<td>0.002691724</td>
</tr>
</tbody>
</table>

Table 23c): Test 7, Step 7.

<table>
<thead>
<tr>
<th>Step 7</th>
<th>Measured components of projected optical lever reflection vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Normalised components of optical lever reflection vectors</td>
</tr>
<tr>
<td></td>
<td>x-axis</td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td>-69.5</td>
<td>-2123</td>
</tr>
<tr>
<td>-153</td>
<td>-2145</td>
</tr>
<tr>
<td>131</td>
<td>-2119</td>
</tr>
</tbody>
</table>

| x-axis                        | y-axis | z-axis | x-axis | y-axis | z-axis | x-axis | y-axis | z-axis |
| -99.66059                    | -0.0755787 | 0.06169086 | -0.9978851 | -0.0204851 |
Table 23d): Test 7, Step 8.

**Step 8**
Normalised optical lever components and steering mirror components substituted into Equation (6.61) to produce \( R'_{17x} \), \( R'_{17y} \), \( R'_{11z} \) reflection vectors

<table>
<thead>
<tr>
<th></th>
<th>( R'_{17x} )</th>
<th>( R'_{17y} )</th>
<th>( R'_{11z} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>0.032625584</td>
<td>0.738438617</td>
<td>-0.66198927</td>
</tr>
<tr>
<td>y-axis</td>
<td>0.758149034</td>
<td>0.053761118</td>
<td>0.749233323</td>
</tr>
<tr>
<td>z-axis</td>
<td>0.651264626</td>
<td>0.672174198</td>
<td>0.020485135</td>
</tr>
</tbody>
</table>

Table 23e): Test 7, Step 9.

**Step 9**
Use Equations (6.71) – (6.73) find both solutions of virtual mirror normal vectors

\[
N'_{Tx} \text{ positive} = \begin{bmatrix} x-axis \n y-axis \n z-axis \end{bmatrix} = \begin{bmatrix} 0.02399431 \\ 0.03753875 \\ 0.99900706 \end{bmatrix}
\]

\[
N'_{Ty} \text{ positive} = \begin{bmatrix} x-axis \n y-axis \n z-axis \end{bmatrix} = \begin{bmatrix} 0.02269298 \\ 0.039838036 \\ 0.998983913 \end{bmatrix}
\]

\[
N'_{Tz} \text{ positive} = \begin{bmatrix} x-axis \n y-axis \n z-axis \end{bmatrix} = \begin{bmatrix} -0.999415191 \\ 0.030751609 \\ 0.014953775 \end{bmatrix}
\]

\[
N'_{Tx} \text{ negative} = \begin{bmatrix} x-axis \n y-axis \n z-axis \end{bmatrix} = \begin{bmatrix} -0.02399431 \\ -0.03753875 \\ -0.99900706 \end{bmatrix}
\]

\[
N'_{Ty} \text{ negative} = \begin{bmatrix} x-axis \n y-axis \n z-axis \end{bmatrix} = \begin{bmatrix} -0.02269298 \\ -0.039838036 \\ -0.998983913 \end{bmatrix}
\]

\[
N'_{Tz} \text{ negative} = \begin{bmatrix} x-axis \n y-axis \n z-axis \end{bmatrix} = \begin{bmatrix} 0.999415191 \\ -0.030751609 \\ -0.014953775 \end{bmatrix}
\]

Table 23f): Test 7, Step 10.

**Step 10**
Use displacement of the optical lever beams relative to “zero position” to determine the correct solution from Step 9

\[
N'_{Tx} = \begin{bmatrix} x-axis \n y-axis \n z-axis \end{bmatrix} = \begin{bmatrix} -0.02399431 \\ -0.03753875 \\ -0.99900706 \end{bmatrix}
\]

\[
N'_{Ty} = \begin{bmatrix} x-axis \n y-axis \n z-axis \end{bmatrix} = \begin{bmatrix} -0.02269298 \\ -0.039838036 \\ -0.998983913 \end{bmatrix}
\]

\[
N'_{Tz} = \begin{bmatrix} x-axis \n y-axis \n z-axis \end{bmatrix} = \begin{bmatrix} 0.999415191 \\ -0.030751609 \\ -0.014953775 \end{bmatrix}
\]
### Appendix A

#### Table 23g): Test 7, Step 11.

<table>
<thead>
<tr>
<th>Dot product of virtual mirror vectors</th>
<th>Convert the components of the virtual mirrors to the actual optical lever tilting mirrors and use the dot product to select the most orthogonal pair of vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_{Tx} \cdot N_{Tz} )</td>
<td>( N_{Ty} \cdot N_{Tz} )</td>
</tr>
<tr>
<td>-0.000152349</td>
<td>-6.96273E-06</td>
</tr>
<tr>
<td>-0.00252144</td>
<td>-0.00432645</td>
</tr>
</tbody>
</table>

#### Table 23h): Test 7, Step 12.

<table>
<thead>
<tr>
<th>( N_{Tz} ) (aligned with X cube mirror components)</th>
<th>( N_{Tz} \times \frac{N_{Tx}}{N_{Ty}} = N_{y} ) (aligned with Y cube mirror components)</th>
<th>( \frac{N_{Tx}}{N_{Tz}} ) (aligned with Z cube mirror components)</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>0.99998376</td>
<td>-0.00512527</td>
<td>-0.00249229</td>
</tr>
</tbody>
</table>
Figure 84: Test 7, a) X interferometer interferogram, b) Y interferometer interferogram, c) Z interferometer interferogram, d) optical lever beam positions.
Table 24a): Test 8, Steps 1 – 4, determining the magnitude of Y interferometer misalignment.

<table>
<thead>
<tr>
<th>Interferometer</th>
<th>Step 1</th>
<th>Step 2</th>
<th>Step 3</th>
<th>Step 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Fringe slope</td>
<td>Fringe spacing mm</td>
<td>Mirror tilt angle</td>
<td>deg°</td>
</tr>
<tr>
<td>X</td>
<td>( \varphi_x )</td>
<td>83.82</td>
<td>0.746</td>
<td>( \alpha_x )</td>
</tr>
<tr>
<td>Y</td>
<td>( \varphi_y )</td>
<td>indet</td>
<td>( \infty )</td>
<td>( \beta_y )</td>
</tr>
<tr>
<td>Z</td>
<td>( \varphi_z )</td>
<td>indet</td>
<td>( \infty )</td>
<td>( \gamma_z )</td>
</tr>
</tbody>
</table>

Table 24b): Test 8, Steps 5 & 6.

| X              | \( \varphi_x \) | 0.9999999893 | 0.000456039 | -7.70944E-05 |
| Y              | \( \varphi_y \) | 0.00050309 | 0.999999872 | -5.76402E-05 |
| Z              | \( \varphi_z \) | 0 | 0 | 1 |

Table 24c): Test 8, Step 7.

