Optical Based Statistical Space Object Tracking for Catalogue Maintenance

A thesis submitted in fulfilment of the requirements
for the degree of Doctor of Philosophy

Han Cai

Bachelor of Electronic Information Science and Technology
Ocean University of China, China
Master of Aeronautical and Astronautical Science and Technology
Beijing Institute of Technology, China

School of Science
College of Science, Engineering and Health
RMIT University

May 2019
Declaration

I certify that except where due acknowledgment has been made, the work is that of the author alone; the work has not been submitted previously, in whole or in part, to qualify for any other academic award; the content of the thesis is the result of work which has been carried out since the official commencement date of the approved research program; and, any editorial work, paid or unpaid, carried out by a third party is acknowledged; and, ethics procedures and guidelines have been followed.

Han Cai

School of Science, RMIT University

9 May 2019
Abstract

The number of space objects has grown substantially in the past decades due to new launches, regular mission activities, and breakup events. This has significantly affected the space environment and the development of the space industry. To ensure safe operation of space assets, Space Situational Awareness (SSA) has attracted considerable attention in recent years. One primary strategy in SSA is to establish and maintain a Space Object Catalogue (SOC) to provide timely updated data for SSA applications, e.g., conjunction analysis, collision avoidance manoeuvring. This thesis investigates three techniques for SOC maintenance, namely the tracklet association method for initial orbit determination, the multi-target tracking method for the refinement of orbital state estimation, and multi-sensor tasking method for the optimisation of sensor resources.

Generally speaking, due to the limited number of optical sensors used to track the large population of space objects, the obtained observational arcs for many targets are very short. Such short arcs, which contain a small number of angular observations, are referred as tracklets. Given such limited data, typical orbit determination methods, e.g., Laplace, Gaussian, Double-R methods, may fail to produce a valid orbital solution. By contrast, tracklet association methods compare and correlate multiple tracklets across time, and following successful association, a reliable initial orbital state can be further determined for SOC maintenance. This thesis proposes an improved initial value problem optimisation method for accurate and efficient tracklet association, and a common ellipse method to distinguish false associations of tracklets from objects in the same constellation. The proposed methods are validated using real optical data collected from the Mount Stromlo Observatory, Canberra, Australia.

Furthermore, another challenging task in SSA is to track multiple objects for the maintenance of a catalog. The Bayesian multi-target tracking filter addresses this issue by associating measurements to initially known or newly detected targets and simultaneously estimating the time-varying number of targets and their orbital states. In order to achieve efficient tracking of the new space objects, a novel birth model using the Boundary Value Problem (BVP) approach is proposed. The proposed BVP birth model is implemented in the Labelled Multi-Bernoulli (LMB) filter, which is an efficient multi-target tracker developed based on the Random Finite Set (RFS) theory, for improved computational efficiency of new space object tracking. Sim-
ulation results indicate that the computational efficiency of the proposed method significantly outperforms the state-of-the-art methods.

Finally, as limited sensors are available for SOC maintenance, an appropriate sensor tasking scheme is essential for the optimisation of sensor resources. The optimal sensor tasking command allocates multiple sensors to take the best action and produce useful measurements for more accurate orbital state estimation. In this thesis, an analytical form is derived for the Rényi divergence of LMB RFS in which each target state density is a single Gaussian component. The obtained analytical Rényi divergence is formulated as a reward function for multi-sensor tasking, which improves the computational efficiency, especially for large-scale space object tracking. In addition, this thesis further investigates the benefits of using the analytical Rényi divergence and various space-based and ground-based sensor networks for accurate tracking of objects in geosynchronous Earth orbit.
Acknowledgements

I would like to acknowledge my senior supervisor Prof. Kefei Zhang, for his initiative of the research area, and endless encouragement and enthusiasm throughout my Ph.D. study, also for providing a pleasant environment for doing research.

I would also like to express my gratitude to my associate supervisors Dr. Steve Gehly and Dr. Yang Yang. Their rich knowledge, valuable discussion, rigorous reviews, and fruitful suggestions, have benefitted and inspired me enormously. Without their steadfast guidance and support, this work would not come to fruition. I would like also to thank my associate supervisor Dr. Suqin Wu for her reviews and comments that helped me to improve the manuscripts.

I am very grateful to Prof. Reza Hoseinnezhad for offering his knowledge, and advice regarding the work on the Random Finite Set theory. I particularly thank him for his contribution to the fundamentals and mathematical derivations in Chapter 5.

I also sincerely acknowledge the China Scholarship Council (CSC) and Cooperative Research Centre for Space Environment Management (SERC Limited) through the Australian Government Cooperative Research Centre Programme for their financial supports.

My deepest appreciation goes to my wife, Stephanie, for all her love and devotion. Thank you for your belief in me and the happiness brought into my life. Special thanks to my parents, for their unconditional love and support, motivating me to overcome all difficulties in my study.
## Contents

Abstract iii

Acknowledgements iv

1 Introduction 1

1.1 Motivation ........................................... 2

1.2 Research Objectives ................................... 5

1.2.1 Tracklet Association ......................... 5

1.2.2 Multi-Target Tracking ....................... 7

1.2.3 Multi-Sensor Tasking ....................... 10

1.3 Thesis Overview .................................... 12

1.4 Contributions of Research ...................... 14

2 Tracklet Association Methods 15

2.1 Admissible Region Methods .................... 16

2.1.1 Attributable Vector ....................... 16

2.1.2 Admissible Region ......................... 18

2.1.3 Constrained Admissible Region .......... 21

2.2 Traditional Tracklet Association Methods .... 24

2.2.1 Optimisation Methods ..................... 24

2.2.2 Hyperplane Intersection Methods ....... 32

2.3 Improved Initial Value Problem Optimisation Method 35

2.3.1 Introduction .................................. 36

2.3.2 Methodology .................................. 37

2.3.3 Assessment and Comparison ............ 46

2.4 Common Ellipse Method for Constellation Tracklets 52

2.4.1 Introduction .................................. 52
CONTENTS

2.4.2 Methodology ......................................................... 53
2.4.3 Results ................................................................. 58
2.5 Summary ................................................................. 59

3 Labelled Random Finite Set Filters 61

3.1 Random Finite Sets ..................................................... 61
  3.1.1 Definition ............................................................. 61
  3.1.2 Common Types of Random Finite Sets ......................... 62
3.2 Multi-Target Bayesian Estimation ................................... 64
3.3 Multi-Target Filtering with Labelled Random Finite Sets .......... 65
  3.3.1 Fundamentals of Labelled Random Finite Sets ................ 65
  3.3.2 The $\delta$-Generalised Labelled Multi-Bernoulli Filter .... 71
  3.3.3 The Joint $\delta$-Generalised Labelled Multi-Bernoulli Filter 75
  3.3.4 The Labelled Multi-Bernoulli Filter ............................ 78
  3.3.5 The Joint Labelled Multi-Bernoulli Filter .................... 82
  3.3.6 Comparison of Labelled RFS Filters for Space Object Tracking 83
3.4 Summary ................................................................. 90

4 A Multi-Target Tracking Method for New Space Object Using a Boundary Value Approach 91

4.1 Introduction ............................................................ 92
4.2 CAR and PAR Methods ................................................ 93
4.3 The BVP Birth Model .................................................. 95
  4.3.1 Covariance Estimation ............................................. 96
  4.3.2 Classification Method ............................................ 98
  4.3.3 Probability of Existence ........................................ 101
4.4 Implementation of The BVP-LMB Filter .............................. 103
4.5 Simulation ............................................................. 108
  4.5.1 Simulation Design and Data Selection ......................... 108
  4.5.2 Case I: Dense Measurements .................................... 111
  4.5.3 Case II: Sparse Measurements ................................. 115
## CONTENTS

4.6 Summary ........................................................................ 117

5 A Multi-Sensor Tasking Method Using Analytical Rényi Divergence ... 119
  5.1 Introduction .................................................................. 120
  5.2 Development of Reward Functions ................................. 122
      5.2.1 Information Functionals for Sensor Tasking ............... 122
      5.2.2 Rényi Divergence for LMBs .................................. 124
      5.2.3 Analytical Form of Rényi Divergence for LMBs .......... 126
      5.2.4 Analysis of Reward Functions ................................ 128
  5.3 Centralised Data Fusion .................................................. 131
  5.4 Sensor Tasking Method ................................................... 134
      5.4.1 Visibility Analysis ................................................. 135
      5.4.2 Sensor Tasking Mode ............................................. 136
      5.4.3 Implementation ..................................................... 139
  5.5 Simulation .................................................................. 142
      5.5.1 Sensor Network Design .......................................... 142
      5.5.2 Simulation Design and Data Selection ....................... 144
      5.5.3 Case I: Large-Scale Tracking ................................. 147
      5.5.4 Case II: Dim Object Tracking ............................... 152
  5.6 Summary .................................................................. 154

6 Summary, Conclusions and Recommendations ......................... 156
  6.1 Summary and Conclusions .............................................. 156
  6.2 Recommendations for Future Work ................................. 158

Appendix A Publications ..................................................... 160
  A.1 Journal Papers ........................................................... 160
  A.2 Conference Papers ...................................................... 161

Appendix B The Algorithms in Labelled Random Finite Set Filters .... 162
  B.1 Truncation Formulation of the Joint δ-GLMB filter ............. 162
  B.2 Gibbs Sampling .......................................................... 164
Appendix C  Percentile Results of OSPA Errors  
C.1 Percentile Results of Case I  
C.2 Percentile Results of Case II

Appendix D  Information Gain Functionals  
D.1 Cauchy-Schwarz Divergence for $\delta$-GLMB  
D.2 The Derivation of The Analytical Rényi Divergence

References
Tables

2.3.1 Classification of the 308 tracklets ............................................. 46
3.3.1 Construction of $\delta$-GLMB hypotheses ........................................ 80
3.3.2 Orbital elements of the simulated GEO object ................................. 84
3.3.3 Parameters of the Mt. Stromlo Observatory ..................................... 84
3.3.4 Orbital parameters of the SSO sensor ............................................. 85
3.3.5 Parameters of the LMB filter ......................................................... 86
3.3.6 Computation times of the tested filters for 10 objects using $p_D = 0.95$ .... 86
3.3.7 Computation times of the tested filters for 10 objects using $p_D = 0.75$ .... 88
3.3.8 Computation times of the tested filters for 100 objects using $p_D = 0.95$ .... 89
3.3.9 Computation times of the tested filters for 100 objects using $p_D = 0.75$ .... 89
4.3.1 Constraint values of the classification ............................................. 99
4.5.1 Orbital elements of the GEO and GTO objects ................................. 109
4.5.2 Parameters of the LMB filter ......................................................... 110
4.5.3 Force models for orbit propagation ................................................ 111
4.5.4 Averaged OSPA errors at the final epoch ........................................ 113
4.5.5 Averaged computation time and parameters of birth tracks .................. 114
4.5.6 Averaged OSPA errors at the final epoch ........................................ 117
4.5.7 Averaged computation time and parameters of birth tracks .................. 118
5.2.1 Orbital elements of the two LEO objects ....................................... 128
5.5.1 Parameters of the ground stations ................................................... 142
5.5.2 Orbital parameters of the GTO and SSO sensor networks .................... 143
5.5.3 Force models for orbit propagation ................................................ 147
B.1.1 The cost matrix of the joint prediction and update assignment problem ...... 164
## List of Figures

1.0.1 Number of various types of objects in Earth orbit ....................... 2
1.3.1 The research steps of the thesis ........................................... 13
2.1.1 Least squares fit to angular observations ................................. 18
2.1.2 Admissible region .......................................................... 21
2.1.3 Constrained admissible region ............................................. 24
2.2.1 The flowchart of the IVP optimisation method ........................... 26
2.2.2 Topography of the IVP loss function ...................................... 27
2.2.3 The flowchart of the BVP optimisation method ........................... 30
2.2.4 The BVP loss function and optimisation process ......................... 31
2.2.5 The flowchart of the BIN method ......................................... 34
2.2.6 Bins of two pairs of tracklets in the Poincaré space, the top three subfigures are the results of tracklets from the same object (NORAD ID: 815), the bottom ones are the results from two different objects (NORAD ID: 815 and 1430) ........... 34
2.3.1 Intersections of a pair of $a$ and $e$ curves in a CAR .................... 38
2.3.2 The topography of the new loss function .................................. 42
2.3.3 Optimisation of the new loss function .................................... 45
2.3.4 Association results of three groups of tracklets with different time intervals of 1 day, 1-3 days and 3-5 days using the improved IVP method ......................... 47
2.3.5 True positive and true negative rates of all tracklets within 5 days resulting from the three methods. Note that the $\log_{10}$ value is adopted for the results of IVP and BVP for better visualisation .................................................. 49
2.3.6 Computation time of the improved IVP, IVP and BVP methods .......... 50
2.4.1 Illustration of the common ellipse method ................................. 55
LIST OF FIGURES

2.4.2 Number of true and false associations for different $\log(M_i)$ values (the dashed line indicates the selected $T_c$ for the common ellipse method) .................. 57
2.4.3 True positive and true negative rates of the improved IVP and common ellipse methods for the 86 Iridium constellation tracklets ......................... 59

3.3.1 An example of label assignment to tracks ........................................ 67
3.3.2 The flowchart of the $\delta$-GLMB filter ........................................... 71
3.3.3 The flowchart of the LMB filter ..................................................... 79
3.3.4 OSPA errors of 10 objects in the case of $p_D = 0.95$ ......................... 87
3.3.5 OSPA errors of 10 objects in the case of $p_D = 0.75$ ......................... 87
3.3.6 OSPA errors of 100 objects in the case of $p_D = 0.95$ ....................... 88
3.3.7 OSPA errors of 100 objects in the case of $p_D = 0.75$ ....................... 89

4.2.1 The PDFs of CAR and PAR via the GM approximation ....................... 95
4.3.1 Comparison of the estimated covariance and initial covariance .............. 98
4.3.2 Multiple admissible regions of two tracklets .................................... 100
4.4.1 The flowchart of the LMB filter and the three birth models .................. 103
4.5.1 Averaged position, velocity OSPA errors and cardinality estimate results of three birth models in the case of tracking 4 GEO and 4 GTO objects for 4 days .... 112
4.5.2 Averaged position, velocity OSPA errors and cardinality estimate results of three birth models in the case of tracking 3 GEO and 3 GTO objects for 7 days .... 116

5.2.1 Reward function test case ............................................................. 130
5.2.2 Tracking results by sensor tasking via maximising different reward functions .. 130
5.3.1 General schematics of centralised multi-sensor fusion and tasking using a network of space- or ground-based sensors or a mix of them, in an SSA application. 132
5.4.1 Multi-sensor multi-target tracking .................................................. 139
5.5.1 GTO, SSO sensor networks and GEO space objects .......................... 143
5.5.2 GEO space objects and FOR of the three ground-based sensors ............. 145
5.5.3 Position accuracy ................................................................. 148
5.5.4 All position errors (grey curves) and averaged position error (black curve) for all sensor networks ......................................................... 149
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.5.5</td>
<td>Number of targets and measurements</td>
<td>150</td>
</tr>
<tr>
<td>5.5.6</td>
<td>Number of visible targets</td>
<td>151</td>
</tr>
<tr>
<td>5.5.7</td>
<td>Run time ratio of the Cauchy-Schwarz divergence and Rényi divergence</td>
<td>152</td>
</tr>
<tr>
<td>5.5.8</td>
<td>Position accuracy and cardinality estimation results</td>
<td>153</td>
</tr>
<tr>
<td>5.5.9</td>
<td>Number of measurements and visible objects</td>
<td>154</td>
</tr>
<tr>
<td>B.3.1</td>
<td>The LMB filter with grouping and gating</td>
<td>166</td>
</tr>
<tr>
<td>C.1.1</td>
<td>10\textsuperscript{th} and 90\textsuperscript{th} percentiles of OSPA errors and cardinality estimate in Case I</td>
<td>170</td>
</tr>
<tr>
<td>C.2.1</td>
<td>10\textsuperscript{th} and 90\textsuperscript{th} percentiles of OSPA errors and cardinality estimate in Case II</td>
<td>171</td>
</tr>
</tbody>
</table>
Chapter 1 Introduction

The number of on-orbit space objects has been growing steadily due to the increasing and frequent space exploration activities. The majority of space objects drifting in Earth orbits are debris objects, including inactive spacecraft, fragmentation debris resulting from spacecraft or launch vehicle break-up events, mission-related debris such as discarded equipment dropped by astronauts, and rocket bodies from rocket-stages separated in the launching process. By 04 October 2018, the total number of the traceable on-orbit space targets catalogued by the United States Space Surveillance Network (SSN) was over 19,173 (as shown in Fig. 1.0.1) [1], of which payloads (including working and failed) comprise about 4,816. Among the catalogued space objects, the debris objects account for around 74.9% of the total population, and this number will continue to grow. It is estimated that there are more than 500,000 Earth-orbiting space objects larger than 1 cm in diameter and more than 100 million larger than 1 mm in diameter [2].

As the number of on-orbit space objects rises, the risk of collision between these objects significantly increases. A small piece of debris with only a few centimetres in diameter involved in a collision can damage or destroy an active satellite due to its high kinetic energy. The debris environment poses a serious threat to the normal operation of all spacecraft, the safety of space assets and the sustainable development of the space industry [3]. On 10 February 2009, an active U.S. satellite, Iridium-33, and a defunct Russian communication satellite, Cosmos 2251, collided at an altitude of 790 km over Siberia [4]. The collision generated two distinct debris clouds extending through Low Earth Orbit (LEO) [5] and they will stay in orbit for a long time.
Chapter 1. Introduction

Figure 1.0.1: Number of various types of objects in Earth orbit
due to their high orbit where atmospheric drag is reduced. This potentially increases a long-term risk in the space environment through a collisional cascading phenomenon [6]. On 27 March 2019, India launched an anti-satellite ballistic missile that destroyed an Indian orbiting satellite in LEO [7]. The launch created a cloud of debris that may last a year in space before disintegrating in Earth’s atmosphere. These events have highlighted the importance of developing effective collision avoidance techniques and international guidelines related to debris creation to protect all spacecraft.

1.1 Motivation

In order to mitigate collisions between active satellites and uncontrolled space objects, or eventually remove space debris objects, Space Situational Awareness (SSA) [8, 9] has attracted more and more attention. SSA refers to the ability to view, understand and predict the physical location of natural and manmade objects in orbit around the Earth, with the objective of avoiding collisions [8]. One critical component in SSA is to establish and maintain a Space Object Catalogue (SOC) [10]. A high-capacity SOC can be used for overall conjunction analysis for all catalogued objects so that accurate forecasts of potential risks and guidelines for collision
1.1. Motivation

avoidance manoeuvres can be performed.

An SOC is a record of the characteristics of the orbital population that is derived from measurements or records. The orbital information from the SOC is one major data source for many critical SSA applications, e.g., conjunction assessment and collision avoidance. There are three commonly used SOCs: the SOC of U.S. Space Command [11], the SOC of the Russian Federation, and the Database and the Information System Characterising Objects in Space (DISCOS) of the European Space Agency [12]. The SOC of U.S. Space Command is the most widely used catalogue, and is maintained by the U.S. SSN using tracking data from radar and optical devices. North American Aerospace Defense Command (NORAD) maintains parts of the catalogue for civilian use, published as a set of Two-Line Elements (TLE).

The SOC needs to be updated in a timely manner to provide reliable orbital information for the prediction of the orbital motion of space objects and the correlation of un catalogued observations. In order to routinely update the TLE catalogue, the U.S. SSN performs regular tracking of all catalogued objects and takes 380,000 - 420,000 observations each day using its global network of SSA sensors, e.g., radar and electro-optical systems [13]. As the power requirements of radar increase significantly at a greater range, it is typically used for tracking objects in LEO. Compared to the radar technique, optical sensors are more cost-effective for observing objects in higher orbital altitude as they are rarely restricted by range limitation. Therefore, the optical techniques are indispensable for monitoring a large number of space objects to perform regular catalogue maintenance. Several optical-based surveillance networks have been developed by the SSA community, including the ground-based electro-optical deep space surveillance system [14], and space-based space surveillance system [15, 16].

A major challenge in SSA is that only a limited number of optical sensors are available to cover a surveillance region [17], e.g., the Geosynchronous Earth Orbit (GEO) and Geosynchronous Transfer Orbit (GTO). Thus, many targets may only be observed within a limited time window, and the obtained observational arc is generally very short. IOD for such a short observational arc (i.e., a tracklet) [18] is a practical challenge because the range and range rate
information remain largely unknown, which refers to the Too-Short Arc (TSA) problem [19]. Classical IOD approaches, including Laplace, Gauss or Double-R methods, may yield poor results for observations from a single tracklet [19, 20]. Alternatively, a reliable orbital solution can be obtained by combining multiple tracklets at different epochs. Intuitively, this results in an extra problem of determining whether two tracklets originate from the same object, namely the tracklet association problem [21]. Following the association process, an IOD solution can be obtained for tracklets that have been successfully associated. Tracklet association methods are essential for cataloguing of uncorrelated observations to guarantee follow-up tracking of the catalogued objects.

Once an IOD solution is obtained, it can be further applied to a statistical estimator, e.g., Batch Least Squares (BLS), Bayesian single-target and/or MTT filters, to achieve a more precise orbital state estimate. The U.S. SSN employs an automated BLS technique [22] to improve the orbital state estimates for the large population of catalogued objects. The BLS methods and single-target filters require all subsequent observations used in the estimation process to originate from the same target, which may result in additional calls of the tracklet association process. The MTT filters are able to account for the measurement-to-track association problem and provide the capability of on-line orbital state estimates for a large number of space objects when considering several practical challenges, e.g., false alarms, missed detections, and new target birth. Therefore, the MTT filters are feasible solutions to reduce uncorrelated observations and produce refined orbital state estimation to improve the accuracy of catalogued objects. In addition, IOD methods can be integrated into the MTT framework for joint tracking of catalogued and newly discovered space object, which provides an effective means to enhance both the capability and capacity of the current catalogue.

In practical catalogue maintenance applications, a set of SSA sensors are required to jointly monitor a large number of catalogued space objects [23]. The major difficulty in this process can be formulated as a resource management problem, in which the major objective is to produce the best sensor tasking command to the most useful measurements for the desired application. The multi-sensor tasking scheme plays an important role in catalogue maintenance. It optimises
sensor resources to collect the best measurements for accurate orbital state estimation and consistently represents the uncertainty of catalogued or first detected targets to enable successful follow-up tracking [24].

Further, due to the upgrades of current sensors, along with the deployment of new sensors with greater capability, available observations will keep increasing in the future. NORAD aims to expand the capacity of the current catalogue to accommodate more than 150,000 objects in the future [15]. This raises the demand for further development of the tracklet association, multi-target tracking, and multi-sensor tasking techniques that provide improved performance for processing the increasing volume of observations. Thus, innovations for the above techniques are crucial to enhance the capacity and capability of the current catalogue, which is the primary focus of this dissertation.

### 1.2 Research Objectives

This section presents the research objectives for the tracklet association, multi-target tracking, and multi-sensor tasking techniques that are developed for accurate and efficient catalogue maintenance. For each investigated technique, a brief literature review is introduced as prerequisite knowledge, and the research objectives for corresponding practical challenges are then summarised.

#### 1.2.1 Tracklet Association

Tracklet association methods mainly assess whether two tracklets originate from the same object or not; if they are, the common orbit solution can be determined [19, 17]. As the initial orbital elements of tracklets are not deterministic, the association process is a challenging task and it has attracted considerable attention from the SSA community [17, 19, 25, 26, 27, 28, 29, 30].

Generally, a tracklet contains angles and angular-rate information, while the ranges and range rates are further required to obtain a complete IOD solution. However, these two quan-
ties are not available from optical observations. The admissible region approach [27] is a feasible IOD method which determines all range and range-rate hypotheses for angles-only observations. Given a tracklet, the admissible region is constructed as a closed region in the range and range-rate plane by considering the two-body energy constraint. The admissible region can be further restricted to a subset by considering several physical constraints (e.g., semi-major axis and eccentricity), which results in the so-called Constraints Admissible Region (CAR) [31]. The range and range-rate hypotheses combined with the angular information are transformed to Cartesian or Keplerian orbital elements space as hypothetical orbits. These hypothetical orbits can be used to test association with other tracklets, or employed to initialise a Bayesian filter for recursive estimation.

Several tracklet association methods have been developed based on the admissible region approach in the last decade [19, 28, 29]. Among the existing efforts, of particular interest in this thesis is the optimisation method because it avoids exhaustively processing all hypotheses. Siminski et al. [32, 17] proposed two representative optimisation methods, i.e., the Initial Value Problem (IVP) approach and the Boundary Value Problem (BVP) approach, that transform the tracklet association process to an optimisation problem. The optimisation results, i.e., the global minimum solutions of the corresponding loss function, are subsequently tested with a threshold. If no results pass this gate, it can be concluded that the observations do not share a common origin. The effectiveness of the IVP and BVP methods has been assessed using real optical observations of GEO objects. Assessment results indicate that the BVP method outperforms the IVP method in terms of robustness and computational efficiency. One drawback of these two methods is that the measurement noise needs to be calibrated before application, which increases the overall computational demand. In addition, the feasibilities of these two optimisation methods still need to be validated using tracklets from objects in different orbit domains. As the uncorrelated observations increase, it is necessary to develop a new tracklet association method that is able to accurately and efficiently associate a large population of tracklets from different orbit domains. Thus, this dissertation aims to develop an improved optimisation method that avoids the calibration process by using a new loss function defined in
1.2. Research Objectives

a non-singular canonical element space. The good association and run-time performance of the improved IVP method are validated using real optical data from different orbit domains.

In addition, the tracklets from different objects in the same constellation (dubbed as constellation tracklets in this thesis for simplicity), e.g., debris objects from a break-up event, remain highly indistinguishable. Fujimoto et al. [19] explained the difficulty of solving this issue as such tracklets share similarity in terms of the Keplerian orbital elements except for the mean anomaly. Furthermore, as the time interval between two constellation tracklets increases, they are more likely to be falsely associated. The erroneously associated constellation tracklets can lead to inaccurate orbital state estimation or even lost custody of catalogued targets. Therefore, developing a tracklet association method to identify false associations of constellation tracklets is essential for accurate catalogue maintenance, and this is considered as another objective of this thesis. A common ellipse method that can effectively reduce false associations of constellation tracklets is proposed in Chapter 2.

1.2.2 Multi-Target Tracking

Traditional methods of MTT applied to SSA include Global Nearest Neighbour (GNN) [33, 34], Multi-Hypothesis Trackers (MHT) [35, 36, 37, 38], and Joint Probabilistic Data Association (JPDA) [39, 40, 41] methods. Alternatively, more recently developed MTT methods leverage the theory of Finite Set Statistics (FISST) [42], yielding an explicit, comprehensive, unified statistical modelling of multi-sensor multi-target systems. It unifies target detection and state estimation into a single, seamless, Bayes-optimal procedure. The basic element of FISST is the Random Finite Set (RFS), an order-independent set of random vectors, that is used to define the multi-target state or measurement set at any given time. The core of the RFS approach is a Bayes multi-target filter that recursively propagates multi-target state density in time. Several efficient approximations to the RFS Bayes multi-target filter have been developed, including the Probability Hypothesis Density (PHD) filter [43], Cardinalised Probability Hypothesis Density (CPHD) filter [44], Multi-Bernoulli (MB) filter [45], $\delta$-Generalised Labelled Multi-Bernoulli ($\delta$-GLMB) filter [46, 47], and Labelled Multi-Bernoulli (LMB) filter [48]. The $\delta$-GLMB fil-
ter and the LMB filter are two of the latest implementations based on the labelled RFS theory. These labelled RFS filters are able to maintain target identities using a labelling strategy, therefore, they are naturally suitable for the maintenance of catalogued objects.

1.2.2.1 Comparison of Labelled RFS Filters

Vo et al. [46] proved that a class of labelled RFS distributions are conjugate, and their conjugate priors are closed under the multi-target Chapman Kolmogorov equation. In Bayes’ Theorem, if the multi-target prior and posterior belong to the same family, then they are regarded as conjugate distributions. Based on this derivation, the $\delta$-GLMB filter is proposed as an analytical solution to the multi-target Bayes recursion. One major contribution of the $\delta$-GLMB filter is that the target identities or labels are incorporated into individual target states. In this way, the target trajectories can be extracted as the target states with the same labels in the time series. The LMB inherits the advantages of the $\delta$-GLMB filter in that it outputs target tracks, while also achieving better computational efficiency. In order to simplify the computational complexity, a dynamic grouping and gating method is further developed by Reuter et al. [48] to enable parallelisation. Compared to the PHD, CPHD filters, the labelled RFS filters achieve more accurate state estimation, and they do not yield a cardinality bias or “spooky effect” \(^{1}\) [48, 47] because improved approximations of multi-target density have been made.

One bottleneck of the $\delta$-GLMB and LMB filter is the high computational complexity due to the exponential increase of hypotheses in the filtering process. In order to avoid exhaustively computing all hypotheses, two truncations of the multi-target densities in the prediction and update steps respectively are developed to generate hypotheses with significant weights. In order to further improve the computational efficiency, Vo et al. [49] proposed an efficient implementation of the $\delta$-GLMB filter using a joint prediction and update strategy, which only yields one truncation in each iteration. In addition, the truncation is implemented using a Gibbs sampling method, which dramatically reduces the computational complexity. The joint prediction and update and the Gibbs sampling method [50] can also be applied to the LMB filter, namely the joint

\(^{1}\)The “spooky effect” refers to a major drawback of the CPHD filter, which indicates the filter tends to shift the probability mass of a target to others as a result of missed detections.
1.2. Research Objectives

LMB filter [51]. Compared to the standard LMB filter, the joint LMB filter is able to further reduce the computational burden by simplifying the truncation process. Other developments that aim to improve the efficiency of the labelled RFS filters can be found in Refs. [52, 53, 54], and will not be elaborated from here.

The labelled RFS filters have been validated as viable approaches for SSA [55, 56, 9, 57]. However, a rigorous comparison of the labelled RFS filters for space object tracking is unavailable in the existing literature. This dissertation provides a quantitative comparison between the above mentioned four labelled RFS filters, i.e., the δ-GLMB filter, the LMB filter, and their joint versions, using two simulated test cases, including closely located space object tracking and large-scale space object tracking. Implementation of the four filters and their comparison results are presented in Chapter 2.

1.2.2.2 Multi-Target Tracking of New Space Object

In the context of multi-target tracking for SSA, tracking new space object is essential for expanding the capacity of the current catalogue. The initiation of new space object is extremely challenging for multi-target tracking using angles-only observations. Typical solutions to determine an IOD solution for optical observations from a single-arc are mainly based on the admissible region approach, e.g., the CAR [31] and Probabilistic Admissible Region (PAR) [58] methods. These two approaches provide a probabilistic representation of the initial orbital state of a tracklet, which can be used to initialise a Bayesian filter for follow-up tracking [31, 59, 40, 60, 58, 61, 62].

Recently, the CAR and PAR methods have been formulated as birth models and incorporated with the RFS filters for new space object tracking using short-arc angles-only observations. The measurement-based birth distribution [63, 48] is commonly used in RFS filters to determine the probability of existence of newly detected targets based on its association probability to existing measurements. Jones et al. [64, 65, 56] demonstrated the effectiveness of using RFS filters in combination with measurement-based CAR and PAR birth models for new space object tracking. Gehly et al. [66] employed the CAR birth model and CPHD filter for
new GEO object tracking, and derived the probability of existence of new tracks by considering the probability of observations originating from clutter or existing targets.

The major drawback of the typical CAR/PAR birth model is that a large number of Gaussian Mixture (GM) components are generated to approximate the initial target state, which introduces significant computational demand for the recursive filtering process. As mentioned in Sec. 1.2.1, the tracklet association methods are able to determine an IOD solution by associating two tracklets. Therefore, this dissertation explores the use of an efficient and robust tracklet association method, i.e., the BVP optimisation method, to model new target birth for improved computational efficiency of the filtering process. Compared with the CAR/PAR method which models the initial target state of a single-arc as a GM form, the BVP optimisation method computes a deterministic IOD solution using an additional arc. In this way, the orbit propagation and measurement-to-track association process can be dramatically simplified. The detailed procedure is presented in Chapter 4.

1.2.3 Multi-Sensor Tasking

Multi-sensor tasking in SSA aims to optimally allocate limited sensor resources to cover a large population of space objects. Traditional multi-sensor tasking methods in SSA can be organised as two subclasses, including the heuristic and information theoretic methods. The heuristic methods generate sensor tasking solutions without using target state information, and mainly focus on one particular task, such as high-quality measurements collection or more frequent coverage of the surveillance region [67, 68, 69, 70, 71, 72]. In the information theoretic sensor tasking process, an information functional is employed to measure the information divergence between the prior and pseudo-posterior multi-target densities. As the higher information divergence indicates a more significant reduction of the posterior uncertainty, it can be expected that the measurement maximising the information divergence also yields more accurate state estimation. Therefore, the information theoretic method enables the sensor tasking scheme to take the most informative measurements and achieve better use of information. Several information functionals have been developed for sensor tasking, including the Kullback-Leibler [73],
1.2. Research Objectives

Rényi [74], Fisher [75], and Cauchy-Schwarz divergences [76]. Based on these information functionals, the information theoretic sensor tasking method has been widely investigated in SSA [75, 77, 78, 79, 80, 81, 82].

Recent advances in Rényi divergence allow new developments of several multi-sensor tasking algorithms for SSA. Gehly et al. [24] designed a multi-sensor tasking scheme by maximising the Rényi divergence. In addition, a multi-step assignment strategy is developed to take the advantage of limited observation opportunities. The obtained sensor tasking solutions are able to account for targets leaving sensor Field of Regard (FOR). Gehly et al. [83] considered target priorities in sensor tasking process by introducing the tactical importance function into the Rényi divergence. Ravagoa et al. [57] employed the Rényi divergence and the LMB filter to conduct the sensor tasking and tracking problem. Note that the Rényi divergence is implemented in the Poisson RFS of the unlabelled LMB distribution, which is identical to the formulation for PHD in Ref. [24]. Herz et al. [84] developed the Heimdall software which employs the Rényi divergence and the CPHD filter for sensor tasking and state estimation problems.

The general form of the Rényi divergence needs numerical integration, and therefore the heavy computational load makes the Rényi divergence a poor choice for practical applications. Gehly et al. [24] derived the analytical forms of the Rényi divergence for Independent Identically Distributed (IID) cluster RFSs and Poisson RFSs based on the assumption that each target state is represented by a single Gaussian component. However, the analytical form of the Rényi divergence has not been derived for labelled RFS families in the existing literature. Therefore, the derivation of an analytical Rényi divergence for labelled RFSs is essential for efficient multi-sensor tasking using the labelled RFS filters. In order to address this issue, this dissertation aims to develop an analytical Rényi divergence for LMB RFSs to simplify the computational complexity. The derived analytical Rényi divergence can be formulated as an objective function to solve the multi-sensor tasking problem.

In addition, sensor tasking for object in the GEO region yields more challenges because the high orbit altitude restricts the use of ground-based radar and laser ranging sensors to produce
range information. Traditional methods for monitoring the GEO region mainly rely on angular observations generated by ground-based optical sensors. Recent works [85, 77] suggest that space-based optical sensors provide improved capability for GEO object tracking because they are not restricted by the weather conditions and atmospheric attenuation. Several space-based optical sensors have been launched into Sun Synchronous Orbit (SSO) [86, 87, 88]. However, space-based optical sensors placed in GTO [85] can theoretically provide better viewing geometry for the GEO region, and are more applicable for tracking small-size debris objects in GEO for improved catalogue maintenance. Space-based sensors at different orbit domains provide different tracking performance, and a quantitative comparison is essential to support practical applications. This dissertation designs several space-based sensor networks and provides a rigorous comparison by assessing several parameters e.g., the orbital state estimation, number of measurements and targets detected. In addition, in the sensor tasking process, several realistic constraints are also considered including illumination and eclipse conditions [89, 90, 91].

1.3 Thesis Overview

The overall organisation of this dissertation is shown in Fig. 1.3.1. Three new methods are developed to support catalogue maintenance and other SSA applications.

Chapter 2 first briefly introduces the research background of the tracklet association methods for IOD, including the definition of the admissible region and two popular tracklet association methods, namely the optimisation and hyperplane intersection methods. Then, an improved IVP optimisation method is proposed and tested using real optical data collected from Mt. Stromlo Observatory. In addition, in order to address the challenging problem of tracklet association for tracklets from objects in a constellation, a common ellipse method is proposed and validated using real optical data.

Chapter 3 introduces the mathematical definition of the RFS theory, and several efficient implementations of the labelled RFS Bayesian multi-target filters are introduced. The performance of four labelled RFS filters are compared in two simulated space object tracking scenar-
Chapter 4 proposes a new multi-target birth model for use in the LMB filter using the BVP optimisation approach. The BVP birth model estimates a reliable IOD solution of the new track using two successive tracklets, and its initial covariance is estimated by processing all individual observations of the two tracklets using the BLS method. The performance of the proposed BVP birth model is validated by comparing against two conventional birth models in two simulated multi-target tracking scenarios.

Chapter 5 proposes a novel multi-sensor tasking method for GEO catalogue maintenance. An analytical formulation of the Rényi divergence for LMB RFSs is derived and utilised as the objective function for sensor tasking. The proposed sensor tasking method is implemented in several space-based and ground-based sensor networks. Two simulated scenarios, including large-scale GEO object tracking and dim GEO object tracking are presented for validation.

The conclusion of this thesis and proposed directions for future research are provided in
the last chapter.

1.4 Contributions of Research

The primary contributions of this thesis are briefly summarised as follows:

1) The multi-target tracking filters based on the latest labelled RFS theory have been demonstrated as effective solutions for space object tracking.

2) An improved IVP optimisation method is proposed that achieves improved tracklet association and run-time performance compared with the traditional IVP and BVP optimisation methods.

3) A common ellipse method is developed to identify the false associations of tracklets from objects in the same constellation for better tracklet association accuracy.

4) A new birth model based on the BVP optimisation is presented for use with the LMB filter. The derived BVP-LMB filter achieves better efficiency for new space object tracking compared with the conventional CAR and PAR methods.

5) An analytical Rényi divergence for LMB RFSs is derived and formulated as the reward function to efficiently and effectively task multiple sensors for accurate tracking of GEO object.
Chapter 2  Tracklet Association Methods

This chapter first introduces the fundamentals of the methodologies investigated throughout the thesis. Section 2.1 briefly introduces the theory of the attributable vector, the admissible region method, and the constrained admissible region method for IOD. Two representative tracklet association methods based on the admissible region approach are then introduced in Sec. 2.2.

In order to further investigate the optimisation-based tracklet association methods for improved association and run-time performance, Sec. 2.3 proposes an improvement to the traditional IVP optimisation approach that determines the association by searching for the global minimum of a new loss function defined in a non-singular canonical space. As all canonical elements have the same unit, a direct comparison of the distance between any hypothetical orbits can be achieved without considering the calibration of the measurement noise. The improved IVP method is validated using optical data of space objects at different altitudes collected from the Mt. Stromlo Observatory and compared with traditional IVP and another popular tracklet association method: the BVP optimisation approach. Results illustrate that the improved IVP method is superior to IVP and BVP in terms of association performance and run-time performance.

In addition, traditional methods are prone to the problem of incorrectly associating tracklets from different objects in the same constellation. To address this issue, a new approach dubbed the common ellipse method is presented in Sec. 2.4. The core of this approach is to find a best fitting common ellipse to all hypothetical orbits of the falsely associated tracklets in a
constellation, if such a common ellipse exists, then the false associations can be identified. The common ellipse method is tested with tracklets from 86 satellites in the Iridium constellation, and results show that it significantly improves the true negative rate for the tested scenario.

2.1 Admissible Region Methods

The IOD problem, as one of the main focuses of SSA, is essential for the maintenance of a catalogue. A good IOD result can also improve the accuracy of orbit prediction and conjunction assessment. IOD for debris objects is particularly challenging [92], especially in the case where only optical observations, i.e., angles-only observations, are available, such as for objects in GEO. As the large population of space debris is tracked only by a limited number of sensors, the observation arcs of debris objects are very short (typically a few minutes). Given a tracklet, classical IOD methods such as Laplace, Gauss, and Double-R tend to produce poor results.

A tracklet from an optical arc consists of angular observations, from which angular rates can be extracted whereas range and range-rate are unavailable. As the ranging information cannot be determined from a single-arc, the admissible region method restricts all possible range and range-rate values to a closed region using constraints on the two-body energy [27]. The admissible region can either be approximated using the Gaussian Mixture Model (GMM) method to initialise a Bayesian sequential estimator, or applied to tracklet association methods to determine the association of two tracklets. The admissible region method is the foundation of the tracklet association method and the birth model developed in this thesis, and therefore its fundamentals are first introduced in this section.

2.1.1 Attributable Vector

Given a tracklet formed by a set of discrete optical observations, its angular data can be organised in an optical attributable vector for further derivation of the range and range-rate hypotheses. The formulation of the optical attributable vector $\mathcal{A}$ used here follows Maruskin [28]. Specifically, the optical attributable vector $\mathcal{A}$ contains two angles and two angular rates at an
2.1. Admissible Region Methods

observing epoch,

\[ \mathcal{A} = (\alpha, \delta, \dot{\alpha}, \dot{\delta}) \in [-\pi, \pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}^2, \]  

(2.1.1)

where \( \alpha \) and \( \delta \) are the topocentric right ascension and declination respectively; \( \dot{\alpha} \) and \( \dot{\delta} \) are their angular rates. These angles and angular rates are defined in an equatorial reference system (e.g., the commonly used EME2000 coordinate system).

