Bifurcation and Stable Fixed Points in the $l_1$-norm Minimization Problem

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Abstract

The topic of this paper is an optimization problem which arises naturally in the design of feedback controllers to achieve optimal robustness. Stated mathematically, the problem imposes an $l_1$-norm objective on the input and output signals of a linear discrete-time dynamic system. In a recent paper an algorithm has been presented which systematically determines initial conditions for which exact solutions can be found. The contribution of this paper is twofold. Firstly, we illustrate the usefulness of the algorithm in understanding optimal dynamic response for a specific example. Secondly, we investigate the creation and apparent disappearance of stable fixed points as an input data parameter is varied. For the first time to our knowledge the conjecture is made that the dynamic evolution of optimal solutions may exhibit chaos.

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1 Introduction

The problem of designing a linear time-invariant controller to optimally reject bounded disturbances for discrete-time linear dynamic systems was introduced in the 1980’s by Vidyasagar [7] in the Western literature and by Barabanov [1] in Russia. A comprehensive early treatment is in [2]. See also [6]. Since that time an open problem in the literature, still unsolved, is to determine whether or not optimal solutions always have a rational Z-transform. In recent work, [5] and [4], the author introduces the idea of describing the time evolution of the solution to the optimization problem as the output of a non-linear system, and gives an algorithm for the determination of periodic fixed points for the non-linear system describing the evolution of the optimal dual variables. In this paper we investigate the stability of these fixed points and show that both the number and type - stable or unstable - can change abruptly with changes in an input data parameter.

We consider an example for which most of the fixed points are unstable, and introduce the hypothesis that for some, though certainly not all, problems and some but not all initial conditions, the time evolution of the optimal solution exhibits chaos. If this turns out to be true, then for some problems the optimal solution will not have a rational Z-transform. Whether or not there is genuine chaos in the system, the results here show that numerical solutions that appear periodic may not be. This only becomes evident through consideration of exact solutions and fixed points.
2 Formulation

2.1 Terminology

Denote by \( \mathbb{R}^n \) the \( n \)-dimensional real space. A \( p \times k \) matrix \( M \) will sometimes have its dimension made explicit by the notation \( M_{p \times k} \). The \( p \times p \) identity matrix is denoted \( I_p \). The set of positive integers is denoted \( \mathbb{N} \). The \( l_1 \)-norm of a vector sequence \( e = (e_k)_{k=1}^\infty \) is defined as \( \|e\|_1 = \sum_{k=1}^\infty |e_k| \) whenever the series exists. The Banach space of absolutely-summable sequences, equipped with the \( l_1 \)-norm, is denoted \( \ell_1 \). The space of continuous linear functionals on \( I_1 \), that is the dual of \( l_1 \), is denoted \( l_\infty \); it is the space of bounded sequences with the norm \( \|e\|_\infty := \sup_k |e_k| \). The \( Z \)-transform of an arbitrary sequence \( e = (e_k)_{k=1}^\infty \) is defined to be \( \hat{e}(z) = \sum_{k=1}^\infty e_k z^{-k} \), where \( z \) lies within the radius of convergence of the series. The function \( \text{sgn} \) of a real number \( x \) is defined to be \( 1 \) if \( x > 0 \), \( -1 \) if \( x < 0 \), and \( 0 \) if \( x = 0 \). The \( \text{sgn} \) of a vector with components \( (e_i)_{i=1}^n \) is the vector with components \( \left( \text{sgn}(e_i) \right)_{i=1}^n \). Given a vector \( e \) and any \( s \in \mathbb{N} \), \( t \in \mathbb{N} \) satisfying \( s < t \), we denote \( (e_s, e_{s+1}, \ldots, e_t) \) by \( e_{(s:t)} \). If \( s, t, q, r \in \mathbb{N} \), \( 1 < s < t \) and \( 1 < q < r \) then \( M_{(s:t),(q:r)} \) is a matrix composed of row \( s \) to row \( t \), and of columns \( q \) to \( r \), of the matrix \( M \) having at least \( t \) rows and at least \( r \) columns. Square brackets are used when it is important to distinguish between row and column vectors. For example, \( e = (e_1, e_2) \in \mathbb{R}^n \times \mathbb{R}^n \) can be written in matrix equations as either

\[
\begin{bmatrix}
e_1 \\
e_2
\end{bmatrix}_{2n \times 1}
\text{ or } [ \begin{array}{c} e_1 \\ e_2 \end{array} ]_{1 \times 2n}.
\]

The concatenation of \( (e_1, u_1) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_1} \) and \( (e_2, u_2) \in \mathbb{R}^{n_2} \times \mathbb{R}^{n_2} \) is defined to be the vector \( (e_3, u_3) \in \mathbb{R}^{n_1+n_2} \times \mathbb{R}^{n_1+n_2} \) for which \( e_3 = [e_1 \\ e_2] \) and \( u_3 = [u_1 \\ u_2] \). The superscript \( T \) denotes transpose.