<table>
<thead>
<tr>
<th>Step 7</th>
<th>Measured components of projected optical lever reflection vectors</th>
<th>Normalised components of optical lever reflection vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>( R_{18x} )</td>
<td>1</td>
<td>-2123</td>
</tr>
<tr>
<td>( R_{18y} )</td>
<td>0.5</td>
<td>-2145</td>
</tr>
<tr>
<td>( R_{12z} )</td>
<td>12.5</td>
<td>-2119</td>
</tr>
</tbody>
</table>
Experimentation Results

Table 24d): Test 8, Step 8.

<table>
<thead>
<tr>
<th></th>
<th>$R'_{17x}$</th>
<th></th>
<th>$R'_{17y}$</th>
<th></th>
<th>$R'_{11z}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>-0.00047103</td>
<td>y-axis</td>
<td>0.708437724</td>
<td>z-axis</td>
<td>0.705773172</td>
</tr>
<tr>
<td></td>
<td>0.706173421</td>
<td>x-axis</td>
<td>0.000988959</td>
<td>y-axis</td>
<td>0.70803822</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>-0.70292324</td>
<td>x-axis</td>
<td>0.711265556</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>y-axis</td>
<td>-0.00047191</td>
</tr>
</tbody>
</table>

Table 24e): Test 8, Step 9.

Step 9

Use Equations (6.71) – (6.73) find both solutions of virtual mirror normal vectors.

<table>
<thead>
<tr>
<th></th>
<th>$N_{Tx}^{'}$ positive</th>
<th></th>
<th>$N_{Ty}^{'}$ positive</th>
<th></th>
<th>$N_{Tz}^{'}$ positive</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>-0.000333383</td>
<td>y-axis</td>
<td>0.000942006</td>
<td>z-axis</td>
<td>0.999999501</td>
</tr>
<tr>
<td></td>
<td>-0.00065955</td>
<td>x-axis</td>
<td>0.000698839</td>
<td>y-axis</td>
<td>0.999999538</td>
</tr>
<tr>
<td></td>
<td>-0.99999559</td>
<td>x-axis</td>
<td>0.00294941</td>
<td>y-axis</td>
<td>-0.00033468</td>
</tr>
<tr>
<td></td>
<td>0.000333383</td>
<td>y-axis</td>
<td>-0.00094200</td>
<td>x-axis</td>
<td>0.999999554</td>
</tr>
<tr>
<td></td>
<td>-0.000698833</td>
<td></td>
<td>-0.99999953</td>
<td></td>
<td>-0.00294941</td>
</tr>
<tr>
<td></td>
<td>0.000334681</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 24f): Test 8, Step 10.

Step 10

Use displacement of the optical lever beams relative to "zero position" to determine the correct solution from Step 9.

<table>
<thead>
<tr>
<th></th>
<th>$N_{Tx}^{'}$</th>
<th></th>
<th>$N_{Ty}^{'}$</th>
<th></th>
<th>$N_{Tz}^{'}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>0.000333383</td>
<td>y-axis</td>
<td>-0.00094200</td>
<td>z-axis</td>
<td>-0.99999950</td>
</tr>
<tr>
<td></td>
<td>0.000659551</td>
<td>x-axis</td>
<td>-0.00069883</td>
<td>z-axis</td>
<td>-0.99999953</td>
</tr>
<tr>
<td></td>
<td>0.999995954</td>
<td>x-axis</td>
<td>-0.00294941</td>
<td>y-axis</td>
<td>0.000334681</td>
</tr>
<tr>
<td></td>
<td>-0.000334681</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**Table 24g): Test 8, Step 11.**

<table>
<thead>
<tr>
<th>Dot product of virtual mirror vectors</th>
<th>Convert the components of the virtual mirrors to the actual optical lever tilting mirrors and use the dot product to select the most orthogonal pair of vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{Tx} \cdot N_{Tz}$</td>
<td>$N_{Ty} \cdot N_{Tz}$</td>
</tr>
<tr>
<td>-1.86862E-05</td>
<td>1.75414E-05</td>
</tr>
<tr>
<td>7.32834E-05</td>
<td>-7.76488E-05</td>
</tr>
</tbody>
</table>

**Table 24h): Test 8, Step 12.**

<table>
<thead>
<tr>
<th>Reverse the direction of the selected X or Y optical lever normal vectors and take the cross product with the Z optical lever normal</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{Tz}$ (aligned with X cube mirror components)</td>
</tr>
<tr>
<td>x-axis</td>
</tr>
<tr>
<td>0.999999878</td>
</tr>
</tbody>
</table>

203
Figure 85: Test 8, a) X interferometer interferogram, b) Y interferometer interferogram, c) Z interferometer interferogram, d) optical lever beam positions.
### Appendix A

**Table 25a):** Test 9, zero position, Steps 1 – 4, determining the magnitude of Y interferometer misalignment.

<table>
<thead>
<tr>
<th>Interferometer</th>
<th>Step 1</th>
<th>Step 2</th>
<th>Step 3</th>
<th>Step 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Fringe slope</td>
<td>Fringe spacing mm</td>
<td>Mirror tilt angle</td>
<td>deg.</td>
</tr>
<tr>
<td>X</td>
<td>$\varphi_x$</td>
<td>62.57</td>
<td>3.784</td>
<td>$\alpha_x$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td>$\varphi_y$</td>
<td>9.43</td>
<td>0.633</td>
<td>$\beta_y$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z</td>
<td>$\varphi_z$</td>
<td>34.36</td>
<td>3.192</td>
<td>$\gamma_z$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table 25b):** Test 9, zero position, Steps 5 & 6.

<table>
<thead>
<tr>
<th>Step 5</th>
<th>Step 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cube mirror X, Y and Z components with $\varphi_x$, $\varphi_y$, $\varphi_z$ after re-aligning Y</td>
<td>Cube mirror X, Y and Z components with $\varphi_x + \pi$, $\varphi_y + \pi$, $\varphi_z + \pi$ after re-aligning Y</td>
</tr>
<tr>
<td>The most orthogonal permutation of cube mirror X, Y and Z components</td>
<td></td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td>--------</td>
<td>--------</td>
</tr>
<tr>
<td>X</td>
<td>0.999999995</td>
</tr>
<tr>
<td>Y</td>
<td>-3.29471E-05</td>
</tr>
<tr>
<td>Z</td>
<td>9.52624E-05</td>
</tr>
</tbody>
</table>

**Table 25c):** Test 9, zero position, Step 7.

**Step 7**

<table>
<thead>
<tr>
<th>Measured components of projected optical lever reflection vectors</th>
<th>Normalised components of optical lever reflection vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td>--------</td>
<td>--------</td>
</tr>
<tr>
<td>$R_{18x}$</td>
<td>-0.5</td>
</tr>
<tr>
<td>$R_{18y}$</td>
<td>0</td>
</tr>
<tr>
<td>$R_{12z}$</td>
<td>-1</td>
</tr>
</tbody>
</table>
Experimentation Results

Table 25d): Test 9, zero position, Step 8.