Other data including the observing time \( t \) and the location of the observing station are also included, forming the extended attributable vector \( \mathbf{x} \) given by:

\[ \mathbf{x} = (\mathcal{A}, t, h, \phi, \varphi), \]  

(2.1.2)

where \( h, \phi \) and \( \varphi \) are the altitude, longitude, and latitude of the observing station respectively.

The angles information \( \alpha \) and \( \delta \) of each discrete observation within the tracklet can be directly extracted from the raw data, while the angular rate information is generally obtained based on estimation methods. Maruskin [28] proposed an approach to model the kinematics of the angular motion of an object, given by

\[ \alpha(t) = \alpha_0 + \dot{\alpha}_0(t - t_0) + \frac{1}{2}\ddot{\alpha}_0(t - t_0)^2, \]  

(2.1.3)

where \( \dot{\alpha}_0 \) and \( \ddot{\alpha}_0 \) are the angular rate and acceleration of \( \alpha_0 \) at time \( t_0 \) respectively. The objective is to estimate the angular position, angular rate, and angular acceleration of the object at time \( t \), which can be solved using the Least squares (LS) estimation method to process all observations within a tracklet. This estimation process is applied assuming one-dimensional angular motion with observations from the tracklet equally spaced and centred on the epoch \( t_0 \) with uncorrelated errors. According to Maruskin [28], if the sensors have 1 arc-second observation uncertainties, then the uncertainty of the angular location and angular rate of the object are on the order of \( 2.62 \times 10^{-6} \text{ rad} \) and \( 6.28 \times 10^{-5} \text{ rad/h} \) respectively.

Fig. 2.1.1 shows an example of the angular rate estimation process. The LS method is
used to fit the real right ascension and declination observations of a tracklet from an object in LEO (NORAD ID 815) collected from the Mt. Stromlo Observatory, Canberra, Australia. In each subplot, the observations are shown as red dots, and the LS estimation results are shown as the black curves. These two types of data are highly overlapped, which indicates that the LS method is able to fit the angular observations.

![Least squares fit to angular observations](image)

**Figure 2.1.1:** Least squares fit to angular observations

### 2.1.2 Admissible Region

The theory of the admissible region was first proposed by Milani et al. [27] for tracking celestial bodies. Tommei et al. [26] applied the theory to the field of angles-only IOD for space debris tracking. The admissible region approach is the basis of several innovations for IOD and tracklet association problems. Given an attributable vector, the admissible region method restricts the possible values of the two undetermined quantities: range $\rho$ and range-rate $\dot{\rho}$. An admissible region can be regarded as a closed region constructed in the $(\rho, \dot{\rho})$ plane using the constraint of two-body energy $\varepsilon$ to ensure a closed orbit about Earth, given by:

$$
\varepsilon = \frac{\| \dot{r} \|^2}{2} - \frac{\mu}{\| r \|} < 0,
$$

(2.1.4)
where $\mu = GM$ is the gravitational parameter of the Earth, $\mathbf{r}$ and $\mathbf{r}$ are the inertial position and velocity of a space object with respect to the mass centre of the Earth.

Generally, the admissible region is approximated as a uniform distribution, based on the assumption that no range and range-rate pairs are more likely than any other \[59\]. Each uniformly discretised point in the $(\rho, \dot{\rho})$ plane is referred to as a Virtual Particle (VP) \[19\], and theoretically, the number of VPs within an admissible region is infinite. Given an attributable vector, each VP can be used to generate a hypothetical orbit $O$. The position and velocity of the object with respect to the Earth centre is defined by

\[
\mathbf{r} = \mathbf{r}_0 + \rho
\]

\[
\dot{\mathbf{r}} = \dot{\mathbf{r}}_0 + \dot{\rho},
\]

where $\mathbf{r}_0$ and $\dot{\mathbf{r}}_0$ are the position and velocity of the observing station with respect to the Earth centre respectively, and $\rho$ and $\dot{\rho}$ are the inertial position and velocity of the object with respect to the station respectively, which can be described using the spherical coordinates, given by

\[
\rho = \rho \mathbf{u}_\rho
\]

\[
\dot{\rho} = \dot{\rho} \mathbf{u}_\rho + \rho \dot{\alpha} \mathbf{u}_\alpha + \rho \dot{\delta} \mathbf{u}_\delta,
\]

where $\mathbf{u}_\rho$, $\mathbf{u}_\alpha$ and $\mathbf{u}_\delta$ are given by

\[
\mathbf{u}_\rho = [\cos \alpha \cos \delta, \sin \alpha \cos \delta, \sin \delta]^T
\]

\[
\mathbf{u}_\alpha = [- \sin \alpha \cos \delta, \cos \alpha \cos \delta, 0]^T
\]

\[
\mathbf{u}_\delta = [- \cos \alpha \sin \delta, - \sin \alpha \sin \delta, \cos \delta]^T.
\]

The topocentric right ascension $\alpha$, declination $\delta$ and their angular rates are calculated using
the following equations

\[
\alpha = \tan^{-1} \left( \frac{r_y - r_{y,0}}{r_x - r_{x,0}} \right) \tag{2.1.12}
\]

\[
\delta = \sin^{-1} \left( \frac{\hat{r}_z - \hat{r}_{z,0}}{\rho} \right) \tag{2.1.13}
\]

\[
\dot{\alpha} = \frac{1}{1 + \left( \frac{r_x - r_{x,0}}{r_y - r_{y,0}} \right)^2} \left[ \frac{(\hat{r}_y - \hat{r}_{y,0})(r_x - r_{x,0}) - (\hat{r}_x - \hat{r}_{x,0})(r_y - r_{y,0})}{(r_x - r_{x,0})^2} \right] \tag{2.1.14}
\]

\[
\dot{\delta} = \frac{1}{1 - \left( \frac{r_x - r_{x,0}}{\rho} \right)^2} \left[ \frac{\dot{r}_z - \dot{r}_{z,0} - (\rho \cdot \dot{\rho})(r_z - r_{z,0})}{\rho^3} \right] \tag{2.1.15}
\]

where all values are defined in the ECI coordinate frame, \( r = (r_x, r_y, r_z) \) and \( \dot{r} = (\dot{r}_x, \dot{r}_y, \dot{r}_z) \) are the position and velocity of the object respectively, and \( r_0 = (r_{x,0}, r_{y,0}, r_{z,0}) \) and \( \dot{r}_0 = (\dot{r}_{x,0}, \dot{r}_{y,0}, \dot{r}_{z,0}) \) are the position and velocity of the ground station respectively.

The squared Euclidean norms of the position and velocity of the object with respect to the Earth centre are given by

\[
\| r \|^2 = \rho^2 + w_5 \rho + w_0 \tag{2.1.16}
\]

\[
\| \dot{r} \|^2 = \dot{\rho}^2 + w_1 \dot{\rho} + w_2 \dot{\rho}^2 + w_3 \rho + w_4, \tag{2.1.17}
\]

where the scalar variables are defined as

\[
w_0 = \| \mathbf{q} \|^2, w_1 = 2 (\mathbf{q} \cdot \mathbf{u}_\rho), w_2 = \dot{\alpha}^2 \cos^2 \delta + \dot{\delta},
\]

\[
w_3 = 2 \dot{\alpha} (\mathbf{q} \cdot \mathbf{u}_\alpha) + 2 \dot{\delta} (\mathbf{q} \cdot \mathbf{u}_\delta), w_4 = \| \mathbf{q} \|^2, w_5 = 2 (\mathbf{q} \cdot \mathbf{u}_\rho).
\]

The energy equation can be rearranged by substituting Eqs. (2.1.16) and (2.1.17) into Eq. (2.1.4), given by

\[
\dot{\rho}^2 + w_1 \dot{\rho} + F(\rho) - 2 \varepsilon = 0, \tag{2.1.19}
\]

where

\[
F(\rho) = w_2 \dot{\rho}^2 + w_3 \rho + w_4 - \frac{2 \mu}{\sqrt{\dot{\rho}^2 + w_5 \rho + w_0}}. \tag{2.1.20}
\]
Given a fixed value of \( \rho \), the value of \( \dot{\rho} \) is obtained by solving the quadratic Eq. (2.1.19), and \( \dot{\rho} \) can be written as a function of \( \rho \) using the following equation.

\[
\dot{\rho} = \frac{w_1}{2} \pm \sqrt{\left(\frac{w_1}{2}\right)^2 - F(\rho) + 2\varepsilon}.
\]  

(2.1.21)

Fig. 2.1.2 shows an example of an admissible region for an attributable vector: \( A = (3.044, 0.099, 0.262, 5.836 \times 10^{-5}) \) of a LEO object (NORAD ID:815). The admissible region is depicted in the range and range-rate plane with units of Earth radius (ER) and Earth radius/hour (ER/h) respectively. The blue curve is the boundary of the admissible region generated using the two-body energy constraint, the red dots represent a set of discretised and uniformly distributed VPs, and the black star is the location of the truth value of range and range-rate.

2.1.3 Constrained Admissible Region

In order to further reduce the number of hypotheses, the admissible region can be confined to a smaller subset by introducing more physical constraints on parameters such as the semi-major axis and eccentricity of the orbit. The obtained subset of the admissible region is called the CAR [31]. Given constraints on semi-major axis and eccentricity, the corresponding values determining the boundary of the CAR are generated in the \((\rho, \dot{\rho})\) plane.
The semi-major axis constraint can be regarded as an energy constraint since the semi-major axis \( a \) is a function of energy \( \varepsilon \), as expressed by

\[
\varepsilon = -\frac{\mu}{2a}.
\] (2.1.22)

Given a semi-major axis constraint value, an energy value is determined using Eq. (2.1.22). Then, the boundary curve corresponding to the semi-major axis constraint can be generated by solving Eq. (2.1.19) using the obtained energy value and all possible range values.

The eccentricity constraint is a function of angular momentum \( h \) and energy \( \varepsilon \), which is given by

\[
e = -\sqrt{1 + \frac{2\varepsilon \|h\|^2}{u^2}},
\] (2.1.23)

where \( h \) is the cross product of the position and velocity of a space object, given by

\[
h = r \times \dot{r}.
\] (2.1.24)

Defining the following vector parameters,

\[
\begin{align*}
h_1 &= q \times u_\rho \\
h_2 &= u_\rho \times (\dot{\alpha}u_\alpha + \dot{\delta}u_\delta) \\
h_3 &= u_\rho \times \dot{q} + q \times (\dot{\alpha}u_\alpha + \dot{\delta}u_\delta) \\
h_4 &= q \times \dot{q},
\end{align*}
\] (2.1.25, 2.1.26, 2.1.27, 2.1.28)

then, Eq. (2.1.24) can be rearranged as

\[
h = h_1 \dot{\rho} + h_2 \rho^2 + h_3 \rho + h_4.
\] (2.1.29)
The squared Euclidean norm of the angular momentum is given by

$$\| h \|^2 = c_0 \dot{\rho}^2 + P \rho \dot{\rho} + U(\rho),$$

(2.1.30)

where

$$P(\rho) = c_1 \rho^2 + c_2 \rho + c_3$$

(2.1.31)

$$U(\rho) = c_4 \rho^4 + c_5 \rho^3 + c_6 \rho^2 + c_7 \rho + c_8,$$

(2.1.32)

and the scalar parameters are given by

$$c_0 = \| h_1 \|^2, c_1 = 2 h_1 \cdot h_2, c_2 = 2 h_1 \cdot h_3,$$

$$c_3 = 2 h_1 \cdot h_4, c_4 = \| h_2 \|^2, c_5 = 2 h_2 \cdot h_3,$$

(2.1.33)

$$c_6 = 2 h_2 \cdot h_4 + \| h_3 \|^2, c_7 = 2 h_3 \cdot h_4, c_8 = \| h_4 \|^2.$$

Thus, substituting Eqs. (2.1.19) and (2.1.30) into the eccentricity equation (2.1.23) results in

$$a_4 \dot{\rho}^4 + a_3 \dot{\rho}^3 + a_2 \dot{\rho}^2 + a_1 \dot{\rho} + a_0 = 0,$$

(2.1.34)

where

$$a_4 = c_0, a_3 = P(\rho) + c_0 w_1, a_2 = U(\rho) + c_0 F(\rho) + w_1 P(\rho),$$

$$a_1 = F(\rho) P(\rho) + w_1 U(\rho), a_0 = F(\rho) U(\rho) + \mu^2 (1 - e^2).$$

(2.1.35)

Given an eccentricity value, a boundary curve of the eccentricity constraint can be obtained by solving the above equations for all possible range values [31]. Then, the intersection of the semi-major axis and eccentricity constraint curve reduces the admissible region to a CAR.

Fig. 2.1.3 shows an example of the CAR constructed using the same attributable vector as employed in Fig. 2.1.2. The constraint range of semi-major axis values is $a \in [0, 6.5]$ (ER), and the constraint range of eccentricity values is $e \in [0, 0.3]$. The blue curve is the semi-major
axis constraint boundary, the red curve is the eccentricity constraint boundary, the black dots represent the uniformly discretised VPs within the CAR, and the star is the true value of range and range-rate. Results indicate that the CAR significantly reduces the admissible region to a compact subregion so that the computational efficiency in subsequent applications, e.g., tracklet association and birth models, can be improved.

2.2 Traditional Tracklet Association Methods

In order to solve the tracklet association problem, several methods based on the admissible region have been explored. Two representative methods, namely the optimisation method and the hyperplane intersection method, are investigated in this section.

2.2.1 Optimisation Methods

The orbital state $x(t) = (r(t), \dot{r}(t))^T$ of a space object follows the first order differential equations, given by

$$\dot{x}(t) = f(x(t), t).$$

(2.2.1)

An orbital state $x(t)$ can be uniquely defined by six independent parameters and described using either an initial value $(r(t_1), \dot{r}(t_1))^T$ at epoch $t_1$, or boundary values $(r(t_1), r(t_2))^T$ at epochs
2.2. Traditional Tracklet Association Methods

$t_1$ and $t_2$ respectively. Siminski et al. [17, 93] presented two solutions to tracklet association based on initial value and boundary value formulations respectively, with proposed optimisation methods for solving both.

In the IVP and BVP methods, the tracklet association is reformulated as an optimisation problem, and a gradient-based optimiser was used to find the global minimum of the loss functions of the two methods for the determination of association. These methods are of particular interest due to their high accuracy and efficiency of association performance.

2.2.1.1 Initial Value Problem Approach

Given two tracklets $\mathbf{x}_1$ and $\mathbf{x}_2$ with attributable vector $\mathbf{A}_1 = (\alpha_1, \delta_1, \dot{\alpha}_1, \dot{\delta}_1)$ and $\mathbf{A}_2 = (\alpha_2, \delta_2, \dot{\alpha}_2, \dot{\delta}_2)$ respectively, the IVP optimisation process determines a best VP $p^* = (\rho^*, \dot{\rho}^*)$ that generates a best fitting hypothetical orbit to both $\mathbf{x}_1$ and $\mathbf{x}_2$.

Specifically, the attributable vector $\mathbf{A}_1 = (\alpha_1, \delta_1, \dot{\alpha}_1, \dot{\delta}_1)^T$ of tracklet $\mathbf{x}_1$ at epoch $t_1$, together with a VP $p_1 = (\rho_1, \dot{\rho}_1)$ are used to define a hypothetical orbit using Eqs. (2.1.5) and (2.1.6). This hypothesis is propagated to the time epoch $t_2$ of tracklet $\mathbf{x}_2$ and transformed to the observation space to obtain the modelled angular observations $\hat{\mathbf{A}}_2 = (\hat{\alpha}_2, \hat{\delta}_2, \hat{\dot{\alpha}}_2, \hat{\dot{\delta}}_2)^T$. A loss function is defined to measure the difference between the actual observation $\mathbf{A}_2 = (\alpha_2, \delta_2, \dot{\alpha}_2, \dot{\delta}_2)^T$ of tracklet $\mathbf{x}_2$ and the modelled value $\hat{\mathbf{A}}_2$ which is scaled by their associated uncertainties (i.e., $C_{\mathbf{A}_2}$ and $C_{\hat{\mathbf{A}}_2}$), given by

$$L_{IVP}(p) = (\mathbf{A}_2 - \hat{\mathbf{A}}_2)^T(C_{\mathbf{A}_2} + C_{\hat{\mathbf{A}}_2})^{-1}(\mathbf{A}_2 - \hat{\mathbf{A}}_2),$$

(2.2.2)

where the covariance $C_{\hat{\mathbf{A}}_2}$ is obtained by linearly propagating the covariance of the initial observation $C_{\mathbf{A}_1}$

$$C_{\hat{\mathbf{A}}_2} = (\frac{\partial \hat{\mathbf{A}}_2}{\partial \mathbf{A}_1})^T C_{\mathbf{A}_1} (\frac{\partial \hat{\mathbf{A}}_2}{\partial \mathbf{A}_1}).$$

(2.2.3)

Optimising Eq. (2.2.2) produces the desired best fitting hypothetical orbit $(r^*, \dot{r}^*)$ with the associated global minimum value of the loss function. The association is decided by checking
if the global minimum value falls below a predefined threshold. The IVP optimisation method is presented in Fig. 2.2.1.

![Flowchart of IVP Optimisation Method](image)

**Figure 2.2.1:** The flowchart of the IVP optimisation method

The IVP loss function can be optimised using a pattern search method [17]. One challenging problem in the IVP optimisation process is the determination of the global minimum despite the existence of several local minima, which render the problem one of Multi-Modal Optimisation (MMO) [17]. In other words, in the IVP optimisation procedure, each local minima needs to be identified and compared to determine the global minimum. Otherwise, the result of IVP optimisation may turn out to be a local minimum and thus yield missed associations.

The loss function of two GEO tracklets separated by a three-day interval is computed and presented in Fig. 2.2.2. The loss function is defined on the plane of the semi-major axis and relative range [17]. The semi-major axis constraint used to bound the hypothetical orbital states is set to $a \in [4, 7]$ ER. There are four valleys as seen in Fig. 2.2.2a, and each needs to be identified for optimisation. In this study, the eccentricity constraint is introduced to IVP to improve the run-time performance by reducing the population of hypothetical orbits. The
2.2. Traditional Tracklet Association Methods

![Graph](image)

(a) \( \log(L_{IVP}) \) in the semi-major axis and relative range plane  
(b) The region inside the green boundary shown in left subfigure

**Figure 2.2.2:** Topography of the IVP loss function

curves in Fig. 2.2.2a represent the boundaries of different eccentricity constraint values, i.e., \( e = 0.1, 0.3, 0.5, 0.7 \). Smaller eccentricity constraint values yield a more compact region of the loss function topography. For GEO tracklets, the eccentricity constraint value can be defined within a small range. Fig. 2.2.2b shows the loss function topography constrained by \( e \in [0, 0.1] \), which is the region within the blue boundary shown in Fig. 2.2.2a. The use of such a small eccentricity constraint range yields only one valley in this case, which simplifies the optimisation process. To further speed up the optimisation, the semi-major axis constraint can also be reduced to a small range for tracklets from a specific orbit region.

### 2.2.1.2 Boundary Value Problem Approach

Given two tracklets \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \) with attributable vectors \( \mathbf{A}_1 = (\alpha_1, \delta_1, \dot{\alpha}_1, \dot{\delta}_1) \) and \( \mathbf{A}_2 = (\alpha_2, \delta_2, \dot{\alpha}_2, \dot{\delta}_2) \) respectively, the angles vector \( z_{LOS} = (\alpha_1, \delta_1, \alpha_2, \delta_2) \) describes the Line-of-Sight (LOS) information at both epochs, and the angular rates \( z_r = (\dot{\alpha}_1, \dot{\delta}_1, \dot{\alpha}_2, \dot{\delta}_2) \) will be employed as the discriminator in the BVP optimisation process.

To obtain the boundary value \( (r(t_1), r(t_2))^T \) representation of a complete orbital state, the LOS vector \( z_{LOS} \) can be augmented by a hypothetical range pair \( \mathbf{r} = (\rho_1, \rho_2)^T \). Then the velocity solution \( (\dot{r}(t_1), \dot{r}(t_2))^T \) determined by a Lambert solver is used to form a hypothetical orbit \( \mathcal{O} \) for the hypothesis \( \mathbf{r} \). The modelled angular rates \( \dot{z}_r = (\dot{\alpha}_1, \dot{\delta}_1, \dot{\alpha}_2, \dot{\delta}_2) \) at both epochs
can be calculated based on the hypothetical orbit $O$ using the measurement model $h(x)$.

The loss function used in BVP to determine the best hypothetical orbit $O^*$ is defined as the difference between the measured $z_r = (\dot{\alpha}_1, \dot{\delta}_1, \dot{\alpha}_2, \dot{\delta}_2)$ and the modelled $\hat{z}_r = (\hat{\alpha}_1, \hat{\delta}_1, \hat{\alpha}_2, \hat{\delta}_2)$ scaled by their uncertainty (i.e., $C_{z_r}$ and $C_{\hat{z}_r}$),

$$L_{BVP}(\tau, n_o) = (z_r - \hat{z}_r)^T(C_{z_r} + C_{\hat{z}_r})^{-1}(z - \hat{z}_r),$$

(2.2.4)

where $n_o$ represents half orbital revolution intervals, and the covariance $C_{\hat{z}_r}$ is obtained by the following formulation

$$C_{\hat{z}_r} = \left(\frac{\partial \hat{z}_r}{\partial z_{\text{LOS}}}\right)^T C_{z_{\text{LOS}}} \left(\frac{\partial \hat{z}_r}{\partial z_{\text{LOS}}}\right).$$

(2.2.5)

The BVP optimisation process determines association by evaluating all possible full and half orbit revolutions $n_o$. The parameter $n_o$ is a function of the time interval $\Delta t = t_2 - t_1$ and orbital period $T_p$. Since the orbital state of a tracklet is unavailable, a possible range of $T_p$ can be obtained based on the semi-major axis constraints to calculate $n_o$. Generally, the Lambert solver yields a short path and a long path for the transfer between two points in space. The short path is selected if the fractional part of $n_o$ is smaller than 0.5, otherwise the long path is chosen. For each $n_o$ interval, the BVP optimisation method determines a global minimum of the loss function. The overall global minimum $L^*$ of two tracklets is the smallest value of all global minimum values corresponding to all possible $n_o$ intervals. One primary advantage of BVP is that only one local minimum exists for a $n_o$ value, which does not yield the MMO problem required for IVP.

In order to reduce the set of possible values of the range hypothesis $\tau$, the smallest perigee radius $r_{\text{min}}$ and largest apogee radius $r_{\text{max}}$ need to be defined corresponding to the bounds on semi-major axis $[a_{\text{min}}, a_{\text{max}}]$ and eccentricity $[e_{\text{min}}, e_{\text{max}}]$. The orbit radius is bound within
2.2. Traditional Tracklet Association Methods

\[ [r_{\text{min}}, r_{\text{max}}] \], where

\[
\begin{align*}
r_{\text{min}} &= a_{\text{min}}(1 - e_{\text{max}}) \\
r_{\text{max}} &= a_{\text{max}}(1 + e_{\text{max}}).
\end{align*}
\] (2.2.6)

Then, the bound of \((\rho_1, \rho_2)\) is given by

\[
\rho_i \in \left[ -\left( r_{0,i} \cdot \mathbf{u}_{\rho,i} \right) + \sqrt{\left( r_{0,i} \cdot \mathbf{u}_{\rho,i} \right)^2 + r_{\text{min}}^2 - w_0}, \quad -\left( r_{0,i} \cdot \mathbf{u}_{\rho,i} \right) + \sqrt{\left( r_{0,i} \cdot \mathbf{u}_{\rho,i} \right)^2 + r_{\text{max}}^2 - w_0} \right],
\] (2.2.7)

where \(i = 1, 2\) is the index of two measurements; the ground station position vector \(r_0\), pointing vector \(\mathbf{u}_\rho\) and scalar parameter \(w_0\) are defined in Sec. 2.1.2.

In addition, the boundary of all possible orbital revolution numbers is determined from the bounded orbital period,

\[
n_o \in \left[ \frac{\Delta t}{T_{p,\text{max}}}, \frac{\Delta t}{T_{p,\text{min}}} \right],
\] (2.2.8)

where the maximum and minimum orbit periods are derived from the semi-major axis values using Kepler’s third law.

Optimisation methods based on the quasi-Newton method, e.g., the popular Broyden Fletcher Goldfarb Shanno (BFGS) method [94], can be employed to search for each local minimum of the loss function. Starting from an initial point \((\rho_1^0, \rho_2^0)\), the BFGS method performs an iterative search along the path of steepest descent derived from the gradient of \(L_{BVP}(\tau, n_o)\). For each value of \(n_o\), the initial point \((\rho_1^0, \rho_2^0)\) can be uniquely defined by

\[
\rho_i^0 = -(r_{0,i} \cdot \mathbf{u}_{\rho,i}) + \sqrt{(r_{0,i} \cdot \mathbf{u}_{\rho,i})^2 + a(n_o)^2 - w_0},
\] (2.2.9)

where \(a(n_o)\) is the semi-major axis value for a given \(n_o\) determined using Kepler’s third law. The optimisation process is terminated if the loss function value is reduced under a predefined threshold or if the number of iterations exceeds the maximum. In addition, if the orbital state of an updated range pair violates the constraints, e.g., semi-major axis and eccentricity, the
iterative search is stopped.

![Flowchart of the BVP optimisation method](image)

**Figure 2.2.3:** The flowchart of the BVP optimisation method

For different $n_o$ intervals, if any global minimum values fall below the association threshold $T_o$, then the corresponding two tracklets are regarded to be associated. In this study, only the hypothetical orbit $O^*$ that yields the overall global minimum loss function value $L^*$ is regarded as the IOD solution in the case that multiple global minimum values meet the threshold criterion. Fig. 2.2.3 shows the flowchart of the BVP optimisation method.

The BVP optimisation method is further illustrated by associating two simulated tracklets from the same GEO object separated by around one day. The semi-major axis and the eccentricity constraint values are defined as $[6.3, 7]$ ER and $[0, 0.06]$ respectively. These parameters result in two possible $n_o$ intervals, i.e., $n_o \in [0.5, 1]$ and $n_o \in [1, 1.5]$, and both need to be optimised. Fig. 2.2.4a and Fig. 2.2.4c show the 3-D topography of the BVP loss function for the two $n_o$ intervals. The colour map represents the log value of the loss function, where the log is used for better visualisation. Fig. 2.2.4a shows a flat topography and large loss function values resulting from $n_o \in [0.5, 1]$. The shape of the topography in Fig. 2.2.4c obtained us-
2.2. Traditional Tracklet Association Methods

(a) \( \log_{10} \) value of the BVP loss function for the interval \( n_o \in [0, 5, 1] \)

(b) Optimisation process for the interval \( n_o \in [0, 5, 1] \)

(c) \( \log_{10} \) value of the BVP loss function for the interval \( n_o \in [1, 1.5] \)

(d) Optimisation process for the interval \( n_o \in [1, 1.5] \)

**Figure 2.2.4:** The BVP loss function and optimisation process

\( n_o \in [1, 1.5] \) resembles a funnel, with only one local minimum, which is also the global minimum, as shown in the figure. Fig. 2.2.4b and Fig. 2.2.4d depict the BFGS optimisation process, in which the background is the contour of the topography, the square indicates the starting point, the red lines show the path of steepest descent, the circles are the updated range pairs in each iteration, and the diamond is the obtained local minimum. The truth is shown as the star, which is partly overlapped with the diamond in Fig. 2.2.4d. The optimisation process of Fig. 2.2.4d indicates that the BVP method quickly reaches the local minimum within a few iterations. However, Fig. 2.2.4b shows that the direction of the steepest descent is opposite to the truth, and the final estimated local minimum yields a large loss function value, much larger
than the assumed threshold $T_a$. Therefore, the local minimum in the case of $n_o \in [1, 1.5]$ is the global minimum for the tested GEO tracklets.

### 2.2.2 Hyperplane Intersection Methods

The hyperplane intersection methods determine the association of two tracklets by transforming all hypothetical orbits to a comparison space to assess the intersection of their hyperplanes. Maruskin et al. [28] proposed a recursive intersection method to determine the unique orbit of two tracklets in the Delaunay space. However, it needs to be performed manually, which is difficult for processing a large amount of data. Maruskin et al. [29] also investigated the strategy of employing a distance metric to determine the correlation between unperturbed Kepler hypothetical orbits. However, it is mainly suitable for the case of zenith observations. Fujimoto et al. [19, 95] transformed the uniform PDF of the admissible region to the Poincaré space, and the association is determined by assessing the overlap of the hypercubes (or bins) in the Poincaré space. This approach is referred to the BIN method in this dissertation. Compared to the intersection approach proposed by Maruskin, the BIN method is more efficient, and it is able to automatically process a large numbers of tracklets to determine association.

The core of the BIN method is to map the PDF of an admissible region to the 6-dimensional Poincaré space, and the association is confirmed if two Poincaré PDFs are overlapped. Given two Poincaré PDFs $f$ and $g$ from two admissible regions, which are propagated to the same epoch, their intersection region is represented by $h$, where $h > 0$ for a bin in the case that both $f > 0$ and $g > 0$. The association of two tracklets can be determined by checking if $h > 0$ over all bins. In addition, multiple tracklets can be grouped together according to the Bayes’ theorem.

Another contribution made by Fujimoto is the development of a linear map from the admissible region to the Poincaré space. In order to ensure a high accuracy of association, a large number of VPs needs to be mapped to the Poincaré space and propagated to the same epoch. The transformation of a discrete VP starts from topocentric spherical coordinates (i.e., $(\alpha, \delta, \dot{\alpha}, \dot{\delta}, \rho, \dot{\rho})$) to Cartesian coordinates $(x, y, z, \dot{x}, \dot{y}, \dot{z})$, then to Keplerian orbital elements.
2.2. Traditional Tracklet Association Methods

\((a, e, i, \Omega, \omega, M)\), and finally to Poincaré variables \((L, I, \Theta, \Phi, H, h)\).

For the unperturbed Keplerian orbit, the orbit propagation of the Poincaré elements from time \(t_0\) to \(t\) is expressed by

\[
\Psi_{\text{poi}}(t, t_0) : (L(t_0), I(t_0), \Theta(t_0), \Phi(t_0), H(t_0), h(t_0)) \rightarrow (L(t_0), I(t_0) + \frac{\mu^2}{L(t_0)^3}(t - t_0), \Theta(t_0), \Phi(t_0), H(t_0), h(t_0)).
\]

The transformation and orbit propagation process can be linearised, and the detailed derivation can be found in Ref. [19].

Mapping numerous points exactly results in large computational demands. Alternatively, a group of uniformly distributed VPs can first be selected, and the linearisation algorithm applied to map the vicinity of each VP to the comparison space. In other words, the algorithm linearly maps a sub-plane in the admissible region to the comparison space with fast speed, and it is able to obtain all bins as the exact map does. The flowchart of the BIN method is shown in Fig. 2.2.5.

The BIN method is evaluated using a test scenario of associating two tracklets from the same object (NORAD ID: 815), and associating two tracklets from two objects (NORAD ID: 815 and 1430) respectively. The former is shown in the top three subfigures in Fig. 2.2.6, and the latter is given in the bottom subfigures. The bins are expressed in three 2-dimensional subspaces of the Poincaré space. In each subfigure, the blue pluses and red stars represent the bins of corresponding tracklets that are propagated to the same epoch, and the black circles represent the overlapped bins between the two tracklets. Results of the top subfigures show 201 bins are overlapped, indicating that these two tracklets are from the same object. In addition, the bottom three subfigures show no overlap bins, and actually, the two tracklets are from different objects. This test case illustrates that the BIN method is able to associate tracklets from the same target, and it is able to distinguish tracklets from different targets.

Note that the accuracy and efficiency of the association are highly dependent on the dis-
Chapter 2. Tracklet Association Methods

Figure 2.2.5: The flowchart of the BIN method

Figure 2.2.6: Bins of two pairs of tracklets in the Poincaré space, the top three subfigures are the results of tracklets from the same object (NORAD ID: 815), the bottom ones are the results from two different objects (NORAD ID: 815 and 1430)
cretisation size of the bins in the comparison space. Decreasing the bin size will result in a more accurate association, but significantly increase the computation time and vice versa. A possible solution is to refine the discretisation size only for the regions that two PDFs overlap. However, how to adaptively tune the discretisation size to balance the accuracy and efficiency of association still needs further investigation.

Generally, compared to the BIN method, the IVP and BVP optimisation methods yield less computational demand. A detailed comparison of the computational efficiencies can be found in Ref. [17]. Therefore, further improvement based on the optimisation method is one major objective of this thesis.

Note that both the hyperplane method and the optimisation method are able to determine a reliable IOD solution for two tracklets after association. Fujimoto et al. [25] validated the feasibility of combining the tracklet association method with a Bayesian LS estimator to refine the orbital state estimation. However, tracklet association methods have not been considered for incorporation with multi-target Bayesian recursive estimation methods for space object tracking. Thus, this thesis seeks to explore the use of tracklet association methods for initialisation of multi-target trackers to achieve an improved computational efficiency. In the following section, a summary of the RFS theory is given, along with an overview of the multi-target Bayesian estimation, four latest labelled RFS filters, and a comparison of their performance for space objects tracking.

2.3 Improved Initial Value Problem Optimisation Method

In this section, an improved IVP method [21] using a new loss function defined in a non-singular canonical space is proposed to achieve better association and run-time performance. The use of the new loss function directly compares two six-dimensional canonical orbits of two tracklets being associated. The new loss function can be efficiently optimised by a genetic algorithm, and it does not require the calibration process of the noise model, as is the case for the traditional IVP optimisation method. Therefore, more efficient computational performance
can be achieved compared to IVP and BVP. The improved IVP method was validated using optical data at different altitudes collected from the Mt. Stromlo Observatory and compared with traditional IVP and BVP approaches.

2.3.1 Introduction

Across a broad range of tracklet association methods, either hypothetical orbits of tracklets or angular information derived from the hypothetical orbits is commonly used as the discriminator in the loss function. The improved IVP method defines a loss function to directly compare two hypothetical orbits rather than only consider the angular information for determination of tracklet association. Two problems need to be addressed to implement the improved IVP method.

The first one is how to assess the difference between two hypothetical orbits. The hypothetical orbits are usually expressed in the geocentric Cartesian frame for orbit determination. However, since the orbital uncertainty spreads out quickly in the Cartesian frame, other spaces, e.g., Delaunay or Poincaré elements [28, 19], can be used for further tracklet association. In this study, VPs in the admissible region are transformed to a non-singular canonical element space to take advantage of the expression of the variables in the same unit, which is convenient to compare two orbits. This study employs the Euclidean distance between two canonical orbits as the loss function to determine tracklet association. The transformation of VPs to the canonical space is introduced in Sec. 2.3.2.2.

The second problem to be addressed is to efficiently find the optimal solution for the given loss function. The brute force approach is to generate two sets of uniformly distributed VPs from the tracklets and transform the VPs to the canonical space to form hypothetical orbits. Then the minimum loss function values of all combinations of hypothetical orbits is selected to determine association. The main drawback is that a large number of hypothetical orbits are needed to ensure accurate association, which dramatically increases the computational burden.

In order to improve the run-time performance, a potential solution to reduce the number of
combinations is to generate candidate VPs. Instead of using uniformly distributed VPs in the
range and range-rate space, the intersections from a set of uniformly sampled $a$ and $e$ values are
used. For an unperturbed Keplerian orbit, the two quantities $a$ and $e$ are constant. Thus, if the
VPs of two tracklets are generated from the same pair of $a$ and $e$ values, then their hypothetical
orbits can be linked together for the subsequent association because the $a$ and $e$ values are
identical. These VPs are regarded as candidate VPs. Using the same $a$ and $e$ for both VPs
is based on the assumption of unperturbed Keplerian motion, which is a poor assumption for
space objects in low Earth orbit. However, the test results indicate that the proposed method
can achieve good association performance for tracklets generated by real data.

Even though using the candidate VPs reduces the number of combinations for association,
there still exist a large number of candidate VPs to be processed. To further improve the run-
time performance, this study uses an optimisation method to iteratively search for the optimal
solution without processing all candidate VPs. The loss function of two canonical orbits can be
reformulated as a function of $a$ and $e$ and optimised by a genetic algorithm [96]. The procedure
is detailed in Sec. 2.3.2.3.

2.3.2 Methodology

The procedure of improved IVP includes three steps: 1) use the intersection of $a$ and $e$
values to generate candidate VPs; 2) linearly transform the candidate VPs to canonical space at
the same time epoch; and 3) use the genetic algorithm to minimise the loss function to determine
the global minimum of the loss function. The algorithm is introduced below.

2.3.2.1 Generation of Candidate VPs

In this study, the intersections of semi-major axis and eccentricity constraint curves gener-
ated in a CAR are used as VPs, according to the following procedure.

The constraint equations of $a$ and $e$ can be expressed by

$$\dot{\rho}^2 + w_1 \dot{\rho} + F(\rho) - 2\varepsilon = 0$$

(2.3.1)
The coefficients used in the above equations can be found in Sec. 2.1.3.

The intersections of $a$ and $e$ values can be determined using the solutions of Eqs. (2.3.1) and (2.3.2). In contrast to numerically solving these two equations, a straightforward and efficient bisection method is developed to search for the intersections [97]. Its procedure is as follows.

Given a pair of semi-major axis and eccentricity values $(a_0, e_0)$ and a set of sparse samples of uniformly distributed $\rho$ values, a set of discrete points on the $a_0$ curve can be determined by solving Eq. (2.3.1). Then, the eccentricity value $e_i$ at the $i$th discrete point can be obtained using Eq. (2.3.2). To roughly determine the bounds of the intersections, a discriminator $d_e$ is defined as the difference between $e_i$ and $e_0$. If the signs of two discriminators from two consecutive points are different, an intersection exists between these two points. Finally, the bisection method is employed to search for the exact location of the intersection. The searching process is terminated if the value of $d_e$ is within a predefined tolerance value.

![Image](image_url)

**Figure 2.3.1:** Intersections of a pair of $a$ and $e$ curves in a CAR

Fig. 2.3.1 shows the intersections of the $a$ and $e$ curves of the CAR for the given attributable vector of a tracklet from a GEO object: $A = (3.044, 0.099, 0.262, 5.836 \times 10^{-5})$, as an example.
The semi-major axis and eccentricity values are set to $a = 6.5$ ER and $e = 0.3$ respectively. The two black circles denote the intersections of the $a$ and $e$ curves, which are used as VPs. Note that multiple intersections may be generated from a single pair of $a$ and $e$ values, which means multiple combinations of hypothetical orbits need to be processed for association. These combinations can be reduced to a unique pair of hypothetical orbits by maintaining only the two orbits with minimum value of the loss function.

If a pair of VPs from two tracklets are generated using the same $a$ and $e$ curves, then they are regarded as candidate VPs. For the subsequent association, the hypothetical orbits of the candidate VPs are derived and transformed to the canonical space.

### 2.3.2.2 Transformation to Canonical Space

The observation space coordinates are represented in the topocentric spherical coordinate system and they need to be transformed into the orbital space represented by six Cartesian coordinates or a set of orbital elements. The six non-singular canonical elements are used in this section because they are formulated with the same units so that they can be compared directly using one distance metric.

Several steps are needed to perform the exact transformation. The first transformation is from the topocentric spherical frame to the Delaunay space at a given epoch, defined as:

$$\Psi_1 : S_{Top} \rightarrow S_{Del}, \quad (2.3.3)$$

where $S_{Top}$ are the topocentric spherical coordinates consisting of range, range-rate and the attributable vector $(\rho, \dot{\rho}, A)$; and $S_{Del} : (L, l, G, g, H, h)$ denotes Delaunay variables, where

$$l = M, \quad g = \omega, \quad h = \Omega$$

$$L = \sqrt{\mu a}, \quad G = L \sqrt{1 - e^2}, \quad H = G \cos i. \quad (2.3.4)$$

The complete transformation of $\Psi_1$ requires several steps, $S_{Top}$ should be transformed to Cartesian space $S_{Cart}$, and then transformed to the Keplerian orbital space $S_{Orb}$ to finalise the trans-
formation to $S_{Del}$, this process can be found in Ref. [28].