2.1.1 Complementary pairs of vectors

A complementarity relation between optimal primal and dual vectors holds in all of the dual pairs of programs we shall consider. Vectors \( (e, u) \) and \( (e^*, u^*) \), both in \( \mathbb{R}^p \times \mathbb{R}^p \), \( p \in \mathbb{N} \), are said to be complementary if (1) holds for \( k = 1, \ldots, p \).

\[
e_k > 0 \implies e_k^* = K_1, \quad u_k > 0 \implies u_k^* = K_2
\]
\[
e_k < 0 \implies e_k^* = -K_1, \quad u_k < 0 \implies u_k^* = -K_2
\]
\[
|e_k^*| < K_1 \implies e_k = 0, \quad |u_k^*| < K_2 \implies u_k = 0.
\]

(1)
3 Problem description

The decision vectors in the space $l_1$ are denoted $e$ and $u$. The cost function is $K_1 \| e \|_1 + K_2 \| u \|_1$, where $K_1$ and $K_2$ are given positive real numbers.

The problem we investigate is

$$
\mathcal{P}(b) : \begin{cases}
\min_{e \in l_1, u \in l_1} K_1 \| e \|_1 + K_2 \| u \|_1 \\
\text{subject to} \\
d \hat{e} + n \hat{u} = \hat{b},
\end{cases}
$$

(2)

where $\hat{n}$, $\hat{d}$ and $\hat{b}$ are polynomials with real coefficients,

$$
\hat{n}(z) = n_1 + n_2 z + n_3 z^2 + \cdots + n_{l+1} z^l \\
\hat{d}(z) = d_1 + d_2 z + d_3 z^2 + \cdots + d_{l+1} z^l \\
\hat{b}(z) = b_1 + b_2 z + \cdots + b_{l+1} z^{l+1},
$$

(3)

$n_{l+1}$ and $d_{l+1}$ are not both zero, and $l \geq 1$ is a positive integer. It is assumed that neither $\hat{n}(z)$ nor $\hat{d}(z)$ have zeros lying on the unit circle in the complex plane, and that $\hat{n}(z)$ and $\hat{d}(z)$ have no zeros in common. The vector $b$ specifies initial conditions for the discrete-time dynamic system represented by the equation $\hat{d} \hat{e} + \hat{n} \hat{u} = \hat{b}$. It can be shown that a solution to $\mathcal{P}(b)$ with finite cost is guaranteed to exist. There is a stronger conjecture, namely that all optimizing vectors $(e, u)$ for $\mathcal{P}(b)$ have rational $Z-$transforms. We put forward the hypothesis in this paper that this conjecture does not hold in general because of chaos in the time evolution of the optimal response of $e$ and $u$. We do not prove the existence of chaos, but give reasons for suggesting the possibility of it occurring. This is interesting because any proof of the validity of the conjecture would necessarily imply the absence of chaos.

3.1 Linear Programming Formulation of Problem $\mathcal{P}(b)$

Using block matrix notation, the problem $\mathcal{P}(b)$ can be written as

$$
\mathcal{P}(b) : \begin{cases}
\min_{e \in l_1, u \in l_1} K_1 \| e \|_1 + K_2 \| u \|_1 \\
\text{subject to} \\
D \begin{bmatrix} e \\ u \end{bmatrix} = \begin{bmatrix} b^T, 0, \ldots \end{bmatrix}^T,
\end{cases}
$$

(4)

where $D$ is the infinite-dimensional lower-triangular toeplitz matrix with $(d_1, \ldots, d_{l+1}, 0, 0, \ldots)$ as its first column, and $N$, defined similarly, has first column $(n_1, \ldots, n_{l+1}, 0, 0, \ldots)$. Also $b := [b_1, \ldots, b_l]^T$. 