<table>
<thead>
<tr>
<th>R'_{17x}</th>
<th>R'_{17y}</th>
<th>R'_{11z}</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>0.000235516</td>
<td>0.706940207</td>
<td>0.707273277</td>
</tr>
</tbody>
</table>

Table 25e): Test 9, zero position, Step 9.

<table>
<thead>
<tr>
<th>N'_T_x positive</th>
<th>N'_T_y positive</th>
<th>N'_T_z positive</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>0.000166515</td>
<td>-0.00011777</td>
<td>0.999999979</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>N'_T_x negative</th>
<th>N'_T_y negative</th>
<th>N'_T_z negative</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>-0.00016652</td>
<td>0.000117772</td>
<td>-0.99999998</td>
</tr>
</tbody>
</table>

Table 25f): Test 9, zero position, Step 10.

<table>
<thead>
<tr>
<th>N'_T_x</th>
<th>N'_T_y</th>
<th>N'_T_z</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>-0.00016652</td>
<td>0.000117772</td>
<td>-0.99999998</td>
</tr>
</tbody>
</table>
### Table 25g): Test 9, zero position, Step 11.

<table>
<thead>
<tr>
<th>Dot product of virtual mirror vectors</th>
<th>Convert the components of the virtual mirrors to the actual optical lever tilting mirrors and use the dot product to select the most orthogonal pair of vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{N}<em>{T_x} \cdot \mathbf{N}</em>{T_z} )</td>
<td>( \mathbf{N}<em>{T_y} \cdot \mathbf{N}</em>{T_z} )</td>
</tr>
<tr>
<td>3.71021E-05</td>
<td>4.64475E-05</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 25h): Test 9, zero position, Step 12.

<table>
<thead>
<tr>
<th>( \mathbf{N}_{T_z} ) (aligned with X cube mirror components)</th>
<th>( \mathbf{N}<em>{T_z} \times \mathbf{N}</em>{T_x}/\mathbf{N}_{T_y} = \mathbf{N}_y ) (aligned with Y cube mirror components)</th>
<th>( \mathbf{N}<em>{T_x}/\mathbf{N}</em>{T_y} ) (aligned with Z cube mirror components)</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>0.999999998</td>
<td>3.9336E-05</td>
<td>-5.56033E-05</td>
</tr>
</tbody>
</table>
Figure 86: Test 9, a) X interferometer interferogram, b) Y interferometer interferogram, c) Z interferometer interferogram, d) optical lever beam positions.
Table 26a): Test 10, Steps 1 – 4, determining the magnitude of Y interferometer misalignment.

<table>
<thead>
<tr>
<th>Interferometer</th>
<th>Step 1</th>
<th>Step 2</th>
<th>Step 3</th>
<th>Step 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Fringe slope</td>
<td>Fringe spacing mm</td>
<td>Mirror tilt angle</td>
<td>deg°</td>
</tr>
<tr>
<td>X</td>
<td>( \varphi_x )</td>
<td>-46.55</td>
<td>0.046</td>
<td>( \alpha_x )</td>
</tr>
<tr>
<td>Y</td>
<td>( \varphi_y )</td>
<td>23.82</td>
<td>0.064</td>
<td>( \beta_y )</td>
</tr>
<tr>
<td>Z</td>
<td>( \varphi_z )</td>
<td>-67.80</td>
<td>0.066</td>
<td>( \gamma_z )</td>
</tr>
</tbody>
</table>

Table 26b): Test 10, Steps 5 & 6.

<table>
<thead>
<tr>
<th>Step 5</th>
<th>Step 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cube mirror ( X, Y ) and ( Z ) components with ( \varphi_x, \varphi_y, \varphi_z ) after re-aligning ( Y )</td>
<td>The most orthogonal permutation of cube mirror ( X, Y ) and ( Z ) components</td>
</tr>
<tr>
<td>Cubic mirror ( X, Y ) and ( Z ) components with ( \varphi_x + \pi, \varphi_y + \pi, \varphi_z + \pi ) after re-aligning ( Y )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Step 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured components of projected optical lever reflection vectors</td>
</tr>
<tr>
<td>( x )-axis</td>
</tr>
<tr>
<td>( R_{18x} )</td>
</tr>
<tr>
<td>( R_{18y} )</td>
</tr>
<tr>
<td>( R_{12z} )</td>
</tr>
</tbody>
</table>
Experimentation Results

Table 26d): Test 10, Step 8.

<table>
<thead>
<tr>
<th></th>
<th>R'_{17x}</th>
<th>R'_{17y}</th>
<th>R'_{11z}</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>0.064810042</td>
<td>0.734238177</td>
<td>0.675791357</td>
</tr>
<tr>
<td>y-axis</td>
<td>0.768475043</td>
<td>0.02871204</td>
<td>0.639235267</td>
</tr>
<tr>
<td>z-axis</td>
<td>-0.65786396</td>
<td>0.752010864</td>
<td>0.041166378</td>
</tr>
</tbody>
</table>

Table 26e): Test 10, Step 9.

<table>
<thead>
<tr>
<th></th>
<th>N'_{Tx} positive</th>
<th></th>
<th>N'_{Ty} positive</th>
<th></th>
<th>N'_{Tz} positive</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>0.046805007</td>
<td>0.019593957</td>
<td>0.998711854</td>
<td></td>
<td></td>
</tr>
<tr>
<td>y-axis</td>
<td>0.045523872</td>
<td>0.02129901</td>
<td>0.998736166</td>
<td></td>
<td></td>
</tr>
<tr>
<td>z-axis</td>
<td>0.99900558</td>
<td>0.032864755</td>
<td>0.030129174</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>N'_{Tx} negative</th>
<th></th>
<th>N'_{Ty} negative</th>
<th></th>
<th>N'_{Tz} negative</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>-0.04680500</td>
<td>-0.01959395</td>
<td>-0.99871185</td>
<td></td>
<td></td>
</tr>
<tr>
<td>y-axis</td>
<td>-0.04552387</td>
<td>-0.02129901</td>
<td>-0.99873616</td>
<td></td>
<td></td>
</tr>
<tr>
<td>z-axis</td>
<td>0.99900556</td>
<td>-0.03286476</td>
<td>-0.03012917</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 26f): Test 10, Step 10.