The second step transforms the Delaunay variables into non-singular canonical space

$$\Psi_2 : S_{Del} \rightarrow S_{Can},$$  \hspace{1cm} (2.3.5)

where $S_{can} : (\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3)$ are canonical variables, given by

$$\begin{align*}
\xi_1 &= \sqrt{2(G + H)} \cos \left( \frac{l + g + h}{2} \right) \hspace{1cm} (2.3.6) \\
\xi_2 &= \sqrt{2(G - H)} \cos \left( \frac{l + g - h}{2} \right) \hspace{1cm} (2.3.7) \\
\xi_3 &= \sqrt{2(L - G)} \cos(l) \hspace{1cm} (2.3.8) \\
\eta_1 &= \sqrt{2(G + H)} \sin \left( \frac{l + g + h}{2} \right) \hspace{1cm} (2.3.9) \\
\eta_2 &= \sqrt{2(G - H)} \cos \left( \frac{l + g - h}{2} \right) \hspace{1cm} (2.3.10) \\
\eta_3 &= \sqrt{2(L - G)} \sin(l). \hspace{1cm} (2.3.11)
\end{align*}$$

For simplicity, the above transformation steps from topocentric spherical coordinates to canonical space are expressed compactly by:

$$\Psi_{can}^{top} = \Psi_2 \circ \Psi_1.$$  \hspace{1cm} (2.3.12)

The last step is the transformation of the canonical elements from time $t_0$ to $t$. For the unperturbed Keplerian orbit, the time-evolved transformation can be simplified to

$$\Psi_{can}(t; t_0) : Can(t_0) \rightarrow Can(t),$$  \hspace{1cm} (2.3.13)

where the time-evolved canonical orbital elements are given by

$$\xi_1(t) = \xi_1(t_0) - \xi_1(t_0) \tan \left( \frac{l_0 + g_0 + h_0}{2} \right) \frac{\mu^2}{L_0} \Delta t$$  \hspace{1cm} (2.3.14)
\[
\xi_2(t) = \xi_2(t_0) - \xi_2(t_0) \tan \left( \frac{l_0 + g_0 - h_0}{2} \right) \frac{\mu^2}{L_0^3} \Delta t 
\]
(2.3.15)

\[
\xi_3(t) = \xi_3(t_0) - \xi_3(t_0) \tan(l_0) \frac{\mu^2}{L_0^3} \Delta t 
\]
(2.3.16)

\[
\eta_1(t) = \eta_1(t_0) + \eta_1(t_0) \tan \left( \frac{l_0 + g_0 + h_0}{2} \right) \frac{\mu^2}{L_0^3} \Delta t 
\]
(2.3.17)

\[
\eta_2(t) = \eta_2(t_0) + \eta_2(t_0) \tan \left( \frac{l_0 + g_0 - h_0}{2} \right) \frac{\mu^2}{L_0^3} \Delta t 
\]
(2.3.18)

\[
\eta_3(t) = \eta_3(t_0) + \eta_3(t_0) \tan(l_0) \frac{\mu^2}{L_0^3} \Delta t. 
\]
(2.3.19)

where \( \Delta t = t - t_0 \) is the time interval.

In short, the exact transformation from the topocentric spherical frame to the time-evolved canonical space can be expressed as

\[
\Psi_{(t; t_0)} = \Psi_{\text{can}}(t; t_0) \circ \Psi_{\text{top}}^{\text{can}}. 
\]
(2.3.20)

### 2.3.2.3 New Loss Function and Optimisation

The loss function of improved IVP can be defined in various forms. In this study, the Euclidean distance between two canonical orbits is used as the loss function

\[
D(\mathbf{O}_1, \mathbf{O}_2) = (\mathbf{O}_1 - \mathbf{O}_2)^2 = \mathbf{O}_1^2 - 2\mathbf{O}_1 \cdot \mathbf{O}_2 + \mathbf{O}_2^2. 
\]
(2.3.21)

Fig. 2.3.2a shows the topography of \( D(\mathbf{O}_1, \mathbf{O}_2) \) for two LEO tracklets separated by a one-day interval, where the colour map represents the \( \log_{10} \) value of the loss function. The semi-major axis constraint is \( a \in [1, 1.2] \) ER, and the eccentricity constraint is \( e \in [0, 0.1] \). The new loss function yields four valleys. In order to determine the global minimum, each valley needs to be identified and optimised separately. The location of each valley depends on the number of orbit revolutions that would occur between the two tracklets if they are associated. In fact, each valley represents a group or family of orbits with a similar orbital period. Any orbit from the family can produce observations to fit the tracklets at the two observing epochs. Siminski et
al. [17] divided the IVP topography into several sub-regions using the semi-major axis bounds by defining a range of allowed orbit revolutions between the observation epochs. This strategy can also be applied to the improved IVP method. In Fig. 2.3.2a, the numbers in the brackets represent the range of orbit revolutions, the red line is the semi-major axis bound.

The complete orbit revolution number $m$ corresponding to each valley is defined as belonging to an integer set $m$, and

$$m = \{ m : m \in [m_{\text{min}}, m_{\text{max}}], m \neq 0 \} \in \mathbb{Z}^{\Delta m},$$  \hspace{1cm} (2.3.22)

where $m_{\text{min}}$ and $m_{\text{max}}$ are the minimum and maximum numbers of complete orbit revolutions respectively; $\Delta m = m_{\text{max}} - m_{\text{min}}$ is the difference in the numbers of orbit revolutions; $\mathbb{Z}$ denotes the set of integers. For a given time interval $\Delta t$, $m_{\text{min}}$ corresponds to the valley with $a_{\text{max}}$ and vice versa, given by

$$m_{\text{min}} = \text{floor} \left( \frac{\vert \Delta t \vert}{2\pi} \sqrt{\frac{\mu}{a_{\text{max}}^3}} \right),$$  \hspace{1cm} (2.3.23)

$$m_{\text{max}} = \text{ceil} \left( \frac{\vert \Delta t \vert}{2\pi} \sqrt{\frac{\mu}{a_{\text{min}}^3}} \right),$$  \hspace{1cm} (2.3.24)

Each complete orbit revolution represents a semi-major axis bound which is defined in
the range of $[m - 0.5, m + 0.5]$. Note that each bound is defined within the range of minimum and maximum orbit revolutions $\left(\frac{\Delta t}{2\pi} \sqrt{\frac{\mu}{a_{\max}^3}}, \frac{\Delta t}{2\pi} \sqrt{\frac{\mu}{a_{\min}^3}}\right)$. Note the distinction between the minimum and maximum orbit revolutions and $m_{\min}$ and $m_{\max}$ (minimum and maximum complete orbit revolutions). For example, the lower boundary value of the bound at the right side of Fig. 2.3.2a is 12.7 because it is the minimum allowed orbit revolution. The loss function within each semi-major axis bound can be optimised separately. After the local minima from all valleys are obtained, the global minimum can easily be identified from them.

Fig. 2.3.2b is the 3-D plot of the loss function. The topography is smooth and only one local minimum in each orbit revolution bound is shown in Fig. 2.3.2b. The gradient-based optimisation methods, e.g., conjugate gradient and Quasi-Newton methods, are potential approaches to quickly reach the global minimum if the loss function being optimised is smooth enough. However, the semi-major axis and eccentricity can yield multiple intersections. In addition, similar to IVP, there are several angular quantities employed in the improved IVP loss function, this may lead to discontinuities in the loss function (e.g., there are a few jumps at the boundary region of Fig. 2.3.2b). In order to address the non-smooth optimisation problem, a genetic algorithm is employed in this study.

Genetic algorithms [96, 98] are randomised population-based heuristic search methods inspired by natural evolution. The evolution is an iterative process, and it needs to be initialised by a population of individuals (i.e., chromosomes). The population at each iteration is a generation, and each individual is a potential solution to the optimisation problem. The fitness of each individual is evaluated using the improved IVP loss function. The more fit individuals are selected from the current population and two biologically inspired operators, i.e., crossover and mutation, are used to generate a new generation. The new generation is used in the next iteration until termination conditions are satisfied, e.g., the number of generations exceeds a given maximum or the highest fitness value reaches a given threshold.

**Initialisation**

Each individual in the population contains two free parameters, i.e., semi-major axis and
eccentricity. The initial semi-major axis value $a_0$ can be randomly generated within each orbit revolution bound. In the case $m \neq \emptyset$, $a_0$ for a valley can be randomly generated from the following range

$$a_0 \in \left[ \sqrt[3]{\frac{\mu \Delta t^2}{(2\pi m_{i+1})^2}}, \sqrt[3]{\frac{\mu \Delta t^2}{(2\pi m_i)^2}} \right]_{1 \leq i < m_{\text{max}} - m_{\text{min}}},$$

(2.3.25)

In the case when $m = \emptyset$, no valley exists, but the global minimum can still be found, and $a_0$ can be randomly selected from the range $(a_{\text{min}}, a_{\text{max}})$.

The initial value $e_0$ is independent of the location of the valley, thus it can be randomly selected from the range of the $e$ constraint values

$$e_0 \in [e_{\text{min}}, e_{\text{max}}].$$

(2.3.26)

The above process is repeated for all the chromosomes in the population.

**Selection**

This study uses a fitness proportionate selection rule to select the individuals (parent chromosomes) which are employed to create a new population at the next generation. Given each successive generation, the fitness of each individual is calculated using Eq. (2.3.21) and the fitter individuals are more likely to be selected. This is to ensure good solutions can be selected for the next generation.

**Crossover**

The crossover operator is used to generate child chromosomes by exchanging information between a pair of parent chromosomes. The generated child chromosomes inherit the characteristics of the parent chromosomes. As only two genes exist in each chromosome, the simple single-point crossover is employed in this study. The crossover operation results in a big change in the search space, and a user defined probability is defined to decide whether to execute such a crossover operation.

**Mutation**
2.3. Improved Initial Value Problem Optimisation Method

(a) Iterative process of the genetic algorithm

(b) An example of using genetic algorithm to search for the global minimum, the triangle is the start point, the circles are updated \((a_i, e_i)\), the star is the achieved global minimum, the diamond is the \((a, e)\) from TLE data, which is regarded as the reference

Figure 2.3.3: Optimisation of the new loss function

Another genetic operator is mutation, which maintains genetic diversity of the generation by applying random changes to create child chromosomes. In mutation, a few genes of a parent chromosome will be altered. A child chromosome that has undergone mutation may be significantly different from its parent. The solution is accepted if the mutation results in an improvement of the fitness. Generally, a low probability value should be defined, otherwise mutation may turn into a random search.

The optimisation process is terminated if the number of iterations \(n\) reaches the predefined maximum value \(n_{max}\) or if the loss function value becomes smaller than a predefined threshold \(T\). The genetic algorithm optimisation process is summarised in Fig. 2.3.3a.

Fig. 2.3.3b gives an example of determining of the local minimum of one valley shown in Fig. 2.3.2b. The colours of the curves represent the values of the loss function; the triangle represents the individual with lowest loss function value in the initial population; the circles are the chromosome \((a_i, e_i)\) with the best fitness in each generation; the red lines connecting chromosomes show the path of the evolution process; the star is the local minimum as determined by
Chapter 2. Tracklet Association Methods

the genetic algorithm; the diamond is the \((a, e)\) pair obtained from TLE data used as reference. The global minimum obtained is within 7 km of the semi-major axis obtained from the TLE data. This discrepancy may be due to the errors in the optical observations, orbit propagation, or the TLE itself. TLEs do not have an associated covariance, making it difficult to characterise the consistency between the two orbital states. However, it is clear that the genetic algorithm iteratively approaches the TLE orbital state, so the solution yielded is taken as reasonable.

2.3.3 Assessment and Comparison

2.3.3.1 Data Selection

The new tracklet association method is tested using optical observations from the Mount Stromlo Observatory in the two-month period from March 1 to May 10, 2013. A total of 308 tracklets are generated from the collected data from various orbit domains, including LEO, medium Earth orbit (MEO), GEO, and GTO. Each tracklet contains five to six observations over a duration of two to three minutes. Among the 308 tracklets, 22 tracklets are from space objects that have only one pass, while the remaining 286 tracklets are from space objects that have at least two passes, and their detailed classification is given in Table 2.3.1.

<table>
<thead>
<tr>
<th></th>
<th>Single-pass</th>
<th>Multi-pass</th>
<th>Total No.</th>
</tr>
</thead>
<tbody>
<tr>
<td>LEO</td>
<td>18</td>
<td>271</td>
<td>289</td>
</tr>
<tr>
<td>MEO</td>
<td>4</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>GEO</td>
<td>0</td>
<td>11</td>
<td>11</td>
</tr>
<tr>
<td>GTO</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Total No.</td>
<td>22</td>
<td>286</td>
<td>308</td>
</tr>
</tbody>
</table>

In this study, all 308 tracklets are tested, and the number of all tracklet pairs is 47278, but only those pairs of tracklets that have a time interval less than five days are tested, because associating tracklets over long time intervals is unreliable and time-consuming. All tracklets have been successfully associated with their objects by the Mount Stromlo Observatory based on TLE data, and this is regarded as reference for comparison.
2.3.3.2 Association Performance

Tracklet association results are indicated by two commonly used parameters in this study — sensitivity and specificity. Sensitivity, also known as the true positive rate, means the percentage of true tracklet associations that are correctly identified as associated. Specificity, also known as the true negative rate, measures the percentage of false tracklet associations that are correctly identified as unassociated. High sensitivity and specificity values mean high accuracy of association. Since the values of these two rates are highly dependent on the predefined threshold \( T \) for association, various values of \( T \) are tested. The association results of the 308 tracklets are divided into three groups, according to their time intervals: 1) less than one day, 2) one to three days and 3) three to five days, see Fig. 2.3.4.

![Graphs showing true positive and true negative rates for different time intervals](image)

**Figure 2.3.4:** Association results of three groups of tracklets with different time intervals of 1 day, 1-3 days and 3-5 days using the improved IVP method

Fig. 2.3.4a shows that the true positive rates of all three groups increase rapidly with the increase of \( T \), meaning that the larger \( T \), the less likely the proposed method will fail to identify true associations. Among these three curves, the one for the one-day interval rises slightly faster, while the other two are almost the same, but this difference is not significant. These results indicate that the improved IVP method can achieve similar true positive rates for tracklets with different time intervals.

The three solid curves for the true negative rates of all three groups in Fig. 2.3.4b decrease
rapidly with the increase of $T$. The fact that tracklet pairs with shorter time intervals result in smaller true negative rates is not consistent with the general theory — longer time intervals increase the difficulty in association and are expected to result in smaller true negative rates than short time intervals. The reason for this discrepancy is that out of the 308 tracklets, 86 tracklets originate from the Iridium constellation, and their intervals are all less than three days. Tracklets from different objects but in the same constellation can be easily and erroneously associated to the same object, which leads to the low true negative rate in this category.

To illustrate the low true negative rates of the 86 tracklets, the true negative rates of the remaining 222 non-constellation tracklets are also shown in Fig. 2.3.4b in dashed curves. It is noted that both the solid and dashed curves of the tracklet pairs with an interval of three to five days overlap since none of the tracklets in this range are from a constellation. The trend of these dashed curves agrees with the above mentioned general theory. Comparing the top two dashed curves against the bottom two solid curves, the true negative rates presented in the dashed curves are much larger. This difference implies that the discrepancy is caused by the low true negative rates of the 86 constellation tracklets. In order to further improve the true negative rate of the constellation tracklets, a new approach named the common ellipse method is proposed in the next section.

The performance of improved IVP is compared with IVP and BVP by assessing the results of associating the same 308 tracklets. The approaches for minimising the loss functions of IVP and BVP are a pattern search algorithm and the BFGS optimisation method respectively. The two body dynamic model is employed to assess the association performance of all three methods.

The association performance of IVP and BVP depends on the measurement noise. More robust association can be achieved if the assumed measurement noise distribution is properly calibrated. The detailed procedure of calibration can be found in Ref [17]. The standard deviation (STD) of the measurement noise is around 1 arc-second for the tested data collected by the Mount Stromlo Observatory. Therefore, the STDs of measurement noise are
2.3. Improved Initial Value Problem Optimisation Method

Figure 2.3.5: True positive and true negative rates of all tracklets within 5 days resulting from the three methods. Note that the log_{10} value is adopted for the results of IVP and BVP for better visualisation.

$\sigma_{\alpha,\delta} = 4.89 \times 10^{-6}$ rad and $\sigma_{\dot{\alpha},\dot{\delta}} = 1.17 \times 10^{-4}$ rad/hour respectively in this study. Compared with IVP and BVP, the improved IVP works robustly and does not need calibration.

Fig. 2.3.5 shows the results of the three methods for all tracklet pairs, where the blue solid curves and red dashed curves represent the true positive and true negative rates respectively, and each of these curves consists of 20 points corresponding to different selections of the threshold value. The improved IVP and IVP yield similar results, with high true positive and negative rates across a range of threshold values. The improved IVP achieves a slightly higher success rate in identifying false associations when its true positive rate is the same as IVP. BVP provides the worst association performance as it yields much lower true negative and true positive rates.
In order to investigate its poor performance, the BVP is further optimised by the pattern search method. The obtained results are similar to Fig. 2.3.5c, meaning that the association performance of BVP is independent of optimisation method. Even if the global minimum can be found, BVP still leads to bad association performance. The generally used Lambert solver only accounts for two-body dynamics. However, the majority of the tested tracklets are from LEO objects, which are significantly affected by non-conservative forces, e.g., atmospheric drag. The use of a typical Lambert solver may yield an inaccurate orbital state and therefore lead to worse association performance. Siminski et al.[17] suggested addressing this issue by using a numerical orbit model for Lambert’s problem that also accounts for perturbations. This is considered beyond the scope of this study and needs to be further investigated in future work.

The comparison results indicate that improved IVP and IVP are more robust, especially for associating tracklets from LEO.

### 2.3.3.3 Run-Time Performance

The run-time performance of improved IVP, IVP and BVP is also evaluated in terms of the computation time required to associate the 308 tracklets across various time intervals. All three methods are implemented in Matlab, and the computation time is based on an Intel core i7 CPU with 64-bit numerics and 2.7 GHz clock rate.
The initial population used in the genetic algorithm contains 20 individuals, and the maximum number of iterations is defined as 20. The use of the above parameters yields the results presented in Fig. 2.3.5a. More individuals and iterations can yield more opportunity to find the global minimum along with more computational burden. These parameters are selected to balance the association and run-time performances based on a large number of tests. In the pattern search method, each search employs 10 initial points and the maximum search step is 1000. In the BFGS optimisation method, the maximum number of steps is defined as 10. For all three optimisation methods, if either the search step number exceeds the maximum or the loss function value satisfies the corresponding threshold, the search process is terminated. Generally, the BFGS method can achieve fast convergence within 3 to 4 steps, while IVP requires many more steps to find the global minimum.

As most tracklets are collected during limited windows each night at the Mount Stromlo Observatory, they are divided into five groups of time intervals from one to five days, with the exact intervals rounded to the closest integer number of days. The mean computation time of all pairs of tracklets in each group are shown in Fig. 2.3.6. The results of IVP and BVP are only for the case of 1 arc-second STD measurement noise. Results indicate that the computation times of all three methods grow quickly with increasing time intervals. The computation time of improved IVP is between BVP and IVP. BVP is the fastest throughout all five groups, while IVP performs worst. Although the figure provides the computation times using one value of the measurement noise uncertainty, it is important to note that the calibration process requires IVP and BVP to be performed several times using different noise models, which significantly increases the computational burden. As the improved IVP method does not need calibration, its run-time performance outperforms the other two methods.

In addition, the results in Fig. 2.3.6 reveal that time interval is the dominant factor affecting the efficiency of all three methods. This is because most tested tracklets in Fig. 2.3.6 are from LEO. According to Eq. (2.3.23), the small semi-major axis $a$ values of the LEO objects result in large differences in the numbers of complete orbit revolutions $\Delta m = m_{\text{max}} - m_{\text{min}}$. The large $\Delta m$ leads to more computational effort because the optimisation algorithm needs to be
Chapter 2. Tracklet Association Methods

performed as many times as $\Delta m$. Theoretically, the $\Delta m$ of MEO and GEO objects does not vary too much with increasing time interval between tracklets.

The run-time performance of tracklet association methods depends on several parameters, e.g., the number of individuals, number of iterations. For a specific method, more accurate association performance generally requires higher computational cost by tuning the parameters. Therefore, it is necessary to achieve a tradeoff between the association and the run-time performance. Solving these two conflicting objectives can be regarded as a multi-objective optimization problem, which is beyond the scope of this thesis.

2.4 Common Ellipse Method for Constellation Tracklets

An issue common to all tracklet association methods is the high rate of erroneous association of tracklets from different objects in the same constellation, leading to a low true negative rate. Thus, a new approach named the common ellipse method is proposed in this section to identify associated tracklets by checking if there is a common ellipse sufficiently close to the hypothetical ellipses of all the tracklets in the group.

2.4.1 Introduction

As discussed in Sec. 2.3.3.2, the limitation of improved IVP is the low true negative rate of the association of constellation tracklets. This issue is noted in previous work [19], and it can be explained using the following example of associating two tracklets from the Iridium constellation. Let tracklet $\mathbf{x}_1$ be from the Iridium satellite $I_1$ (NORAD ID: 24795) at epoch $t_1$, and tracklet $\mathbf{x}_2$ be from the Iridium satellite $I_2$ (NORAD ID: 24793) at epoch $t_2$, 0.89 days later than $\mathbf{x}_1$. The two satellites have similar Keplerian orbital elements, but the mean anomaly of $I_1$ is about 72.3 degrees ahead of $I_2$. Therefore, $I_1$ has already passed out of the field of view of the telescope at epoch $t_2$. However, the large range uncertainty encompassed in the admissible region means that VPs generated from $\mathbf{x}_1$ may be visible in the vicinity of $I_2$ at time $t_2$. In this case, the two tracklets can be erroneously associated. If several tracklets from a space object are available, it may lead to more reliable association. Unfortunately, most of
the Iridium constellation objects tested in this study were tracked only twice, which introduces more difficulty for association.

2.4.2 Methodology

To address this issue, a common ellipse method is developed in this study. Theoretically, the ellipse of a space object consists of five Keplerian orbital elements \((a, e, i, \Omega, \omega)\). The two elements \(a\) and \(e\) define the size and shape of the ellipse respectively, \(i\) and \(\Omega\) define the orientation of the orbital plane and \(\omega\) defines the orientation of the ellipse in the orbital plane. Often, constellation space objects can be regarded as located at different locations in their common ellipse (e.g., a string of pearls formation). If several tracklets are identified to be associated with the same tracklet, but the five Keplerian orbital elements (not including mean anomaly) obtained from the IOD process are significantly different, then at least some of the associations must be incorrect. The principle can be used to prescreen the constellation tracklets. Given a group of constellation tracklets, a common ellipse can be generated from the IOD states computed from any two tracklets in the group. If this common ellipse is close to the orbital elements of other tracklets (not including mean anomaly), the two selected tracklets are truly associated and the other tracklets correspond to a string of pearls constellation. In order to compute the orbital elements for the additional constellation tracklets, a hypothetical ellipse can be generated for each tracklet using the semi-major axis and eccentricity values obtained from the IOD process above. A distance metric is then used to assess the difference between the common ellipse and the hypothetical ellipses. If the distance is within a predefined threshold \(T_c\), then the true association is determined. As a result, the true negative rate is improved because the false positives are reduced. This common ellipse method is implemented by three steps introduced below.

In the first step, the common ellipse of two tracklets is generated. For example, given a tracklet \(\mathbf{x}_0\) and a group of constellation tracklets \((\mathbf{x}_1, \cdots, \mathbf{x}_n)\), where \(\mathbf{x}_0\) and \(\mathbf{x}_i\) are correctly associated, the IOD state \(\mathbf{x}^{\text{Cart}}_i\) in a Cartesian coordinate system is obtained from the improved IVP method. To obtain an estimate of the covariance \(P^{\text{Cart}}_i\) for later determination of the \((a, e)\) constraints, a Bayesian Kalman filter method (i.e., Extended Kalman Filter (EKF)) is used. The
EKF is initialised by $x_{i}^{\text{Cart}}$ and a predefined covariance $P_{0}^{\text{pre}}$, and the measurements $Z_{i}$ are taken from the tracklets. This process yields a more accurate covariance $P_{i}^{\text{Cart}}$ than simply adopting $P_{0}^{\text{pre}}$ with the note the $x_{i}^{\text{Cart}}$ is still taken from the improved IVP solution. Subsequently, the estimated Cartesian state $x_{i}^{\text{Cart}}$ and $P_{i}^{\text{Cart}}$ are mapped to the Keplerian orbital elements space using the unscented transform (UT) method to obtain the state $x_{i}^{\text{Ell}}$ and covariance $P_{i}^{\text{Ell}}$ of the common ellipse $\text{Ell}_{i}$.

The second step is to generate hypothetical ellipses $\text{Ell}_{j}, (j = 1, 2, \ldots, n \cap j \neq i)$ for the other $n - 1$ tracklets. The three-sigma bounds of $a$ and $e$ determined from $P_{i}^{\text{Ell}}$ are used as the constraints $C_{i}$ to generate CARs for these tracklets. Then, the CAR$_{j}$ of the $j$th tracklet is approximated using the GMM method, which is a weighted mixture of Gaussian components [31]:

$$p(\rho, \dot{\rho}) = \sum_{k=1}^{L_{j}} w_{k} p_{g}([\rho, \dot{\rho}]^{T}; \mu_{\rho, \dot{\rho}, k}; P_{\rho, \dot{\rho}, k}),$$

(2.4.1)

where $L_{j}$ is the number of GM components; $p_{g}$ denotes normal distribution; $w_{k}, \mu_{\rho, \dot{\rho}, k}$ and $P_{\rho, \dot{\rho}, k}$ are the weight, mean and covariance of each GM component respectively. For the detailed derivation of the GM approximation refer to DeMars et al. [31]. The GM components are then transformed from topocentric spherical coordinates to the Keplerian orbit element space by the UT method. Then the state $x_{i}^{\text{Ell}}$ and covariance $P_{i}^{\text{Ell}}$ of $\text{Ell}_{i}$ are extracted from the transformed GM components. The second step is repeated until hypothetical ellipses of all $n - 1$ tracklets are determined.

In the last step, the Mahalanobis distance is employed to calculate the distance between each of the hypothetical ellipses determined from the second step and the common ellipse $\text{Ell}_{i}$. Note that the units of the five orbital elements in an ellipse are different. Thus, the Mahalanobis distance of only the angular quantities $(i, \Omega, \omega)$ of the common ellipse and hypothetical ellipses are calculated because all $\text{Ell}_{j}$ are generated from the same $(a, e)$ constraint values derived from $\text{Ell}_{i}$. Then the mean of a weighted sum of the Mahalanobis distances $M_{i}$ of $(i, \Omega, \omega)$ between
all hypothetical ellipses and $Ell_i$ is obtained by:

$$
M_i = \frac{1}{n} \sum_{j,j \neq i}^{n} \sum_{k} L_j w_{j,k}(x_{j,k}^{Ell} - x_{i}^{Ell})(P_{j,k}^{Ell} + P_{i}^{Ell})^{-1}(x_{j,k}^{Ell} - x_{i}^{Ell})^T, \tag{2.4.2}
$$

where $w_{j,k}$, $x_{j,k}^{Ell}$ and $P_{j,k}^{Ell}$ are the weight, mean and covariance of the $k$th GM component from the $j$th tracklet respectively. The $M_i$ value is computed for each combination of $\mathbf{x}_0$ with $\mathbf{x}_i$ to determine if the common ellipse $Ell_i$ agrees with the hypothetical ellipses of all tested tracklets. If the $M_i$ value is within $T_c$, then $\mathbf{x}_0$ and $\mathbf{x}_i$ are regarded as associated.

It is noted that the common ellipse method assumes that all tested tracklets are from the same constellation. The robustness of the method for processing a group of tracklets containing both constellation and non-constellation tracklets needs to be further assessed in future work.

The pseudo-code of the above common ellipse algorithm is given in Table 2.4.1. A case study is used here to further explain the process, which considers 10 randomly selected constellation tracklets: $\mathbf{x}_0$ and $\mathbf{x}_i$, $i = 1, \cdots, 9$. Only $\mathbf{x}_0$ and $\mathbf{x}_1$ are from the same Iridium satellite: NORAD ID 24795, whereas the other eight tracklets are from different satellites in the Iridium constellation at different epochs. All ten tracklets are observed within one day from the Mount Stromlo Observatory. In each subfigure of Fig. 2.4.1a, the blue circles are the common ellipses

![Figure 2.4.1: Illustration of the common ellipse method](image)

(a) Nine groups of common ellipses and hypothetical ellipses with nine different pairs of tracklets assumed to be associated

(b) $\log_{10}$ values of the Mahalanobis distances corresponding to the left subfigures

The pseudo-code of the above common ellipse algorithm is given in Table 2.4.1. A case study is used here to further explain the process, which considers 10 randomly selected constellation tracklets: $\mathbf{x}_0$ and $\mathbf{x}_i, \cdots, \mathbf{x}_9$. Only $\mathbf{x}_0$ and $\mathbf{x}_1$ are from the same Iridium satellite: NORAD ID 24795, whereas the other eight tracklets are from different satellites in the Iridium constellation at different epochs. All ten tracklets are observed within one day from the Mount Stromlo Observatory. In each subfigure of Fig. 2.4.1a, the blue circles are the common ellipses
Chapter 2. Tracklet Association Methods

from the tracklets shown with NORAD IDs at the top of the figure and the grey circles are the hypothetical ellipses of the rest of the tracklets. From all subfigures, we can see that the common ellipse and hypothetical ellipses of the first subfigure are much closer than the rest, which indicates that the two tracklets \( \mathbf{x}_0 \) and \( \mathbf{x}_1 \) are more likely than other tracklets to be from the same space object, as confirmed using the Mahalanobis distance.

---

**Algorithm 1**: The common ellipse algorithm

**INPUT**: \( \mathbf{x}_0, \mathbf{x}_i, P^\text{pre}, T_c \)

**OUTPUT**: Association results

\[
\text{for } i = 1 : n \quad \text{do}
\]

**Obtain the IOD solution from the improved IVP**

\[
(\mathbf{x}_0, \mathbf{x}_i) \rightarrow x^\text{Cart}_i
\]

**Estimate covariance of the IOD**

\[
\text{EKF} \left(x^\text{Cart}_i, P^\text{pre}, Z_{\mathbf{x}_0, \mathbf{x}_i}\right) \rightarrow P^\text{Cart}_i
\]

**Transform IOD to orbital elements space**

\[
\text{UT} \left(x^\text{Cart}_i, P^\text{Cart}_i\right) \rightarrow (x^\text{Orb}_i, P^\text{Orb}_i)
\]

**Generate the common ellipse**

\[
(x^\text{Orb}_i, P^\text{Orb}_i) \rightarrow (x^\text{Ell}_i, P^\text{Ell}_i)
\]

**Generate constraints for CAR**

**3 sigma boundary** \( (a, e) \rightarrow C_i \)

\[
\text{for } j = 1 : n \& \ j \neq i \quad \text{do}
\]

**Generate CAR**

\[
(\mathbf{x}_j, C_i) \rightarrow CAR_j
\]

**Approximate CAR using GMMs**

\[
(CAR_j, \sigma_\rho, \sigma_\dot{\rho}) \rightarrow GMM^{\text{Top}}
\]

**Transform GMMs to orbital elements space**

\[
\text{UT} \left(GMM^{\text{Top}}\right) \rightarrow GMM^{\text{Orb}}
\]

**Generate the hypothetical ellipse**
2.4. Common Ellipse Method for Constellation Tracklets

\[ GMM^{\text{Orb}} \rightarrow GMM_j^{\text{Ell}} \]

end

Calculate Mahalanobis distance

\[ M(GMM_j^{\text{Ell}}, (x_i^{\text{Ell}}, P_i^{\text{Ell}})) \rightarrow M_i \]

end

Determine association

\[ \text{find } (M_i < T_c) \rightarrow \text{Association} \]

Fig. 2.4.1b shows the \( \log(M_i) \) values corresponding to each subfigure of Fig. 2.4.1a and the green dashed line indicates the threshold \( T_c \) for associating these tracklets. The use of \( \log(M_i) \), rather than \( M_i \), is for better visualisation due to the extremely large differences among the nine \( M_i \) values. However, \( M_i \) is still used in the association process. \( T_c \) is determined based on the statistical results of the \( M_i \) values of the 86 Iridium constellation tracklets by the following procedure.

\[ \text{Figure 2.4.2: Number of true and false associations for different } \log(M_i) \text{ values (the dashed line indicates the selected } T_c \text{ for the common ellipse method)} \]

For a constellation tracklet, different improved IVP association thresholds \( T \) result in different groups of associated tracklets. The common ellipse threshold \( T_c \) depends on the tracklets
in the group because the $M_i$ value for each pair of tracklets depends on all the tracklets in the group. Thus, although only 86 Iridium tracklets are tested, a large number of $M_i$ values for all tracklet pairs resulting from different $T$ values can be obtained, i.e., the total number of $M$ values for the true and false associations are 1661 and 7058 respectively, for which their associations are already known. These known associations can be used to determine an appropriate value for $T_c$. The $M_i$ values of these associations are shown in Fig. 2.4.2. An optimal $T_c$ value needs to be selected to ensure that high true negative and true positive rates can be achieved. The vertical dotted line indicates the final $T_c$ value, which is located around $T_c = 600$ (i.e., $\log(T_c) \approx 6.40$). This $T_c$ value is used in Fig. 2.4.1b. The result is only $M_1$ being less than $T_c$, which agrees with the real case.

2.4.3 Results

The use of $T_c = 600$ in the common ellipse methods can improve the association accuracy of all 86 constellation tracklets compared to the improved IVP method. The solid and dotted curves in Fig. 2.4.3 represent the results of the improved IVP and common ellipse methods respectively. Compared with the red solid curve, the red dotted curve (indicating true negative rate) from the common ellipse method is significantly higher throughout all $T$ values, while the blue dotted curve indicates a reduction of true positive rate compared to the blue solid curve. This is mainly because some true associations are rejected, i.e., the red bars in the right side of the threshold line in Fig. 2.4.2. These results suggest that the common ellipse method significantly improves the true negative rate with a small reduction of true positive rate for associating the 86 Iridium constellation tracklets compared to the improved IVP method.

The threshold $T_c = 600$ is determined based on the statistical results of the association of the 86 Iridium constellation tracklets because only these data are available. For associating other constellation tracklets, the optimal threshold values may be different, which needs to be further investigated using more data.

The common ellipse method should be tested using tracklets of objects not from the same constellation. However, only tracklets in the Iridium constellation were tested because the ob-
2.5 Summary

This chapter introduces the fundamentals of the attributable vector, and the construction of the admissible region and CAR. Then, two commonly used tracklet association methods, i.e., the optimisation and hyperplane intersection methods, are investigated. The optimisation method are superior in terms of run-time performance, while their loss functions yield an additional need for the measurement noise calibration process.

The improved IVP tracklet association method is proposed for associating tracklets from space objects covering various orbital domains, e.g., LEO, MEO and GEO. A new loss function is defined in the non-singular canonical space, which does not require a measurement noise calibration process.

Figure 2.4.3: True positive and true negative rates of the improved IVP and common ellipse methods for the 86 Iridium constellation tracklets.

The Mount Stromlo Observatory tracked 29 Iridium satellites for 7 days, and no other space objects were observed during this time window. The rest of the data were collected after a very long time gap, making it impossible to compute associations with the observations from the Iridium satellites. Additional testing of the method using real data from multiple constellation and non-constellation objects is considered a task for future work.
calibration process as opposed to traditional IVP and BVP methods. The method is tested using optical data from the Mt. Stromlo Observatory, and the results indicate that the improved IVP method achieves better association and run-time performance compared with IVP and BVP for the tested data.

In addition, this chapter presents a common ellipse algorithm for discrimination of false associations for constellation tracklets. The algorithm determines if a common ellipse is consistent with the hypothetical orbits of several constellation tracklets. Results indicate significant improvement in the true negative rate compared to the results given in Sec. 2.3.3.2. It can be an attractive solution for the constellation problem especially when the number of tracklets from a single object in a constellation is limited.
Chapter 3  Labelled Random Finite Set Filters

This chapter introduces the basic definition of the labelled RFS filters. The mathematical definitions of the RFS theory are elaborated in Sec. 3.1, followed by a brief introduction of the Bayesian multi-target filter in Sec. 3.2. Section 3.3 provides an overview of four labelled RFS filters which are capable of recursively estimating the orbital state of space objects using optical observations as well as enhancing the capability of track management. The accuracy and run-time performance of these filters is assessed using two simulated space object tracking scenarios. A chapter summary is presented in the last section.

3.1 Random Finite Sets

This section briefly introduces the definition of the RFS theory, as well as a few types of RFSs that commonly used in multi-target filtering.

3.1.1 Definition

The random finite set is a finite set-valued random variable. In contrast to a random vector that has an exactly known number of elements with fixed order, the RFS has random cardinality and unordered elements. The cardinality distribution of a random finite set is random and modelled by a discrete distribution \( \rho(n) = \Pr\{|X| = n\} \), where \( n \in \mathbb{N} \), the non-negative integers. The joint distribution of \( n \) points is defined as \( P_n(\cdot) \).

An RFS \( X \) in space \( X \) is defined as a measurable mapping from a sample space \( \Omega \) to \( \mathcal{F}(X) \),
the space of a finite subset of $\mathbb{X}$, which is defined as a measurable mapping

$$X : \Omega \rightarrow \mathcal{F}(\mathbb{X}).$$

(3.1.1)

An RFS is completely described by its probability distribution, which is the probability measure $P$ on $\mathcal{F}(\mathbb{X})$

$$P(T) = \mathbb{P}(\{X \in T\}),$$

(3.1.2)

where $T$ is the Borel subset of $\mathcal{F}(\mathbb{X})$.

An RFS can be described using the belief mass function. The belief mass function of an RFS $X$ for any closed subset $S \subseteq \mathbb{X}$ is given by

$$B(S) = \int_S \pi(X) \delta X.$$

(3.1.3)

The belief density $\pi(\cdot)$ is not a probability density, but both the belief density and probability density are the probabilistic representation of an RFS. Based on Mahler’s set integral, an alternative notion of the belief density is defined. The belief mass function can be a more useful option than the probability distribution for modelling multi-target systems because it is defined on the closed subsets of $\mathbb{X}$.

3.1.2 Common Types of Random Finite Sets

There are several commonly used types of RFSs, including the Poisson, IID cluster, Bernoulli and multi-Bernoulli. A brief description of these RFS families is given below.

**Poisson**

The cardinality of a Poisson RFS follows a Poisson distribution with intensity function $\nu(\cdot)$, and its mean is $\lambda = \int \nu(x)dx$. The cardinality distribution of a Poisson RFS is specified as

$$\rho(n) = \frac{e^{-\lambda} \lambda^n}{n!}.$$

(3.1.4)
3.1. Random Finite Sets

The intensity function \( \nu(\cdot) \) is also referred to as the PHD of an RFS \( X \), which is the first statistical moment. The probability density of a Poisson RFS \( X \) is defined as

\[
\pi(X) = e^{-\lambda} \prod_{x \in X} \nu(x).
\]

(3.1.5)

**Independent Identically Distributed Cluster**

An IID cluster RFS \( X \) is described by an intensity function \( \nu(\cdot) \) and cardinality distribution \( \rho(\cdot) \). The cardinality satisfies \( N = \sum_{n=0}^{\infty} n \rho(n) = \int \nu(x) dx \). The probability distribution of an IID cluster RFS is given by

\[
\pi(X) = n! \rho(|X|) \prod_{x \in X} \frac{\nu(x)}{N}.
\]

(3.1.6)

Note that the Poisson RFS is a special case of the IID cluster RFS with the restriction of the Poisson cardinality distribution.

**Bernoulli**

A Bernoulli RFS \( X \) contains a single element distributed based on the spatial distribution \( p(\cdot) \) with the probability \( r \), and containing the empty set \( \emptyset \) with probability \( 1 - r \). The probability density of a Bernoulli RFS can be expressed as follows

\[
\pi(X) = \begin{cases} 
  r \cdot p(x) & X = \{x\} \\
  1 - r & X = \emptyset \\
  0 & |X| > 1,
\end{cases}
\]

(3.1.7)

The cardinality distribution of a Bernoulli RFS is a Bernoulli distribution with parameter \( r \).

**Multi-Bernoulli**

A multi-Bernoulli RFS \( X \) contains a set of finite independent Bernoulli RFSs \( X^{(i)} \), and each has the probability of existence \( r^{(i)} \) and probability density \( p^{(i)} \), where \( i = 1, \cdots, M \).
Thus, a multi-Bernoulli RFS \( X \) can be defined by the parameter set \( \{ (r^{(i)}, p^{(i)}) \}_{i=1}^{M} \). The probability density of a multi-Bernoulli RFS is expressed by

\[
\pi(X) = \begin{cases} 
\prod_{j=1}^{M} (1 - r^{(j)}) & n = 0 \\
\prod_{j=1}^{M} (1 - r^{(j)}) \sum_{1 \leq i_1 \neq i_n \leq M} \prod_{j=1}^{n} \frac{r^{(i)_j} p^{(i)_j}(x_j)}{1 - r^{(i)_j}} & 1 \leq n \leq M \\
0 & n > M,
\end{cases}
\] (3.1.8)

The cardinality of a multi-Bernoulli RFS is \( \sum_{i=1}^{n} r^{(i)} \). The probability density is generally represented by the abbreviation \( \pi(X) = \{ (r^{(i)}, p^{(i)}) \}_{i=1}^{M} \) for convenience. Compared with the Bernoulli RFS, the multi-Bernoulli RFS can be utilised to model the multi-target state and cardinality.