4
3.2 Toeplitz and circulant matrix notation

Define
\[
D_{UT} := \begin{bmatrix}
d_{l+1} & d_l & \ldots & d_2 \\
0 & d_{l+1} & \ddots & \vdots \\
0 & 0 & \ddots & d_{l+1} \\
0 & 0 & 0 & d_{l+1}
\end{bmatrix}_{l \times l}
\quad \text{and} \quad
D_{LT} := \begin{bmatrix}
d_1 & 0 & 0 & 0 \\
\vdots & d_1 & 0 & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
d_{l-1} & d_1 & 0 & d_1
\end{bmatrix}_{l \times l}.
\]

Then for any integer \( p \geq 2l \) the North-West corner submatrix of \( D \) with dimension \( p \times p \) can be written as
\[
D_{(1:p,1:p)} = \begin{bmatrix}
D_{LT} & D_{LT} & \ldots & D_{LT} \\
D_{LT} & D_{LT} & \ldots & D_{LT} \\
\vdots & \vdots & \ddots & \vdots \\
D_{LT} & D_{LT} & \ldots & D_{LT}
\end{bmatrix}_{p \times p}.
\]

Denote by \( D_C(p) \) the circulant matrix of dimension \( p \times p \) whose first column is \((d_1, d_2, \ldots, d_{l+1}, 0, \ldots, 0)\). That is
\[
D_C(p) := \begin{bmatrix}
D_{LT} & D_{LT} & \ldots & D_{LT} \\
D_{LT} & D_{LT} & \ldots & D_{LT} \\
\vdots & \vdots & \ddots & \vdots \\
D_{LT} & D_{LT} & \ldots & D_{LT}
\end{bmatrix}_{p \times p}.
\]

The matrices \( N_{UT}, N_{LT} \) and \( N_C(p) \) are defined similarly. Then
\[
S(\hat{d}, \hat{n}) := \begin{bmatrix}
D_{LT} & N_{LT} \\
D_{LT} & N_{LT}
\end{bmatrix}
\]
is the Sylvester matrix for the polynomials \( \hat{d}(z) \) and \( \hat{n}(z) \). It is well-known that \( S \) is non-singular if and only if \( \hat{d}(z) \) and \( \hat{n}(z) \) are coprime.

3.3 Matching terminal and initial conditions for sub-problems

The constraints to \( \mathcal{P}(b) \) given in (4) can be written as
\[
\begin{bmatrix}
D_{LT} & D_{LT} \\
D_{LT} & D_{LT} \\
\vdots & \vdots
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2 \\
\vdots
\end{bmatrix}
+ \begin{bmatrix}
N_{LT} & N_{LT} \\
N_{LT} & N_{LT} \\
\vdots & \vdots
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
\vdots
\end{bmatrix}
= \begin{bmatrix}
b_1 \\
b_2 \\
0 \\
\vdots
\end{bmatrix}.
\]
Any vector pair \((e, u) \in l_1 \times l_1\) satisfying (5) will be termed feasible for \(\mathcal{P}(b)\). Let \((e, u)\) be feasible for \(\mathcal{P}(b)\) and let \(M \geq 2l\) be an integer. Define the initial condition at time \(M\) for \((e, u)\) by

\[
b^{(M)}(e, u) := - \left[ D_{UT} e_{(M-l+1:M)} + N_{UT} u_{(M-l+1:M)} \right].
\]  

(6)

Then it is straightforward to show that the concatenation of \(\begin{bmatrix} e^{(M)} \\ e^{(1)} \end{bmatrix}, \begin{bmatrix} u^{(M)} \\ u^{(1)} \end{bmatrix}\) is feasible for \(\mathcal{P}(b)\) if and only if \((e^{(M)}, u^{(M)})\) satisfies \(D_{(1:M,1:M)} e^{(M)} + N_{(1:M,1:M)} u^{(M)} = \begin{bmatrix} b^T, 0, \ldots, 0 \end{bmatrix}^T\) and \((e^{(1)}, u^{(1)})\) is feasible for \(\mathcal{P}(b^{(M)}(e, u))\).