<table>
<thead>
<tr>
<th></th>
<th>N'_{Tx}</th>
<th></th>
<th>N'_{Ty}</th>
<th></th>
<th>N'_{Tz}</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>-0.04680500</td>
<td>-0.01959395</td>
<td>-0.99871185</td>
<td></td>
<td></td>
</tr>
<tr>
<td>y-axis</td>
<td>0.04552387</td>
<td>-0.02129901</td>
<td>-0.99873616</td>
<td></td>
<td></td>
</tr>
<tr>
<td>z-axis</td>
<td>0.99900557</td>
<td>-0.03286476</td>
<td>-0.03012917</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Appendix A

Table 26g): Test 10, Step 11.

| Step 11 |
|-------------------|-------------------|
| Dot product of virtual mirror vectors | Convert the components of the virtual mirrors to the actual optical lever tilting mirrors and use the dot product to select the most orthogonal pair of vectors |
| $N_{Tx} \cdot N_{Tz}$ | $N_{Ty} \cdot N_{Tz}$ |
| $N_{Tx} \cdot N_{Tz}$ | $N_{Tx} \cdot N_{Tz}$ |
| x-axis | y-axis | z-axis | x-axis | y-axis | z-axis |
| -0.000167039 | -2.36549E-05 | 
| -0.00505821 | -0.00236656 | -0.99998441 |
| 0.99997239 | -0.00547746 | -0.00502153 |

Table 26h): Test 10, Step 12.

<table>
<thead>
<tr>
<th>Step 12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reverse the direction of the selected X or Y optical lever normal vectors and take the cross product with the Z optical lever normal</td>
</tr>
<tr>
<td>$N_{Tz}$ (aligned with X cube mirror components)</td>
</tr>
<tr>
<td>x-axis</td>
</tr>
<tr>
<td>0.99997239</td>
</tr>
</tbody>
</table>
Experimentation Results

Figure 87: Test 10, a) X interferometer interferogram, b) Y interferometer interferogram, c) Z interferometer interferogram, d) optical lever beam positions.
Appendix A

Table 27a): Test 11, Steps 1 – 4, determining the magnitude of Y interferometer misalignment.

<table>
<thead>
<tr>
<th>Interferometer</th>
<th>Step 1</th>
<th>Step 2</th>
<th>Step 3</th>
<th>Step 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Fringe slope</td>
<td>deg°</td>
<td>Fringe spacing mm</td>
<td>Mirror tilt angle</td>
</tr>
<tr>
<td>X</td>
<td>-3.80</td>
<td>0.661</td>
<td>( \alpha_x )</td>
<td>0.0294</td>
</tr>
<tr>
<td>Y</td>
<td>16.76</td>
<td>0.644</td>
<td>( \beta_y )</td>
<td>0.0302</td>
</tr>
<tr>
<td>Z</td>
<td>-74.49</td>
<td>0.719</td>
<td>( \gamma_z )</td>
<td>0.0270</td>
</tr>
</tbody>
</table>

Table 27b): Test 11, Steps 5 & 6.

<table>
<thead>
<tr>
<th>Step 5</th>
<th>Step 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cube mirror X, Y and Z components with ( \varphi_x, \varphi_y, \varphi_z ) after re-aligning Y</td>
<td>The most orthogonal permutation of cube mirror X, Y and Z components</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>0.999999674</td>
<td>-3.4313E-05</td>
<td>-0.00080730</td>
<td>0.999999674</td>
<td>3.4313E-05</td>
<td>0.000807308</td>
<td>0.999999868</td>
<td>3.40515E-05</td>
</tr>
<tr>
<td>Y</td>
<td>-8.12012E-06</td>
<td>0.999999985</td>
<td>0.000171335</td>
<td>0.001014301</td>
<td>0.999999445</td>
<td>-0.00028661</td>
<td>0.999999993</td>
<td>0.000113818</td>
</tr>
<tr>
<td>Z</td>
<td>-0.00072179</td>
<td>-0.00019756</td>
<td>0.99999972</td>
<td>0.000721798</td>
<td>0.9999976</td>
<td>0.00019756</td>
<td>0.99999972</td>
<td>-0.00045520</td>
</tr>
</tbody>
</table>

Table 27c): Test 11, Step 7.

<table>
<thead>
<tr>
<th>Step 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured components of projected optical lever reflection vectors</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R_{18x} )</td>
<td>13</td>
<td>-2123</td>
<td>2</td>
<td>( R_{18y} )</td>
<td>0.0612329</td>
</tr>
<tr>
<td>( R_{18y} )</td>
<td>15</td>
<td>-2145</td>
<td>-12.5</td>
<td>( R_{18z} )</td>
<td>0.0069927</td>
</tr>
<tr>
<td>( R_{12z} )</td>
<td>0</td>
<td>-2119</td>
<td>8.5</td>
<td>( R_{12y} )</td>
<td>0</td>
</tr>
</tbody>
</table>

213
Experimentation Results

Table 27d): Test 11, Step 8.

<table>
<thead>
<tr>
<th>Step 8</th>
<th>Normalised optical lever components and steering mirror components substituted into Equation (6.61) to produce $R'<em>{17}/R'</em>{11}$ reflection vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$R'_{17x}$</td>
</tr>
<tr>
<td>x-axis</td>
<td>0.00612329</td>
</tr>
<tr>
<td>y-axis</td>
<td>0.707759337</td>
</tr>
<tr>
<td>z-axis</td>
<td>-0.00401129</td>
</tr>
</tbody>
</table>

Table 27e): Test 11, Step 9.

<table>
<thead>
<tr>
<th>Step 9</th>
<th>Use Equations (6.71) – (6.73) find both solutions of virtual mirror normal vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N'_{T_x}$ positive</td>
<td>$N'_{T_y}$ positive</td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td>-0.00432778</td>
<td>-0.00048039</td>
</tr>
<tr>
<td>$N'_{T_x}$ negative</td>
<td>$N'_{T_y}$ negative</td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td>0.004327784</td>
<td>0.000480392</td>
</tr>
</tbody>
</table>

Table 27f): Test 11, Step 10.

<table>
<thead>
<tr>
<th>Step 10</th>
<th>Use displacement of the optical lever beams relative to &quot;zero position&quot; to determine the correct solution from Step 9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N'_{T_x}$</td>
<td>$N'_{T_y}$</td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td>0.004327784</td>
<td>0.000480392</td>
</tr>
</tbody>
</table>
Table 27g): Test 11, Step 11.