### 3.2 Multi-Target Bayesian Estimation

The multi-target recursive Bayes filter is the theoretical fundamental for multi-target detection, tracking, and identification. The multi-target filter consists of a prediction and measurement update of the multi-target PDF. The prior multi-target probability density is given by the multi-target Chapman Kolmogorov equation

\[
\pi_{k|k-1}(X_{k-1}) = \int f_{k|k-1}(X_{k}|X_{k-1}) \pi_k(X_{k-1}|Z_{k-1}) \delta X_{k-1},
\] (3.2.1)

where \( f_{k|k-1}(X_{k}|X_{k-1}) \) is the multi-target state transition density to time \( t_k \); \( f_{k|k-1}(X_{k}|X_{k-1}) \) accounts for the propagation of survival targets as well as birth and spawn targets. Each target \( x \in X \) survives with a probability \( p_s(x) \), and new birth targets are assumed to follow an MB process parameterised by \( \pi_k = \{(r_B, p_B(x))\} \). Eq. (3.2.1) is also known as the time update of a Bayes filter. The integral in the above equation is known as a set integral, given by

\[
\int f(X) \delta X = \sum_{i=0}^{\infty} \frac{1}{i!} \int f(\{x_1, \ldots, x_i\}) d(x_1, \ldots, x_i).
\] (3.2.2)
The derivation of the posterior multi-target probability density is given by the following Bayes’ theorem

\[
\pi_k(X_k|Z_k) = \frac{g_k(Z_k|X_k)\pi_{k|k-1}(X_k)}{\int g_k(Z_k|X_k)\pi_{k|k-1}(X_k)\delta X}, \tag{3.2.3}
\]

where \(g(Z|X)\) is the multi-target likelihood function, which models the measurement process of the sensor. In the multi-target tracking scheme, \(g(Z|X)\) accounts for several realistic conditions including measurement noise, clutter, and false alarms, where the clutter is assumed to be a Poisson process with intensity function \(\kappa(\cdot)\). A single state \(x \in X\) is detected by a sensor with the probability of detection \(p_D(x)\), which models the sensor detection profile. Each target generates a measurement \(z \in Z\) according to the single-target likelihood \(g(z|x)\).

### 3.3 Multi-Target Filtering with Labelled Random Finite Sets

This section introduces the fundamentals of the labelled RFS theory, including notations and definitions of the labelled RFS, and the labelled multi-target dynamic model and observation model. Then, the detailed implementations of four labelled RFS filters are given, and a comparison is conducted by assessing their accuracy and computational efficiency for space object tracking.

#### 3.3.1 Fundamentals of Labelled Random Finite Sets

**3.3.1.1 Notations**

A common set of notations is used for the convenience of derivation and explanation: single-target states are denoted as lower-case letters (e.g., \(x\)), multi-target states are represented by upper-case letters (e.g., \(X\)), and the labelled states and distributions are denoted by bold-type symbols (e.g., \(x\), \(X\) and \(\pi\)). The black-board bold letters indicate spaces (e.g., the state space \(X\) and the measurement space \(Z\)). \(\mathcal{F}(X)\) represents all finite subsets of space \(X\), and \(\mathcal{F}_n(X)\) indicates all subsets comprising \(n\) elements. Furthermore, several abbreviations are defined as
well. The inner product of two functions \( f(x) \) and \( g(x) \) is

\[
\langle f, g \rangle \overset{\Delta}{=} \int f(x)g(x)dx,
\]

and the multi-object exponential is defined as

\[
f^X \overset{\Delta}{=} \prod_{x \in X} f(x),
\]

where \( f^0 = 1 \). The generalised Kronecker delta function \( \delta_Y(X) \) and inclusion function \( 1_Y(X) \) supporting sets, vectors and integers are given by

\[
\delta_Y(X) = \begin{cases} 
1, & \text{if } X = Y \\
0, & \text{otherwise}
\end{cases}
\]

(3.3.3)

\[
1_Y(X) = \begin{cases} 
1, & \text{if } X \subseteq Y \\
0, & \text{otherwise}
\end{cases}
\]

(3.3.4)

The multi-target state \( X_k \) and observation \( Z_k \) at epoch \( t_k \) can be modeled by RFSs

\[
X_k = \{x_1, x_2, \ldots, x_n\}
\]

\[
Z_k = \{z_1, z_2, \ldots, z_m\},
\]

(3.3.5)

where \( n \) and \( m \) are the number of states and observations.

Representing the multi-target state with the labels attached enables us to estimate not only the target states but also their trajectories. This representation is usually performed using the labelled RFS notation in which the labelled multi-target state is denoted by

\[
X = \{x_1, x_2, \ldots, x_n\},
\]

(3.3.6)

where each element includes the target state augmented by a label denoted by \( x = (x, \ell) \in \)
\(X \times \mathbb{L}\). The set of labels of a labelled RFS \(X\) is denoted by

\[
\mathcal{L}(X) = \{\mathcal{L}(x) : (x \in X)\},
\]

where \(\mathcal{L} : X \times \mathbb{L} \rightarrow \mathbb{L}\) defines the projection from the labelled RFS space to the label space. To ensure the labels of tracks are distinct, a distinct label indicator can be defined as

\[
\Delta(X) = \delta|X|(|\mathcal{L}(X)|),
\]

which requires the number of elements \(|X|\) in the labelled RFS to be equal to the number of RFS labels.

![Label Assignment to Tracks](image)

**Figure 3.3.1:** An example of label assignment to tracks

To ensure the uniqueness of target labels, the labels are constructed in the form of \(\ell = (k_B, i)\) where \(k_B\) specifies the time of a target birth and \(i \in \mathbb{N}\) is included to distinguish between targets born at the same epoch. The label space evolves with time, and to emphasise on variations with time, at any epoch \(t_k\), the label space is denoted by \(\mathbb{L}_{0:k}\) accounts for all labels born at epoch \(t_k\) or before. Denoting the space of all labels born at epoch \(t_k\) by \(\mathbb{L}_k\), the complete label space at epoch \(t_k\) is constructed by \(\mathbb{L}_{0:k} = \mathbb{L}_{0:k-1} \cup \mathbb{L}_k\). Fig. 3.3.1 shows an example of label assignment to tracks. The label \((3, 1)\) indicates this track is the first one born at the third time epoch. The trajectory of a target can be straightforwardly obtained by extracting the tracks with
the same label in a time series.

### 3.3.1.2 Definition of Labelled Random Finite Sets

The LMB RFS is a particular form of multi-target distribution that is completely characterised by two parameters for each target label \( \ell \in \mathbb{L} \): the probability of existence of a target with that label denoted by \( r(\ell) \), and its single-object density conditional on its existence, denoted by \( p(\ell)(\cdot) \). A complete LMB multi-target density that is characterised by these parameters is denoted by \( \pi \), and the density of an LMB RFS parametrised by \( \{(r(\ell), p(\ell))\}_{\ell \in \mathbb{L}} \) is given by [46]

\[
\pi(X) = \Delta(X) w(L(X)) p^X, \quad (3.3.9)
\]

where

\[
w(L) = [1 - r]^L \setminus [1 - L \cap L] \quad (3.3.10)
\]

\[
p(x, \ell) = p(\ell)(x), \quad (3.3.11)
\]

where \( L \) is a set of labels.

A GLMB RFS is a labelled RFS defined by the following parameter set:

\[
\pi(X) = \Delta(X) \sum_{c \in C} w(c)(L(X)) [p(c)]^X, \quad (3.3.12)
\]

where \( C \) is a discrete index set, which introduces the hypothesis in the multi-target tracking problem. The weights and the spatial distribution are

\[
\sum_{L \subseteq L} \sum_{c \in C} w(c)(L) = 1 \quad (3.3.13)
\]

\[
\int p(c)(x, \ell) dx = 1. \quad (3.3.14)
\]

The LMB RFS is a special case of the GLMB RFS that has only one component, and thus the superscript \( (c) \) is omitted. The multiple-component and single-component is the main
3.3. Multi-Target Filtering with Labelled Random Finite Sets

The $\delta$-GLMB RFS with the state and label space $X \times \mathbb{L}$ is another special form of the GLMB RFS. In the theory of $\delta$-GLMB, the index set $C$ is defined over all subsets of $\mathbb{L}$ and the space $\Xi$ is defined to represent the history of the track to measurement association.

\[
C = \mathcal{F(\mathbb{L})} \times \Xi \tag{3.3.15}
\]

\[
w^{(c)}(L) = w^{(I,\xi)}\delta_I(L) \tag{3.3.16}
\]

\[
p^{(c)} = p^{(I,\xi)} = p^{(\xi)}, \tag{3.3.17}
\]

where $\xi$ is a realisation of $\Xi$, and $I$ indicates a set of track labels $I \in \mathbb{L}$.

The probability distribution of the $\delta$-GLMB RFS is defined as

\[
\pi(X) = \Delta(X) \sum_{(I,\xi) \in \mathcal{F(\mathbb{L})} \times \Xi} w^{(I,\xi)}\delta_I(L(X))[p^{(\xi)}]^X, \tag{3.3.18}
\]

where $(I, \xi) \in \mathcal{F(\mathbb{L})} \times \Theta$ represents a hypothesis, and $w^{(I,\xi)}$ can be interpreted as the probability of hypothesis $(I, \xi)$. The hypothesis $(I, \xi)$ indicates a set of targets with labels $I$ and association history $\xi$. The GLMB can be transformed to $\delta$-GLMB using the identity $w^{(\xi)}(J) = \sum_{I \in \mathcal{F(\mathbb{L})}} w^{(\xi)}(I)\delta_I(J)$.

The prior and posterior densities are $\delta$-GLMB densities for time epoch $t_k$

\[
\pi_{k|k-1}(X) = \Delta(X) \sum_{(I,\xi) \in \mathcal{F(\mathbb{L})} \times \Theta} w^{(I,\xi)}_{k|k-1}\delta_I(L(X))[p^{(\xi)}_{k|k-1}]^{X_{k|k-1}} \tag{3.3.19}
\]

\[
\pi_k(X|Z) = \Delta(X) \sum_{(I,\xi) \in \mathcal{F(\mathbb{L}_k)} \times \Theta_k} w^{(I,\xi)}_k\delta_I(L(X))[p^{(\xi)}_k(X|Z_k)]^{X_k}, \tag{3.3.20}
\]

where $p^{(\xi)}_{k|k-1}(\cdot, \ell)$ and $p^{(\xi)}_k(\cdot, \ell)$ are the prior and posterior densities of track $\ell$. 


3.3.1.3 Labelled Multi-Target Dynamic Model

For a given labelled multi-target state $X$, each state $(x, \ell) \in X$ has a probability $p_S(x, \ell)$ to survive to the next epoch as a new state $(x_k, \ell_k)$ with probability density $f(x_k|x, \ell)\delta_{\ell}(\ell_k)$, where $f(x_k|x, \ell)$ is the transition kernel. In addition, it also has probability $q_S(x, \ell) = 1 - p_S(x, \ell)$ of death and being eliminated.

The survival targets at the next epoch are distributed as

$$f_S(S|X) = \Delta(S)\Delta(X)1_{\mathcal{L}(X)}(\mathcal{L}(S))[\Phi(S; :)]^X,$$

(3.3.21)

where

$$\Phi(S; x, \ell) = \begin{cases} p_S(x, \ell)f(x_k|x, \ell), & \theta(\ell) > 0 \\ q_S(x, \ell), & \theta(\ell) = 0. \end{cases}$$

(3.3.22)

The newborn targets at the next epoch are modelled as a labelled multi-Bernoulli RFS with distribution

$$f_B(Y) = \Delta(Y)w_B(\mathcal{L}(Y))[p_B]^Y,$$

(3.3.23)

where $f_B$ is defined on $\mathbb{X} \times \mathbb{L}_k$, and $w_B(\cdot)$ and $p_B(\cdot, \ell)$ are the weight and single target density corresponding to target $\ell$ respectively. Note that although the above birth model is presented for the LMB RFS, it can be straightforwardly extended to the $\delta$-GLMB RFS.

Target spawning is not considered in this dissertation, but a detailed description can be found in Bryant et al. [99]. As such, the multi-target state at the time epoch $t_k$ is the combination of surviving and new birth tracks, and the multi-target transition kernel is given by

$$f(X_k|X_{k-1}) = f_S(X_k \cap (\mathbb{X} \times \mathbb{L}|X_{k-1}))f_B(X_k - (\mathbb{X} \times \mathbb{L}|X_{k-1})).$$

(3.3.24)
3.3. Multi-Target Filtering with Labelled Random Finite Sets

3.3.1.4 Multi-Target Observation Model

Given a labelled RFS $X$ representing the existing targets, each target $x \in (x, \ell)$ has the probability of detection $p_D$ producing a measurement $z$ with likelihood $g(z|x, \ell)$, as well as the probability $q_D = 1 - p_D$ of a missed detection.

The measurement to track association can be described using an association map function $\theta : \mathbb{L} \rightarrow \{0, 1, \ldots, |Z|\}$, and $\theta \in \Theta$, where $\Theta$ denotes the association map space and $\Theta(I)$ denotes the subset of association maps of $I$.

The multi-target likelihood function is defined as

$$g(Z|X) = e^{-(\kappa, 1)} \kappa^Z \sum_{\theta \in \Theta(L(X))} [\psi_Z(\cdot; \theta)]^X,$$  \hspace{1cm} (3.3.25)

where $e^{-(\kappa, 1)} \kappa^Z$ is the distribution of the clutter, and the function $\psi_Z(\cdot; \theta)$ is given by

$$\psi_Z(x, \ell; \theta) = \begin{cases} p_D(x, \ell) g(z_\theta|x, \ell) \kappa(Z_{\theta(\ell)}) & \text{if } \theta(\ell) > 0 \\ 1 - p_D(x, \ell) & \text{if } \theta(\ell) = 0 \end{cases}$$ \hspace{1cm} (3.3.26)

The above multi-target likelihood function is defined based on the assumption that each target generates at most one measurement, i.e., $\theta(i) = \theta(i') > 0$ indicates $i = i'$.

3.3.2 The $\delta$-Generalised Labelled Multi-Bernoulli Filter

![Figure 3.3.2: The flowchart of the $\delta$-GLMB filter](image)

The probability density of $\delta$-GLMB is a closed form approximation to the Bayes multi-
target filter, and the \(\delta\)-GLMB filter outperforms PHD, CPHD, and MB filters in terms of state and cardinality estimation [47]. The flowchart of the \(\delta\)-GLMB filter is shown in Fig. 3.3.2.

3.3.2.1 Prediction

In the \(\delta\)-GLMB filter, the multi-target state is modelled as a \(\delta\)-GLMB RFS. The posterior \(\delta\)-GLMB density is recursively predicted and corrected using equations (3.2.1) and (3.2.3).

**Proposition 2.1** If the posterior multi-target density is a \(\delta\)-GLMB, then the predicted multi-target density is a \(\delta\)-GLMB given by [46]

\[
\pi_{k|k-1}(X_{k|k-1}) = \Delta(X_{k|k-1}) \sum_{(I,\xi) \in \mathcal{P}(L_{0:k-1})} w_{k|k-1}^{(I,\xi)} \delta_{L_k}(\mathcal{L}(X_{k|k-1}))[P_{k|k-1}] X_{k|k-1}, \tag{3.3.27}
\]

where

\[
w_{k|k-1}^{(I,\xi)} = w_{S}^{(I)}(I_k \cap L_{0:k-1}) w_B(I_k \cap L_k) \tag{3.3.28}
\]

\[
w_{S}^{(\xi)}(L) = \left[\eta_{S}^{(\xi)}\right]_{L} \sum_{L_{\geq L}} [1 - \eta_{S}^{(\xi)}]^{1-L} w_{S}^{(I,\xi)} \tag{3.3.29}
\]

\[
\eta_{S}^{(\xi)}(\ell) = \langle p_S(\cdot, \ell), p^{(\xi)}(\cdot, \ell) \rangle \tag{3.3.30}
\]

\[
p_{k|k-1}^{(\xi)}(x, \ell) = 1_{L_{0:k-1}}(\ell) p_{S}^{(\xi)}(x, \ell) + 1_{L_k}(\ell) p_B(x, \ell) \tag{3.3.31}
\]

\[
p_{S}^{(\xi)}(x, \ell) = \langle p_S(\cdot, \ell) f(x|\cdot, \ell), p^{(\xi)}(\cdot, \ell) \rangle / \eta_{S}^{(\xi)}(\ell). \tag{3.3.32}
\]

In order to effectively implement the \(\delta\)-GLMB prediction, an equivalent form of Eq. (3.3.27) is expressed by

\[
\pi_{k|k-1}(X_{k|k-1}) = \Delta(X_{k|k-1}) \sum_{(I,\xi) \in \mathcal{P}(L_{0:k-1})} w_{k|k-1}^{(I,\xi)} \sum_{J \in \mathcal{P}(I)} \left[\xi_{S}^{J}\right] [1 - \eta_{S}^{\xi}]^{1-J} \times \sum_{L \in \mathcal{P}(L_k)} w_B(L) \delta_{J \cup L}(\mathcal{L}(X_{k|k-1}))[P_{k|k-1}] X_{k|k-1}. \tag{3.3.33}
\]
The weight of a survival hypothesis \((\ell, \xi)\) is given by

\[
\omega_S^{(I, \xi)} = \omega^{(I, \xi)} [\eta_S^{(\xi)} J | 1 - \eta_S^{(\xi)} I \setminus J].
\] (3.3.34)

For a given hypothesis \((I, \xi)\), the weight of the predicted hypothesis \((J \cup L, \xi)\) is denoted as

\[
\omega^{(I, \xi)} = \omega_S^{(I, \xi)} (J \cup L, L).
\] (3.3.35)

The prediction process involves target birth, death, and survival, which results in a large population of predicted \(\delta\)-GLMB hypotheses. In order to improve the computational efficiency, the \(K\)-shortest path algorithm [100] is employed to determine the \(K\) most important hypotheses to truncate the predicted \(\delta\)-GLMB hypotheses. This algorithm determines the \(K\) desired paths in non-decreasing order of cost, which can be formulated based on the probability of existence of each hypotheses. Note that a fairly large \(K\) is required to avoid dropping new tracks. Additionally, if the maximum number of target that die at each epoch can be specified based on prior knowledge, then the number of predicted \(\delta\)-GLMB hypotheses can be reduced significantly [49].

### 3.3.2.2 Update

As the number of updated hypotheses increases exponentially with time, a truncation approach is required to reduce the weak hypotheses. An efficient truncation method is to solve a ranked assignment problem by constructing a cost matrix. The solution to this ranked assignment problem is the desired number of hypotheses with the highest weights. For a set of measurements \(Z = \{z_1, \ldots, z_{|Z|}\}\), and a set of targets with labels \(I = \{\ell_1, \ldots, \ell_{|I|}\}\), the cost matrix is given by

\[
C_Z^{(I, \xi)} = \begin{bmatrix}
C_{1,1} & \cdots & C_{1,|Z|} \\
\vdots & \ddots & \vdots \\
C_{|I|,1} & \cdots & C_{|I|,|Z|}
\end{bmatrix}.
\] (3.3.36)
where \( i \in \{1, \cdots, |I|\}, \ j \in \{1, \cdots, |Z|\} \). Each column of the matrix represents a measurement, and each row represents a target. Each item in the cost matrix is expressed by

\[
C_{i,j} = - \ln \left( \frac{\langle p(\xi, \ell_i) \cdot g(z_j | \cdot, \ell_i) \rangle}{\langle p(\xi, \ell_i), 1 - p(\cdot, \ell_i) \rangle \kappa(z_j)} \right),
\]

which can be interpreted at the cost of assigning the measurement \( j \) to track \( \ell_i \).

The ranked assignment problem is the enumeration of the \( T \) least cost assignments. A similar problem can also be found in the multiple hypothesis tracker (MHT), and one efficient solution to this ranked assignment problem is Murty’s algorithm [101]. It generates the cheapest one-to-one assignments of the cost matrix in an increasing order.

**Proposition 2.2** If the predicted multi-target density is a \( \delta \)-GLMB, then the posterior multi-target density is also a \( \delta \)-GLMB given by [46]

\[
\pi_k(X_k | Z_k) = \Delta(X_k) \sum_{(I, \xi) \in \mathcal{F}(L_0, k) \times \mathcal{Z} \in \Theta_k(I)} \sum_{\theta_k \in \Theta_k(I)} w^{(I, \xi, \theta_k)}(Z_k) \delta_i(L(X_k)) \left[ p^{(\xi, \theta_k)}(\cdot | Z_k) \right] X_k,
\]

where

\[
w^{(I, \xi, \theta_k)}(Z_k) \propto w^{(I, \xi)}[\eta_{Z_k}]^I
\]

\[
\eta_{Z_k}^{(\xi, \theta_k)}(\ell) = \langle p(\xi, \cdot), \psi_{Z_k}(\cdot, \ell; \theta_k) \rangle
\]

\[
p^{(\xi, \theta_k)}(x, \ell | Z_k) = \frac{p(\xi)(x, \ell) \psi_{Z_k}(x, \ell; \theta_k)}{\eta_{Z_k}^{(\xi, \theta_k)}(\ell)}.
\]

For a hypothesis \((I, \xi)\) with a weight \( \omega^{(I, \xi)} \), the \( \delta \)-GLMB update yields a set of new hypotheses \((I, \xi, \theta_k)\) based on measurement to track association, \( \theta_k \in \Theta_k(I) \). Each updated hypothesis has a weight of \( \omega^{(I, \xi, \theta_k)}(Z_k) \propto \omega^{(I, \xi)}[\eta_{Z_k}]^I \).

### 3.3.2.3 Multi-target State Estimation

Several methods can be used to extract target states from the posterior multi-target state modelled by the \( \delta \)-GLMB RFS. The multi-Bernoulli estimator is an efficient state extraction
method for the $\delta$-GLMB density. The tracks with probability of existence higher than a threshold are selected, and the maximum a posteriori (MAP) from the density of each state is calculated as the state of each track. The probability of existence of track $\ell$ is the sum of the weights of all hypotheses which contain this track.

$$r_{\ell} = \sum_{(I, \xi) \in F(L) \times \Xi} \omega^{(I, \xi)}_I \delta_1(\ell).$$ \hspace{1cm} (3.3.42)

The MAP cardinality estimate of the $\delta$-GLMB filter is given by [47]

$$\rho(n) = \sum_{(I, \xi) \in F_n(L) \times \Xi} \omega^{(I, \xi)}.$$ \hspace{1cm} (3.3.43)

The highest weighted hypothesis that has the same cardinality as $\rho(n)$ is selected, and its labels and mean estimates of the states are extracted as the multi-target state.

### 3.3.3 The Joint $\delta$-Generalised Labelled Multi-Bernoulli Filter

One primary drawback of the $\delta$-GLMB filter is the high computational complexity because two truncations are required for the prediction and update. Vo et al. [49] proposed an efficient implementation of the $\delta$-GLMB filter by combining the prediction and update steps together, which is also referred to as the joint $\delta$-GLMB Filter. This joint prediction and update approach significantly reduces the computational complexity in the truncation procedures. In addition, a Gibbs sampling method [50] is further proposed to efficiently solve the ranked assignment problem.

#### 3.3.3.1 Joint Prediction and Update

**Proposition 2.3** Given the $\delta$-GLMB posterior density at time $t_{k-1}$, the $\delta$-GLMB posterior density at time epoch $t_k$ is given by [49]

$$
\pi_k(X_k|Z_k) \propto \Delta(X_k) \sum_{I, \xi, I_k, \theta_k} w^{(I, \xi)}_{Z_k} w^{(I, \xi, I_k, \theta_k)}_k \delta_{I_k} [\mathcal{L}(X_k)] \left[p^{(\xi, \theta_k)}(\cdot|Z_k)\right]^{X_k},
$$ \hspace{1cm} (3.3.44)
where \( I \in \mathcal{F}(\mathbb{L}) \), \( \xi \in \Xi \), \( I_k \in \mathcal{F}(\mathbb{L}_{0:k}) \), \( \theta_k \in \Theta_k \), and

\[
\begin{align*}
\omega^{(I_k, \xi, \theta_k)}_{Z_k} &= 1_{\Theta_k(I_k)}(\theta_k) \left[ 1 - \eta_S^{(\xi)} \right]^{I_k \setminus I_k} \left[ \eta_S^{(\xi)} \right]^{I_k \cap I_k} \times \left[ 1 - r_{B,k} \right]^{I_k \setminus I_k} r_{B,k}^{I_k \cap I_k} \left[ \eta_{Z_k}^{(\xi, \theta_k)} \right]^{I_k} \\
\eta_S^{(\xi)}(\ell) &= \langle p^{(\xi)}(\cdot, \ell), P_S(\cdot, \ell) \rangle \\
\eta_{Z_k}^{(\xi, \theta_k)}(\ell) &= \langle p_k^{(\xi)}(\cdot, \ell), \eta_{Z_k}^{(\theta_k(\ell))}(\cdot, \ell) \rangle \\
p_k^{(\xi)}(x_k, \ell) &= 1_{L_{0:k-1}}(\ell) \left\{ P_S(\cdot, \ell) f_k(x_k|\cdot, \ell), p^{(\xi)}(\cdot, \ell) \right\} / \eta_S^{(\xi)}(\ell) + 1_{L_k}(\ell) p_B(x_k, \ell) \\
p^{(\xi, \theta_k)}(x_k|Z_k) &= p_k^{(\xi)}(x_k, \ell) \eta_{Z_k}^{(\theta_k(\ell))}(x_k, \ell) / \eta_S^{(\xi)}(\ell).
\end{align*}
\]

Eq. (3.3.44) indicates the summation of the enumeration of all possible combinations of births, deaths, and survivals with associations of new measurements to hypothesised labels.

Note that Eq. (3.3.44) can be expressed as the \( \delta \)-GLMB form by writing the weight as the following form

\[
\omega^{(I_k, \xi, \theta_k)}_{Z_k} \propto \sum_I \omega^{(I_k, \xi)} \omega^{(I_k, \xi, \theta_k)}_{Z_k}.
\]

The number of hypotheses of the \( \delta \)-GLMB increases exponentially with time, and thus a truncation is required to reduce the weak hypotheses. The detailed truncation formulation is given in Appendix B.1.

Note that the ranked assignment problem needs to be address in the truncation process, which is to generate the \( T \)-best positive 1-1 vectors, and it can be solved using the Murty’s algorithm employed in the \( \delta \)-GLMB filter. However, the computational complexity of the Murty’s algorithm is extremely high if a large number of targets are present within sensor Field of View (FOV). Therefore, the Gibbs sampling algorithm was introduced by Vo et al. [49] to efficiently address the ranked assignment problem. The Gibbs sampling algorithm is presented in Appendix B.2.
3.3.3.2 Implementation

Following the derivation of Vo et al. [49], the δ-GLMB can be rewritten as the following formula by defining $h = (I, \xi)$

$$
\pi(X) = \Delta(X) \sum_{h=1}^{H} \omega^{(h)}(\delta_{f(h)}[F(X)]\left[p^{(\xi(h))}\right])^X.
$$

The δ-GLMB RFS can be characterised by the parameter set $\{(I^{(h)}), \xi^{(h)}, \omega^{(h)}, p^{(h)}\}_{h=1}^{H}$.

As the association history is not employed in the calculation, the joint GLMB filter only propagates the parameter set $\{(I^{(h)}), \omega^{(h)}, p^{(h)}\}_{h=1}^{H}$ forward, and the major task is to calculate the parameter set $\{(I^{(h)}), \omega^{(h)}, p^{(h)}\}_{h=1}^{H}$ at time epoch $k$.

The first step is to generate a set of ‘children’ hypotheses $\{(I^{(h)}), \xi^{(h)}, I^{(h)}_{k}, \theta^{(h)}_{k}\}_{h=1}^{H_{k}}$ with significant weights by sampling from the distribution $\pi$, which is given by

$$
\pi(I, \xi, I_{k}, \theta_{k}) \propto \omega^{(I, \xi)} Z_k^{(I, \xi, I_{k}, \theta_{k})}. \tag{3.3.52}
$$

Specifically, $H_{k}$ hypotheses $(I^{(h)}, \xi^{(h)})$ are sampled from $\pi(I, \xi) \propto \omega^{(I, \xi)}$, and then for each hypothesis, the Gibbs sampler is used to generate $T_{k}^{(h)}$ samples $(I^{(h)}, \xi^{(h)}, \theta^{(h)}_{k})$ from $\pi(I_{k}, \theta_{k}|I^{(h)}, \xi^{(h)})$, where $t \in [1 : T_{k}^{(h)}]$. The Gibbs sampler generates potential hypotheses represented by the positive 1-1 vector $\gamma^{(h,t)}$, and the repeated vectors need to be eliminated. An initial $\gamma^{(h,1)}$ is required to feed to the Gibbs sampler, which can be any 1-1 vectors. In this study, an all-zeros 1-1 vector is used to initialise the Gibbs sampler to avoid extra computation.

Then, the set of intermediate parameters $\{(I^{(h)}, I^{(h)}_{k}, \omega^{(h)}_{k}, p^{(h)}_{k})\}_{h=1}^{H_{k}}$ needs to be calculated. For each $h \in H$, the parameter set $\{(I^{(h)}, I^{(h)}_{k}, \omega^{(h)}_{k}, p^{(h)}_{k})\}_{h=1}^{H_{k}}$ can be calculated using $\gamma^{(h,t)}$, which is given by

$$
I^{(h)}_{k} = \{\ell_{i} \in I^{(h)} \cup I_{k} : \gamma^{(h,t)}_{i} \geq 0\}. \tag{3.3.53}
$$
Chapter 3. Labelled Random Finite Set Filters

\[ \omega_k^{(h,t)} \propto \omega(h) \prod_{i=1}^{\|\ell_h\|} \eta_i^{(h)}(\gamma_i^{(h,t)}) \]  

(3.3.54)

\[ p_k^{(h,t)}(\cdot, \ell_i) = p_k^{(h)}(\cdot, \ell_i) \eta_i^{(h,t)}(\cdot, \ell_i) / \eta_i^{(h,t)}(\ell_i). \]  

(3.3.55)

Finally, the parameter set \( \{(I_k^{(h_k)}, \omega_k^{(h_k)}, p_k^{(h_k)}\})_{h_k=1}^H \) is calculated using the intermediate parameters which are marginalised via Eq. (3.3.50). The algorithm of the joint \( \delta \)-GLMB filter is given in Algorithm 2.2.

---

**Algorithm 2.2: Joint prediction and update**

**Input:** \( \{(I^{(h)}, \omega^{(h)}, p^{(h)}\})_{h=1}^H, Z_k \)

**Output:** \( \{(I_k^{(h_k)}, \omega_k^{(h_k)}, p_k^{(h_k)}\})_{h_k=1}^H \)

for \( h = 1 : H \)

Initialise \( \gamma^{(h,1)} \)

Compute the cost matrix \( \eta^{(h)} \) using Eq. (B.1.3)

\( \{\gamma^{(h,t)}\}_{t=1}^{T_k^h} = \text{Unique}(\text{Gibbs}(\gamma^{(h,1)}, T_k^{(h)}, \eta^{(h)}) \)

for \( t = 1 : T_k^h \)

Calculate \( I_k^{(h,t)}, \omega_k^{(h,t)}, p_k^{(h,t)} \) based on \( \gamma^{(h,t)} \) using Eqs. (3.3.53) – (3.3.55)

end

\( \{I_k^{(h_k)}, p_k^{(h_k)}\}_{h_k=1}^H = \text{Unique}\left(\{(I_k^{(h,t)}, p_k^{(h,t)})\}_{(h,t)=(1,1)}^{H,T_k^h}\right) \)

for \( h_k = 1 : H_k \)

\( \omega_k^{(h_k)} = \sum_{h,t} \omega_k^{(h,t)} \)

normalise weights \( \{w_k^{(h_k)}\}_{h_k=1}^H \)

---

### 3.3.4 The Labelled Multi-Bernoulli Filter

According to Reuter et al. [48], the LMB filter can be interpreted as an efficient approximation of the \( \delta \)-GLMB filter. Compared with the standard \( \delta \)-GLMB filter, which exhibits exponential growth in the number of posterior components, the LMB filter can be more efficient since the number of components approximated by the LMB RFS only increases linearly.

In the LMB filter, an LMB RFS is propagated through two major steps, prediction and
3.3. Multi-Target Filtering with Labelled Random Finite Sets

update, to produce the LMB posterior. The prediction step incorporates all the information available about possible random changes that can occur to each single-target state including target birth and death. The information is mathematically modelled using the probability of survival, \( p_s(\cdot) \), and a birth process which itself is modelled as an LMB denoted by \( \pi_B = \{(r_B^{(\ell)} : p_B^{(\ell)})\}_{\ell \in \mathbb{L}_k} \) where \( \mathbb{L}_k \) is the label of newly born targets at each time (indeed, it is \( \mathbb{L}_k \) at time \( t_k \)). In the update step, the information provided by sensor measurements is utilised in a Bayesian inference framework. This information includes a sensor model formulated as a single-target measurement likelihood function \( g(z|x, \ell) \), the intensity function of a Poisson RFS model for clutter measurements, denoted by \( \kappa(z) \), and a detection profile model for the sensor formulated as a state-dependent probability of detection \( p_D(x, \ell) \). The flowchart of the LMB filter is detailed in Fig. 3.3.3.

### 3.3.4.1 Prediction

Following the development of Reuter et al. [48], the density of new born targets is modelled by an LMB RFS with parameter set \( \{(r_B^{(\ell)} : p_B^{(\ell)})\}_{\ell \in \mathbb{L}_k} \)

\[
\pi_B(X) = \Delta(X)w_B(\mathcal{L}(X))[p_B]^X, \tag{3.3.56}
\]

where

\[
w_B(L) = [1 - r_{B,k}]^{\mathbb{L}_k \setminus L}[r_{B,k}]^{\mathbb{L}_k \cap L} \tag{3.3.57}
\]

Let the multi-target posterior state at time \( t_{k-1} \) be approximated by an LMB RFS denoted
by \( \pi_{k-1} \). The predicted multi-target density is still an LMB RFS parameterised by

\[
\{ (r^{(\ell)}_{k|k-1}, p^{(\ell)}_{k|k-1}) \}_{\ell \in L_{0:k}} = \bigcup_{\ell \in L_{0:k-1}} \{ (r^{(\ell)}_{S,k|k-1}, p^{(\ell)}_{S,k|k-1}) \} \cup \{ (r^{(\ell)}_{B}, p^{(\ell)}_{B}) \}_{\ell \in L_k},
\]

(3.3.58)

where the label space is \( L_{0:k} = L_{0:k-1} \cup L_k \), with \( L_{0:k-1} \cap L_k = \emptyset \) and the predicted parameters of the survival track \( \ell \) are given by

\[
r^{(\ell)}_{S,k|k-1} = \eta_S r^{(\ell)}_{k-1}, \quad p^{(\ell)}_{S,k|k-1} = \frac{\langle p_s(\cdot, \ell)f(x|\cdot, \ell), p_{k-1}(\cdot, \ell) \rangle}{\eta_S(\ell)},
\]

\[
\eta_S(\ell) = \langle p_s(\cdot, \ell), p_{k-1}(\cdot, \ell) \rangle,
\]

(3.3.59), (3.3.60), (3.3.61)

where \( \eta_S(\ell) \) is the survival probability for track \( \ell \).

The predicted LMB RFS is required to be converted to the \( \delta \)-GLMB form to perform the full \( \delta \)-GLMB update. This conversion is obtained by

\[
\pi_{k|k-1}(X_{k|k-1}) = \Delta(X_{k|k-1}) \sum_{I_k \in \mathcal{F}(L_{0:k})} w^{(I_k)}_{k|k-1} \delta_1(\mathcal{L}(X_{k|k-1})) [p_{k|k-1}] X_{k|k-1},
\]

(3.3.62)

where \( I \) represents a set of track labels at time \( t_k \), and

\[
w^{(I_k)}_{k|k-1} = \prod_{\ell \in L_{0:k} \setminus I_k} (1 - r^{(\ell)}_{k|k-1}) \prod_{\ell' \in I_k} 1_{L}(\ell') r^{(\ell')}_{k|k-1}.
\]

(3.3.63)

<table>
<thead>
<tr>
<th>Cardinality</th>
<th>Hypotheses</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>(( \ell_1, \ell_2, \ell_3 ))</td>
</tr>
<tr>
<td>2</td>
<td>(( \ell_1, \ell_2 ))</td>
</tr>
<tr>
<td></td>
<td>(( \ell_1, \ell_3 ))</td>
</tr>
<tr>
<td></td>
<td>(( \ell_2, \ell_3 ))</td>
</tr>
<tr>
<td>1</td>
<td>( \ell_1 )</td>
</tr>
<tr>
<td></td>
<td>( \ell_2 )</td>
</tr>
<tr>
<td></td>
<td>( \ell_3 )</td>
</tr>
<tr>
<td>0</td>
<td>\emptyset</td>
</tr>
</tbody>
</table>

| Table 3.3.1: Construction of \( \delta \)-GLMB hypotheses |

Table 3.3.1 shows an example of transformation from LMB to \( \delta \)-GLMB. The LMB contains three tracks represented by their labels \( I = (\ell_1, \ell_2, \ell_3) \). All permutations of the label set \( I \)
are listed in Table 3.3.1 based on the cardinality of the hypotheses. The weight of a hypothesis depends on the probability of existence of each Bernoulli component. For example, the weight of hypothesis \((\ell_1, \ell_2)\) can be expressed as 
\[
\omega(\ell_1, \ell_2) = r(\ell_1) \cdot r(\ell_2) \cdot (1 - r(\ell_3)).
\]

### 3.3.4.2 Update

Given the predicted \(\delta\)-GLMB, the updated \(\delta\)-GLMB is given by [48]

\[
\pi_k(X_k | Z_k) = \Delta(X_k) \sum_{(I_k, \theta_k) \in \mathcal{F}(L_0:k \times \Theta_k)} w^{(I_k, \theta_k)} \delta_{I_k} (\mathcal{L}(X)) \left[ p_k^{(\theta_k)} (\cdot | Z) \right] X_k,
\]

where

\[
\omega^{(I_k, \theta_k)}(Z_k) \propto w_k^{(I_k)} \left[ \eta_{Z_k}^{(\theta_k)} \right]_{I_k}
\]

\[
p_k^{(\theta_k)} (x, \ell) = \frac{p_{k|k-1}^o (x, \ell) \psi_{Z_k} (x, \ell; \theta_k)}{\eta_{Z_k}^{(\theta_k)} (\ell)}
\]

\[
\eta_{Z_k}^{(\theta_k)} (\ell) = \left\{ \begin{array}{ll}
p_{k|k-1}^o (\cdot, \ell), & \psi_{Z_k} (\cdot, \ell; \theta_k) \\
1 - p_D (x, \ell) & \text{if} \ \theta_k (\ell) = 0.
\end{array} \right.
\]

The posterior \(\delta\)-GLMB must be transformed to the LMB \(\pi_k(X_k | Z_k)\) form to perform the recursive estimation. The posterior probability of existence \(r_k^{(\ell)}\) and probability density \(p_k^{(\ell)} (\cdot)\) of track \(\ell\) are given by [48]

\[
r_k^{(\ell)} = \sum_{(I_k, \theta_k) \in \mathcal{F}(L_0:k \times \Theta_k)} 1_{I_k}(\ell) \omega^{(I_k, \theta_k)}(Z_k),
\]

\[
p_k^{(\ell)} (x) = \frac{1}{r_k^{(\ell)}} \sum_{(I_k, \theta_k) \in \mathcal{F}(L_0:k \times \Theta_k)} 1_{I_k}(\ell) \omega^{(I_k, \theta_k)}(Z_k) p_k^{(\theta_k)}(x, \ell).
\]

The parameters of the updated LMB density are obtained by Eqs. (3.3.69) and (3.3.70). In addition, the tracks with the probability of existence lower than a pruning threshold are removed, and tracks that are closely spaced are merged. The tracks with the probability of
existence higher than a specific threshold are extracted as the posterior state estimate.

$$\hat{X} = \{ (\hat{x}, \ell) : r^{(\ell)} > T_r \}, \quad (3.3.71)$$

where $$\hat{x} = \arg \max_x p^{(\ell)}(x)$$.

The grouping and gating method is an efficient implementation of the LMB filter which can significantly reduce the computational burden. The principle is to partition the LMB and measurements into several groups, such that each contains closely spaced targets and corresponding associated measurements. Compared with the full set of targets and measurements, each constructed group has a reduced number of components. In addition, the LMB update of the constructed groups can be performed in parallel to further speed up the computation. The detailed procedure of the grouping and gating technique is given in Appendix B.3.

**Remark** The grouping and gating method is extremely suitable for optical space object tracking because the small FOV of SSA sensors, e.g., telescopes, only cover a small portion of the surveillance region which yields a natural partition. Even though a tracking campaign may conduct a large population of space objects, only a few of them can be detected at a single time epoch. Thus, all targets and measurements within the sensor FOV are grouped together, and only this group requires performance of the measurement update step.

### 3.3.5 The Joint Labelled Multi-Bernoulli Filter

The LMB filter with grouping and gating is an efficient solution for space object tracking using optical sensors which have a small FOV. In the case of a large number of closely space objects, e.g., space debris objects from break-up event, the group and gating approach may still result in a heavy computational burden. Motivated by the joint prediction and update $\delta$-GLMB filter, the joint LMB filter was developed by Reuter et al. [48] to further reduce the computation cost. The truncation of the joint LMB filter can be implemented using either the traditional Murty’s algorithm or the more recent Gibbs sampling method. The comparison results demonstrate that more efficient performance is achieved by the Gibbs sampling imple-
mentation. Therefore, the joint LMB filter via the Gibbs sampling implementation is introduced in this section.

The prediction of the joint LMB filter resembles the regular LMB filter. The prior multi-target state is approximated using the LMB RFS as the union of the predicted existing targets and the newborn targets. The update of the joint LMB filter is similar to the joint δ-GLMB filter. The use of the joint prediction and update avoids to calculating the predicted hypotheses with significant weights using the \( K \)-shortest path algorithm and therefore speeds up the calculation, especially for a large number of targets.