Suppose now we are given an optimal (hence feasible) solution to \(\mathcal{P}(b)\), denoted \((e^{P(b)}, u^{P(b)})\). That is \((e^{P(b)}, u^{P(b)}) \in \arg\min \mathcal{P}(b)\). The initial condition at time \(M\) for \((e^{P(b)}, u^{P(b)}), b^{(M)}(e^{P(b)}, u^{P(b)})\), is given by (6). It follows from the discussion above and Bellman’s Principle of optimality that \((e^{(2)}, u^{(2)}) \in \arg\min \mathcal{P}(b^{(M)}(e^{P(b)}, u^{P(b)}))\) if and only if \(\begin{bmatrix} e^{P(b)} \\ e^{(1:M)}^{(2)} \end{bmatrix}, \begin{bmatrix} u^{P(b)} \\ u^{(1:M)}^{(2)} \end{bmatrix}\) \(\in \arg\min \mathcal{P}(b)\).

This observation motivates consideration of the time evolution of \(b^{(M)}(e^{P(b)}, u^{P(b)})\) as \(M\) tends to infinity. We will show that, for some initial conditions \(b\), either \(b^{(M)}(e^{P(b)}, u^{P(b)})\) is a decaying periodic vector (perhaps after some finite length initial transient), or \(b^{(M)}(e^{P(b)}, u^{P(b)})\) is a finite sum of decaying periodic vectors (again perhaps after some finite length initial transient). For some problem data \(b^{(M)}(e^{P(b)}, u^{P(b)})\) will exhibit this behaviour for all initial conditions \(b\). For some other problems, however, their exist initial conditions which do not seem to produce any form of decaying periodicity in \(b^{(M)}(e^{P(b)}, u^{P(b)})\). There is the interesting possibility that \(b^{(M)}(e^{P(b)}, u^{P(b)})\) wanders around for ever on a strange attractor.

The rest of this Section sets up a framework for the analysis of the dynamics of the time evolution of \(b^{(M)}(e^{P(b)}, u^{P(b)})\) when there is periodicity in both the pattern of the locations of the zero values of \(e^{P(b)}\) and \(u^{P(b)}\), and in the sign pattern of the non-zero values of \(e^{P(b)}\) and \(u^{P(b)}\). Such periodicity will be related in Sections 3.4 and 4.3 to basis periodicity, where the term basis is the familiar one used in linear programming theory.

### 3.4 Notation for a basis

Consider the set of equations \(Ae + Bu = b\), or in block matrix notation

\[
\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} e \\ u \end{bmatrix} = b,
\]

(7)
where $A$ and $B$ are any real $p \times p$ matrices, and $e$, $u$ and $b$ are real $p$-dimensional column vectors. Let $\left[ A \ B \right]_B$ be any non-singular $p \times p$ sub-matrix made up of the columns of the $p \times 2p$ matrix $\left[ A \ B \right]$. Thus the $p$ integers $i_1, i_2, \ldots, i_p$ from $1, 2, \ldots, 2p$ identify the columns of $\left[ A \ B \right]$ that have been retained in $\left[ A \ B \right]_B$. The set $\{i_1, i_2, \ldots, i_p\}$ determines the basis $\left[ A \ B \right]_B$ and the notation $B = [i_1, i_2, \ldots, i_p]$ will be used to identify the basis, where $i_1, \ldots, i_p$ have been ordered by increasing size. The vector $B$ will be referred to as the basis vector. If the $p$ components of $\left[ e \ u \right]_B$ not associated with columns of $\left[ A \ B \right]_B$ are set equal to zero, the solution to the resulting set of equations is said to be a basic solution to (7) with respect to the basis vector, $B$, denoted $\left[ e \ u \right]_B^{bsol}$, a $2p \times 1$ vector. We shall also use the notation $(e^{bsol}(B), u^{bsol}(B))$ to denote the basic solution with respect to $B$. The components of $\left[ e \ u \right]$ associated with $B$ are called basic variables and will be denoted by $\left[ e \ u \right]_B^{bsol}$, a $p \times 1$ vector.

Thus $\left[ e \ u \right]_B^{bsol} = \left[ A \ B \right]_B^{-1}b$, where $\left[ A \ B \right]_B^{-1}$ denotes the inverse of the matrix $\left[ A \ B \right]_B$.