<table>
<thead>
<tr>
<th>Step 11</th>
<th>Dot product of virtual mirror vectors</th>
<th>Convert the components of the virtual mirrors to the actual optical lever tilting mirrors and use the dot product to select the most orthogonal pair of vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$N_{T_x} \cdot N_{T_z}$</td>
<td>$N_{T_y} \cdot N_{T_z}$</td>
</tr>
<tr>
<td></td>
<td>8.1294E-06</td>
<td>3.09049E-05</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 27h): Test 11, Step 12.

<table>
<thead>
<tr>
<th>Step 12</th>
<th>Reverse the direction of the selected X or Y optical lever normal vectors and take the cross product with the Z optical lever normal</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$N_{T_z}$ (aligned with X cube mirror components)</td>
</tr>
<tr>
<td></td>
<td>$x$-axis</td>
</tr>
<tr>
<td></td>
<td>0.9999999888</td>
</tr>
</tbody>
</table>
Figure 88: Test 11, a) X interferometer interferogram, b) Y interferometer interferogram, c) Z interferometer interferogram, d) optical lever beam positions.
### Table 28a): Test 12, Steps 1 – 4, determining the magnitude of Y interferometer misalignment.

<table>
<thead>
<tr>
<th>Interfer</th>
<th>Step 1</th>
<th>Step 2</th>
<th>Step 3</th>
<th>Step 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Fringe slope</td>
<td>Fringe spacing</td>
<td>Mirror tilt angle</td>
<td>deg^\circ</td>
</tr>
<tr>
<td>X</td>
<td>-39.17</td>
<td>1.507</td>
<td>α_x</td>
<td>0.0129</td>
</tr>
<tr>
<td>Y</td>
<td>19.53</td>
<td>0.541</td>
<td>β_y</td>
<td>0.0359</td>
</tr>
<tr>
<td>Z</td>
<td>-36.03</td>
<td>1.338</td>
<td>γ_z</td>
<td>0.0145</td>
</tr>
</tbody>
</table>

### Table 28b): Test 12, Steps 5 & 6.

<table>
<thead>
<tr>
<th>Step 5</th>
<th>Step 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cube mirror X, Y and Z components with φ_x, φ_y, φ_z after re-aligning Y</td>
<td>Cube mirror X, Y and Z components with φ_x + π, φ_y + π, φ_z + π after re-aligning Y</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>0.999999952</td>
<td>-0.00014352</td>
<td>-0.00027537</td>
<td>0.999999952</td>
<td>0.000143528</td>
<td>0.000275371</td>
<td>0.999999975</td>
<td>0.000142434</td>
<td>0.000174819</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td>-9.55906E-05</td>
<td>0.999999962</td>
<td>0.000258138</td>
<td>0.001101771</td>
<td>0.999999323</td>
<td>-0.00037341</td>
<td>-9.4423E-05</td>
<td>0.9999999377</td>
<td>-0.00024806</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z</td>
<td>-0.00023704</td>
<td>-0.00032123</td>
<td>0.99999992</td>
<td>0.000237047</td>
<td>0.000321236</td>
<td>0.99999992</td>
<td>-0.000149494</td>
<td>-0.000205359</td>
<td>0.999999968</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 28c): Test 12, Step 7.

<table>
<thead>
<tr>
<th>Step 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured components of projected optical lever reflection vectors</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>R_{18x}</td>
<td>4.5</td>
<td>-2123</td>
<td>4.5</td>
<td>R_{18x_n}</td>
<td>0.00211963</td>
<td>-0.999955</td>
</tr>
<tr>
<td>R_{18y}</td>
<td>7</td>
<td>-2145</td>
<td>-2.5</td>
<td>R_{18y_n}</td>
<td>0.00326338</td>
<td>-0.999939</td>
</tr>
<tr>
<td>R_{12z}</td>
<td>-3</td>
<td>-2119</td>
<td>2.5</td>
<td>R_{12z_n}</td>
<td>-0.0014158</td>
<td>-0.9999983</td>
</tr>
</tbody>
</table>
Experimentation Results

Table 28d): Test 12, Step 8.

<table>
<thead>
<tr>
<th>R_{17x}'</th>
<th>R_{17y}'</th>
<th>R_{11z}'</th>
<th>R_{17x}'</th>
<th>R_{17y}'</th>
<th>R_{11z}'</th>
<th>R_{17x}'</th>
<th>y-axis</th>
<th>z-axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.00211963</td>
<td>0.705604798</td>
<td>0.708602411</td>
<td>0.704888097</td>
<td>-0.00148343</td>
<td>0.709316975</td>
<td>-0.70810667</td>
<td>0.70610449</td>
<td>-0.00117980</td>
</tr>
</tbody>
</table>

Table 28e): Test 12, Step 9.

<table>
<thead>
<tr>
<th>N_{T_x}' positive</th>
<th>N_{T_y}' positive</th>
<th>N_{T_z}' positive</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>-0.00149722</td>
<td>-0.00106093</td>
<td>0.999998316</td>
</tr>
<tr>
<td>N_{T_x}' negative</td>
<td>N_{T_y}' negative</td>
<td>N_{T_z}' negative</td>
</tr>
<tr>
<td>-------------------</td>
<td>-------------------</td>
<td>-------------------</td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>0.001497221</td>
<td>0.001060939</td>
<td>-0.99999831</td>
</tr>
</tbody>
</table>

Table 28f): Test 12, Step 10.

<table>
<thead>
<tr>
<th>N_{T_x}'</th>
<th>N_{T_y}'</th>
<th>N_{T_z}'</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>0.001497221</td>
<td>0.001060939</td>
<td>-0.99999831</td>
</tr>
</tbody>
</table>
Appendix A

Table 28g): Test 12, Step 11.

<table>
<thead>
<tr>
<th>Dot product of virtual mirror vectors</th>
<th>Convert the components of the virtual mirrors to the actual optical lever tilting mirrors and use the dot product to select the most orthogonal pair of vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{T_x} \cdot N_{T_z}$</td>
<td>$N_{T_y} \cdot N_{T_z}$</td>
</tr>
<tr>
<td>2.74293E-05</td>
<td>3.51153E-05</td>
</tr>
<tr>
<td>0.999999983</td>
<td>0.000118038</td>
</tr>
</tbody>
</table>

Table 28h): Test 12, Step 12.

<table>
<thead>
<tr>
<th>$N_{T_z}$ (aligned with X cube mirror components)</th>
<th>$N_{T_x} \times N_{T_z}/N_{T_y}$ (aligned with Y cube mirror components)</th>
<th>$N_{T_x}/N_{T_y}$ (aligned with Z cube mirror components)</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>0.999999983</td>
<td>0.000118038</td>
<td>0.000138942</td>
</tr>
</tbody>
</table>
Figure 89: Test 12, a) X interferometer interferogram, b) Y interferometer interferogram, c) Z interferometer interferogram, d) optical lever beam positions.
Table 29a): Test 13, Steps 1 – 4, determining the magnitude of Y interferometer misalignment.