Note that the assignment cost of the joint LMB slightly differs from the joint δ-GLMB, and is given by

\[
\eta_i(j) = \begin{cases} 
1 - \eta_S(\ell_i)r(\ell_i), & 1 \leq i \leq R, j < 0 \\
\eta_S(\ell_i)r(\ell_i)\eta_Z(\ell_i), & 1 \leq i \leq R, j \geq 0 \\
1 - r_{B,k}(\ell_i), & R + 1 \leq i \leq P, j < 0 \\
r_{B,k}(\ell_i)\eta_Z(\ell_i), & R + 1 \leq i \leq P, j \geq 0.
\end{cases}
\]  

(3.3.72)

The major differences are that the association history \( \xi \) is not considered in the calculation due to the LMB approximation at each recursion, and \( \eta_S(\ell_i)r(\ell_i) \) is used to replace \( \eta_S(\ell_i) \). The ranked assignment problem is efficiently solved by using the Gibbs sampling method rather than Murty’s algorithm. The detailed implementation is exactly the same as the joint δ-GLMB filter.

### 3.3.6 Comparison of Labelled RFS Filters for Space Object Tracking

In order to evaluate the performance of the labelled RFS filters introduced in this chapter, i.e., the δ-GLMB filter, the joint δ-GLMB filter, the LMB filter, and the joint LMB filter, two simulated scenarios are designed and used for validation.
3.3.6.1 Simulation Design

The first test is to track 10 closely spaced GEO objects using a ground-based telescope. The second test considers a more complicated scenario, which tracks a catalogue of 100 GEO objects using a space-based optical sensor.

<table>
<thead>
<tr>
<th>Table 3.3.2: Orbital elements of the simulated GEO object</th>
</tr>
</thead>
<tbody>
<tr>
<td>a (km)</td>
</tr>
<tr>
<td>--------</td>
</tr>
<tr>
<td>42160</td>
</tr>
</tbody>
</table>

In the first test scenario, the 10 GEO objects are generated by randomly perturbing a simulated GEO object with Keplerian orbital elements shown in Table 3.3.2. The random perturbation is assumed as zero-mean Gaussian with standard deviations in semi-major axis, inclination, and true anomaly of 1 km, 0.1 deg, and 0.1 deg respectively. Both the mean and covariance are transformed to Cartesian coordinates using the unscented transform method.

The ground-based telescope is assumed as located at the Mt. Stromlo Observatory, the geodetic location and detailed system parameters are listed in Table 3.3.3. The ground-based sensor generates 50 scans in the 10-hour total time window. As the targets are closely spaced, all targets can be detected at every single scan.

<table>
<thead>
<tr>
<th>Table 3.3.3: Parameters of the Mt. Stromlo Observatory</th>
</tr>
</thead>
<tbody>
<tr>
<td>Latitude (deg)</td>
</tr>
<tr>
<td>----------------</td>
</tr>
<tr>
<td>-35.32</td>
</tr>
</tbody>
</table>

In the second test scenario, the 100 GEO objects are randomly selected from the NORAD TLE public catalogue \(^1\) for the date 10 November 2017 using the following constraints

\[
0 \leq e \leq 0.1; \quad 0 \text{ deg} \leq \text{inc} \leq 70 \text{ deg}; \quad 0.9 \text{ day} \leq n_m \leq 1.1 \text{ day},
\]

(3.3.73)

where \(n_m\) is the orbital period. The orbital states of the 100 tested objects are randomly per-

\(^1\)www.space-track.org, 10/11/2017
3.3. Multi-Target Filtering with Labelled Random Finite Sets

turbed from the truth using the same uncertainty parameters in the first case. In addition, the mean and covariance are transformed to Cartesian coordinates using the unscented transform method.

The 100 GEO objects are tracked using an optical space-based sensor placed on an SSO platform with orbital parameters given in Table 3.3.4. Due to the assumed uncertainty of the SSO orbital state estimate, the measurement noise assigned to the SSO sensor is slightly larger than the ground-based telescope.

**Table 3.3.4:** Orbital parameters of the SSO sensor

<table>
<thead>
<tr>
<th>$a$ (km)</th>
<th>$e$</th>
<th>$inc$ (deg)</th>
<th>$\omega$ (deg)</th>
<th>$\Omega$ (deg)</th>
<th>$M$ (deg)</th>
<th>FOV size ($\alpha, \delta$)</th>
<th>Noise ($\alpha, \delta$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7162.17</td>
<td>$7.6 \times 10^{-4}$</td>
<td>98.54</td>
<td>159.97</td>
<td>133.10</td>
<td>106.96</td>
<td>[2 deg, 2 deg]</td>
<td>[3&quot;, 3&quot;]</td>
</tr>
</tbody>
</table>

A total number of 600 scans in the 24-hour tracking time window are generated. Only a small portion of the 100 objects can be detected at each scan due to the small sensor FOV. Therefore, the sensor tasking problem needs to be considered to conduct this scenario. A simple sensor tasking method is used in this test. The space-based sensor is tasked to routinely point to an object at each epoch until the entire catalogue is covered, and this process is repeated. The pointing direction of the sensor at each epoch is the calculated angular direction of the corresponding target.

Both the ground-based and space-based optical sensor measures right ascension and declination observations. Measurement noise values of the ground-based and space-based sensors are provided in Table 3.3.3 and Table 3.3.4 respectively. The process noise covariance is a $6 \times 6$ diagonal matrix, which assumes 1 $\sigma$ values of 1 m and $10^{-3}$ m/s for position and velocity respectively. The measurement and process noises are constant for all objects.

The target birth process is not considered in both tests for simplicity. The orbital state of all tested targets is assumed as known *a priori*. The parameters used in the LMB filter are shown in Table 3.3.5.

The orbital motion is assumed as unperturbed two-body dynamics for simplicity. All la-
Table 3.3.5: Parameters of the LMB filter

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability of survival</td>
<td>0.99</td>
</tr>
<tr>
<td>Probability of detection</td>
<td>0.95</td>
</tr>
<tr>
<td>Mean clutter return</td>
<td>1</td>
</tr>
<tr>
<td>Pruning threshold of LMB track</td>
<td>1e-4</td>
</tr>
<tr>
<td>Merging threshold of GM components</td>
<td>3</td>
</tr>
<tr>
<td>Maximum number of surviving hypotheses</td>
<td>100</td>
</tr>
<tr>
<td>Maximum number of updated hypotheses</td>
<td>100</td>
</tr>
<tr>
<td>Maximum number of posterior hypotheses</td>
<td>100</td>
</tr>
</tbody>
</table>

labelled RFS filters are implemented in Matlab®, and the computation times are based on an Intel Core i7 CPU with 64-bit numerics and a 2.7 GHz clock rate. The accuracy is assessed by the second-order OSPA error [102], with cutoff values 100 km and 2 km/s for position and velocity respectively.

3.3.6.2 Case I

Figure 3.3.4 shows the averaged OSPA position and velocity errors of the 10 GEO objects using the four labelled RFS filters using $p_D = 0.95$ and 50 MC runs. It can be observed that all four filters can significantly reduce the OSPA state errors at the start of the simulation, while the LMB and $\delta$-GLMB filters outperform the joint LMB and joint $\delta$-GLMB filters at the middle stage of tracking. Finally, similar accuracy is achieved by the four filters at the end of the simulation. As all filters detected all targets throughout the entire simulation, their cardinality estimates equal the true number of targets, and these results are not shown here.

The computation times of the four filters are shown in Table 3.3.6. The joint LMB filter is the most efficient and significantly outperforms the other three filters, and the $\delta$-GLMB filter results in the most computation time.

Table 3.3.6: Computation times of the tested filters for 10 objects using $p_D = 0.95$

<table>
<thead>
<tr>
<th>Filters</th>
<th>LMB group</th>
<th>joint LMB</th>
<th>$\delta$-GLMB</th>
<th>joint $\delta$-GLMB</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time (s)</td>
<td>11.54</td>
<td>3.74</td>
<td>14.54</td>
<td>5.62</td>
</tr>
</tbody>
</table>
Figure 3.3.4: OSPA errors of 10 objects in the case of $p_D = 0.95$

Figure 3.3.5 shows the averaged OSPA position and velocity errors from the four labelled RFS filters using $p_D = 0.75$ and 50 MC runs. The OSPA errors of the four filters are slightly larger than the previous test due to more missed detections. Although the accuracy at the final stage of tracking is similar, the LMB and $\delta$-GLMB filters are able to provide more accurate state estimation than the other two joint filters during the first half of tracking. The computation times of the four filters are shown in Table 3.3.7. Again, the joint LMB filter achieves the most efficient performance, and the computation times of different filters resemble the results of $p_D = 0.95$ in Table 3.3.6.
Table 3.3.7: Computation times of the tested filters for 10 objects using $p_d = 0.75$

<table>
<thead>
<tr>
<th>Filters</th>
<th>LMB group</th>
<th>joint LMB</th>
<th>$\delta$-GLMB</th>
<th>joint $\delta$-GLMB</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time (s)</td>
<td>12.43</td>
<td>4.17</td>
<td>15.94</td>
<td>4.80</td>
</tr>
</tbody>
</table>

3.3.6.3 Case II

Figure 3.3.6 shows the OSPA position and velocity errors of the 100 GEO objects for the four labelled RFS filters using $p_d = 0.95$. It can be seen from the figure that the four curves are highly overlapped, indicating the similar performance of the four filters. In addition, results show dramatic reduction of the OSPA errors cover the course of the simulation for all filters. The cardinality estimates of all four filters equal the true number of targets, so these results are again omitted.

![Figure 3.3.6: OSPA Position error](image1)

![Figure 3.3.6: OSPA Velocity error](image2)

The computation times of the four filters are shown in Table 3.3.8. The LMB and joint LMB filters achieve similar performance and significantly outperform the others, and the $\delta$-GLMB filter results in the largest computation time. This is mainly because a large number of weak hypotheses are maintained in the $\delta$-GLMB filter, which significantly increases the computational effort. Results also demonstrate that the joint $\delta$-GLMB filter provides improved computational efficiency as compared to the standard $\delta$-GLMB filter.

Figure 3.3.7 shows the OSPA position and velocity errors of the 100 GEO objects for the
Table 3.3.8: Computation times of the tested filters for 100 objects using $p_D = 0.95$

<table>
<thead>
<tr>
<th>Filters</th>
<th>LMB group</th>
<th>joint LMB</th>
<th>$\delta$-GLMB</th>
<th>joint $\delta$-GLMB</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time (s)</td>
<td>85.66</td>
<td>82.20</td>
<td>2423.83</td>
<td>436.19</td>
</tr>
</tbody>
</table>

four labelled RFS filters using $p_D = 0.75$. The accuracies of the four filters are slightly worse than the case of $p_D = 0.95$ due to more missed detections. The LMB filter and $\delta$-GLMB filters converge much quicker than the others, indicating they are more robust to the scenario of low signal-to-noise (SNR) ratio, e.g., more missed detections. In contrast, as fewer hypotheses are kept by the joint LMB filter and joint $\delta$-GLMB filter, their performance is more sensitive to missed detections caused by low SNR.

![Graphs showing OSPA errors](image)

(a) OSPA Position error  
(b) OSPA Velocity error

Figure 3.3.7: OSPA errors of 100 objects in the case of $p_D = 0.75$

The computation times of the four filters are shown in Table 3.3.9. The joint LMB filter performs best as expected, and the $\delta$-GLMB filter again yields the highest computation time.

Table 3.3.9: Computation times of the tested filters for 100 objects using $p_D = 0.75$

<table>
<thead>
<tr>
<th>Filters</th>
<th>LMB group</th>
<th>joint LMB</th>
<th>$\delta$-GLMB</th>
<th>joint $\delta$-GLMB</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time (s)</td>
<td>90.10</td>
<td>86.66</td>
<td>2609.95</td>
<td>615.06</td>
</tr>
</tbody>
</table>
3.4 Summary

This chapter introduces the labelled RFS theory and several efficient implementations. The accuracy and efficiency of four labelled RFS filters are compared using two test scenarios of space object tracking. Simulation results validated the effectiveness of the labelled RFS filters for multiple space object tracking in different scenarios. Among the four tested filters, the LMB filter provides fast convergence and accurate orbital state estimation. In addition, its efficiency significantly outperforms the standard $\delta$-GLMB filter, especially when tracking a large population of space objects. Even though the joint LMB filter slightly outperforms the LMB filter in terms of efficiency, it is more sensitive to the case of low SNR and converges much slower. Therefore, the LMB filter is considered as a practical solution for accurate and efficient space object tracking, and the LMB filter is employed in Chapter 4 and Chapter 5 to conduct the orbital state estimation and target identity management of space objects.
Chapter 4  A Multi-Target Tracking Method for New Space Object Using a Boundary Value Approach

As discussed in Sec. 3.3.6, the labelled RFS filters, especially the LMB filter, can provide effective and efficient performance for multiple space object tracking. One objective of this thesis is to investigate the labelled RFS filters for tracking both cataloged object and newly discovered object to improve the capacity and capability of the current catalog. In the context of MTT, modelling new space object birth can be interpreted as an IOD process. Birth models using the CAR and PAR methods have been implemented with RFS filters for tracking of new space objects. In order to better approximate the large initial uncertainty of short-arc optical observations, these admissible region birth models often produce a large number of GM components, which leads to increased computational demand in both orbit propagation and measurement-to-track association in the recursive estimation process. In order to achieve efficient filtering, this chapter proposes a new birth model based on the BVP optimisation method for use with the LMB filter. Compared with the CAR and PAR birth model, the BVP birth model determines a reliable orbital state represented by a single Gaussian component, and thus simplifies the overall computational complexity. Section 4.3 details the implementation of the BVP birth models, including the estimation of initial covariance, the classification approach, and the calculation of the probability of existence for new targets. In Sec. 4.5, the BVP birth model is validated by comparing against the CAR and PAR birth models using two simulated
4.1 Introduction

A challenging task in SSA is to track multiple space objects for the SOC maintenance. The Bayesian MTT filter addresses this issue by associating measurements to initially known or newly detected tracks and simultaneously estimating the time-varying number of targets and their orbital states. In addition, several challenges including missed detections and clutter measurements need to be considered in MTT filters.

In the scenario of multi-target tracking for space objects, modelling new target birth is similar to a process of IOD. Determining an orbital solution from a single arc of measurement data is a challenging task in SSA, and is commonly referred to as the too-short arc problem [19]. The methods of CAR and PAR have been developed to approximate the PDF of the initial orbit state given the measurements and physical constraints on the solution. Recently, the CAR and PAR have been successfully incorporated in RFS filters for modelling target birth, e.g., the CAR-CPHD [66], PAR-CPHD [64] and CAR-LMB [56] filters. However, one drawback of the CAR and PAR birth model is the large computational burden resulting from the Gaussian mixture representation of the orbital state. The population of GM components from a birth model is one of the dominant factors in the computational demand of RFS filters. The large number of GM components results in more computation time for orbit propagation in the prediction step; in addition, they also lead to more computational effort in the track-to-measurement associations in the update step. The computational efficiency of RFS filters can also be adversely affected by dense clutter.

Another solution to the too-short arc problem is the tracklet association method, which determines if multiple tracklets originate from the same object and computes the initial orbital state following the association. The tracklet association problem has been investigated in Chapter 2. Motivated by previous efforts, this chapter presents a BVP birth model for improved computational efficiency of the LMB filter. The proposed BVP birth model employs a classi-
4.2 CAR and PAR Methods

The CAR method [31] determines all hypothetical orbits of a tracklet by mapping the constraints on orbital parameters, i.e., the semi-major axis $a$ and eccentricity $e$, to the undetermined state space of $\rho$ and $\dot{\rho}$. The constraints equations can be expressed below [31]

$$\dot{\rho}^2 + w_1 \dot{\rho} + F(\rho) - 2\varepsilon = 0 \quad (4.2.1)$$
$$a_4 \dot{\rho}^4 + a_3 \dot{\rho}^3 + a_2 \rho^2 + a_1 \dot{\rho} + a_0 = 0 \quad (4.2.2)$$

The equations of the coefficients used in the above equations can be found in Sec. 2.1.3. Solving these two polynomials produces the desired CAR on the range and range-rate space.

One primary drawback of the CAR birth model is the low computational efficiency. The number of GM components affects the computational complexity of both the prediction and update steps of RFS filters. According to Jones et al. [65], small $(\sigma_\rho, \sigma_{\dot{\rho}})$ can result in high computational demand without significant reduction of the OSPA position error in certain cases. However, this consideration does not apply to the case of sparse data and long gaps between measurements, in which small values for design parameters $(\sigma_\rho, \sigma_{\dot{\rho}})$ are necessary to guarantee an accurate representation of the initial uncertainty to facilitate accurate estimation in the filter. In short, the design parameters need to be well-selected according to the characteristics of the measurements.
The PAR method was developed by Hussein et al. [58] for probabilistically interpreting the CAR by considering the uncertainties of the observations and constraints. The observation noise is assumed to be Gaussian, while the constrained parameters $a$ and $e$ are uniformly distributed. Let $a$ and $e$ be independent of each other, and independent of the observations $(\alpha, \delta, \dot{\alpha}, \dot{\delta})$ as well. The joint distribution of $(\alpha, \delta, \dot{\alpha}, \dot{\delta}, a, e)$ is

$$p(\alpha, \delta, \dot{\alpha}, \dot{\delta}, a, e) = p(\alpha, \delta, \dot{\alpha}, \dot{\delta}) p(a) p(e),$$

which can be mapped to $p(\rho, \dot{\rho})$ using Eqs. (4.2.1) and (4.2.2). Given a group of Monte Carlo samples of $(\alpha, \delta, \dot{\alpha}, \dot{\delta}, a, e)$, a group of non-uniformly distributed particles in the $(\rho, \dot{\rho})$ plane are determined to represent the uncertainty of $(\rho, \dot{\rho})$. These particles can be approximated by the GM method for analytical computations, e.g., the initialisation of a Kalman filter and the representation of birth tracks in a RFS MTT framework.

Theoretically, the PAR particles can be generated from Eqs. (4.2.1) and (4.2.2). Hussein et al. [58] developed an analytical method to calculate the particles by using an alternative formulation of CAR. Jones et al. [103] suggested a Newton-Raphson method to solve the above two equations, but this approach yields a large computational cost, especially for a rough initial guess. The PAR particles can be regarded as the intersections of $a$ and $e$ constraint curves. Thus, in this study, a straightforward and efficient bisection method is developed to search for the intersections. The procedure can be found in Sec. 2.3.2.1.

The PAR particles can be either used directly in a particle-based filter [61, 62], or approximated by the GM method [64]. The GM-formed PAR for initialisation of the LMB filter is further investigated in simulation to compare against the CAR and the proposed BVP method. Note that the generation of PAR particles is highly parallelisable, but the GM approximation of the particles leads to high computational demand compared to CAR.

According to Hussein [58], the optimal number of GM components for representing the PAR particles can be determined by an Expectation-Maximisation (EM) algorithm [104] within the user defined maximum value. Compared with CAR, the PAR birth model consumes more
time in the process of modelling target birth due to the high computational demand of PAR generation and GM approximation. In order to improve the computational efficiency of the PAR birth model, the number of GM components can be reduced by specifying an exact value to use, or by reducing the maximum number of components considered for use. However, the problem of selecting an appropriate number of GM components to reduce computational burden and achieve the desired accuracy in subsequent RFS filtering is still being investigated.

Fig. 4.2.1 gives an example of range and range-rate PDFs generated by the GM approximation of the CAR and PAR for a simulated GEO object. The constraint values are $a \in [4, 7]$ ER and $e \in [0, 0.4]$, and result in 225 GM components generated to approximate the CAR PDF. The set of parameters $(\alpha, \delta, \dot{\alpha}, \dot{\delta}, a, e)$ is sampled 10,000 times, which results in 12,847 PAR particles. Note that a pair of $a$ and $e$ constraint curves may yield multiple intersections. These are then converted to 50 GM components using the EM method. Fig. 4.2.1a indicates that the PDF approximates a uniform distribution within the CAR, while Fig. 4.2.1b shows that the PDF of the PAR is obviously non-uniform.

### 4.3 The BVP Birth Model

The CAR and PAR methods have been applied to RFS filters to model new target birth [66, 64, 31, 58], while the BVP optimisation method has not previously been investigated for this
Chapter 4. A Multi-Target Tracking Method for New Space Object Using a Boundary Value Approach

application. Compared to the admissible region GM approaches, the BVP method provides a reliable IOD solution using boundary values at two points while applying the same constraints. As a result, the use of BVP has the potential to reduce the computation time of the LMB filter birth model. This section presents a new BVP-based birth method for improved computational efficiency of the multi-target filtering. The detailed implementation of the BVP optimisation method can be found in Sec. 2.2.1.2.

4.3.1 Covariance Estimation

The mean orbital state and covariance of a birth track are needed for the generation of the birth LMB density. The BVP method produces a reliable IOD solution that can be used as the mean of the initial state of two tracklets, but it does not yield an estimate of the covariance. One possible approach is to generate a covariance based on the accuracy of the estimated state using BVP. However, such a covariance may lead to large errors in the state if the variances are too large, while if the variances are too small it may lead to missed associations of subsequent measurements to birth tracks, causing the LMB filter to lose custody of the new object. Alternatively, this study employs a Batch Least Squares (BLS) method to estimate the covariance of a birth track by utilizing all measurements from the two observed tracklets used in BVP.

Given two observed tracklets $\mathbf{x}_1$ and $\mathbf{x}_2$, their IOD solution $x_{BVP}$ from BVP and an initial covariance $P_{ini}$ assumed from prior knowledge are used to initialise the BLS processor. Generally, a tracklet consists of several independent measurements of angles and angular rates [28]. The BLS processor is applied to all measurements in $\mathbf{x}_1$ and $\mathbf{x}_2$. The BLS processor outputs both the estimated orbital state $x_{BLS}$ and covariance $P_{BLS}$. The estimation process is summarised as

\[
(x_{BVP}, P_{ini}, \mathbf{x}_1, \mathbf{x}_2) \xrightarrow{\text{BLS}} (x_{BLS}, P_{BLS}).
\]

Note that a further improvement of $x_{BVP}$ cannot be achieved by BLS because all information of $\mathbf{x}_1$ and $\mathbf{x}_2$ has been considered in the generation of $x_{BVP}$. Therefore, $x_{BVP}$ at epoch $t_2$ is still used as the mean birth state in the LMB filter in order to avoid double counting information.
4.3. The BVP Birth Model

Then, the covariance $P_{BLS}$ is propagated to the time epoch of $t_2$ to approximate the orbital state uncertainty. The PDF $p_B(\cdot)$ of the birth track can then be represented as a Gaussian distribution with mean $x_{BVP}$ at epoch $t_2$ and covariance $P_{BLS}$, given by

$$p_B(x) = p_g(x; x_{BVP}, P_{BLS}),$$

(4.3.2)

where $p_g(\cdot)$ denotes the Gaussian PDF.

**Remark 4.1** Even though the birth track is modelled by the output of BLS, the BVP optimisation method is indispensable in the BVP birth model. The reasons are twofold. First, the BVP optimisation determines if a valid IOD solution can be generated from two tracklets. Naturally, it is unnecessary to perform BLS for two tracklets if they do not yield a valid orbit. In this manner, the BVP birth model also serves as an effective means to reduce clutter birth tracks. Therefore, the use of BVP effectively reduces the clutter birth tracks. In addition, a good prior orbital state provided by BVP is required to guarantee the convergence of the BLS processor, especially in the sparse data scenario.

The BLS method is illustrated using a case of estimating mean orbital state and covariance for a birth track generated from two GEO tracklets separated by around two hours. The initial covariance $P_{ini}$ of the orbital state in Cartesian space is

$$P_{ini} = \text{diag} [100, 100, 10, 10^{-4}, 10^{-4}, 10^{-5}],$$

(4.3.3)

where the units of position and velocity are km and km/s respectively.

The BLS solution is computed, and the estimated covariance $P_{BLS}$ and initial covariance $P_{ini}$ are transformed to the topocentric space, and their $3\sigma$ ($\rho, \dot{\rho}$) components are shown as the red solid ellipse and black dash ellipse in Fig 4.3.1 respectively. The range and range rate results from $(x_{BLS}, P_{BLS})$ and $(x_{BVP}, P_{ini})$ using the unscented transform are depicted as a red diamond and black triangle respectively, which are too close to be distinguished in this case. The similar range and range rate results indicates that either $x_{BVP}$ or $x_{BLS}$ can be used as the orbital state of
the birth track. The blue curves denote the CAR generated based on a small range of constraint values, i.e., \( a \in [6.55, 6.65] \) ER and \( e \in [0, 0.06] \), which is merely for better visualising the size of the covariance. The position traces of \( P_{\text{ini}} \) and \( P_{\text{BLS}} \) are around 210 and 134.5 respectively. Obviously, the estimated covariance is significantly reduced and more concentrated around the truth (yellow star).

### 4.3.2 Classification Method

In the scenario of modelling the birth of multiple object from different orbital domains, a wide range of semi-major axis constraint values need to be considered. As a result, a large number of orbit revolution intervals \( n_o \) is generated by the BVP solver, requiring additional calls to the optimisation routine.

To address this issue, this section develops a classification method to partition the constraint values into multiple subsets for objects from different domains. The partitions are determined based on the altitude and eccentricity classification of the geocentric orbits, i.e., LEO, GEO and GTO. The constraint values of these two parameters are derived based on the a priori information obtained from Space-Track\(^1\). Table 4.3.1 outlines the constraint values of each classification used in this study. Although this classification covers the majority of cataloged object, a more comprehensive and rigorous classification could be implemented based on user

\(^1\)https://www.space-track.org/
requirements. A more detailed discussion about the classification can be found in Ref. [89].

<table>
<thead>
<tr>
<th>Classification</th>
<th>Semi-major axis (ER)</th>
<th>Eccentricity</th>
</tr>
</thead>
<tbody>
<tr>
<td>LEO</td>
<td>[1, 1.3]</td>
<td>[0, 0.24]</td>
</tr>
<tr>
<td>GEO</td>
<td>[6.3, 7.0]</td>
<td>[0, 0.06]</td>
</tr>
<tr>
<td>GTO</td>
<td>[3.6, 4.3]</td>
<td>[0.7, 0.75]</td>
</tr>
</tbody>
</table>

Specifically, let $C$ be a set of multiple constraints, partitioned into a group of independent subsets

$$C = \{C_1, \ldots, C_i, \ldots, C_{nc}\}$$

where $n_C$ denotes the number of partitions. For a classification $C_i$, the BVP optimisation is applied if it yields valid admissible regions for both tracklets. Otherwise, the classification $C_i$ will be ignored. In addition, an IOD solution is only obtained if the global minimum of the loss function is smaller than the association threshold $T_a$. Therefore, the number of generated IOD solutions $m_C$ is no more than the number of classifications $n_C$.

If multiple classifications satisfy the threshold $T_a$, then only the one $C_i^*$ with the smallest BVP loss function value is employed in the BVP birth model, given by

$$C_i^* = \arg\min_{C_i} L(C_i).$$

The PDF $p_B$ of a birth track can then be represented by a single Gaussian component

$$p_B(x) = p_{g}(x; x_{BVP,C_i^*}, P_{BLS,C_i^*}),$$

where $x_{BVP,C_i^*}$ and $P_{BLS,C_i^*}$ are the mean and covariance of the orbital state from the best classification $C_i^*$.

The proposed classification method is illustrated using a simple scenario of processing two tracklets from a GTO object separated by around 16.5 hours. The admissible regions of each tracklet are generated using all constraint values shown in Table 4.3.1. The obtained admissible
regions are depicted in the two subfigures of Fig. 4.3.2. In each subfigure, the black curve is an admissible region established only using a semi-major axis constraint: \( a \in [1, 8] \) (ER) to involve all potential orbital solutions in different altitudes. As the first tracklet is observed near perigee and the second is around apogee, the range and range-rate values of the two admissible regions are significantly different. Both tracklets yield an admissible region for GTO, which is shown as the region within the red boundary in each subfigure. The red boundary in Fig. 4.3.2a is too narrow to be distinguished, but actually it represents an admissible region rather than a curve. Even though the two tracklets are from the same GTO object, different admissible regions can be generated using the multiple constraint classifications. In Fig. 4.3.2a, the blue boundary is the admissible region for LEO, while the region within the green boundary in Fig. 4.3.2b represents GEO.

In the two subfigures, only the GTO classification results in valid admissible regions for both tracklets, meaning that the hypothetical orbits from other classifications cannot fit the two tracklets. Therefore, these two tracklets must be from a GTO object. As confirmation of this analysis, the true values of range and range-rate for the two tracklets are shown as the black stars, and both are located within the GTO admissible regions. In this scenario, the BVP optimisation needs to be executed once as the GTO classification results in one possible \( n_o \).
interval: \( n_o \in [1, 1.5] \). However, if these two tracklets are processed using the semi-major axis constraint: \( a \in [1, 8] \) ER, then 23 possible \( n_o \) intervals (i.e., from \( n_o \in [0.5, 1] \) to \( n_o \in [11.5, 12] \)) are produced and need to be optimised. Therefore, the classification significantly reduces the computational efforts for processing tracklets from space objects in different regions.

### 4.3.3 Probability of Existence

According to the literature [48, 47], there are two commonly used birth distributions in the RFS framework. The first assumes fixed birth locations with small spatial uncertainties. At each epoch, birth components are instantiated only in the vicinity of the birth locations, making the method effective in high clutter environments. However, the trajectories of birth tracks need to be known \textit{a priori}. Alternatively, an adaptive birth intensity can be used to generate a birth LMB PDF concentrated around measurements which do not originate from existing tracks. This approach is also known as a measurement-based birth model, and has been applied for several RFS filters [48, 56].

The birth LMB RFS at epoch \( t_k \) is parameterised by \( \pi_{B,k} = \{(r^{(l)}_{B,k}, p^{(l)}_{B,k})\}_{l \in L_k} \), where \( p^{(l)}_{B,k} \) is obtained using Eq. (4.3.6), and the probability of existence \( r^{(l)}_{B,k} \) is needed to form the complete birth LMB RFS.

In the CAR/PAR birth model, the probability of existence \( r_{B,k}(z_{k-1,i}) \) for a single measurement \( z_{k-1,i} \in \{1, \cdots, |z_{k-1}|\} \) can be calculated based on the adaptive birth distribution [48], given by

\[
r_{B,k}(z_{k-1,i}) = \min \left( r_{B,\text{max}}, \frac{1 - r_{A,k-1}(z_{k-1,i})}{\sum_{z' \in Z_{k-1}} \left(1 - r_{A,k-1}(z')\right)} \cdot \lambda_{B,k} \right),
\]

where \( r_{B,\text{max}} \) is the maximum probability of existence defined by the prior knowledge; \( \lambda_{B,k} \) is the mean number of new target birth; \( r_{A,k-1} \) is the probability of the measurement associated to the existing targets, given by [48]

\[
r_{A,k-1}(z_{k-1,i}) = \sum_{(I,\theta) \in F \times \Theta} 1_{\theta}(z_{k-1,i}) w_k^{(I,\theta)}. \tag{4.3.8}
\]
Chapter 4. A Multi-Target Tracking Method for New Space Object Using a Boundary Value Approach

The association probability can be interpreted as the sum of hypotheses weights that use the same measurement for update. Note that $\lambda_{B,k}$ is always defined as a fixed, small value to avoid accumulation of false tracks. However, this also delays new track confirmation and leads to biased cardinality estimates. Gehly et al. [66] proposed a means to adaptively compute the weight of birth PHD components using a ratio of Gaussian likelihoods for existing targets and GM clutter. This approach is suitable for sensor allocation schemes to effectively schedule follow-on measurement for new targets.

In addition, the adaptation of $r_{B,k}(z_{k-1,i})$ to the BVP birth model is not straightforward because the BVP birth model relies on two sequential measurements $z_{k-1,i}$ and $z_{k,j}, j \in \{1, \cdots, |z_k|\}$. In this study, the probability of existence of a birth track from $z_{k-1,i}$ and $z_{k,j}$ is regarded as the probability that the two measurement arcs originate from a common object, which depends on the probability that both $z_{k-1,i}$ and $z_{k,j}$ are not associated to existing targets, and the probability that $z_{k-1,i}$ and $z_{k,j}$ are associated to each other $r_{TA}(z_{k-1,i}, z_{k,j})$ through the BVP optimisation, given by

$$
 r_{B,k+1}(z_{k-1,i}, z_{k,j}) = (1 - r_{A,k-1}(z_{k-1,i}))(1 - r_{A,k}(z_{k,j}))r_{TA}(z_{k-1,i}, z_{k,j}). \quad (4.3.9)
$$

There is no explicit formulation for the measurement association probability $r_{TA}(z_{k-1,i}, z_{k,j})$. Omitting $r_{TA}(z_{k-1,i}, z_{k,j})$, the probability that $z_{k-1,i}$ and $z_{k,j}$ are not associated to existing targets can be used to determine an upper bound on $r_{B,k+1}(z_{k-1,i}, z_{k,j})$ because cases in which $z_{k-1,i}$ and $z_{k,j}$ do not form a valid orbit (e.g., if one measurement originate from clutter) yield a value $r_{TA}(z_{k-1,i}, z_{k,j}) < 1$. In practice, the BVP birth model effectively reduces the number of clutter birth tracks because two clutter measurements can rarely produce a valid IOD solution via the optimisation process. Therefore, clutter measurements are not considered to be a significant issue when computing the probability of existence for new targets. Then, the probability of existence $r_{B,k+1}(z_{k-1,i}, z_{k,j})$ can be assumed equal to the upper bound, given by:

$$
 r_{B,k+1}(z_{k-1,i}, z_{k,j}) \leq (1 - r_{A,k-1}(z_{k-1,i}))(1 - r_{A,k}(z_{k,j})). \quad (4.3.10)
$$
Even though the use of the upper bound for \( r_{B,k+1}^{B}(z_{k-1,i}, z_{k,j}) \) may lead to an overestimate of cardinality when clutter birth tracks are generated, the clutter tracks can be quickly pruned due to the lack of subsequent measurements to confirm their presence.

### 4.4 Implementation of The BVP-LMB Filter

General schematics of the LMB filter with a birth model are shown in Fig. 4.4.1a. The two orange blocks are the prediction and update of the LMB filter; the blue blocks represent the birth model. The implementation of the LMB filter with a birth model is briefly introduced as follows.

![Diagram of the LMB filter and birth models](image)

**Figure 4.4.1:** The flowchart of the LMB filter and the three birth models

1. An IOD technique (e.g., BVP, CAR or PAR) is used to generate the birth LMB \( \pi_k^B \) based on measurements in the last one or two epochs depending on which birth model is used.
(2) The birth LMB RFS and the posterior LMB RFS are assumed to be independent. The predicted multi-target state is an LMB RFS parameterised by \( \pi_{k+1|k} = \{ (r_{k+1|k}^{(f)}, P_{k+1|k}^{(f)}) \}_{\ell \in L_{0,k}} \), which is the union of the surviving LMB RFS and birth LMB RFS.

\[
\{ (r_{k+1|k}^{(f)}, P_{k+1|k}^{(f)}) \}_{\ell \in L_{0,k}} = \{ (r_{S,k+1|k}^{(f)}, P_{S,k+1|k}^{(f)}) \}_{\ell \in L_{0,k-1}} \bigcup \{ (r_{B,k+1|k}^{(f)}, P_{B,k+1|k}^{(f)}) \}_{\ell \in L_{k}}. \tag{4.4.1}
\]

The predicted probability of existence \( r_{S,k+1|k}^{(f)} \) and the probability density \( p_{S,k+1|k}^{(f)} \) of the orbital state of a survival track \( \ell \) via the GM implementation are expressed by

\[
\begin{align*}
r_{S,k+1|k}^{(f)} &= P_{k}^{(f)} r_{S,k}^{(f)} \tag{4.4.2} \\
p_{S,k+1|k}^{(f)} &= 1_L(\ell) \sum_{j=1}^{J} w_{k+1|k,j}^{(f)} p_{g}\left( x; m_{k+1|k,j}^{(f)}, P_{k+1|k,j}^{(f)} \right) \tag{4.4.3}
\end{align*}
\]

where \( p_{S} (\cdot) \) is the survival probability, \( m_{k+1|k,j}^{(f)} \) and \( P_{k+1|k,j}^{(f)} \) are obtained by propagating \( m_{k,j}^{(f)} \) and \( P_{k+1|k,j}^{(f)} \) to time \( t_{k+1} \) using the unscented transform, and \( J \) denotes the number of GM components. The weight \( w_{k+1|k,j}^{(f)} \) is a constant in the prediction process. The predicted mean and covariance \( m_{k+1|k,j}^{(f)} \) and \( P_{k+1|k,j}^{(f)} \) computed via the unscented transform are given by

\[
\begin{align*}
m_{k+1|k,j}^{(f)} &= \sum_{i=0}^{2n} w_{i}^{m} \chi_{k+1|k,j,i}^{(f)} \tag{4.4.4} \\
P_{k+1|k,j}^{(f)} &= \sum_{i=0}^{2n} w_{i}^{p} (\chi_{k+1|k,j,i}^{(f)} - m_{k+1|k,j}^{(f)}) (\chi_{k+1|k,j,i}^{(f)} - m_{k+1|k,j}^{(f)})^T \tag{4.4.5}
\end{align*}
\]

where \( \chi_{k+1|k,j,i}^{(f)} \) denotes the \( i \)th sigma point, the total number of sigma points is \( 2n + 1 \), and \( n \) is the dimension of the orbital state. The sigma point \( \chi_{k+1|k,j,i}^{(f)} \) is given by

\[
\chi_{k+1|k,j,i}^{(f)} = f(\chi_{k,j,i}^{(f)}, \Delta t) + Q_{k+1} \tag{4.4.6}
\]

where \( f (\cdot) \) represents the nonlinear dynamics of the system, and \( Q_{k+1} \) is the process noise. The
4.4. Implementation of The BVP-LMB Filter

The sigma point set $\chi_{k,j}^{(l)}$ is given by

$$
\chi_{k,j}^{(l)} = \left[ m_{k,j}^{(l)} m_{k,j}^{(l)} + \gamma S_{k,j}^{(l)} \right] \left[ m_{k,j}^{(l)} m_{k,j}^{(l)} - \gamma S_{k,j}^{(l)} \right]
$$

(4.4.7)

where $S_{k,j}^{(l)}$ is the lower-triangular matrix of the covariance $P_{k,j}^{(l)}$ calculated using the Cholesky factorisation: $P_{k,j}^{(l)} = S_{k,j}^{(l)} S_{k,j}^{(l)T}$, and $\gamma = \sqrt{n + \lambda}$ and $\lambda = \alpha_c^2 (n + \kappa) - n$ are the scaling parameters. The constant $\alpha_c$ determines the distribution of the sigma points and $\alpha_c = 1$ is used in this work, and $\kappa$ is set to $3 - nL$. Given these parameters, the weights $w_i^m$ and $w_i^p$ of the sigma points for mean and covariance respectively are defined by

$$
w_i^m = \begin{cases} 
\frac{\lambda}{n+\lambda}, & i = 0 \\
\frac{1}{2(n+\lambda)}, & i = 1, \ldots, 2n 
\end{cases}
$$

(4.4.8)

$$
w_i^p = \begin{cases} 
\frac{\lambda}{n+\lambda} + \beta, & i = 0 \\
\frac{1}{2(n+\lambda)}, & i = 1, \ldots, 2n 
\end{cases}
$$

(4.4.9)

where $\beta = 2$ is the known optimal choice for Gaussian distribution [105].

In order to perform the full $\delta$-GLMB update, the predicted LMB needs to be converted to a $\delta$-GLMB.