<table>
<thead>
<tr>
<th>Interferometer</th>
<th>Step 1</th>
<th>Step 2</th>
<th>Step 3</th>
<th>Step 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fringe slope</td>
<td>deg°</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fringe spacing</td>
<td>mm</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mirror tilt</td>
<td>angle</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>deg°</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>X</td>
<td>φ_x</td>
<td>17.92</td>
<td>1.203</td>
<td>α_x</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td>φ_y</td>
<td>8.59</td>
<td>0.612</td>
<td>β_y</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z</td>
<td>φ_z</td>
<td>77.89</td>
<td>1.086</td>
<td>γ_z</td>
</tr>
</tbody>
</table>

Table 29b): Test 13, Steps 5 & 6.

<table>
<thead>
<tr>
<th>Step 5</th>
<th>Step 6</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Cube mirror X, Y and Z components with φ_x, φ_y, φ_z after re-aligning Y</td>
<td>The most orthogonal permutation of cube mirror X, Y and Z components</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td>0.999999906</td>
<td>8.76503E-05</td>
</tr>
<tr>
<td>-5.25732E-05</td>
<td>0.999999996</td>
</tr>
<tr>
<td>0.000485069</td>
<td>-0.00010261</td>
</tr>
</tbody>
</table>

Table 29c): Test 13, Step 7.

<table>
<thead>
<tr>
<th>Step 7</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>Measured components of projected optical lever reflection vectors</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>x-axis</td>
</tr>
<tr>
<td>R_{18x}</td>
</tr>
<tr>
<td>R_{18y}</td>
</tr>
<tr>
<td>R_{12z}</td>
</tr>
</tbody>
</table>
Experimentation Results

**Table 29d): Test 13, Step 8.**

<table>
<thead>
<tr>
<th></th>
<th>$R'_{17x}$</th>
<th></th>
<th>$R'_{17y}$</th>
<th></th>
<th>$R'_{11z}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>0.003532715</td>
<td>y-axis</td>
<td>0.707102369</td>
<td>z-axis</td>
<td>0.707102369</td>
</tr>
<tr>
<td></td>
<td>0.710478608</td>
<td></td>
<td>0.000164825</td>
<td></td>
<td>0.703718779</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>-0.70743802</td>
<td></td>
<td>0.706770623</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.002595555</td>
</tr>
</tbody>
</table>

**Table 29e): Test 13, Step 9.**

<table>
<thead>
<tr>
<th></th>
<th>$N'_{Tx}$ positive</th>
<th></th>
<th>$N'_{Ty}$ positive</th>
<th></th>
<th>$N'_{Tz}$ positive</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>0.002498006</td>
<td>y-axis</td>
<td>-3.12003E-06</td>
<td>z-axis</td>
<td>0.99999688</td>
</tr>
<tr>
<td></td>
<td>0.002389961</td>
<td></td>
<td>0.000116828</td>
<td></td>
<td>0.999997137</td>
</tr>
<tr>
<td></td>
<td>-0.999998289</td>
<td></td>
<td>-0.00023764</td>
<td></td>
<td>0.001834902</td>
</tr>
<tr>
<td></td>
<td>0.999997137</td>
<td></td>
<td>-0.999998289</td>
<td></td>
<td>0.001834902</td>
</tr>
</tbody>
</table>

**Table 29f): Test 13, Step 10.**

<table>
<thead>
<tr>
<th></th>
<th>$N'_{Tx}$</th>
<th></th>
<th>$N'_{Ty}$</th>
<th></th>
<th>$N'_{Tz}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>-0.002498006</td>
<td>y-axis</td>
<td>3.12003E-06</td>
<td>z-axis</td>
<td>-0.99999688</td>
</tr>
<tr>
<td></td>
<td>-0.00238996</td>
<td></td>
<td>-0.00011682</td>
<td></td>
<td>-0.999997137</td>
</tr>
<tr>
<td></td>
<td>0.999998288</td>
<td></td>
<td>0.000237643</td>
<td></td>
<td>-0.00183490</td>
</tr>
</tbody>
</table>
Appendix A

Table 29g): Test 13, Step 11.

<table>
<thead>
<tr>
<th>Dot product of virtual mirror vectors</th>
<th>Convert the components of the virtual mirrors to the actual optical lever tilting mirrors and use the dot product to select the most orthogonal pair of vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{Tx} \cdot N_{Tz}$</td>
<td>$N_{Ty} \cdot N_{Tz}$</td>
</tr>
<tr>
<td>2.82607E-05</td>
<td>4.02652E-05</td>
</tr>
<tr>
<td>-0.00027756</td>
<td>3.4667E-07</td>
</tr>
</tbody>
</table>

Table 29h): Test 13, Step 12.

<table>
<thead>
<tr>
<th>$N_{Tz}$ (aligned with X cube mirror components)</th>
<th>Reverse the direction of the selected X or Y optical lever normal vectors and take the cross product with the Z optical lever normal</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{Tx}/N_{Tz}$</td>
<td>$N_{Tx}/N_{Tz}$ (aligned with Z cube mirror components)</td>
</tr>
<tr>
<td>$N_{Tz} \times N_{Tx}/N_{Ty} = N_{y}$ (aligned with Y cube mirror components)</td>
<td></td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td>0.999999952</td>
<td>3.96072E-05</td>
</tr>
</tbody>
</table>
Experimentation Results

Figure 90: Test 13, a) X interferometer interferogram, b) Y interferometer interferogram, c) Z interferometer interferogram, d) optical lever beam positions.
Appendix A

Table 30a): Test 14, Steps 1 – 4, determining the magnitude of Y interferometer misalignment.

<table>
<thead>
<tr>
<th>Interferometer</th>
<th>Step 1</th>
<th>Step 2</th>
<th>Step 3</th>
<th>Step 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Fringe slope</td>
<td>deg°</td>
<td>Fringe spacing mm</td>
<td>Mirror tilt angle</td>
</tr>
<tr>
<td>X</td>
<td>( \varphi_x )</td>
<td>-4.98</td>
<td>0.648</td>
<td>( \alpha_x )</td>
</tr>
<tr>
<td>Y</td>
<td>( \varphi_y )</td>
<td>40.51</td>
<td>0.500</td>
<td>( \beta_y )</td>
</tr>
<tr>
<td>Z</td>
<td>( \varphi_z )</td>
<td>-50.36</td>
<td>0.529</td>
<td>( \gamma_z )</td>
</tr>
</tbody>
</table>

Table 30b): Test 14, Steps 5 & 6.