(3) The prior $\delta$-GLMB density is updated using measurements $z_{k+1}$, and the obtained posterior $\delta$-GLMB needs to be converted back to the LMB form $\pi_{k+1}$ for sequential filtering. The updated parameter $\eta_{Z_{k+1}}^{(\theta_{k+1})} (\ell)$ and the posterior single target density of track $\ell$ via the GM implementation are expressed as

$$
\eta_{Z_{k+1}}^{(\theta_{k+1})} (\ell) = \sum_{j=1}^{J} \omega_{Z_{k+1}, j}^{(\theta_{k+1})} (\ell)
$$

(4.4.10)

$$
p_{(\theta_{k+1})} (x; \ell) = \sum_{j=1}^{J} \omega_{k+1,j}^{(\theta_{k+1})} (\ell) p_{g}\left( x; m_{Z,k+1,j}^{(\theta_{k+1})} (\ell), P_{k+1,j}^{(\theta_{k+1})} (\ell) \right)
$$

(4.4.11)
and

\[
\omega_{k,j}^{(\theta_{k+1})}(\ell) = \omega_{k+1|j}^{(\ell)} \begin{cases} 
\frac{p_D(\ell)g_{z_{\theta_{k+1}(\ell)}}}{\kappa_{z_{\theta_{k+1}(\ell)}}} & \text{if } \theta_{k+1}(\ell) > 0 \\
1 - p_D(\ell) & \text{if } \theta_{k+1}(\ell) = 0
\end{cases}
\]  

(4.4.12)

\[
q_j(z; \ell) = p_j(z, h(m_{k+1|j}^{(\ell)}, P_{zz, j}^{(\ell)})
\]  

(4.4.13)

\[
m_{k+1, j}^{(\ell)}(\ell) = \begin{cases} 
m_{k+1|j}^{(\ell)}(z_{\theta_{k+1}(\ell)}), & \text{if } \theta_{k+1}(\ell) > 0 \\
m_{k+1|j}^{(\ell)}, & \text{if } \theta_{k}(\ell) = 0
\end{cases}
\]  

(4.4.14)

\[
P_{k,j}^{(\theta_{k+1})}(\ell) = \begin{cases} 
P_{k+1, j}^{(\ell)}, & \text{if } \theta_{k+1}(\ell) > 0 \\
P_{k+1|j}^{(\ell)}, & \text{if } \theta_{k+1}(\ell) = 0
\end{cases}
\]  

(4.4.15)

where \(m_{k+1,j}^{(\ell)}, P_{k+1,j}^{(\ell)}\) and \(P_{zz,j}^{(\ell)}\) are the mean, covariance and innovation covariance calculated using the sigma points of the predicted mean and covariance \(m_{k+1|j}^{(\ell)}, P_{k+1|j}^{(\ell)}\), which are given by

\[
m_{k+1,j}^{(\ell)} = m_{k+1|j}^{(\ell)}(z) + K_{k+1,j}^{(\ell)}(z_{\theta_{k+1}(\ell)} - \hat{z}_{k+1,j}^{(\ell)})
\]  

(4.4.16)

\[
P_{k+1,j}^{(\ell)} = P_{k+1|j}^{(\ell)} - K_{k+1,j}^{(\ell)}P_{zz,j}^{(\ell)}K_{k+1,j}^{(\ell)T}
\]  

(4.4.17)

\[
P_{zz,j}^{(\ell)} = \sum_{i=0}^{2L} w_i^{m}(\hat{z}_{k+1,j,i}^{(\ell)} - \hat{z}_{\theta_{k+1}(\ell)}) (\hat{z}_{k+1,j,i}^{(\ell)} - \hat{z}_{\theta_{k+1}(\ell)})^T + R_{k+1}
\]  

(4.4.18)

where \(R_{k+1}\) is the measurement noise, and

\[
K_{k+1,j}^{(\ell)} = P_{zz,j}^{(\ell)}P_{zz,j}^{(\ell)}^{-1}
\]  

(4.4.19)

\[
P_{zz,j}^{(\ell)} = \sum_{i=0}^{2L} w_i^{m}(\hat{z}_{k+1,j,i}^{(\ell)} - m_{k+1|j}^{(\ell)}) (\hat{z}_{k+1,j,i}^{(\ell)} - \hat{z}_{\theta_{k+1}(\ell)})^T
\]  

(4.4.20)

\[
\hat{z}_{k+1,j}^{(\ell)} = \sum_{i=0}^{2L} w_i^{m}\hat{z}_{k+1,j,i}^{(\ell)}
\]  

(4.4.21)

where \(\chi_{k+1|j,i}^{(\ell)}\) is obtained from Eq. (4.4.6), and the measurement-transformed sigma points are calculated through

\[
\hat{z}_{k+1,j,i}^{(\ell)} = h(\chi_{k+1|j,i}^{(\ell)})
\]  

(4.4.22)
where $h(\cdot)$ is the measurement model.

(4) Finally, the probability of existence $r_{B,k+1}(\cdot)$ is calculated using Eq. (4.3.7) or (4.3.10) for CAR/PAR or BVP respectively. The measurement set $Z_{B,k+1}$ used for birth tracks is the collection of measurements with association probabilities $r_{A,k+1}$ less than one.

The detailed procedure of the BVP, CAR and PAR birth models are shown in Fig. 4.4.1b. The major difference is that the BVP birth model uses two sets of sequential measurements $Z_{B,k-1}$ and $Z_{B,k}$, while CAR and PAR only employ measurement set $Z_{B,k-1}$. Each combination of measurement pairs from sets $Z_{B,k-1} = \{z_{B,k-1,1}, \cdots, z_{B,k-1,|Z_{B,k-1}|}\}$ and $Z_{B,k} = \{z_{B,k,1}, \cdots, z_{B,k,|Z_{B,k}|}\}$ needs to be processed by the BVP birth model. Since the standard measurement model assumes a track can generate at most one measurement at any epoch, each measurement is allowed to generate only one birth track. Even though the enumerations of $z_{B,k-1,i}$ and $Z_{B,k}$ may yield multiple IOD solutions, only the one with the lowest loss function value is selected to represent the birth target.

As specified in Sec. 4.3.1, the initial orbital state $x_{BVP}$ generated from the BVP optimisation method and an initial covariance $P_{ini}$ are selected to initialise a BLS estimator, and a refined covariance $P_{BLS}$ can be obtained by processing all independent measurements within the two associated observations at $t_k$ and $t_{k-1}$. The IOD solution $x_{BVP}$ from BVP and the estimated covariance $P_{BLS}$ from the BLS method are regarded as the mean and covariance of the birth track. The probability of existence of the birth track is calculated using Eq. (4.3.10).

The CAR/PAR-LMB filter uses measurements at the previous epoch $t_{k-1}$ to determine orbital states of birth tracks. Therefore, the presence of a new birth target using CAR/PAR requires at least one epoch lag, and this leads to a biased cardinality estimate at $t_{k-1}$. However, the BVP-LMB filter uses measurements from the last two consecutive epochs $t_{k-1}$ and $t_k$, which results in two-epoch lags when confirming birth targets. Compared to CAR/PAR, the additional lag in the BVP birth model leads to slower convergence of the LMB filter. It may be possible to reduce the lag to one epoch by using the measurement set $Z^B_k$ to generate birth tracks at time $t_k$, however, this poses additional challenges in the measurement update step to avoid double
counting measurements.

Regardless of the cardinality estimation lag, the computational complexity of the BVP-LMB filter is reduced compared to the CAR/PAR LMB filter. A sufficient number of GM components are required in the CAR and PAR birth models to achieve a desired approximation of the orbital state, while the BVP birth model only yields one Gaussian component to approximate the initial target state. More GM components result in more computational effort during orbit propagation, especially when employing a high fidelity numerical orbit propagation model in the filter. Thus, the BVP-LMB filter is a useful option for large-scale space object cataloging due to its reduced computational demand.

As the typical Lambert solver only accounts for two-body dynamics, the accuracy of the BVP IOD solution degrades as the time interval between tracklets increases, especially for objects greatly perturbed by J2 perturbations. A potential solution to alleviate this issue is to use a shooting technique to solve the Lambert problem that takes into account the J2 perturbation, and further investigation will be considered for future work.

4.5 Simulation

One of the major concerns in SSA is the safety of active GEO spacecraft. These precious space assets are threatened by a large population of space objects in nearby orbital domains, e.g., GTO, highly elliptical orbit (HEO), and near GEO. The GTO is a Hohmann transfer orbit which is used to attain GEO altitude, and it is occupied by many large size space debris objects, e.g., rocket bodies. Some GTO objects are very close to the GEO ring near their apogees, which may yield potential risk of collision with GEO objects. Therefore, it is advantageous to track GEO and GTO objects that have close trajectories.

4.5.1 Simulation Design and Data Selection

Two case studies of GEO and GTO object tracking based on simulated optical measurements are proposed to validate the BVP-LMB filter. The first test case considers the scenario of
simultaneously tracking four GEO objects and four GTO objects using dense measurements for four days. The observing time window is set at 8 hours each day, with 5 evenly spaced measurement arcs collected within the window. Case II tests three GEO objects and three GTO objects tracked using sparse measurements with longer time gaps. These objects are randomly selected from the eight objects in the first case. In this more realistic scenario, only two measurement arcs are generated each night during the time window of 7 days, and the time interval between two measurement arcs in a single night is around 6 hours. In both test cases, one GEO object and one GTO object are initially known, the others need to be generated using a birth model.

The Keplerian orbital elements of the initially known GEO and GTO objects are listed in Table 4.5.1. These two objects are selected because their trajectories nearly intersect around the apogee of the GTO object, meaning that potential danger of collision may exist. The rest of the GEO and GTO objects are generated by adding zero mean Gaussian perturbations to the state vectors of the known GEO and GTO objects. The standard deviations of the Gaussian random perturbations in semi-major axis and mean anomaly are 200 km and 2 deg respectively.

**Table 4.5.1: Orbital elements of the GEO and GTO objects**

<table>
<thead>
<tr>
<th></th>
<th>a (km)</th>
<th>e</th>
<th>i (deg)</th>
<th>ω (deg)</th>
<th>Ω (deg)</th>
<th>M (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>GEO 1</td>
<td>42160</td>
<td>10^{-4}</td>
<td>0.01</td>
<td>0</td>
<td>0</td>
<td>270</td>
</tr>
<tr>
<td>GTO 1</td>
<td>25510</td>
<td>0.71</td>
<td>0.01</td>
<td>0</td>
<td>0</td>
<td>90</td>
</tr>
</tbody>
</table>

All tested space objects are simulated and tracked by a set of identical and collocated telescopes that allows observation of all targets simultaneously. Their orbit states are modelled by the inertial position and velocity. The initial covariance of known targets is given by

\[ P_{\text{ini}} = \text{diag} [100, 100, 10, 10^{-4}, 10^{-4}, 10^{-5}], \]

where the units of position and velocity are km and km/s respectively. The measurements used in the filter update step are topocentric right ascension \( \alpha \), declination \( \delta \) and their angular rates. Each measurement arc is 2 minutes long with 10 independent measurements. The measurement noise is 1 arc-second and 0.08 arc-second/s for both angles and angular rates. This study as-
assumes the measurement noise to be initially known, and therefore calibration of noise for the BVP optimisation is not necessary [17].

The parameters of the LMB filter are given in Table 4.5.2. The clutter measurements are assumed to be uniformly distributed over the observation spaces, with the number of clutter returns given by a Poisson distribution with mean $\lambda_c$. The intensity of the clutter distribution is expressed as $\kappa(z) = \lambda_c \cdot \mathcal{U}(z)$, where $\mathcal{U}$ denotes a uniform distribution over the sensor field of view. Two constant $p_D$ values are applied in this study to model missed detections in different scenarios. Both test cases consider four pairs of $p_D$ and $\lambda_c$ values: (1) $p_D = 0.95$ and $\lambda_c = 0.1$, (2) $p_D = 0.95$ and $\lambda_c = 1$, (3) $p_D = 0.75$ and $\lambda_c = 0.1$, (4) $p_D = 0.75$ and $\lambda_c = 1$.

<table>
<thead>
<tr>
<th>Table 4.5.2: Parameters of the LMB filter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter</td>
</tr>
<tr>
<td>Probability of survival</td>
</tr>
<tr>
<td>Probability of detection</td>
</tr>
<tr>
<td>Mean clutter return</td>
</tr>
<tr>
<td>Pruning threshold of LMB track</td>
</tr>
<tr>
<td>Merging threshold of GM components</td>
</tr>
</tbody>
</table>

Several parameters and threshold values are employed in the BVP, CAR and PAR birth models. In the BVP birth model, the maximum existence probability of target birth is $r_{B,\text{max}} = 0.1$; the measurement association probability threshold used to determine the measurement set for the BVP birth model at the next epoch is $T_r = 10^{-4}$. The tracklet association threshold value used in BVP optimisation is $T_a = 1$. The selection of parameters used for the CAR and PAR birth models affects both efficiency and accuracy. For different test cases, the CAR and PAR birth model employs different design parameters values. In the first case, $\sigma_{\rho} = 0.01$ ER and $\sigma_{\dot{\rho}} = 0.15$ ER/h are used for CAR and 10 GM components are defined for PAR. In the second case, accurate representation of the initial uncertainty using more GM components is necessary due to the sparse measurements and long time gap for orbit propagation. Thus, smaller values of the design parameters $\sigma_{\rho} = 2 \times 10^{-3}$ ER and $\sigma_{\dot{\rho}} = 0.08$ ER/h are employed for CAR, and 50 GM components are used for PAR. Further improvement of accuracy of the CAR/PAR-LMB filter can be achieved by using more GM components, though this increases computational demand.
The constraint values of the semi-major axis and eccentricity in Table 4.3.1 are applied for the three birth models in all test cases.

### Table 4.5.3: Force models for orbit propagation

<table>
<thead>
<tr>
<th>Force models</th>
<th>Truth</th>
<th>Filter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass</td>
<td>487 kg</td>
<td>487 kg</td>
</tr>
<tr>
<td>Earth gravity model</td>
<td>WGS84 EGM96</td>
<td>WGS84 EGM96</td>
</tr>
<tr>
<td>Gravity degree/order</td>
<td>$10 \times 10$</td>
<td>$2 \times 2$</td>
</tr>
<tr>
<td>Third body</td>
<td>Sun and Moon</td>
<td>Sun and Moon</td>
</tr>
<tr>
<td>Solar and lunar gravity</td>
<td>on</td>
<td>on</td>
</tr>
<tr>
<td>Solar radiation pressure</td>
<td>on</td>
<td>off</td>
</tr>
<tr>
<td>Coefficient of reflection</td>
<td>0.7</td>
<td>N/A</td>
</tr>
<tr>
<td>Area-to-mass ratio</td>
<td>0.005 m$^2$/kg</td>
<td>N/A</td>
</tr>
<tr>
<td>Atmospheric drag</td>
<td>NRL MISESE-00</td>
<td>off</td>
</tr>
<tr>
<td>Coefficient of drag</td>
<td>2.3</td>
<td>N/A</td>
</tr>
</tbody>
</table>

The force models used in truth generation and the LMB filters are shown in Table 4.5.3. The truth generation employs precise force models given in the left column of Table 4.5.3, while a simplified force model for GEO and GTO objects is used in the LMB filter. The orbit propagation is performed by the Orbit Determination Toolbox (ODTBX) 6.5 based on Matlab®, developed by NASA Goddard Space Flight Center. Each orbital state is propagated using the unscented transform and a linear process noise. The process noise is an identity matrix, and the standard deviations are $10^{-3}$ km and $10^{-6}$ km/s for position and velocity respectively.

All LMB algorithms based on the three birth models and the LMB filter are implemented in Matlab®, and the computation times are based on an Intel core i7 CPU with 64-bit numerics and a 2.7 GHz clock rate. The results are assessed by the second-order OSPA error [102], with cutoff values 100 km and 2 km/s for position and velocity respectively for all test cases.

#### 4.5.2 Case I: Dense Measurements

This case validates the accuracy and efficiency of the BVP-LMB filter in the case of dense measurements. The state and cardinality OSPA errors of the LMB filter using all three birth models with $p_D = 0.95$ and $\lambda_c = 0.1$ are shown in Fig. 4.5.1a. Results show that all the position and velocity OSPA errors of the three birth models have a sharp increase at the second
and sixth epochs when new targets are introduced to the scene. This is mainly because the three birth models lead to at least one epoch lag of target birth, so the OSPA position errors include the penalty 100 km value for true targets not yet added to the filter. After these two epochs, all OSPA errors converge quickly because the new targets can be immediately confirmed, see the cardinality estimate in the third subfigure of Fig. 4.5.1a. The only difference is that BVP needs one more epoch to confirm new targets because it has two epochs lag. In addition, all cardinality estimates are very close to the truth after all new targets are confirmed.

Figure 4.5.1: Averaged position, velocity OSPA errors and cardinality estimate results of three birth models in the case of tracking 4 GEO and 4 GTO objects for 4 days

Fig. 4.5.1b depicts the OSPA error results from the same $P_D = 0.95$, $\lambda_c = 1$ value and a larger mean clutter rate $\lambda_c = 1$. The results indicate a similar trend for all birth models compared to Fig. 4.5.1a. More clutter measurements do not increase the cardinality estimate of the LMB filter because
new generated clutter tracks cannot be confirmed by follow on measurements. However, the errors at the end of tracking are slightly larger than the previous case due to statistical variations, see Table 4.5.4 for detail.

The results of the three methods using \( p_D = 0.75 \) and \( \lambda_c = 0.1 \) are given in Fig. 4.5.1c. The lower \( p_D \) value results in larger state OSPA errors for all three methods throughout the entire simulation compared with the two previous cases. The third subfigure in Fig. 4.5.1b shows that all three methods need more time to confirm new tracks, but the cardinality estimate converges to the truth during the last two days of tracking. In addition, the BVP birth model requires more time to confirm all birth targets because the low \( p_D \) value reduces the probability of detecting measurements of a target for two consecutive epochs. Fig. 4.5.1d presents the OSPA errors of each method using \( p_D = 0.75 \) and \( \lambda_c = 1 \). These results again illustrate that the three birth models provide similar performance in terms of state and cardinality estimation.

<table>
<thead>
<tr>
<th>( p_D )</th>
<th>( \lambda_c )</th>
<th>results</th>
<th>BVP</th>
<th>CAR</th>
<th>PAR</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.95</td>
<td>0.1</td>
<td>Position OSPA (km)</td>
<td>1.39</td>
<td>1.29</td>
<td>1.36</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Velocity OSPA (m/s)</td>
<td>14.21</td>
<td>14.20</td>
<td>14.20</td>
</tr>
<tr>
<td>0.95</td>
<td>1</td>
<td>Position OSPA (km)</td>
<td>1.31</td>
<td>1.99</td>
<td>1.30</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Velocity OSPA (m/s)</td>
<td>14.21</td>
<td>28.35</td>
<td>14.20</td>
</tr>
<tr>
<td>0.75</td>
<td>0.1</td>
<td>Position OSPA (km)</td>
<td>7.39</td>
<td>7.53</td>
<td>6.34</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Velocity OSPA (m/s)</td>
<td>71.65</td>
<td>73.19</td>
<td>78.43</td>
</tr>
<tr>
<td>0.75</td>
<td>1</td>
<td>Position OSPA (km)</td>
<td>8.10</td>
<td>10.68</td>
<td>8.14</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Velocity OSPA (m/s)</td>
<td>113.67</td>
<td>134.72</td>
<td>114.24</td>
</tr>
</tbody>
</table>

The position and velocity OSPA errors of the above four cases at the final epoch are given in Table 4.5.4. Results indicate that BVP can achieve similar accuracy compared to the CAR and PAR for all test parameters, and the difference of the three methods results mainly from statistical variations [56]. The 10th and 90th percentiles of state and cardinality errors are also computed and shown in Fig. C.1.1, see Appendix C.1 for detailed discussion.

Table 4.5.5 provides the averaged number of birth tracks, the averaged number of GM
Table 4.5.5: Averaged computation time and parameters of birth tracks

<table>
<thead>
<tr>
<th>$p_D$</th>
<th>$\lambda_c$</th>
<th>results</th>
<th>BVP</th>
<th>CAR</th>
<th>PAR</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.95</td>
<td>0.1</td>
<td>Number of birth tracks</td>
<td>6.64</td>
<td>10.26</td>
<td>9.32</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Number of GM</td>
<td>1</td>
<td>12.04</td>
<td>15.45</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Time consumption (s)</td>
<td>459.14</td>
<td>973.27</td>
<td>1092.44</td>
</tr>
<tr>
<td>0.95</td>
<td>1</td>
<td>Number of birth tracks</td>
<td>6.6</td>
<td>24.86</td>
<td>22.92</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Number of GM</td>
<td>1</td>
<td>13.29</td>
<td>13.09</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Time consumption (s)</td>
<td>466.43</td>
<td>1811.89</td>
<td>1743.19</td>
</tr>
<tr>
<td>0.75</td>
<td>0.1</td>
<td>Number of birth tracks</td>
<td>6.5</td>
<td>11.80</td>
<td>11.34</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Number of GM</td>
<td>1</td>
<td>13.92</td>
<td>15.26</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Time consumption (s)</td>
<td>406.68</td>
<td>1142.56</td>
<td>1291.04</td>
</tr>
<tr>
<td>0.75</td>
<td>1</td>
<td>Number of birth tracks</td>
<td>6.38</td>
<td>26.94</td>
<td>26.06</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Number of GM</td>
<td>1</td>
<td>14.02</td>
<td>12.99</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Time consumption (s)</td>
<td>457.22</td>
<td>2411.20</td>
<td>2314.59</td>
</tr>
</tbody>
</table>

components of each birth track, and the averaged computation time of the LMB filters using the three birth models based on 50 Monte Carlo simulations. As expected, the BVP-LMB filter provides the most efficient performance in all test cases. Compared with the large number of birth tracks and GM components generated by CAR and PAR, BVP only generates around 6 to 7 birth tracks for all test cases, which are very close to the true number of new targets (6 GEO and GTO targets), and each birth track only contains one Gaussian component. Thus, the computational demand of the orbit propagation process of the BVP-LMB filter is dramatically reduced.

The larger mean clutter rate, $\lambda_c = 1$, results in more computation time for the CAR/PAR-LMB filter. Even though the majority of clutter tracks can be quickly removed by the LMB filter due to the lack of follow on measurements, the orbit propagation of these clutter tracks significantly increases the overall computational demand of the CAR/PAR-LMB filter. In contrast, the BVP birth model yields a similar number of birth tracks for different scenarios because the clutter measurements rarely produce a valid solution to the BVP birth model. Therefore, using more clutter measurements does not significantly affect the computational efficiency of the
4.5. Simulation

BVP-LMB filter.

In the case of \( p_D = 0.75 \), the computation time of the BVP-LMB filter is slightly decreased. The major reasons are a lower \( p_D \) value results in fewer measurements that need to be processed by a birth model, as well as fewer hypotheses in the \( \delta \)-GLMB update. However, the CAR and PAR birth models generate more birth tracks even if there are fewer measurements. This is because more missed detections lead to the LMB filter losing custody of new tracks, which requires the target birth process to be performed more frequently.

4.5.3 Case II: Sparse Measurements

In this case, the three birth models are tested by a more realistic scenario using sparse measurements of three GEO and three GTO objects. The sparse measurements and long time gap increase the difficulty of data association and state estimation.

The averaged state and cardinality OSPA errors of the three methods in the case of \( P_D = 0.95, \lambda_c = 0.1 \) and \( P_D = 0.95, \lambda_c = 1 \) are presented in Fig. 4.5.2a and Fig. 4.5.2b respectively. Using different mean clutter rate values results in similar trends of OSPA errors for all three methods. Compared with CAR/PAR-LMB filter, the BVP-LMB filter converges slightly slower due to the fact that it takes one more epoch to confirm the presence of target birth, but its state OSPA errors are reduced to similar levels as the CAR and PAR by the second half of the simulation. The cardinality OSPA errors demonstrate that the cardinality estimate of the BVP-LMB filter can catch up with the CAR/PAR birth models after three days tracking.

The results for all three birth models in the case of \( P_D = 0.75, \lambda_c = 0.1 \) and \( P_D = 0.75, \lambda_c = 1 \) are presented in Fig. 4.5.2c and Fig. 4.5.2d respectively. Compared with Fig. 4.5.2a and Fig. 4.5.2b, the smaller \( P_D \) value results in a large increase in averaged state errors because fewer measurements are available to improve the state estimate. The BVP-LMB filter gradually reduces the state OSPA errors to the same level of the CAR/PAR-LMB filters. The cardinality OSPA errors indicate that all birth models incur longer time delays for the cardinality estimate of new birth tracks, but all converge on the correct estimate of the number of targets during the
Figure 4.5.2: Averaged position, velocity OSPA errors and cardinality estimate results of three birth models in the case of tracking 3 GEO and 3 GTO objects for 7 days.

The OSPA errors of the three methods at the final epoch are given in Table 4.5.6. The BVP-LMB filter slightly outperforms the other two methods across all test cases. The PAR-LMB filter provides the worst performance, especially when using a lower $p_D$ value. A significant increase in error occurs for all three methods for the $p_D = 0.75$ case due to a reduced number of measurements and increased time gaps between measurements. In this scenario, the uncertainty cannot be well described by a fixed number of GM components. Recent development using the Adaptive Entropy-based Gaussian Information Synthesis (AEGIS) [108] theory can better account for the nonlinearity of orbital dynamics. It splits a Gaussian PDF into GM components triggered by an entropy-based detection of nonlinearity during the evolution of the PDF.
4.6 Summary

This chapter proposes a BVP-LMB filter for high efficiency tracking of multiple space object. Compared to the typical CAR/PAR birth model, the BVP birth model approximates the orbital state of a new target by a single Gaussian component. In addition, most clutter tracks can be effectively rejected by the BVP optimisation process to further reduce the computational

Results in Table 4.5.7 validate that the best computational efficiency is again achieved by the BVP-LMB filter. The BVP birth model produces around 5 to 6 new birth tracks for different combinations of $P_D$ and $\lambda_c$ values, which is very close to the true number of birth tracks. This demonstrates that the BVP birth model is able to identify and prune most clutter measurements. In addition, BVP determines an accurate initial state and uncertainty of a birth track using only one Gaussian component, while a large number of GM components are required in the CAR and PAR birth models to ensure the successful convergence of the LMB filter. More GM components and long time gaps for orbit propagation result in the large computation time for the CAR/PAR-LMB filter.

Table 4.5.6: Averaged OSPA errors at the final epoch

<table>
<thead>
<tr>
<th>$p_D$</th>
<th>$\lambda_c$</th>
<th>results</th>
<th>BVP</th>
<th>CAR</th>
<th>PAR</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.95</td>
<td>0.1</td>
<td>Position OSPA (km)</td>
<td>0.67</td>
<td>1.34</td>
<td>1.64</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Velocity OSPA (m/s)</td>
<td>0.09</td>
<td>0.16</td>
<td>15.23</td>
</tr>
<tr>
<td>0.95</td>
<td>1</td>
<td>Position OSPA (km)</td>
<td>0.63</td>
<td>0.56</td>
<td>2.19</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Velocity OSPA (m/s)</td>
<td>0.09</td>
<td>0.08</td>
<td>17.16</td>
</tr>
<tr>
<td>0.75</td>
<td>0.1</td>
<td>Position OSPA (km)</td>
<td>5.68</td>
<td>8.49</td>
<td>9.66</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Velocity OSPA (m/s)</td>
<td>49.42</td>
<td>97.30</td>
<td>93.50</td>
</tr>
<tr>
<td>0.75</td>
<td>1</td>
<td>Position OSPA (km)</td>
<td>6.19</td>
<td>11.14</td>
<td>14.23</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Velocity OSPA (m/s)</td>
<td>33.36</td>
<td>114.47</td>
<td>176.75</td>
</tr>
</tbody>
</table>

Introducing AEGIS into the BVP-LMB filter to better approximate the PDF when nonlinearity is encountered needs further investigation. In addition, the 10th and 90th percentile of state and cardinality errors are shown in Fig. C.2.1, see Appendix C.2 for detail.
Chapter 4. A Multi-Target Tracking Method for New Space Object Using a Boundary Value Approach

Table 4.5.7: Averaged computation time and parameters of birth tracks

<table>
<thead>
<tr>
<th>$p_D$</th>
<th>$\lambda_c$</th>
<th>results</th>
<th>BVP</th>
<th>CAR</th>
<th>PAR</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Number of birth tracks</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.95</td>
<td>0.1</td>
<td>5.14</td>
<td>8.32</td>
<td>7.86</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Number of GM</td>
<td>1</td>
<td>55.80</td>
<td>50.01</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Time consumption (s)</td>
<td>588.61</td>
<td>4350.18</td>
<td>4189.29</td>
</tr>
<tr>
<td>0.95</td>
<td>1</td>
<td>5.42</td>
<td>18.04</td>
<td>17.94</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Number of GM</td>
<td>1</td>
<td>52.19</td>
<td>46.28</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Time consumption (s)</td>
<td>645.05</td>
<td>7424.28</td>
<td>7386.38</td>
</tr>
<tr>
<td>0.75</td>
<td>0.1</td>
<td>5.88</td>
<td>7.40</td>
<td>8.76</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Number of GM</td>
<td>1</td>
<td>47.90</td>
<td>50.45</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Time consumption (s)</td>
<td>556.88</td>
<td>3703.57</td>
<td>3920.33</td>
</tr>
<tr>
<td>0.75</td>
<td>1</td>
<td>5.7</td>
<td>17.48</td>
<td>17.80</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Number of GM</td>
<td>1</td>
<td>46.42</td>
<td>47.90</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Time consumption (s)</td>
<td>603.25</td>
<td>7037.87</td>
<td>7274.66</td>
</tr>
</tbody>
</table>

efforts. The proposed method is tested by tracking multiple GEO and GTO objects through simulation. Results validate that the BVP-LMB filter provides similar accuracy of state and cardinality estimation compared to the CAR/PAR-LMB filter, and it achieves the most efficient computational performance in all test cases.
Chapter 5  A Multi-Sensor Tasking Method Using Analytical Rényi Divergence

This chapter presents a novel solution for tasking multiple SSA sensors to acquire the most informative measurements for tracking multiple objects in GEO for the purpose of catalog maintenance. The proposed method is formulated based on utilising the LMB filter for object tracking, complementary data fusion rule for multi-sensor fusion, and an information theoretic reward function for sensor tasking. The proposed sensor tasking solution assigns sensor tasks by maximising a reward function that can be interpreted as information gain. This work employs the Rényi divergence as the reward function and derives its analytical form for LMB RFSs. The developed algorithm is implemented for sensor tasking with various types of space-based sensor networks, including one based on geosynchronous transfer orbit, one based on Sun-synchronous orbit, and two hybrid sensor networks using both space- and ground-based sensors. The proposed method is validated by two simulated case studies, involving large-scale GEO object tracking and dim GEO object tracking. The results indicate that the method is able to maintain custody of all objects successfully, and the space-based sensor networks generally perform better than traditional ground-based sensor networks in terms of state and cardinality estimation errors.
5.1 Introduction

The traditional approach for GEO space object tracking mainly uses the optical measurements provided by ground-based sensor (GBS) networks [24, 75, 109]. Recently, optical space-based sensor (SBS) networks have been applied for GEO object tracking [85, 77]. An SBS provides better observing capability than a GBS. No restrictions due to weather conditions and improved viewing geometry enable the SBS to observe space objects at any longitude. In addition, an SBS is not interrupted by the day/night cycle and can continuously collect measurements. Most of the recently launched SBS platforms for monitoring GEO, such as Sapphire, NEOSSat, and SBSS Block 10 [86, 87, 88], are placed in SSO. Bradley et al. [85] investigated the benefits of using GTO-based SBS for an accurate orbit determination of objects in GEO. In addition, GTO-based SBS is applicable to detect dim and small space debris objects in GEO because it can approach the GEO belt near its apogee.

Motivated by previous efforts, two space-based sensor networks are presented in this chapter: one including three SSO-based SBSs, and one including three GTO-based SBSs. In addition, two hybrid sensor networks are developed and investigated as a complementary approach: one including two GBSs and one SSO-based SBS, and the other including two GBSs and one GTO-based SBS.

In all the proposed space-based sensor networks, the multi-target tracking and sensor tasking solutions are designed in the Random Finite Set framework. The LMB filter used in this study is a principled simplification of the $\delta$-GLMB filter [48]. Beyond that, it not only outperforms the PHD, CPHD and MB filters in terms of accuracy but also outputs targets trajectories. Due to these merits, the LMB filter is a suitable tool for multi-target tracking in the SSA domain [9, 56].

As limited sensor resources are available to collect measurements of the large number of space objects, an appropriate sensor tasking scheme is essential for maintaining an accurate space object catalog. Sensor tasking aims to allocate single or multiple sensors to take the
5.1. Introduction

best action to produce useful measurements, and it can be interpreted as a decision-making process. The information theoretic measures have been widely used as a criterion to drive the sensor tasking problem. Commonly used information functionals include the Kullback-Leibler divergence [73], Rényi divergence [74], and Cauchy-Schwarz divergence [110]. In this study, we present an analytical form derived for the Rényi divergence of two LMB densities in which each target state density is a single Gaussian component. The obtained formula improves the computational efficiency of sensor tasking, especially for large-scale space object tracking.

For multi-sensor applications, the local posteriors from multiple sensor nodes need to be combined using a data fusion method. The recently developed complementary Generalised Covariance Intersection (GCI) data fusion method [111] is useful for fusion of LMB posteriors from sensors with different FOVs. This method computes an adaptive weight for each possibly existing object in each local posterior based on the information content quantified by an information divergence function. This approach is adopted in this study to achieve statistical multi-sensor fusion of SSA sensors, and the analytically derived Rényi divergence is used to generate the adaptive weights.

In summary, the major contributions of this chapter are:

1. The analytical form of the Rényi divergence between the prior and posterior LMB densities is derived under the assumption of single Gaussian component representation of each target state density. The proposed analytically derived Rényi divergence provides a tractable measure of the difference between the LMBs and is applicable for efficient tasking of multiple sensors to track a large population of space objects.

2. A framework of multi-sensor multi-target tracking is presented using the LMB filter and an efficient step-by-step sensor tasking algorithm, and its Gaussian mixture implementation is introduced. Each Gaussian mixture is approximated with a single Gaussian only at the time of computing the Rényi divergences.

3. Several space-based sensor networks and hybrid sensor networks are developed and investigated for accurate GEO object tracking. The results highlight that the GTO-based sensor
network provides better capability for dim GEO object tracking compared to the other types of sensor networks.

The rest of the chapter is organised as follows. The derivation of the Rényi divergence between two LMBs is elaborated in Sec. 5.2. Section 5.3 briefly introduces the complementary data fusion method with adaptive weights that are tuned using Rényi divergence. Section 5.4 details the proposed step-by-step sensor tasking algorithm. Section 5.5 provides the description of the sensor network design. Simulation results for two test cases are also presented to evaluate the performance of the proposed method. Conclusions are drawn in the last section.

5.2 Development of Reward Functions

5.2.1 Information Functionals for Sensor Tasking

Several information functionals have been introduced into the framework of RFS filters for sensor tasking [112, 24, 113, 114]. The Cauchy-Schwarz divergence is based on the Cauchy-Schwarz inequality for the inner product between two probability densities functions $\pi_0$ and $\pi_1$ [114], given by

$$C(\pi_0, \pi_1) = -\log \left( \frac{\int \pi_0(X)\pi_1(X)\delta X}{\sqrt{\int \pi_0(X)^2\delta X \int \pi_1(X)^2\delta X}} \right).$$

Due to the possibility of deriving closed-form formulas for the Cauchy-Schwarz divergence between some RFS densities (such as the PHD and $\delta$-GLMB), it has recently attracted attention in the stochastic sensor control literature [112, 114]. The derived results for the Cauchy-Schwarz divergence for $\delta$-GLMB RFSs are given in Appendix D.1. These equations have been used in the numerical experiments for comparison purposes.

The Rényi divergence has been employed as a reward function in conjunction with many realisations of RFS filters [115, 112] for sensor tasking. The general form of the Rényi diver-
gence between two PDFs $\pi_0$ and $\pi_1$ is [74]

$$R(\pi_0, \pi_1) = \frac{1}{\alpha - 1} \log \int \pi_1(X)^{\alpha} \pi_0(X)^{1-\alpha} \delta X,$$

(5.2.2)

where $0 < \alpha < \infty$, and $\alpha \neq 1$. When $\alpha \to 1$, the Rényi divergence becomes the well-known Kullback-Leibler divergence [116]. If $\alpha = 0.5$, then the Rényi divergence is equivalent to the Bhattacharyya coefficient [117] multiplied by a scale factor of $-2$.

The above information functionals can be applied as reward functions to solve the sensor tasking problem. In such problems, the information functional measures the information divergence between the prior and posterior multi-target densities. As the higher information gain indicates a more significant reduction of the posterior uncertainty, it can be expected that the measurement set maximising the information gain yields more accurate state estimation. In order to obtain the posterior multi-target density, the prior multi-target density is first propagated to the desired time $k$, and then updated based on the LMB update equation using measurement set $Z_k(u)$ generated from a specific sensor control vector $u$. However, the measurement set $Z_k(u)$ cannot be obtained before a particular sensor tasking action is executed.

An efficient approach used in this paper is to generate pseudo-measurements using the Predicted Ideal Measurement Set (PIMS), which is dependent on the prior LMB density with an ideal assumption of no clutter measurements, no measurement noise and applying probability of detection $p_D = 1$ for all objects in the sensor FOV. Indeed, for each predicted target state, $x$, the corresponding PIMS measurement is \( \arg \max_z g(z|x, u) \). Note that the additional parameter $u$ in the measurement likelihood model is the sensor control command, and is included to emphasise the dependence of measurements acquired from a sensor on the control command that is sent to it. Thus, given a control vector $u$ at a specific epoch $k$, the PIMS is given by

$$Z_{k|k-1}^{\text{PIMS}}(u) = \bigcup_{z \in \text{FOV}, x \in \hat{X}_{k|k-1}} \left\{ \arg \max_z g(z|\mathbf{x}, u) \right\}.$$  

(5.2.3)

Using the PIMS measurement, the information functional, in the case of Rényi divergence, can
be expressed as $R(u) \equiv R(\pi_{k|k-1}, \mathcal{Z}_{k|k-1}^{\text{PIMS}}(u))$.

The Rényi divergence has been introduced into the LMB filter for sensor tasking by approximating the multi-target density by its PHD [63, 57]. Note that the approximated labelled RFS is a Poisson RFS $X$ on $\mathbb{X}$ augmented with labels on $\mathbb{L}$, which cannot be regarded as a Poisson RFS and approximated by the PHD equation directly [46]. Thus, these derivations transform the LMB to an unlabelled PHD, and the obtained Rényi divergence is the same as the formula for the Poisson PHD [24]. Generally, the Rényi divergence for the PHD needs numerical integration, and the analytical form is available only if the PHD contains one Gaussian component. Therefore, the heavy computational cost makes the Rényi divergence a poor choice for practical applications.

This paper derives the analytical form of the Rényi divergence for the prior and posterior LMB PDFs, which enables the Rényi divergence to be implemented with the LMB filter for sensor tasking with improved accuracy compared to the state-of-the-art. The explicit expression of the Rényi divergence for two LMBs using a GM implementation is first derived in Sec. 5.2.2. The obtained formulation can be further simplified to an analytical solution based on the assumption of a single Gaussian component representation of the single target state. Detailed derivation is elaborated in Sec. 5.2.3.

### 5.2.2 Rényi Divergence for LMBs

**Proposition 1**: The Rényi divergence between the following LMB RFSs with parameter sets:

\begin{align*}
\pi_0(X) &= \Delta(X)w_0(L(X))[p_0(\cdot)]^X \\
\pi_1(X) &= \Delta(X)w_1(L(X))[p_1(\cdot)]^X,
\end{align*}

is given by

\begin{equation}
R(u) = \frac{1}{\alpha - 1} \log \sum_{L \subseteq L} (w_0(L))^{1-\alpha}(w_1(L))^\alpha \left[ \int (p_0(x, \cdot))^{1-\alpha}(p_1(x, \cdot))^\alpha dx \right]^L.
\end{equation}
5.2. Development of Reward Functions

**Proof.** Substituting Eqs. (5.2.4) and 5.2.5 into Eq. (5.2.2) yields

\[
R(u) = \frac{1}{\alpha - 1} \log \int \Delta(X) \left( w_0(L(X)) \right)^{1-\alpha} \left[ (p_0(\cdot))^{1-\alpha} \right]^X \times \left( w_1(L(X)) \right)^\alpha \left[ (p_1(\cdot))^\alpha \right]^X \delta X
\]

\[
= \frac{1}{\alpha - 1} \log \int \Delta(X) \left( w_0(L(X)) \right)^{1-\alpha} \left( w_1(L(X)) \right)^\alpha \times \left[ (p_0(\cdot))^{1-\alpha} (p_1(\cdot))^\alpha \right]^X \delta X.
\]

(5.2.7)

In contrast to the Cauchy-Schwarz divergence, the Rényi and Kullback-Leibler divergences can be reformulated using set integrals. The set integral for the labelled RFS is given by [46]

\[
\int f(X) \delta X = \sum_{n \geq 0} \frac{1}{n!} \sum_{(\ell_1, \ldots, \ell_n) \in L^n} \int f(\{ (x_1, \ell_1), \ldots, (x_n, \ell_n) \}) d(x_1, \ldots, x_n).
\]

(5.2.8)

The standard (Lebesgue) integral in Eq. (5.2.7) can be replaced by the set integral using Eq. (5.2.8), and according to Vo et al. [46, Lemma 3]

\[
\int \Delta(X) h(L(X)) g^X \delta X = \sum_{L \subseteq L} h(L) \left[ \int g(x, \cdot) dx \right]_L.
\]

(5.2.9)

Thus, the Rényi divergence can be rearranged as follows

\[
R(u) = \frac{1}{\alpha - 1} \log \sum_{L \subseteq L} \left( w_0(L) \right)^{1-\alpha} \left( w_1(L) \right)^\alpha \left[ \int (p_0(x, \cdot))^{1-\alpha} (p_1(x, \cdot))^\alpha dx \right]_L.
\]

(5.2.10)

Gaussian Mixture Implementation

Let the single-target density in \( \pi_0(X) \) and \( \pi_1(X) \) be approximated by the following GMM formulation

\[
p_0^{(L)}(x) = \sum_{i=1}^{N_0} w_0^{(L)} g(x; m_0^{(L)}, P_{0,i}^{(L)})
\]

(5.2.11)

\[
p_1^{(L)}(x) = \sum_{j=1}^{N_1} w_1^{(L)} g(x; m_1^{(L)}, P_{1,j}^{(L)}).
\]

(5.2.12)
With the selection of $\alpha = 0.5$, which provides the best discrimination between densities \cite{118, 119}, then Eq. (5.2.6) can be written explicitly as

$$ R(u) = -2 \log \sum_{L \subseteq L} \left( w_0(L) \right)^{\frac{1}{2}} \left( w_1(L) \right)^{\frac{1}{2}} \prod_{\ell \in L} \gamma_{\pi_0, \pi_1}^{(\ell)} $$

(5.2.13)

$$ \gamma_{\pi_0, \pi_1}^{(\ell)} = \left( \sum_{i=1}^{N_0} \sum_{j=1}^{N_1} w_{0,i}^{(\ell)} w_{1,j}^{(\ell)} K_{i,j}^{(\ell)} P g(x; \mu_{i,j}^{(\ell)}, \Sigma_{i,j}^{(\ell)}) \right)^{\frac{1}{2}} $$

(5.2.14)

where

$$ K_{i,j}^{(\ell)} = p g(m_{0,i}^{(\ell)}, m_{1,j}^{(\ell)}, P_{0,i}^{(\ell)} + P_{1,j}^{(\ell)}) $$

(5.2.15)

$$ \Sigma_{i,j}^{(\ell)} = \left[ (P_{0,i}^{(\ell)})^{-1} + (P_{1,j}^{(\ell)})^{-1} \right]^{-1} $$

(5.2.16)

$$ \mu_{i,j}^{(\ell)} = \Sigma_{i,j}^{(\ell)} \left[ (P_{0,i}^{(\ell)})^{-1} m_{0,i}^{(\ell)} + (P_{1,j}^{(\ell)})^{-1} m_{1,j}^{(\ell)} \right]. $$

(5.2.17)

### 5.2.3 Analytical Form of Rényi Divergence for LMBs

The proposed GM implementation of the Rényi divergence for two LMBs requires the solution of the numerical integral in Eq. (5.2.14). In this section, the analytical form of the Rényi divergence for two LMBs is derived by approximating each single target state density with a single Gaussian PDF. The form of the Rényi divergence does not require numerical integration and therefore reduces the computation.

Let each target density of $\pi_0(X)$ and $\pi_1(X)$ be represented by a single Gaussian PDF

$$ p_0^{(\ell)}(x) = p g(x; m_0^{(\ell)}, P_0^{(\ell)}) $$

$$ p_1^{(\ell)}(x) = p g(x; m_1^{(\ell)}, P_1^{(\ell)}). $$

(5.2.18)

Then, the equation of $\gamma_{\pi_0, \pi_1}^{(\ell)}$ of two single Gaussian components can be rewritten as a simpler
5.2. Development of Reward Functions

The Eq. (5.2.19) can be rearranged as

\[ \gamma(\ell)_{\pi_0,\pi_1} = \left( K_{0,1}^{(\ell)} \right)^{1/2} |8\pi \Sigma_{0,1}^{(\ell)}|^{1/4}. \]  

(5.2.21)

The detailed derivation is given in Appendix D.2. Finally, the analytical form of the Rényi divergence is obtained by substituting Eq. (5.2.21) into Eq. (5.2.13)

\[ R(u) = -2 \log \sum_{L \subseteq \mathcal{L}} (w_0(L))^{1/2} (w_1(L))^{1/2} \prod_{\ell \in L} \left( K_{0,1}^{(\ell)} \right)^{1/2} |8\pi \Sigma_{0,1}^{(\ell)}|^{1/4}. \]  

(5.2.22)

The analytical form of the Rényi divergence is derived based on the assumption of a single Gaussian component representation of each target state. In the case of GM representation of the single target state, the analytical formula of the Rényi divergence can still be applied. This is achieved by merging all GM components of a target state into one Gaussian PDF. The merging equation for the GM form of \( p_0^{(\ell)}(x) \) is given by

\[ m_0^{(\ell)} = \sum_{i=1}^{N_0} w_{0,i} m_0^{(\ell)} \]  

(5.2.23)

\[ P_0^{(\ell)} = \sum_{i=1}^{N_0} w_{0,i} \left[ I_{0,i} + (m_0^{(\ell)} - m_0^{(\ell)})(m_0^{(\ell)} - m_0^{(\ell)})^T \right]. \]  

(5.2.24)

**Remark 5.1** Note that the merging of the GMM may yield an inaccurate approximation in highly non-Gaussian cases. This study considers circular GEO orbit, and assumes that state
uncertainties remain close to Gaussian. Therefore, it is not expected that merging components will have a significant impact in this study. The merged Gaussian PDF is only used to analytically compute the Rényi divergence for sensor tasking and adaptively tune the data fusion weights. The GM form of target state is still employed in the measurement update of the LMB filter for state estimation. The calculation of adaptive weights for data fusion is introduced in Sec. 5.3.

5.2.4 Analysis of Reward Functions

In order to evaluate the performance of the proposed analytical Rényi divergence, a simple case of tracking two LEO space objects is presented. In this case study, the desired sensor tasking behaviour is to observe one out of two objects at each epoch depending on the information gain of each object. The initial means of the two objects are characterised by the Keplerian orbital elements of semi-major axis \( a \), eccentricity \( e \), inclination \( inc \), right ascension of the ascending node \( \Omega \), argument of periapsis \( \omega \), and mean anomaly \( M \), which are shown in Table 5.2.1. The mean anomalies of these two objects are 180 deg apart, and the other five Keplerian orbital elements \((a, e, inc, \Omega, \omega)\) are the same.

<table>
<thead>
<tr>
<th></th>
<th>( a ) (km)</th>
<th>( e )</th>
<th>( inc ) (deg)</th>
<th>( \omega ) (deg)</th>
<th>( \Omega ) (deg)</th>
<th>( M ) (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>object 1</td>
<td>7653.76</td>
<td>0</td>
<td>45</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>object 2</td>
<td>7653.76</td>
<td>0</td>
<td>45</td>
<td>0</td>
<td>0</td>
<td>180</td>
</tr>
</tbody>
</table>

An optical sensor is assumed to be located at the centre of the Earth, which generates angular measurements, i.e., right ascension and declination. The simulation time is half of the orbital period of these two objects, and the sensor is tasked to collect 20 evenly spaced observations within this time window. The FOV size of the sensor is 2 deg in both length and width, and only one object is considered in the sensor FOV at each epoch. Since both orbits are circular and the sensor located at the Earth centre, the distance between each object and the sensor is the same, i.e., equal to the semi-major axis during the entire time window. The same initial covariance \( P = \text{diag}[10^2, 10^2, 10^2, 10^{-6}, 10^{-6}, 10^{-6}] \) with position and velocity
5.2. Development of Reward Functions

units in km and km/s is employed for both objects. As both objects are initialised using the same covariance, and they have the same distance from the sensor, the expected sensor action is to alternately observe each object and yield an equal number of measurements and similar uncertainty values at the final epoch.

The orbital state and covariance are propagated using the unscented transform, and the equations of motion are those of the two-body problem. Given the multi-target state extracted from the prior LMB, the PIMS of each sensor control command is calculated using Eq. (5.2.3). Then the pseudo posterior multi-target density can be obtained using the LMB update and PIMS. The decision-making process depends on maximizing the information gain between the prior and pseudo posterior multi-target densities. Three information divergences are tested, including the analytical Rényi divergence (Eq. (5.2.22)) for LMBs, the numerical Rényi divergence (Eq. (5.2.6)) for LMBs, and the Cauchy-Schwarz divergence (Eq. (5.2.1)) for δ-GLMBs. Note that the Cauchy-Schwarz divergence of a sensor tasking command is calculated using the prior and pseudo posterior δ-GLMBs, and all information of the two objects are used in the calculation. For example, the Cauchy-Schwarz divergence computed for object 1 depends on the state and uncertainty of both objects. In contrast, the analytical and numerical Rényi divergences are calculated only using the prior and pseudo posterior density of objects within the sensor FOV. In addition, the integral (Eq. (5.2.14)) in the numerical Rényi divergence is computed using a Monte Carlo integration method. The sensor control command that yields the highest information gain is selected as the best solution, and the sensor is allocated to collect measurements. The measurement update is performed using the LMB update by assuming the measurement noise is 1 arc-second for both angles.

In Fig. 5.2.1, the black circle represents the trajectory of the two tested objects. The star located at the centre of the circle is the sensor. The solid blue dots and orange triangles are the initial positions of the two objects, the hollow blue dot and orange triangle are the positions of the two objects at each epoch. The blue and orange arrows indicate the corresponding motion of the two objects. The objects are separated such that the sensor can only choose one to detect over the course of the 20 time steps.
Chapter 5. A Multi-Sensor Tasking Method Using Analytical Rényi Divergence

Figure 5.2.1: Reward function test case

(a) Analytical Rényi divergence

(b) Numerical Rényi divergence

(c) Cauchy-Schwarz divergence

Figure 5.2.2: Tracking results by sensor tasking via maximising different reward functions
Fig. 5.2.2 shows the log trace of the covariance of the test objects, as well as the observed target ID using all three formulations. The log value used here is merely for better visualisation. The results indicate that all reward functions are able to provide the desired sensor tasking performance, i.e., the two objects can be alternately observed during the entire simulation. The major difference is that the two Rényi divergences choose to observe target 1 at the first epoch, while the Cauchy-Schwarz divergence selects target 2 to start. Nonetheless, the covariances of the two objects decrease rapidly in all test cases. Moreover, the uncertainties of the two objects are significantly reduced. It can be concluded that the proposed analytical Rényi divergence provides similar performance compared to the numerical Rényi divergence and the Cauchy-Schwarz divergence.

5.3 Centralised Data Fusion

The sensor networks investigated in this study are managed by a centralised processing strategy [120]. It maintains a central catalog using measurements from all sensors and determines the best sensor control vector using all information obtained from each sensor. In this way, the sensor tasking solutions, which are determined based on all available information from all sensor nodes, approximate a globally optimal solution. The general schematics of the centralised network in an SSA application is shown in Figure 5.3.1.

In the proposed sensor network design, in each sensor node, an LMB filter is running. The multi-target posteriors formed in each sensor node are communicated to a ground centre where they are combined into a fused multi-object posterior. After the multi-sensor fusion is performed, the system prepares for the next epoch by running the prediction step of an LMB filter centrally. The predicted LMB prior is then fed back to the sensor nodes. At the centre, the predicted prior is further used for tasking the sensors, and the information theoretic method is used to produce the best sensor tasking commands. The resulting commands are also communicated back to the sensor nodes. Depending on those commands, the sensors are tasked to scan particular regions of space and acquire measurements related to space objects in those
regions. The measurements are then utilised to run local LMB update steps in each sensor node, resulting in LMB posteriors.

In centralised sensor networks, the GCI rule [42] is commonly used to fuse several localised LMB densities. Given the LMB RFS $\pi_i = \{(r_i^{(\ell)}, p_i^{(\ell)})\}_{\ell \in L}, i \in [1, n_s]$, where $n_s$ is the number of sensors, and normalised fusion weights $w_i^{(\ell)} \in (0, 1)$, the fused distribution is computed using the weighted geometric mean [121]

$$\pi^f = \{(r_f^{(\ell)}, p_f^{(\ell)})\}_{\ell \in L},$$

(5.3.1)
where

\[
\begin{align*}
 r_f^{(\ell)} &= \int \prod_{i \in n_s} (r_i^{(\ell)} p_i^{(\ell)}(x))^{w_i^{(\ell)}} \, dx \\
 p_f^{(\ell)}(x) &= \frac{\prod_{i \in n_s} (p_i^{(\ell)}(x))^{w_i^{(\ell)}}}{\int \prod_{i \in n_s} (p_i^{(\ell)}(x'))^{w_i^{(\ell)}} \, dx'}. 
\end{align*}
\] (5.3.2)

Note that the data fusion rule based on the traditional GCI method produces consensus fusion, intended for use when all objects are observed by each sensor at every single epoch. However, it is not applicable for space object tracking in SSA because different sensors usually have different FOVs. Following the work by Wang et al. [111], the complementary fusion solution is adopted in this study. This method adaptively tunes the weight \( w_i^{(\ell)} \) of track \( \ell \) from the local posterior of sensor node \( s_i \), in such a way that it exponentially increases with the information content yielded by the sensor for the track. Provided the labels from multiple sensor nodes are consistent, the complementary fusion is implemented separately for each label, meaning that only the Bernoulli components with the same label are fused.

In this study, the information content of a Bernoulli component is quantified by the Rényi divergence, and the label-dependent fusion weight \( w_i^{(\ell)} \) is given by

\[
w_i^{(\ell)} = \frac{\Omega_i^{(\ell)}}{\sum_{i' \in [1,n_s]} \Omega_{i'}^{(\ell)}} \] (5.3.4)

where \( \Omega_i^{(\ell)} \triangleq \exp \left( R_i^{(\ell)} / \min_{i' \in [1,n_s]} R_{i'}^{(\ell)} \right) \), and \( \min_{i' \in [1,n_s]} R_{i'}^{(\ell)} \) represents the minimum information divergence. If only one sensor observes an object, then no information gain can be obtained from other sensors. In this case, a value of \( R_i^{(\ell)} = 0.01 \) is used as the weight for any sensor with zero information gain [111].

The Rényi divergence \( R_i^{(\ell)} \) between the prior Bernoulli RFS and posterior Bernoulli RFS
Chapter 5. A Multi-Sensor Tasking Method Using Analytical Rényi Divergence

of track $\ell$, namely $\pi_{k|k-1}^{(\ell)} = (r_{k|k-1}, p_{k|k-1})$ and $\pi_{k,i}^{(\ell)} = (r_{k,i}, p_{k,i})$, for sensor $s_i$ can be obtained based on Eq. (5.2.22)

$$R_i^{(\ell)} = -2 \log \left( r_{k|k-1,i} \right)^{\frac{1}{2}} \left( r_{k,i}^{(\ell)} \right)^{\frac{1}{2}} \left( K_{0,1}^{(\ell)} \right)^{\frac{1}{2}} |8\pi \Sigma_{0,1}^{(\ell)}|^\frac{1}{4}. \quad (5.3.5)$$

Remark 5.2 Constructing a fully centralised sensor network may not be feasible in real applications due to the limitation on data transmission between multiple sensor nodes. Therefore, the data fusion between sensor nodes should be carried out in a distributed architecture. In the distributed network, each sensor node can only exchange information with its neighbour nodes. Thus, the sensor tasking solutions from a distributed network are generally suboptimal. Note that the complementary fusion can still be applied to the distributed sensor network. Gehly et al. [120] employed the complementary fusion method to both centralised and distributed sensor networks to monitor space objects from different orbit domains. Results indicate that the state estimation accuracy of the distributed network approaches the centralised network.

Remark 5.3 One prerequisite of the complementary GCI fusion is that all sensors need to share the same label space. However, this assumption may not always be valid in practical scenarios. If the labels in different posteriors are mismatching, the GCI fusion rule will lead to poor performance. For example, in the case that a target in different posteriors is assigned different labels, both consensus and complementary GCI fusions will not correctly fuse information for the target due to the mismatching labels. The label mismatching phenomenon has attracted attention by the fusion community [122, 123, 124, 125, 126]. Further investigation of the label mismatching problem is beyond the scope of this dissertation.

5.4 Sensor Tasking Method

This section details the procedure of the developed sensor tasking method for SSA. The visibility analysis is first presented, then the principle of the step-by-step sensor tasking method
5.4. Sensor Tasking Method

and its implementation are elaborated.

5.4.1 Visibility Analysis

In this study, both GBS and SBS measure angular observations from the east-north-up (ENU) frame. Each sensor control vector specifies a sensor pointing direction defined in the ENU frame using azimuth and elevation (az, el) [24]. Note that not all space objects can be detected by an SSA sensor due to the illumination condition and several observational constraints. For both GBS and SBS, the visibility of a space object is mainly dependent on two factors, the brightness and detection loss mode [89], where the former can be quantified using the apparent magnitude and the latter is mainly based on the viewing geometry of the sensor.

(1) The apparent magnitude for a diffuse spherical object is defined using the following equation [85]

\[
m_v = -26.74 - 2.5 \log \left( \frac{1.5(C_r - 1) \cdot S \cdot F(\psi)}{\rho^2} \right)
\]

where -26.74 is the apparent magnitude of the Sun, \(C_r\) is the coefficient of reflectivity, \(S\) is the cross-sectional area of the sphere, \(F(\psi)\) is the phase function, and \(\rho\) is the distance between sensor and target. The \(S\) used in this study is the radar cross section (RCS) \(^1\) data obtained from CelesTrak \(^2\). The phase function depends on the solar phase angle \(\psi\), which is expressed as

\[
F(\psi) = \frac{2}{3\pi^2} \left\{ (\pi - \psi) \cos(\psi) + \sin(\psi) \right\}.
\]

If the apparent magnitude of a space object is larger than a predefined threshold \(T_{\text{am}}\), it is regarded as too dark to be detected.

(2) The elevation mask is a general detection loss mode employed for the GBS to avoid obstructions and atmospheric attenuation [24]. In contrast, the elevation mask is not applicable for SBS which is not affected by atmospheric conditions. Following the research by

\(^1\)Note that RCS is not the same as physical size and objects will have a different apparent size in radar and optical frequencies. In spite of this, the use of RCS as a proxy for size allows for a realistic sampling of catalogued GEO objects.

\(^2\)CelesTrak: https://celestrak.com/pub/satcat.txt, 10 November 2017
Holzinger et al. [89], another reasonable detection loss mode, transit of a space object out of sensor LOS, is adopted in this study. It can be interpreted as the LOS from sensor to object being blocked by Earth. This LOS loss mode can be further developed to probabilistically determine the time left to detect of space object.

One simple way to implement this LOS loss mode is to calculate the distance $d_{\text{LOS}}$ between the Earth centre and the LOS of an SBS. If the $d_{\text{LOS}}$ is shorter than the Earth radius $R_e$, then the object can be considered to be obscured by the Earth. The equation of $d_{\text{LOS}}$ is given by

$$d_{\text{LOS}} = \frac{\| (x_s - x_t) \times x_s \|}{\| x_s - x_t \|},$$

(5.4.3)

where $x_s, x_t$ are the locations of the sensor and target respectively in the Earth-centred inertial (ECI) coordinate frame. The Earth radius $R_e$ is regarded as the threshold of this detection loss mode.

The eclipse season is another detection loss mode, in which the GEO object can get eclipsed by the shadow of Earth. Note that the initial epoch of our simulation is 10 November 2017, which is out of the eclipse season. Though some objects at higher inclination may still experience eclipse, this loss mode is expected to have negligible impact and is therefore not considered in this study.

Based on the above two visibility constraints, the label set of all visible tracks $L^v$ can be determined as following

$$L^v = \{ \ell \mid m_v(\ell) < T_{am} \cap d_{\text{LOS}}(\ell) > R_e \},$$

(5.4.4)

and all visible objects are denoted as $\hat{X}_k^v$, where the hat refers to states extracted from the LMB.

### 5.4.2 Sensor Tasking Mode

There are several competing objectives for sensor tasking, including searching for new objects, dedicated tracking for catalog maintenance, object characterisation, and manoeuvre
5.4. Sensor Tasking Method

detection [66, 24, 67, 77]. The focus of this chapter is catalogue maintenance to be performed with an information theoretic reward function. The information theoretic sensor tasking can be implemented in either a single-step (i.e., step-by-step) or multiple-step assignment strategy [24]. The step-by-step algorithm utilises predicted information for the next time step to decide which target to observe, whereas the multiple-step method assigns sensor tasks through a future time window (different time durations up to the whole tracking period), and therefore accounts for targets entering and leaving the sensor FOR. The visible windows for GEO objects from either SBS or GBS are sufficiently long that it is not necessary to account for frequently entering and leaving the sensor FOR. Furthermore, the step-by-step method is more applicable for real-time filtering as it is able to reschedule a missed detection immediately, while the multiple-step method needs to wait to reassign a task until the whole assignment time window is completed. Therefore, the more straightforward and flexible step-by-step assignment method is utilised in this study.

For the multi-sensor tasking problem, different control vectors \( u = (u_1, u_2, \cdots, u_{ns}) \) of multiple sensors result in various combinations of local posteriors, and each combination generates a fused pseudo posterior. The general task of information-driven sensor tasking is to find the optimal control vectors \( u^* = (u^*_1, u^*_2, \cdots, u^*_{ns}) \) from all combinations in the control vector space \( U_{ns} \), which maximises the information gain between the prior and fused pseudo posterior LMBs

\[
(u^*_1, u^*_2, \cdots, u^*_ns) = \arg \max_{(u_1, u_2, \cdots, u_{ns})} R(u_1, u_2, \cdots, u_{ns}). \tag{5.4.5}
\]

The conventional method to find the optimal control vectors is to perform a brute-force search in the control vector space \( U_{ns} \). The latest innovation to speed up this optimisation process is the use of a coordinate descent method, and the optimisation needs to be repeated with multiple initialisations to improve robustness [127]. However, this approach is still computationally intractable in the case of large-scale space object tracking.

In order to efficiently generate the best control vector, this study simplifies the objective of the optimisation to determination of the group of sensor control vectors that maximises the
sum of information divergences for the prior and the local pseudo posterior of each sensor node, given by
\[
(u_1^*, u_2^*, \cdots, u_n^*) = \arg \max_{(u_1, u_2, \cdots, u_n)} \sum_{i=1}^{n_s} R(u_i). \tag{5.4.6}
\]

Compared with Eq. (5.4.5), which requires the information divergence between the prior \(\pi_{k|k-1}\) and the fused pseudo posterior \(\tilde{\pi}_{k}^{\text{Pseudo}}\) of all sensor nodes, the simplified objective function Eq. (5.4.6) employs the information divergence for each sensor node using its local pseudo posterior \(\tilde{\pi}_{k,i}^{\text{Pseudo}}\). The information gain can be quickly obtained using the proposed analytical Rényi divergence for LMBs. The computation of an information divergence is carried out only for the visible tracks \(\hat{X}_k^v\) corresponding to sensor \(s_i\) because it is unnecessary to involve the non-visible tracks in decision-making.

The best control vector of a sensor is then the one with the highest information gain. In this way, the computational complexity is significantly reduced. However, this approach is suboptimal, and the obtained best control vector may be a local optimum because the data fusion is omitted in the sensor tasking process. For example, the reward values of control vectors are added together even if they observe the same target. Generating the best control vectors without yielding repetitive observations of the same target can be formulated as an optimal assignment problem, and potential solutions like auction algorithm [24] need to be further investigated. Nonetheless, the proposed reward function has exhibited excellent empirical performance in sensor tasking applications.

Given a specific control vector \(u\), the sensor tasks are scheduled to collect measurements. The probability of detection \(p_D\) is an essential parameter to account for missed detections, and it is important for the LMB filter to correctly maintain custody of all objects. Gehly et al. [24] calculated \(p_D\) based on a robust split GM algorithm to account for objects near the edge of the sensor FOV. Wang et al. [111] assumed \(p_D\) is both range-dependent and angle-dependent and calculated it using a Rayleigh fading signal model. Generally, \(p_D^{(t)}\) depends on whether the track
\( \ell \) is within the sensor FOV. In this study, a simplified formula is used:

\[
P_D^{(\ell)} = P_{D,c} 1_{\text{z_{\text{FOV}}}}(z_{k|k-1}^{(\ell)}).
\] (5.4.7)

Here \( P_{D,c} \) is a constant value to model the detection capability of the sensor, \( z_{k|k-1}^{(\ell)} \) is the predicted measurement of track \( \ell \) calculated from sigma points, the inclusion function \( 1_{\text{z_{\text{FOV}}}}(z_{k|k-1}^{(\ell)}) \) is 1 only when \( z_{k|k-1}^{(\ell)} \in z_{\text{FOV}} \), where \( z_{\text{FOV}} \) represents the bounds of the FOV.

### 5.4.3 Implementation

The overall structure of the multi-sensor multi-target tracking is illustrated in Fig. 5.4.1.

![Figure 5.4.1: Multi-sensor multi-target tracking](image)

Given the multi-target posterior at time \( t_{k-1} \) modelled by an LMB RFS with parameter set \( \pi_{k-1} \), and assuming no target birth, the multi-target prior at time \( t_k \) is an LMB RFS equal to the
propagated survival LMB RFS with the parameter set given by

$$\pi_{k|k-1} = \{ (r_{S,k|k-1}^{(l)}, p_{S,k|k-1}^{(l)}) \}_{l \in \mathbb{L}}.$$  \hfill (5.4.8)

Note that modelling new target birth without prior knowledge is a challenging task in SSA, especially when using too-short arc measurements [21]. Jointly tracking new and cataloged objects for SOC maintenance and expansion is a challenging task which requires further investigation.

The predicted probability of existence $r_{S,k|k-1}^{(l)}$ and the probability density $p_{S,k|k-1}^{(l)}$ of the orbital state of a survival track $\ell$ via the GM implementation are expressed by

$$r_{S,k|k-1}^{(l)} = p_s^{(l)} r_{S,k-1}^{(l)}$$

$$p_{S,k|k-1}^{(l)} = 1_L(\ell) \sum_{j=1}^{J} w_{(j)}^{(l)} p_s(x; m_{k|k-1,j}^{(l)}, P_{k|k-1,j}^{(l)}).$$ \hfill (5.4.10)

where $p_s(\cdot)$ is the survival probability, $m_{k|k-1,j}^{(l)}$ and $P_{k|k-1,j}^{(l)}$ are obtained by propagating $m_{k-1,j}^{(l)}$ and $P_{k-1,j}^{(l)}$ to time $k$ using the unscented transform, and $J$ is the number of GM components. The weight $w_{(j)}^{(l)}$ is constant in the prediction step. The $m_{k|k-1,j}^{(l)}$ and $P_{k|k-1,j}^{(l)}$ via the unscented transform can be found in Sec. 4.4.

The multi-target prior $\pi_{k|k-1}$ needs to be transformed to a $\delta$-GLMB for the measurement update and sensor tasking steps. The number of hypotheses in the predicted $\delta$-GLMB increases exponentially for a large number of tracks. In order to improve efficiency, a $K$-shortest path algorithm can be used to only maintain the $K$ most important hypotheses with the largest weights [47].

To compute the visibility and information divergence for sensor tasking, the multi-target state needs to be extracted from the prior $\delta$-GLMB density. The number of objects $\hat{n}$ can be determined using the maximum a posterior (MAP) value of the cardinality distribution $\rho(n)$

$$\hat{n} = \arg \max_n \rho(n).$$ \hfill (5.4.11)
The tracks $\hat{X}_k$ with first $\hat{n}$ highest probabilities of existence are then extracted. For a track $\ell$, the expected a posterior (EAP) state estimate is given by

$$\hat{x}_k(\ell) = \sum_{j=1}^{J} w_{k|k-1,j}^{(\ell)} m_{k|k-1,j}^{(\ell)}.$$  \hspace{1cm} (5.4.12)

The EAP state is utilised to determine the objects $\hat{X}_{v,k,i}$ visible to sensor $s_i$. Each visible object represents a potential sensor control task. For a given task $u_j$, where $j = \{1, 2, \cdots, |\hat{X}_{k,i}|\}$ and $|\hat{X}_{k,i}|$ is the number of visible targets, the PIMS $Z_{k|k-1,ij}^{\text{PIMS}}$ is calculated including all visible objects in the FOV. Then, the pseudo LMB update is performed using $Z_{k|k-1,ij}^{\text{PIMS}}$ and the prior LMB RFS $\pi_{k|k-1}$. The obtained pseudo-posterior $\tilde{\pi}_{k,ij}^{\text{Pseudo}}$ of each sensor along with the prior LMB RFS are used to calculate the Rényi divergence $R_i(u_j)$. This process is repeated until the Rényi divergences of all sensor tasks are obtained. The best sensor control vector $u_i^*$ of sensor $s_i$ is determined as the one with the highest information gain. With the selected sensor task, the measurements $Z_{k,i}$ are collected by sensor $s_i$ and applied for the LMB update.

In the LMB update step, the $\eta_{Z_k}(\theta_k)(\ell)$ and the single target density of track $\ell$ via the GM implementation are expressed as

$$\eta_{Z_k}(\theta_k)(\ell) = \sum_{j=1}^{J} \omega_{k,j}(\theta_k)(\ell)$$  \hspace{1cm} (5.4.13)

$$p(\theta_k)(x, \ell) = \sum_{j=1}^{J} \frac{\omega_{k,j}(\theta_k)(\ell)}{\eta_{Z_k}(\theta_k)(\ell)} p_g(x; m_{Z,k,j}(\theta_k)(\ell), P_{k,j}^{(\theta_k)}(\ell))$$  \hspace{1cm} (5.4.14)

The parameters of the above two equations can be found in Sec. 4.4.

Each local posterior $\delta$-GLMB RFS is then transformed back to the local posterior LMB RFS $\pi_{k,i}(\cdot|Z_{k,i}) = \{(r_{k,i}^{(\ell)}, p_{k,i}^{(\ell)})\}_{\ell \in L}$ for data fusion, where $i \in [1, n_s]$ and $n_s$ is the number of sensors. All localised LMB RFSs from each sensor are fused to obtain the posterior LMB RFS. The complementary data fusion rule is performed for each track $\ell$ separately, and the fusion weight $w_i^{(\ell)}$ is determined using the Rényi divergence between the prior Bernoulli component $\pi_{k|k-1}^{(\ell)}$ and local posterior Bernoulli component $\pi_{k,i}^{(\ell)}$ of the sensor node $s_i$, which is calculated.
using Eq. (5.3.5). Note that the Rényi divergence used in sensor tasking is calculated between the prior and the pseudo local posterior from PIMS, while the true posterior from the actual measurements is used to calculate the Rényi divergence in fusion. Finally, the multi-target posterior is approximated by the fused LMB RFS, which is employed to perform the SOC maintenance, and propagated to the next time epoch; then, the process of sensor tasking and multi-target tracking is repeated as shown in Fig. 5.4.1.

5.5 Simulation

5.5.1 Sensor Network Design

The developed sensor tasking algorithm is implemented using various types of sensor networks developed in this study, including one based on three GTO sensors, one based on three SSO sensors, and two hybrid sensor networks using both SBS and GBS. Furthermore, the proposed sensor networks are applied to two test cases to assess the performance with respect to the traditional GBS sensor network. Following the research of Gehly et al. [24], the GBS network used in this study is constructed by three ground stations, i.e., Socorro NM, Maui HI, and Diego Garcia. Their geodetic locations and detailed parameters are given in Table 5.5.1. Note that the double prime denotes the units of arc-seconds.

<table>
<thead>
<tr>
<th>Table 5.5.1: Parameters of the ground stations</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>GBS</strong></td>
</tr>
<tr>
<td>Socorro, NM</td>
</tr>
<tr>
<td>Maui, HI</td>
</tr>
<tr>
<td>Diego Garcia</td>
</tr>
</tbody>
</table>

The orbital parameters of the three GTO orbits are generated based on three reference orbits selected from the NORAD TLE public catalog \(^3\). In order to cover the whole GEO belt, the apogees of the GTO orbits are well separated to guarantee each can approach around 1/3 of the population of GEO objects. The three SSO orbits are generated based on the orbit of NEOSSat, which is also obtained from the NORAD TLE public catalog. The major difference

\(^3\)www.space-track.org
of these three SSO orbits is that their mean anomalies $M$ are around 120 deg apart. The Keplerian orbital parameters and sensor FOVs of these two types of sensor networks can be found in Table 5.5.2.

**Table 5.5.2: Orbital parameters of the GTO and SSO sensor networks**

<table>
<thead>
<tr>
<th>SBS</th>
<th>$a$ (km)</th>
<th>$e$</th>
<th>$inc$ (deg)</th>
<th>$\omega$ (deg)</th>
<th>$\Omega$ (deg)</th>
<th>$M$ (deg)</th>
<th>FOV size (az, el)</th>
<th>Noise (az, el)</th>
</tr>
</thead>
<tbody>
<tr>
<td>GTO-A</td>
<td>26558.41</td>
<td>0.64</td>
<td>17.87</td>
<td>280.13</td>
<td>167.21</td>
<td>225.49</td>
<td>[2 deg, 2 deg]</td>
<td>[5”, 5”]</td>
</tr>
<tr>
<td>GTO-B</td>
<td>26099.83</td>
<td>0.60</td>
<td>27.16</td>
<td>9.91</td>
<td>346.23</td>
<td>52.81</td>
<td>[2 deg, 2 deg]</td>
<td>[5”, 5”]</td>
</tr>
<tr>
<td>GTO-C</td>
<td>24729.35</td>
<td>0.72</td>
<td>1.24</td>
<td>46.21</td>
<td>172.28</td>
<td>195.86</td>
<td>[2 deg, 2 deg]</td>
<td>[5”, 5”]</td>
</tr>
<tr>
<td>SSO-A</td>
<td>7162.17 $\times 10^{-4}$</td>
<td>98.54</td>
<td>159.97</td>
<td>133.10</td>
<td>106.96</td>
<td>[2 deg, 2 deg]</td>
<td>[5”, 5”]</td>
<td></td>
</tr>
<tr>
<td>SSO-B</td>
<td>7161.13 $\times 10^{-4}$</td>
<td>98.54</td>
<td>160.38</td>
<td>131.81</td>
<td>228.25</td>
<td>[2 deg, 2 deg]</td>
<td>[5”, 5”]</td>
<td></td>
</tr>
<tr>
<td>SSO-C</td>
<td>7159.62 $\times 10^{-4}$</td>
<td>98.53</td>
<td>161.46</td>
<td>132.46</td>
<td>347.61</td>
<td>[2 deg, 2 deg]</td>
<td>[5”, 5”]</td>
<td></td>
</tr>
</tbody>
</table>

**Figure 5.5.1: GTO, SSO sensor networks and GEO space objects**

The established GTO and SSO sensor networks are depicted in Fig. 5.5.1 in the ECI frame. The sphere located at the centre represents the Earth. The three large blue dots represent the GTO sensors, and their corresponding orbits are shown by the three black ellipses. The three large red dots represent the SSO sensors which are located on three low Earth orbits; the orbits are too close to be distinguished in the figure. The small black dots are 1000 GEO objects which are randomly selected from the NORAD public catalog. From the figure, it appears the GTO sensor network has a better opportunity to closely observe the GEO objects compared with the SSO sensor network.
In addition, two hybrid sensor networks, i.e., GBS-SSO and GBS-GTO, using both SBS and GBS are presented as a complementary approach. Both GBS-SSO and GBS-GTO sensor networks utilise two ground-based stations, Socorro and Diego Garcia, while the former employs SSO-A and the latter uses GTO-A. Theoretically, the SBS is not restricted by the day-night gap and thus can work 24 hours every day. Therefore, the hybrid sensor network is expected to exceed the traditional GBS network by persistently generating measurements to refine the orbital state estimation.

5.5.2 Simulation Design and Data Selection

To validate the performance of the proposed method, two simulated scenarios including a large population of GEO objects and a set of dim GEO objects are considered in this study.

In the first case study, a catalog containing 905 GEO objects is established, and all objects are tracked for three days using different sensor networks (i.e., GBS, SBS, and hybrid) and reward functions (i.e., Rényi and Cauchy-Schwarz divergences). The 905 objects are selected from the NORAD public catalog for the date 10 November 2017 using the following constraints

\[ 0 \leq e \leq 0.1; \quad 0 \, \text{deg} \leq inc \leq 70 \, \text{deg}; \quad 0.9 \, \text{day} \leq n_m \leq 1.1 \, \text{day}, \tag{5.5.1} \]

where \( n_m \) is the orbital period. The RCS values of these 905 objects are taken from CelesTrak.

The ground-based stations and their corresponding FORs are depicted as big black dots and dashed curves respectively in Fig. 5.5.2. Note that the FORs of the three GBS cannot cover the whole GEO belt due to the use of the 20 deg elevation mask. This results in two gaps between the three FORs, i.e., a narrow gap around 135 deg to 155 deg longitude, and a larger gap above the Atlantic Ocean. Therefore, for the purpose of a reasonable comparison between the GBS and SBS approaches, only detectable objects which appear in the FOR of GBS at some epochs within the tracking time window are selected. All test objects are drawn as the small black dots in Fig. 5.5.2. It is worth mentioning that there is a considerable overlap region between the FORs of the Socorro and the Maui stations, meaning that some objects can be jointly detected.
by these two stations, whereas the Diego Garcia station has to solely conduct tracking for the objects above the Indian Ocean.

In Case II, the proposed method is assessed by tracking 100 dim GEO objects. These dim objects (red circles in Fig. 5.5.2) are randomly selected from the 905 objects used in Case I. However, their RCS values are uniformly generated from the small range of \([0.01, 0.2]\) m\(^2\). These constraint values are selected because objects with RCS values larger than the upper bound 0.2 m\(^2\) can be easily detected by all sensor networks, while objects with RCS values smaller than the lower bound 0.01 m\(^2\) may be too faint to be observed. Note that the physical area of a space object is assumed to be equal to RCS in this study. Therefore a sphere with RCS of 0.2 m\(^2\) has a similar size as a basketball, while an object of 0.01 m\(^2\) is as small as a tennis ball. As the tested 100 dim objects are selected from the catalog in Case I, they are able to appear in the FORs of the three GBSs during at least one epoch in the course of the simulation.

In order to achieve accurate state estimation, dense measurements and long observational arcs are necessary for traditional space object tracking campaigns. However, this is not feasible when tracking a large population of space objects. In this study, the measurements are generated from sparse data in a short observational arc, which enables more sensor resources to be allocated to maintain custody of cataloged objects. Both GBS and SBS are assumed to generate
one measurement each minute. This assumption results in 4320 epochs for each sensor network in a three-day time window. The sparse data and short observing arc introduce difficulty to accurate state estimation and data association for the LMB filter.

The dynamics of all tested objects are modelled in the Cartesian coordinate space. The initial covariance is given by

$$P_{\text{ini}} = \text{diag} [100, 100, 10, 10^{-4}, 10^{-4}, 10^{-5}], \quad (5.5.2)$$

where the units of position and velocity are km and km/s respectively. The mean estimated state used in the LMB filter is randomly perturbed from the truth using $P_{\text{ini}}$.

The measurements used in the filter update step are topocentric right ascension and declination, which are defined in Eqs. (2.1.12) - (2.1.15). The measurement noise values of GBS and SBS are provided in Table 5.5.1 and Table 5.5.2 respectively. Process noise is incorporated as a simple additive model to avoid filter saturation, assuming a constant $6 \times 6$ diagonal matrix with $1 \sigma$ values of 1 m and $10^{-3}$ m/s for position and velocity respectively.

The force models used for truth generation and orbit propagation in the LMB filter are shown in Table 5.5.3. The truth of the orbital state is generated using a high-fidelity orbit propagation model including perturbation forces given in the second column of Table 5.5.3. A simplified force model given in the third column of Table 5.5.3 is employed in the LMB filter. The orbit propagation is performed by the ODTBX 6.5 based on MATLAB®. All algorithms including the LMB filter, sensor tasking, and data fusion are implemented in MATLAB®, and the computations are based on an Intel Core i7 CPU with 64-bit numerics and a 2.7 GHz clock rate.

This simulation does not include the target birth and death processes. The LMB filter employs a constant probability of detection value $p_{D,c} = 0.98$ to quantify the capability of sensors. The clutter measurements are assumed to be uniformly distributed over the observation space; the number of returns follows a Poisson distribution with mean $\lambda_c = 0.2$. The intensity of
5.5. Simulation

Table 5.5.3: Force models for orbit propagation

<table>
<thead>
<tr>
<th>Force models</th>
<th>Truth</th>
<th>Filter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass</td>
<td>487 kg</td>
<td>487 kg</td>
</tr>
<tr>
<td>Earth gravity model</td>
<td>WGS84 EGM96</td>
<td>WGS84 EGM96</td>
</tr>
<tr>
<td>Gravity degree/order</td>
<td>$10 \times 10$</td>
<td>$2 \times 2$</td>
</tr>
<tr>
<td>Solar and lunar gravity</td>
<td>on</td>
<td>on</td>
</tr>
<tr>
<td>Solar radiation pressure</td>
<td>on</td>
<td>off</td>
</tr>
<tr>
<td>Coefficient of reflection</td>
<td>0.7</td>
<td>N/A</td>
</tr>
<tr>
<td>Area-to-mass ratio</td>
<td>0.005 m$^2$/kg</td>
<td>N/A</td>
</tr>
</tbody>
</table>

the clutter measurement is expressed as $\kappa(z) = \lambda \mathcal{U}(z)$, where $\mathcal{U}$ denotes a uniform distribution over the sensor FOV. In addition, the pruning threshold of $10^{-5}$ and merging threshold of 3 are used to improve the computational efficiency of the LMB filter. For the visibility analysis, all tested objects are assumed to be spherical; the apparent magnitude threshold is $T_{am} = 16$; the elevation mask for GBS is assumed as 20 deg; the Earth radius used for detection loss mode is $R_e = 6378.137$ km. The positions of all investigated sensor nodes are initially known and without uncertainty. All sensor nodes are assumed to have a fast slewing rate which can be tasked to point to any direction within one minute.

5.5.3 Case I: Large-Scale Tracking

This case tests the proposed methods by tracking 905 GEO space objects for three days using five different sensor networks, i.e., GBS, GTO, SSO, GBS-GTO, and GBS-SSO. The analytical Rényi divergence and Cauchy-Schwarz divergence are used with the above five sensor networks for comparison. Accordingly, the combinations of the two reward functions and the five sensor networks result in ten groups of methods, and each is examined individually to assess its performance.