<table>
<thead>
<tr>
<th>Step 5</th>
<th>Step 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cube mirror X, Y and Z components with ( \varphi_x, \varphi_y, \varphi_z ) after re-aligning Y</td>
<td>The most orthogonal permutation of cube mirror X, Y and Z components</td>
</tr>
<tr>
<td>Cube mirror X, Y and Z components with ( \varphi_x + \pi, \varphi_y + \pi, \varphi_z + \pi ) after re-aligning Y</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>0.99999966</td>
<td>-4.59433E-05</td>
<td>-0.00082313</td>
<td>0.99999966</td>
<td>4.59433E-05</td>
<td>0.000823137</td>
<td>0.999999862</td>
<td>4.55932E-05</td>
<td>0.000522567</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td>-1.93386E-05</td>
<td>0.999999816</td>
<td>0.00606254</td>
<td>0.001025519</td>
<td>0.999999919</td>
<td>-0.00072153</td>
<td>0.001012992</td>
<td>0.999999372</td>
<td>-0.00047931</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z</td>
<td>-0.00078505</td>
<td>-0.00064119</td>
<td>0.999999486</td>
<td>0.000785058</td>
<td>0.000641196</td>
<td>0.999999486</td>
<td>0.000495098</td>
<td>0.000409902</td>
<td>0.999999793</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 30c): Test 14, Step 7.

<table>
<thead>
<tr>
<th>Step 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured components of projected optical lever reflection vectors</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
<th>x-axis</th>
<th>y-axis</th>
<th>z-axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R_{18x} )</td>
<td>14</td>
<td>-2123</td>
<td>12.5</td>
<td>( R_{18x_n} )</td>
<td>0.00659418</td>
</tr>
<tr>
<td>( R_{18y} )</td>
<td>21</td>
<td>-2145</td>
<td>-7.5</td>
<td>( R_{18y_n} )</td>
<td>0.00978968</td>
</tr>
<tr>
<td>( R_{12z} )</td>
<td>-1</td>
<td>-2119</td>
<td>8.5</td>
<td>( R_{12z_n} )</td>
<td>-0.0004719</td>
</tr>
</tbody>
</table>

225
Experimentation Results

Table 30d): Test 14, Step 8.

<table>
<thead>
<tr>
<th></th>
<th>R'_{17x}</th>
<th></th>
<th>R'_{17y}</th>
<th></th>
<th>R'_{11z}</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>-0.00659418</td>
<td>y-axis</td>
<td>0.702915944</td>
<td>z-axis</td>
<td>0.711242359</td>
</tr>
<tr>
<td>y-axis</td>
<td>0.700425577</td>
<td>x-axis</td>
<td>-0.00445008</td>
<td>y-axis</td>
<td>0.713711572</td>
</tr>
<tr>
<td>z-axis</td>
<td>-0.00401129</td>
<td>x-axis</td>
<td>-0.003132006</td>
<td>y-axis</td>
<td>-0.70743471</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.706767318</td>
</tr>
</tbody>
</table>

Table 30e): Test 14, Step 9.

<table>
<thead>
<tr>
<th></th>
<th>N'_{Tx} positive</th>
<th></th>
<th>N'_{Ty} positive</th>
<th></th>
<th>N'_{Tz} positive</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>-0.00464913</td>
<td>y-axis</td>
<td>0.999984828</td>
<td>z-axis</td>
<td>-0.00470229</td>
</tr>
<tr>
<td>y-axis</td>
<td>-0.00313201</td>
<td>x-axis</td>
<td>0.999984039</td>
<td>y-axis</td>
<td>-0.99999595</td>
</tr>
<tr>
<td>z-axis</td>
<td>0.00023998</td>
<td>x-axis</td>
<td>-0.99998403</td>
<td>y-axis</td>
<td>0.999984039</td>
</tr>
<tr>
<td></td>
<td>-0.00283574</td>
<td></td>
<td>0.99999595</td>
<td></td>
<td>-0.00283574</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>N'_{Tx} negative</th>
<th></th>
<th>N'_{Ty} negative</th>
<th></th>
<th>N'_{Tz} negative</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>0.004649126</td>
<td>y-axis</td>
<td>-0.99998482</td>
<td>z-axis</td>
<td>0.004702288</td>
</tr>
<tr>
<td>y-axis</td>
<td>0.003132006</td>
<td>x-axis</td>
<td>-0.99999403</td>
<td>y-axis</td>
<td>0.999984039</td>
</tr>
<tr>
<td>z-axis</td>
<td>0.00023998</td>
<td>x-axis</td>
<td>-0.99999595</td>
<td>y-axis</td>
<td>0.99999595</td>
</tr>
<tr>
<td></td>
<td>0.002835744</td>
<td></td>
<td>0.99998403</td>
<td></td>
<td>0.002835744</td>
</tr>
</tbody>
</table>

Table 30f): Test 14, Step 10.

<table>
<thead>
<tr>
<th></th>
<th>N'_{Tx}</th>
<th></th>
<th>N'_{Ty}</th>
<th></th>
<th>N'_{Tz}</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>0.004649126</td>
<td>y-axis</td>
<td>-0.99998482</td>
<td>z-axis</td>
<td>0.004702288</td>
</tr>
<tr>
<td>y-axis</td>
<td>0.003132006</td>
<td>x-axis</td>
<td>-0.99999403</td>
<td>y-axis</td>
<td>0.999984039</td>
</tr>
<tr>
<td>z-axis</td>
<td>0.00023998</td>
<td>x-axis</td>
<td>-0.99999595</td>
<td>y-axis</td>
<td>0.99999595</td>
</tr>
<tr>
<td></td>
<td>0.002835744</td>
<td></td>
<td>0.99998403</td>
<td></td>
<td>0.002835744</td>
</tr>
</tbody>
</table>
Appendix A

**Table 30g): Test 14, Step 11.**

<table>
<thead>
<tr>
<th>Step 11</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Dot product of virtual mirror vectors</td>
<td>Convert the components of the virtual mirrors to the actual optical lever tilting mirrors and use the dot product to select the most orthogonal pair of vectors</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N_{Tx} \cdot N_{Tz}$</td>
<td>$N_{Ty} \cdot N_{Tz}$</td>
<td>$N_{Tx}/N_{Tz}$</td>
<td>$N_{Ty}$</td>
<td></td>
</tr>
<tr>
<td>4.39588E-05</td>
<td>4.98665E-05</td>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
</tr>
<tr>
<td>0.00051657</td>
<td>0.000328298</td>
<td>-0.99999981</td>
<td>0.999999888</td>
<td>3.99967E-05</td>
</tr>
</tbody>
</table>