The accuracy of the test results is assessed by the averaged state errors of the estimated orbital states of all 905 objects. The averaged position errors by different methods are depicted in Fig. 5.5.3a, and the results of the velocity error are omitted because their trends are similar to the position error results. A comparison of results between the Rényi and Cauchy-Schwarz divergences shows no significant differences across all variations of the sensor network. Both
information measures achieve similar accuracy for a given network and result in significant reduction of averaged position errors in all cases. Specifically, the errors of GTO and SSO directly decrease to around 500 m after 40 hours, and GTO outperforms SSO as it converges slightly quicker during the first half of tracking. The errors of the two hybrid sensor networks all experience two slight variations at around 4-7 and 22-28 hours respectively. This is because only one GBS remains active during these two time periods, which results in an insufficient number of measurements to be collected to suppress the growth of uncertainty and state errors. Nonetheless, the two hybrid sensor networks reach a similar level of accuracy compared to SBS after 40 hours.

GBS performs the worst among all tested sensor networks. It yields a rapid increase of the averaged position errors during hours 4-12. Without sufficient measurements collected during this period, the position errors of several objects evolve quickly and dominate the averaged position error. In addition, the errors of GBS are much larger than other sensor networks throughout the entire simulation. The accuracy of GBS can be further improved by increasing the timespan of the simulation. However, due to limited observing capability, some dim GEO objects cannot be frequently observed by GBS. For this reason, the state errors of these objects may remain large, leading to large averaged state error compared to other sensor networks. This is illustrated in the two subfigures in the first column of Fig. 5.5.4, where several targets still have large errors
at the end of the simulation when using GBS. Hence, even if using a longer simulation, GBS is not likely to achieve similar accuracy as the SBS or hybrid networks.

Fig. 5.5.3b shows the enlarged results of Fig. 5.5.3a during the second half of the tracking. The errors of GBS gradually converge to under 1.5 km after 45 hours, but they still exhibit a large variation. Compared with GBS, the SBS and hybrid sensor networks provide more accurate state estimation as expected, and the averaged position error eventually converges to around 500 m. The results suggest that all sensor networks provide good performance, while the GTO sensor network slightly outperforms others.

**Figure 5.5.4:** All position errors (grey curves) and averaged position error (black curve) for all sensor networks

In order to study the performance of the two tested information divergences and the five sensor networks in more detail, the position errors of all tested targets using different methods are depicted in the ten subfigures of Fig. 5.5.4. In each subfigure, the title shows the employed sensor network and reward function; the grey curves represent the position errors of the 905 objects, and the black curve is their averaged position error. The results of the five sensor networks using the analytical Rényi divergence and the Cauchy-Schwarz divergence are displayed in the first and second row respectively. It can be seen from Fig. 5.5.4 that both divergences yield
similar results for all targets in the course of the entire simulation. The SBS sensor networks result in a quick reduction in the number of targets with large errors, and all targets errors are reduced to under 2 km. The hybrid sensor networks are able to provide steady low position errors for most targets after 40 hours, while the errors of a few targets still vary by around 5 km during the last day. In contrast, the GBS sensor network takes more time to reduce the errors of the majority of targets, and several targets are still varying by tens of kilometres during the last day, which dominates the significant variation of the averaged position error.

![Number of targets and measurements](image)

**Figure 5.5.5:** Number of targets and measurements

The numbers of targets detected by different sensor networks during the first 24 hours are displayed in Fig. 5.5.5a, and all methods detect all targets by the final time. It can be observed that results from the two information divergences are nearly identical, but different sensor networks yield varying performance. The GTO sensor network achieves the best performance and observes all object within 5 hours. The GTO sensors have more opportunity to observe GEO object as their trajectories are closer to the GEO ring. Therefore, GTO sensors are suitable for dim GEO object tracking, and this will be further investigated in Case II. Both hybrid sensor networks outperform GBS, and GBS-SSO is slightly superior to GBS-GTO. While performing the worst, the GBS network can observe all targets by the end of the first day.

Fig. 5.5.5b shows the number of measurements collected by the different methods. The Rényi divergence again yields similar results compared to the Cauchy-Schwarz divergence.
Among different sensor networks, the SBS approach produces the most measurements. SSO is the most productive sensor network, and GTO can also provide a large number of measurements. SSO and GTO sensor networks yield a similar number of measurements during the first 10 hours, but GTO consumes much less time to detect all 905 objects due to its better observing capability for GEO objects. The GBS sensor network still performs worst, and both GBS-GTO and GBS-SSO outperform GBS.

![Figure 5.5.6: Number of visible targets](image)

The number of visible targets is another essential parameter to assess the viewing capability of different sensor networks, and the results of all sensor networks are depicted in Fig. 5.5.6. Since the visibility of an object is independent of information divergence, the results displayed are for the five different sensor networks. As expected, the GTO sensor network has most visible targets throughout the entire simulation because the sensor node is closer to the GEO belt. The SSO sensor network yields a frequent periodic variation because their short orbital revolutions result in a frequent change of apparent magnitude values of GEO objects. The GBS-SSO sensor network also exhibits a similar trend due to the use of an SSO sensor node. The GBS sensor network yields the fewest number of visible targets throughout the simulation because of constraints preventing daytime operations and limited viewing geometry.

The run-time ratio between the calculation of the Cauchy-Schwarz divergence and the
Chapter 5. A Multi-Sensor Tasking Method Using Analytical Rényi Divergence

Figure 5.5.7: Run time ratio of the Cauchy-Schwarz divergence and Rényi divergence

Rényi divergence using GBS is shown in Fig. 5.5.7. The results from other sensor networks are omitted because their trends are similar to GBS. Results indicate that the Rényi divergence significantly outperforms the Cauchy-Schwarz divergence. The computational complexity of the Cauchy-Schwarz divergence is highly dependent on the number of hypotheses in the prior and pseudo-posterior $\delta$-GLMB, and the results presented are based on applying 10 maximum hypotheses for both. Note that if the number of hypotheses is reduced to 1, then the $\delta$-GLMB can be regarded as an LMB, and the computation time of the Cauchy-Schwarz divergence should be similar to the proposed Rényi divergence. The derivation of the Cauchy-Schwarz divergence for LMB is beyond the scope of this paper and further investigation is needed in future research.

As described, the SBS and hybrid sensor networks can provide better accuracy for monitoring a large population of GEO objects compared with the GBS approach. Further, the derived analytical Rényi divergence is validated as a practical approach to the sensor tasking problem. It yields a similar level of state estimate accuracy as compared to the Cauchy-Schwarz divergence with reduced computational complexity.

5.5.4 Case II: Dim Object Tracking

Three sensor networks, i.e., GTO, SSO, and GBS, are tested by tracking 100 dim GEO objects for three days. Only the analytical Rényi divergence is employed in this test case to
5.5. Simulation
determine the sensor control vector. Theoretically, both the Rényi divergence and the Cauchy-
Schwarz divergence can produce similar results.

\[ \text{Fig. 5.5.8a shows the averaged position errors of the 100 dim GEO space objects. The} \]
\[ \text{velocity results are again omitted because the trends are similar to the position error. As shown} \]
\[ \text{in the figure, the GTO sensor network reduces the position error to around 5 km after 20 hours} \]
\[ \text{of tracking and the errors are maintained around 5 km for the rest of the simulation. However,} \]
\[ \text{the position errors of the other two sensor networks fail to converge, and the SSO provides the} \]
\[ \text{worst accuracy in state estimation.} \]

\[ \text{Fig. 5.5.8b shows the number of dim objects detected by the three methods. The result of} \]
\[ \text{the GBS is surprisingly better than SSO. Compared with GBS, SSO has a larger relative distance} \]
\[ \text{to most GEO objects because the orbital plane of the SSO is nearly vertical to the equator and} \]
\[ \text{most objects in GEO have low inclinations. The final results illustrate that both GBS and SSO} \]
\[ \text{can only detect a small portion of the 100 tested objects due to the low brightness. The GTO} \]
\[ \text{approach can detect 95 out of 100 objects, and the other five undetected objects all have RCS} \]
\[ \text{values lower than 0.03. The GTO sensor network is generally effective for tracking dim GEO} \]
\[ \text{object and substantially increases the range of detectable object sizes as compared to other} \]
\[ \text{networks, though there is still a lower limit depending on sensor constraints.} \]

\[ \text{Additionally, the number of measurements collected by each network is shown in Fig. 5.5.9a.} \]
It is observed from the figure that only the GTO sensor network can steadily produce measurements over the entire simulation, while GBS and SSO are often unable to produce any new measurements. Fig. 5.5.9b shows the number of objects visible to each sensor network. There are around 20 to 40 objects visible to the GTO sensor network at most epochs. On the contrary, the other two sensor networks can only provide access to a few targets due to the larger relative distance between the GEO targets and sensors. As discussed previously, the GTO sensors are able to approach the GEO belt, and therefore have more opportunity to detect the dim GEO object than the SSO and GBS sensor networks. The significant difference further demonstrates the better performance of the GTO sensor network for dim GEO object detection.

5.6 Summary

This chapter investigates the use of space-based multi-sensor networks for multi-target estimation of GEO space objects. Two SBS sensor networks (three GTO and three SSO) and two hybrid sensor networks (GBS-GTO and GBS-SSO) are presented and tested. The analytical formulation of Rényi divergence for two LMB densities is derived to measure the information gain, and a simplified reward function of multi-sensor tasking is defined as the sum of information divergences of each sensor node to improve the efficiency of large-scale GEO object tracking. Two numerical simulations including 905 GEO objects and 100 dim GEO objects
are designed for validation. Results indicate that the SBS and hybrid sensor networks outperform the GBS approach in terms of orbital state estimation, the number of targets detected and the number of measurements collected. The derived Rényi divergence provides similar sensor tasking performance compared to the Cauchy-Schwarz divergence with a better computational performance. In addition, the GTO sensor network significantly outperforms other sensor networks for tracking dim GEO object because of its better viewing geometry for GEO objects.
Chapter 6  Summary, Conclusions and Recommendations

6.1  Summary and Conclusions

This dissertation was aimed to develop novel methods of tracklet association and RFS-based statistical multi-target tracking and multi-sensor tasking to improve the capability and capacity of the current space object cataloguing. The major difficulties and challenges of these techniques were first investigated. To achieve accurate catalogue maintenance, two improved tracklet association methods were proposed for efficient and effective processing of increasing numbers of uncorrelated tracklets. The performance of these new methods was studied using real measurements. A comprehensive comparison of four recently-developed labelled RFS filters was carried out to assess their performance for different space object tracking tasks. This thesis then explored the use of the BVP tracklet association method for the initialisation of the LMB filter to achieve a more efficient initialisation process for new space object tracking. A rigorous comparison between the BVP and the traditional CAR and PAR birth models was conducted using objects from different orbital domains. Finally, to allocate sensor resources to make better use of limited information, the analytical formulation for the Rényi divergence of LMB RFSs was derived and formulated as an objective function to address the multi-sensor tasking problem. The proposed method allocates a set of space-based and ground-based sensor networks for GEO object tracking, and their performances were studied using different scenar-
6.1. Summary and Conclusions

ios of space object tracking.

For the tracklet association problem, a new algorithm called the improved IVP method was proposed, which determines the association by optimising a new loss function defined in the non-singular canonical space. Simulations were carried out using real optical data from various orbital regions. The results indicated that the improved IVP method achieves better association performance compared with the IVP and BVP methods for the tested real data, and it provides the most efficient computational performance because the measurement noise calibration process does not need to be considered. However, the improved IVP method yields low true negative rate for the association of tracklets in the same constellation. As an effective solution, a common ellipse algorithm was proposed to distinguish false associations by determining if a best fitting common ellipse to all hypothetical ellipses of the constellation tracklets can be found. Results indicated a significant improvement in the true negative rate of the association results.

For the multiple space object tracking problem, the multi-target Bayesian recursion approach based on the labelled RFS theory was validated as a viable solution through various simulated scenarios. The LMB filter is suited for optical SSA sensors because the small sensor FOV yields a natural grouping and gating that reduces the computational complexity. The comparison results of four labelled RFS filters indicated that the LMB filter is a viable solution for tracking space objects because it can achieve good performance for both accuracy and efficiency.

For the multi-target tracking of new space objects, the BVP optimisation method was formulated as a new birth model for the initialisation of the LMB filter for recursive filtering and estimation. The major advantage this new birth model has over the previously developed CAR/PAR birth model is that the initial target state is approximated by a single Gaussian component, while a large number of GM components are generally produced by the CAR/PAR method. As a result, the BVP birth model can result in significant improvement of the computational demand for the filtering process. The proposed method was tested by tracking multiple
GEO and GTO objects in various scenarios. Results validated that the BVP-LMB filter provides similar accuracy of state and cardinality estimation compared to the CAR/PAR-LMB filter, and it achieves superior results of computational efficiency in all test cases.

For the multi-sensor tasking problem, the analytical formulation of the Rényi divergence for LMB RFSs was derived and formulated as the reward function for this problem. The proposed multi-sensor tasking method is implemented using several space-based and ground-based sensor networks to monitor GEO. Results indicated that the Rényi divergence is highly effectively in reducing estimation errors for a large population of space objects, and it is more computationally efficient than the Cauchy-Schwarz divergence for $\delta$-GLMB RFSs. In addition, the GTO sensor network was validated as a viable solution for tracking dim objects in GEO due to its better viewing geometry for the GEO domain.

The work in this thesis has demonstrated that the proposed new methods are able to provide improved performance in terms of accuracy, effectiveness, and/or efficiency compared to the state-of-the-art methods. These are beneficial for robust SOC maintenance as well as the expansion of the capacity of the current SOC for accommodating more uncorrelated and undiscovered space objects.

### 6.2 Recommendations for Future Work

Future research in the terms of tracklet associations, multi-target tracking and multi-sensor tasking are organised as follows.

1) The improved IVP method is based on the assumption that target motion follows unperturbed two-body dynamics. Further development to remove this assumption is beneficial to improve the association accuracy of tracklets with longer time gaps and/or greatly perturbed by the J2 perturbation. In addition, the standard Lambert solver used in the BVP optimisation process only accounts for the two-body dynamics, and this may result in inaccurate IOD solutions for space objects greatly perturbed by J2 perturbation. Using a shooting method [128] to solve the Lambert problem that considers J2 perturbation is expected to improve the robustness of the
6.2. Recommendations for Future Work

current BVP optimisation.

2) Incorporating the machine learning methods, (e.g., the deep neural network, reinforcement learning), into the tracklet association problem is a promising way to deal with several challenging issues (e.g., the MMO problem in the IVP approach, and the constellation tracklet association problem). Several machine learning algorithms have been validated as effective solutions for clustering and decision-making. Formulating the tracklet association loss function as learning reward, and employing the machine learning method to efficiently determine the association may provide improved performance in comparison to the conventional methods.

3) The multi-sensor tasking algorithm proposed in this thesis is mainly for the catalogued GEO objects with known initial orbital state information, while the joint tracking of catalogued objects and searching for new targets without any prior information is a more complex task. This can be formulated as a joint search and track sensor tasking problem [66, 90], which requires the objective function to account for multiple tasks. Development of automatic joint search and track sensor tasking methods based on the RFS framework may be fruitful.

4) Only the centralised multi-sensor tasking and data fusion are considered in this thesis, and the distributed architecture is more close to the generally used approach in practice. The implementation of the distributed system is a challenging issue as several practical problems need to be considered, e.g., distributed data fusion [120, 111], label mismatching [122, 126, 124], and multi-dimensional assignment [129, 130].
Appendix A  Publications

A.1  Journal Papers


6) He, Changyong, Yang Yang, Brett Carter, Emma Kerr, Suqin Wu, Florent Deleflie, Han

A.2 Conference Papers


Appendix B The Algorithms in Labelled Random Finite Set Filters

B.1 Truncation Formulation of the Joint $\delta$-GLMB filter

For a predicted hypothesis $(I, \xi)$, a set of ‘children’ hypotheses $(I, \xi, I_k, \theta_k)$ are truncated to retain those with significant weights. Given a hypothesis $(I, \xi)$ and measurement set $Z_k$, the measurement to track association considers permutations of measurements $Z_k = \{z_{1:M}\}$, with label sets $I = \{\ell_{1:R}\}$ and $\mathbb{L}_k = \{\ell_{R+1:N_p}\}$, where $M$ is the number of measurements, $R$ is the number of existing targets and $N_p$ is the total number of targets. The truncation reduces the full set of permutations to a set of pairs $(I_k, \theta_k) \in \mathcal{F}(\mathbb{L}_k) \times \Theta_k(I_k)$ with significant weights $w_{Z_k}^{(I, \xi, I_k, \theta_k)}$.

In order to perform truncation, a $N_p$-tuple $\gamma = (\gamma_{1:N_p}) \in \{-1:M\}^{N_p}$ is first defined for each pair $(I_k, \theta_k)$, given by

$$
\gamma_i = \begin{cases} 
\theta_k(\ell_i), & \text{if } \ell_i \in I_k \\
-1, & \text{otherwise}.
\end{cases}
$$

(B.1.1)

There are no distinct $i, i' \in \{1: N_p\}$ with $\gamma_i = \gamma_{i'} > 0$. The set of all 1-1 elements of $\{-1:M\}^{N_p}$ is denoted by $\Gamma$, and $1_{\Gamma}(\gamma) = 1_{\Theta_k(I_k)}(\theta_k)$. $I_k$ and $\theta_k : I_k \rightarrow \{0:M\}$ can be recovered by

$$
I_k = \{\ell_i \in I \cup \mathbb{L}_k : \gamma_i \geq 0\}, \quad \theta_k(\ell_i) = \gamma_i.
$$

(B.1.2)
Then, for all \( i \in \{1 : N_p \} \), the cost of assigning track \( i \) to measurement \( j \), denoted by \( \eta_i(j) \), can be defined as

\[
\eta_i(j) = \begin{cases} 
1 - \eta_S^{(\xi)}(\ell_i), & 1 \leq i \leq R, j < 0 \\
\eta_S^{(\xi)}(\ell_i)\eta_Z^{(\xi,j)}(\ell_i), & 1 \leq i \leq R, j \geq 0 \\
1 - r_{B,k}(\ell_i), & R + 1 \leq i \leq N_p, j < 0 \\
r_{B,k}(\ell_i)\eta_Z^{(\xi,j)}(\ell_i), & R + 1 \leq i \leq N_p, j \geq 0 
\end{cases}
\]  

(B.1.3)

where \( \eta_Z^{(\xi,j)}(\ell_i) = (p_k^{(\xi)}(\cdot, \ell_i), \eta_Z^{(\gamma_i,j)}(\cdot, \ell_i)) \), and \( j \in \{-1 : M\} \), with \( j = 0 \) indicating that \( \ell_i \) is not detected, and \( j = -1 \) indicating that \( \ell_i \) is dead.

As \( \theta_k(\ell_i) = \gamma_i \), \( \eta_Z^{(\xi,\gamma_i)}(\ell_i) = \eta_Z^{(\xi,\theta_k)}(\ell_i) \), Eq. (3.3.50) can be expressed as follows

\[
w^{(\ell,\xi,\ell,\theta_k)}_Z = 1_{\Gamma(\gamma)} \prod_{i=1}^{N_p} \eta_i(\gamma_i).
\]  

(B.1.4)

Therefore, the truncation can also be interpreted as determining a set of positive 1-1 vectors \( \gamma \) with significant \( \prod_{i=1}^{N_p} \eta_i(\gamma_i) \).

The ranked assignment problem is to select the best \( T \) positive \( \gamma \) in non-increasing order of \( \prod_{i=1}^{N_p} \eta_i(\gamma_i) \). Similar to the \( \delta \)-GLMB filter, a cost matrix \( C \) constructed with the dimension of \( N_p \times (M + 2N_p) \), is given by

\[
C_{i,j} = \begin{cases} 
-\ln \eta_i(j), & j \in \{1 : M\} \\
-\ln \eta_i(0), & j = M + i \\
-\ln \eta_i(-1), & j = M + N_p + i \\
\infty, & \text{otherwise.}
\end{cases}
\]  

(B.1.5)

The cost matrix is shown in Table B.1.1, which is divided into three sections, i.e., survived and detected, survived and mis-detected, and died.
### Table B.1.1: The cost matrix of the joint prediction and update assignment problem

<table>
<thead>
<tr>
<th></th>
<th>Survived and detected</th>
<th>Survived and mis-detected</th>
<th>Died</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_1$</td>
<td>$-\ln \eta_1(1)$</td>
<td>$-\ln \eta_1(M)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$\cdots$</td>
<td>$\cdots$</td>
<td>$\cdots$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$z_m$</td>
<td>$-\ln \eta_1(M)$</td>
<td>$-\ln \eta_1(0)$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

$\ell_i$,$\xi$

$- \ln \eta_1(1)$

$\cdots$

$- \ln \eta_1^{(M)}$

$\ell_{i_j}$,$\xi$

$- \ln \eta_i(j)$

$\cdots$

$- \ln \eta_i^{(M)}$

$\ell_{N_p}$,$\xi$

$- \ln \eta_{N_p}(1)$

$\cdots$

$- \ln \eta_{N_p}(M)$

$\ell_{N_p}$,$\xi$

$- \ln \eta_{N_p}(1)$

$\cdots$

$- \ln \eta_{N_p}(M)$

$\ell_{N_p}$,$\xi$

$- \ln \eta_{N_p}(1)$

$\cdots$

$- \ln \eta_{N_p}(M)$

$\ell_{N_p}$,$\xi$

$- \ln \eta_{N_p}(1)$

$\cdots$

$- \ln \eta_{N_p}(M)$

$\ell_{N_p}$,$\xi$

$- \ln \eta_{N_p}(1)$

$\cdots$

$- \ln \eta_{N_p}(M)$

$\ell_{N_p}$,$\xi$

$- \ln \eta_{N_p}(1)$

$\cdots$

$- \ln \eta_{N_p}(M)$


## B.2 Gibbs Sampling

The Gibbs sampling method is able to significantly reduce the dimension of the ranked assignment problem. Compared with other ranked assignment algorithms which have cubic complexity, the Gibbs sampler has quadratic computational complexity. This is achieved by defining a probability distribution $\pi$ on $\{-1 : M\}$, from which a potential positive 1-1 vector $\gamma$ can be directly sampled. The $\pi$ assigns each positive 1-1 vector a probability proportional to its weight

$$
\pi(\gamma) \propto \prod_{i=1}^{N_p} \eta_i(\gamma_i),
$$

meaning that vectors with high weights are more likely to be selected than those with low weights.

The sampling process is to generate the $T$ best hypotheses. For a hypothesis $t \in \{2 : T\}$, the Gibbs sampling can be denoted as sampling from the conditionals $\pi(\cdot | \cdot)$, given by

$$
\gamma^{(t)}_n \sim \pi_n(\cdot | \gamma^{(t)}_1, \gamma^{(t-1)}_{(n+1):N_p}),
$$

where $\pi_n(\gamma'_{(n+1):N_p}) \propto \pi_n(\gamma'_n, \gamma_{(n+1):N_p})$.
According to Lemma 2 in Ref. [49], for each track \( n \in \{1 : N_p\} \)

\[
\pi_n(\gamma_n | \gamma_{1:n-1}, \gamma_{n+1:N_p}) \propto \eta_n(\gamma_n) \prod_{i \in \{1:N_p\} \setminus \{n\}} (1 - 1_{\{1:M\}}(\gamma_n)\delta_{\gamma_n}[\gamma_i]). \tag{B.2.4}
\]

Therefore, for each measurement \( j \in \{1 : M\} \),

\[
\pi_n(j | \gamma_{1:n-1}, \gamma_{n+1:N_p}) \propto \eta_n(j)(1 - 1_{\{\gamma_{1:n-1}, \gamma_{n+1:N_p}\}}(j)). \tag{B.2.5}
\]

The Gibbs sampling for \( n \in \{1 : N_p\} \) can be achieved using a categorical distribution. A categorical distribution describes the potential results of a random variable that can take one of \( K \) possible categories, with the probability of each category separately specified. The Gibbs sampling using a categorical distribution is expressed by

\[
\gamma_n^{(t)} \sim \text{Categorical}(\{-1 : M\}, \eta_n), \tag{B.2.6}
\]

where \( \eta_n \) is the union of \( \eta_n(j) \) for \( j \in [1 : M] \), and each component is expressed by

\[
\eta_n(j) = \eta_n(j)(1 - 1_{\{\gamma_{1:n-1}^{(t)}, \gamma_{n+1:N_p}^{(t-1)}\}}(j)). \tag{B.2.7}
\]

Note that \( \eta_n(j) = 0 \) if \( j \in \{\gamma_{1:n-1}^{(t)}, \gamma_{n+1:N_p}^{(t-1)}\} \), meaning that the \( \eta_n(j) \) of measurement \( j \) is not considered for sampling of track \( n \) if it has been assigned to tracks in the current hypothesis \( \{\gamma_{1:n-1}^{(t)}, \gamma_{n+1:N_p}^{(t-1)}\} \).

The overall process of the Gibbs sampling is presented in Algorithm 2.1.

**B.3 Grouping and Gating**

The detailed procedure of the parallel LMB update is shown in Fig. B.3.1. In each group, the predicted LMB is first transformed to the \( \delta \)-GLMB form for the standard \( \delta \)-GLMB update, and the obtained \( \delta \)-GLMB posterior needs to be approximated via an LMB form. The LMBs
Algorithm 2.1: Gibbs sampling

Input: $\gamma^{(1)}, T, N_p, M, \eta = [\eta_{n}(j)]$

Output: $\gamma^{(1)}, \ldots, \gamma^{(T)}$

for $t = 2 : T$
$\gamma^{(t)} = []$
for $n = 1 : N_p$
for $j = 1 : M$
$\eta_{n}(j) = \eta_{n}(j)(1 - \gamma_{1:n-1,\gamma_{n+1:N_p}^t}(j))$
end
$\gamma_{n}^{(t)} \sim \text{Categorical}([-1 : M], \eta_{n}); \gamma^{(t)} = [\gamma_{1}^{(t)}, \gamma_{n}^{(t)}]$
end
end

from each group are recombined for recursive estimation.

Figure B.3.1: The LMB filter with grouping and gating

The partitioned LMB in each group should be mutually exclusive to avoid influencing each other in the update process. This can be achieved by using the gating of measurements. The gating method can be interpreted as a measurement-to-track association, with measurements assigned to targets according to the gate. Targets that share at least one potential measurement association are grouped together. A group may not have an associated measurement due to missed detections or large errors of the predicted measurements.

Let the partition of the labels be $\mathcal{L}_{0:k} = \{\mathcal{L}_{0:k}^{(1)}, \ldots, \mathcal{L}_{0:k}^{(N)}\}$ and the partition of measurements be $\mathcal{Z}_{k} = \{Z_{k}^{(0)}, Z_{k}^{(1)}, \ldots, Z_{k}^{(N)}\}$, then a group is defined as $\mathcal{G}^{(n)} = (\mathcal{L}_{0:k}^{(n)}, Z_{k}^{(n)})$. The grouping of tracks and measurements is an iterative search process starting from an initial group. An
initial group for a track $\ell$ and the associated measurements is given by

$$G(\ell) = \left\{ \ell, \left\{ z_k : M(\hat{z}_{k|k-1}^{(\ell)}, z_k) < \sqrt{T_\gamma} \right\} \right\},$$

(B.3.1)

where $M(\hat{z}_{k|k-1}^{(\ell)}, z_k)$ denotes the Mahalanobis distance between the predicted measurement $\hat{z}_{k|k-1}^{(\ell)}$ of track $\ell$ and the measurement $z_k \in Z_k$, and $T_\gamma$ is the threshold of the gating distance which is defined using the inverse Chi-squared cumulative distribution.

Any groups with common measurements are combined and this process is repeated until no common measurements exist. Each LMB group is given by $\pi_{k|k-1}^{(i)}$. In this way, the predicted LMB RFS can be rewritten as the union of all groups

$$\pi_{k|k-1} = \bigcup_{i=1}^{N} \pi_{k|k-1}^{(i)}.$$  

(B.3.2)

Each group is an LMB RFS, and it needs to be transformed to the $\delta$-GLMB form. The transformation for the group $G^{(i)} = (L_{0:k}^{(i)}, Z_k^{(i)})$ is obtained as

$$\pi_{k|k-1}^{(i)}(X_{k|k-1}^{(i)}) = \Delta(X_{k|k-1}^{(i)}) \sum_{I_k \in \mathcal{F}(L_{0:k})^{(i)}} w^{(i)}_{k|k-1, i} \delta_{I_k}^{(i)}(L(X_{k|k-1}^{(i)}))[p_{k|k-1}]^X_{k|k-1}^{(i)},$$

(B.3.3)

and the weight is given by

$$w^{(i)}_{k|k-1, i} = \prod_{\ell \in L_{0:k}\setminus I_k} (1 - r^{(i)}_{k|k-1}) \prod_{\ell' \in I_k} 1_{L_{0:k}^{(i)}(\ell')} r^{(i)}_{k|k-1}.$$  

(B.3.4)

The $\delta$-GLMB update for the group $G^{(i)} = (L_{0:k}^{(i)}, Z_k^{(i)})$ is expressed by

$$\pi_{k}^{(i)}(X_{k}^{(i)}|Z_{k}^{(i)}) = \Delta(X_{k}^{(i)}) \times \sum_{(I_k, \theta_k) \in \mathcal{F}(L_{0:k}^{(i)} \times \Theta_k)} w^{(i)}_{k, \theta_k} (Z_{k}^{(i)}) \delta_{I_k}^{(i)}(L(X_{k}^{(i)}))[p_{k|k-1}^{(i)}](\cdot | Z_{k}^{(i)}) X_{k}^{(i)},$$

(B.3.5)
where

\[ w^{(I_k, \theta_k)}(Z_k^{(i)}) \propto w_{k|k+1,i}^{(I_k)}[\eta_{Z_k^{(i)}}] \]  
(B.3.6)

\[ p^{(\theta_k)}(x, \ell|Z_k^{(i)}) = \frac{p_{k|k-1,i}(x, \ell)\psi_{Z_k^{(i)}}(x, \ell; \theta_k)}{\eta_{Z_k^{(i)}}(\ell)} \]  
(B.3.7)

\[ \eta_{Z_k^{(i)}}(\ell) = \langle p_{k|k-1,i}(x, \ell), \psi_{Z_k^{(i)}}(\cdot, \ell; \theta_k) \rangle \]  
(B.3.8)

\[ \psi_{Z_k^{(i)}}(x, \ell; \theta_k) = \begin{cases} 
\frac{p_D(x, \ell) p_G(z_{\theta_k}(\ell)|x, \ell)}{\alpha(z_{\theta_k}(\ell))} & \text{if } \theta_k(\ell) > 0 \\
1 - p_D(x, \ell) p_G & \text{if } \theta_k(\ell) = 0,
\end{cases} \]  
(B.3.9)

where \( p_G \) is the probability of gating.

Since the measurement update of each group is a standard full \( \delta \)-GLMB update, the number of hypotheses increases exponentially. The truncation is performed by solving the ranked assignment problem using Murty’s algorithm.

The updated \( \delta \)-GLMB \( \tilde{\pi}_k^{(i)}(\cdot|Z_k^{(i)}) \) of each group needs to be converted back to the LMB form

\[ \pi_k^{(i)}(\cdot|Z_k^{(i)}) \approx \tilde{\pi}_k^{(i)}(\cdot|Z_k^{(i)}) = \left\{ (r^{(\ell,i)}, p^{(\ell,i)}) \right\}_{\ell \in \mathcal{L}_{0,k}^{(i)}}, \]  
(B.3.10)

The multi-target posterior union is approximated by the LMB approximation using the union of the LMB groups

\[ \pi_k(\cdot|Z_k) \approx \tilde{\pi}_k(\cdot|Z_k) = \bigcup_{i=1}^{N} \left\{ (r^{(\ell,i)}, p^{(\ell,i)}) \right\}_{\ell \in \mathcal{L}_{0,k}^{(i)}}, \]  
(B.3.11)
Appendix C  Percentile Results of OSPA Errors

C.1 Percentile Results of Case I

The 10th and 90th percentiles of state and cardinality errors of Case I are shown in Fig. C.1.1. Generally, a few missed detections may significantly increase the averaged error at a single epoch in the LMB filter. The trends of the 90th percentile of all three methods in the case of \( p_D = 0.95 \) are similar, and all methods have large variations in some epochs due to missed detections. Note that CAR and PAR overestimate the cardinality in the case of \( p_D = 0.75 \), see the right column in Fig. C.1.1c and Fig. C.1.1d. Too many missed detections of a particular target may result in inaccurate state estimation and then lead to false measurement-to-track association. As a result, the measurement from this existing target will be erroneously used to generate a new birth target, which temporarily increases the cardinality. However, the probability of existence of the poorly localised existing target will be gradually reduced by the LMB filter until it is permanently eliminated. Then the cardinality can be rapidly corrected to the truth.

C.2 Percentile Results of Case II

Fig. C.2.1 depicts the 10th and 90th percentiles of state and cardinality errors of Case II. The percentiles in the case of \( p_D = 0.95 \) show that PAR performs the worst, which provides the largest 90th percentile error. Even though BVP takes more time to converge, it produces
Appendix C. Percentile Results of OSPA Errors

![Graphs showing percentile results of OSPA errors and cardinality estimate in Case I](image)

**Figure C.1.1:** 10th and 90th percentiles of OSPA errors and cardinality estimate in Case I

similar state and cardinality estimates during the last three days. In the scenario of $p_D = 0.75$, all three methods exhibit similar trends for both 10th and 90th percentiles. The 90th percentile of PAR illustrates that it overestimates the cardinality in some cases due to sparse measurements and missed detections. Compared with CAR, BVP provides slightly larger 90th percentile state estimation errors and smaller 10th percentile cardinality errors. This is mainly because the low $p_D$ value reduces the chance of a sensor to continuously observe measurements from a new target, which is necessary for the BVP birth model to generate birth tracks. Nonetheless, results validate that BVP is able to produce similar state and cardinality estimation in the scenario of missed detections and long time gaps of orbit propagation. In addition, the large 90th percentile errors of all methods in the case of $p_D = 0.75$ suggest that using a sensor with a high $p_D$ value is necessary to pursue accurate state estimation in the scenario of sparse measurements and long...
C.2. Percentile Results of Case II

time gaps for orbit propagation.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{10\textsuperscript{th} and 90\textsuperscript{th} percentiles of OSPA errors and cardinality estimate in Case II}
\end{figure}
Appendix D  Information Gain Functionals

D.1 Cauchy-Schwarz Divergence for δ-GLMB

Let two δ-GLMB densities be $\pi_0(X)$ and $\pi_1(X)$

$$
\pi_0(X) = \Delta(X) \sum_{\{I,c\} \in \mathcal{F}(L) \times \Xi} w_0^{(I,c)} \delta_I(\mathcal{L}(X)) [p_0^{(c)}(\cdot)]^X \tag{D.1.1}
$$

$$
\pi_1(X) = \Delta(X) \sum_{\{J,d\} \in \mathcal{F}(L) \times \Theta} w_1^{(J,d)} \delta_J(\mathcal{L}(X)) [p_1^{(d)}(\cdot)]^X, \tag{D.1.2}
$$

where $\Xi$ and $\Theta$ are discrete spaces of association map histories.

The Cauchy-Schwarz divergence between $\pi_0(X)$ and $\pi_1(X)$ is

$$
C(u) = -\ln \left( \frac{\langle \pi_0(X), \pi_1(X) \rangle}{\sqrt{\langle \pi_0(X), \pi_0(X) \rangle \langle \pi_1(X), \pi_1(X) \rangle}} \right). \tag{D.1.3}
$$

Given that each single target density is represented by a GMM, the closed form of the Cauchy-Schwarz divergence is written as

$$
\langle \pi_0(X), \pi_1(X) \rangle = \sum_{L \subseteq L} \sum_{\{I,c\} \in \mathcal{F}(L) \times \Xi} \sum_{\{J,d\} \in \mathcal{F}(L) \times \Theta} w_0^{(L,c)} w_1^{(L,d)} \prod_{t \in L} \gamma^{(t)}_{\pi_0, \pi_1} \tag{D.1.4}
$$

$$
\langle \pi_0(X), \pi_0(X) \rangle = \sum_{L \subseteq L} \sum_{\{I,c\} \in \mathcal{F}(L) \times \Xi} \sum_{\{J,d\} \in \mathcal{F}(L) \times \Theta} w_0^{(L,c)} w_0^{(L,d)} \prod_{t \in L} \gamma^{(t)}_{\pi_0, \pi_0} \tag{D.1.5}
$$
D.2 The Derivation of The Analytical Rényi Divergence

\[
\langle \pi_1(X), \pi_1(X) \rangle = \sum_{L \subseteq L} \sum_{(L,c) \in F(L)} \sum_{(L,d) \in F(L) \times \Theta} w_1^{(L,c)} w_1^{(L,d)} \prod_{\ell \in L} \gamma_{\pi_1,\pi_1}(\ell),
\]
(D.1.6)

\[
\text{and}
\]
\[
\gamma_{\pi_0,\pi_1}^{(\ell)} = \sum_{i=1}^{N_0^{(c)}} \sum_{j=1}^{N_1^{(d)}} w_{0,i}^{(c,\ell)} w_{1,j}^{(d,\ell)} \mathcal{N}(m_{0,i}^{(c,\ell)}; m_{1,j}^{(d,\ell)}, P_{0,i} + P_{1,j}^{(d,\ell)})
\]
(D.1.8)

\[
\gamma_{\pi_0,\pi_0}^{(\ell)} = \sum_{i=1}^{N_0^{(c)}} \sum_{j=1}^{N_0^{(d)}} w_{0,i}^{(c,\ell)} w_{0,j}^{(d,\ell)} \mathcal{N}(m_{0,i}^{(c,\ell)}; m_{0,j}^{(d,\ell)}, P_{0,i} + P_{0,j}^{(d,\ell)})
\]
(D.1.9)

\[
\gamma_{\pi_1,\pi_1}^{(\ell)} = \sum_{i=1}^{N_1^{(c)}} \sum_{j=1}^{N_1^{(d)}} w_{1,i}^{(c,\ell)} w_{1,j}^{(d,\ell)} \mathcal{N}(m_{1,i}^{(c,\ell)}; m_{1,j}^{(d,\ell)}, P_{1,i} + P_{1,j}^{(d,\ell)})
\]
(D.1.10)

D.2 The Derivation of The Analytical Rényi Divergence

Given the equation of \(\gamma_{\pi_0,\pi_1}^{(\ell)}\) for two LMBs using a single Gaussian component representation of each target state:

\[
\gamma_{\pi_0,\pi_1}^{(\ell)} = \int K_{0,1}^{(\ell)} \mathcal{N}(x; \mu_{0,1}^{(\ell)}, \Sigma_{0,1}^{(\ell)}) \frac{1}{2} dx,
\]
(D.2.1)

where \(K_{0,1}^{(\ell)}\) is expressed as the following form

\[
K_{0,1}^{(\ell)} = \frac{1}{\sqrt{|2\pi(P_{0}^{(\ell)} + P_{1}^{(\ell)})|}} \exp \left[ -\frac{1}{2} (m_{0}^{(\ell)} - m_{1}^{(\ell)})^T (P_{0}^{(\ell)} + P_{1}^{(\ell)})^{-1} (m_{0}^{(\ell)} - m_{1}^{(\ell)}) \right].
\]
(D.2.2)

Eq. (D.2.1) can be rearranged as follows

\[
\gamma_{\pi_0,\pi_1}^{(\ell)} = \left(K_{0,1}^{(\ell)}\right)^{\frac{1}{2}} \int \left( \frac{1}{\sqrt{2\pi \Sigma_{0,1}^{(\ell)}}} \exp \left[ -\frac{1}{2} (x - \mu_{0,1}^{(\ell)})^T \Sigma_{0,1}^{(\ell)}^{-1} (x - \mu_{0,1}^{(\ell)}) \right] \right)^{\frac{1}{2}} dx
\]
(D.2.3)
\[
\gamma^{(\ell)}_{\pi_0, \pi_1} = \left( K_{0,1} \right)^{\frac{1}{2}} \left( \frac{1}{\sqrt{2\pi \Sigma_{0,1}^{(\ell)}}} \right)^{\frac{1}{2}} \int \exp \left[ -\frac{1}{4} (x - \mu_{0,1}^{(\ell)})^T \left( \Sigma_{0,1}^{(\ell)} \right)^{-1} (x - \mu_{0,1}^{(\ell)}) \right] dx. \quad (D.2.4)
\]

The integral in \( \gamma^{(\ell)}_{\pi_0, \pi_1} \) then yields an analytic solution

\[
\int \exp \left[ -\frac{1}{4} (x - \mu_{0,1}^{(\ell)})^T \left( \Sigma_{0,1}^{(\ell)} \right)^{-1} (x - \mu_{0,1}^{(\ell)}) \right] dx = |4\pi \Sigma_{0,1}^{(\ell)}|^{\frac{1}{2}}. \quad (D.2.5)
\]

Substituting Eq. (D.2.2) and Eq. (D.2.5) into Eq. (D.2.3) produces the analytical form of \( \gamma^{(\ell)}_{\pi_0, \pi_1} \)

\[
\gamma^{(\ell)}_{\pi_0, \pi_1} = \left( K_{0,1} \right)^{\frac{1}{2}} \left( \frac{1}{\sqrt{2\pi \Sigma_{0,1}^{(\ell)}}} \right)^{\frac{1}{2}} \left| 4\pi \Sigma_{0,1}^{(\ell)} \right|^{\frac{1}{2}} = \left( K_{0,1} \right)^{\frac{1}{2}} \left| 8\pi \Sigma_{0,1}^{(\ell)} \right|^{\frac{1}{2}}. \quad (D.2.6)
\]
References