**Table 30h): Test 14, Step 12.**

<table>
<thead>
<tr>
<th>Step 12</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Reverse the direction of the selected X or Y optical lever normal vectors and take the cross product with the Z optical lever normal</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N_{Tz}$ (aligned with X cube mirror components)</td>
<td>$N_{Tx} \times N_{Tz}/N_{Ty} = N_{Y}$ (aligned with Y cube mirror components)</td>
<td>$N_{Tz}/N_{Ty}$ (aligned with Z cube mirror components)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>x-axis</td>
<td>y-axis</td>
<td>z-axis</td>
<td>x-axis</td>
</tr>
<tr>
<td>0.999999888</td>
<td>3.99967E-05</td>
<td>0.000472624</td>
<td>-4.01518E-05</td>
<td>0.999999944</td>
</tr>
</tbody>
</table>
Figure 91: Test 14, a) X interferometer interferogram, b) Y interferometer interferogram, c) Z interferometer interferogram, d) optical lever beam positions.
### Appendix A

**Table 31a): Test 15, zero position, Steps 1 – 4, determining the magnitude of Y interferometer misalignment.**

<table>
<thead>
<tr>
<th>Interferometer</th>
<th>Step 1</th>
<th>Step 2</th>
<th>Step 3</th>
<th>Step 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fringe slope</td>
<td>deg°</td>
<td>Fringe spacing mm</td>
<td>Mirror tilt angle</td>
<td>deg°</td>
</tr>
<tr>
<td>X</td>
<td>61.93</td>
<td>5.133</td>
<td>α_x 0.0037</td>
<td>0.999999998</td>
</tr>
<tr>
<td>Y</td>
<td>11.30</td>
<td>0.699</td>
<td>β_y 0.0278</td>
<td>2.00885E-05</td>
</tr>
<tr>
<td>Z</td>
<td>23.12</td>
<td>3.040</td>
<td>γ_z 0.0064</td>
<td>4.38647E-05</td>
</tr>
</tbody>
</table>

**Table 31b): Test 15, zero position, Steps 5 & 6.**

<table>
<thead>
<tr>
<th>Step 5</th>
<th>Step 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cube mirror X, Y and Z components with ( \varphi_x, \varphi_y, \varphi_z ) after re-aligning Y</td>
<td>The most orthogonal permutation of cube mirror X, Y and Z components</td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td>X</td>
<td>0.999999997</td>
</tr>
<tr>
<td>Y</td>
<td>2.03369E-05</td>
</tr>
<tr>
<td>Z</td>
<td>6.95545E-05</td>
</tr>
</tbody>
</table>

**Table 31c): Test 15, zero position, Step 7.**

<table>
<thead>
<tr>
<th>Step 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normalised components of projected optical lever reflection vectors</td>
</tr>
<tr>
<td>x-axis</td>
</tr>
<tr>
<td>( R_{18x} )</td>
</tr>
<tr>
<td>( R_{18y} )</td>
</tr>
<tr>
<td>( R_{12z} )</td>
</tr>
</tbody>
</table>
## Experimentation Results

**Table 31d): Test 15, zero position, Step 8.**

<table>
<thead>
<tr>
<th></th>
<th>$R'_{17x}$</th>
<th>$R'_{17y}$</th>
<th>$R'_{11z}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>0.000235516</td>
<td>0.707106762</td>
<td>0.707106762</td>
</tr>
<tr>
<td>y-axis</td>
<td>0.707106762</td>
<td>-2.22045E-16</td>
<td>0.707106781</td>
</tr>
<tr>
<td>z-axis</td>
<td>0.707106762</td>
<td>0.707106781</td>
<td>-0.70710670</td>
</tr>
</tbody>
</table>

**Table 31e): Test 15, zero position, Step 9.**

<table>
<thead>
<tr>
<th></th>
<th>$N'_{Tx}$ positive</th>
<th>$N'_{Ty}$ positive</th>
<th>$N'_{Tz}$ positive</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>0.000166535</td>
<td>-1.38669E-08</td>
<td>0.999999986</td>
</tr>
<tr>
<td>y-axis</td>
<td>1.57009E-16</td>
<td>-1.57009E-16</td>
<td>-0.999999944</td>
</tr>
<tr>
<td>z-axis</td>
<td>-1.57009E-16</td>
<td>1</td>
<td>5.56773E-08</td>
</tr>
</tbody>
</table>

**Table 31f): Test 15, zero position, Step 10.**

<table>
<thead>
<tr>
<th></th>
<th>$N'_{Tx}$ negative</th>
<th>$N'_{Ty}$ negative</th>
<th>$N'_{Tz}$ negative</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-axis</td>
<td>-0.00016653</td>
<td>1.38669E-08</td>
<td>0.999999944</td>
</tr>
<tr>
<td>y-axis</td>
<td>-0.99999998</td>
<td>1.57009E-16</td>
<td>5.56773E-08</td>
</tr>
<tr>
<td>z-axis</td>
<td>1.57009E-16</td>
<td>-1</td>
<td>-0.00033369</td>
</tr>
</tbody>
</table>

**Step 8**
Normalised optical lever components and steering mirror components substituted into Equation (6.61) to produce $R'_{17}/R'_{11}$ reflection vectors

**Step 9**
Use Equations (6.71) – (6.73) find both solutions of virtual mirror normal vectors

**Step 10**
Use displacement of the optical lever beams relative to "zero position" to determine the correct solution from Step 9
Appendix A

Table 31g): Test 15, zero position, Step 11.

<table>
<thead>
<tr>
<th>Dot product of virtual mirror vectors</th>
<th>Convert the components of the virtual mirrors to the actual optical lever tilting mirrors and use the dot product to select the most orthogonal pair of vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_{Tx} \cdot N_{Tx} )</td>
<td>( N_{Tz} \cdot N_{Tz} )</td>
</tr>
<tr>
<td>2.59892E-05</td>
<td>4.44931E-05</td>
</tr>
<tr>
<td>-1.85039E-05</td>
<td>1.54077E-09</td>
</tr>
</tbody>
</table>

Table 31h): Test 15, zero position, Step 12.

<table>
<thead>
<tr>
<th>Step 12</th>
<th>Reverse the direction of the selected X or Y optical lever normal vectors and take the cross product with the Z optical lever normal</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_{Tz} ) (aligned with X cube mirror components)</td>
<td>( N_{Tz} \times N_{Tx}/N_{Ty} = N_{y} ) (aligned with Y cube mirror components)</td>
</tr>
<tr>
<td>x-axis</td>
<td>y-axis</td>
</tr>
<tr>
<td>0.999999999</td>
<td>7.42364E-09</td>
</tr>
</tbody>
</table>
Experimentation Results

Figure 92: Test 15, a) X interferometer interferogram, b) Y interferometer interferogram, c) Z interferometer interferogram, d) optical lever beam positions.