Suppose that, for some integer $p \geq 2l$, a $p$-dimensional basis vector, $B$, is given. Define

\[
Z(B) := \left[ D_{(1:p,1:p)} \ N_{(1:p,1:p)} \right]_B \\
\bar{Z}(B) := [Z^{-1}]_{(1:p,1:l)} \\
Y(B) := \left[ D_{C}(p) \ N_{C}(p) \right]_B \\
F(p) := \left[ D_{(1:p,1:p)} \ N_{(1:p,1:p)} \right] - \left[ D_{C}(p) \ N_{C}(p) \right] \\
H(B) := I_{p} - YZ^{-1} = \left[ F(p) \right]_BZ^{-1} \\
G(B) := H_{(1:l,1:l)}
\]

The significance of the $l \times l$ matrix $G(B)$ will now be explained. First recall from (6) the definition, for any $(e, u)$ feasible for for $P(b)$, of the initial condition at time $M$, denoted by $b^{(M)}(e, u)$. For any $p$-dimensional basis $B$ there is the basic solution to $D_{(1:p,1:p)}e + N_{(1:p,1:p)}u = \left[ \frac{b_{l \times 1}}{0_{(p-l) \times 1}} \right]$, denoted $(e^{bsol}(B), u^{bsol}(B))$. Then it is readily shown that $G(B)$ maps $b$ to the initial condition at time $p$ for $(e^{bsol}(B), u^{bsol}(B))$. That is, for any $b$,

\[
G(B) : b \mapsto b^{(p)}(e^{bsol}(B), u^{bsol}(B)). \tag{9}
\]
Let \((e^{bsol(n)}, u^{bsol(n)})\) denote the basic solution, with respect to \(B\), to the equations \(D_{(1:p,1:p)}e + N_{(1:p,1:p)}u = \begin{bmatrix} G^n(B)b_{l \times 1} \\ 0_{(p-1) \times 1} \end{bmatrix}\). Then by (9) and the discussion in Section 3.3, the concatenation \(\begin{bmatrix} e^{bsol(0)} \\ e^{bsol(1)} \\ \vdots \\ u^{bsol(0)} \\ u^{bsol(1)} \end{bmatrix}\) is feasible for \(De + Nu = \begin{bmatrix} b \\ 0 \\ \vdots \end{bmatrix}\). In the terminology of nonlinear systems, \(G^n(B)b\) acts as a stroboscopic return map of period \(p\) for the trajectory \((e^{feas}, u^{feas})\). The pattern of zeros in \((e^{feas}, u^{feas})\) is necessarily periodic. Periodicity also in the sign pattern of the non-zero elements of \((e^{feas}, u^{feas})\) is required for our main optimization result. A sufficient condition for this is that \(b\) is an eigenvector of \(G(B)\) with corresponding real and positive eigenvalue having magnitude less than one.

The above discussion explains some of the conditions required of so-called fixed-point bases, to be defined in Section 4.3, from which optimal solutions for \(P\) can be constructed. Fixed-point bases, in addition to having periodicity properties, are also required to optimize certain finite-dimensional programs. These will be described next.

4 Duality

The infinite-dimensional dual to \(P(b)\), denoted \(D(b)\), has been derived in [3]. It can be expressed in the form

\[
\begin{align*}
\mathcal{D}(b) : & \quad \max_{e^* \in \ell_\infty, u^* \in \ell_\infty} \left[ e_1^*, \ldots, e_l^*, u_1^*, \ldots, u_l^* \right] S^{-1} \begin{bmatrix} b_{l \times 1} \\ 0_{l \times 1} \end{bmatrix} \\
& \text{subject to } \left\| e^* \right\|_\infty \leq K_1, \left\| u^* \right\|_\infty \leq K_2 \\
& \quad \text{and } D^T u^* = N^T e^*
\end{align*}
\]

4.1 \(MD(b, p)\) -a finite-dimensional modification of \(D(b)\)

Let \(p \geq 2l\) be an integer. We construct a finite-dimensional convex programming problem related to \(D(b)\) that has \(2p\) variables and \(p\) equality constraints. The constraints for \(MD(b, p)\), the problem \(D\) constrained in a manner consistent with its variables \(e^*\) and \(u^*\) being periodic of period \(p\), can be written
\[ D^T_C(p)u^* = N^T_C(p)e^*. \]

\begin{equation}
MD(b,p) : \begin{cases}
\max_{e^* \in \mathbb{R}^p, u^* \in \mathbb{R}^p} [e^*_1, \ldots, e^*_l, u^*_1, \ldots, u^*_l] S^{-1} \begin{bmatrix} b_l \\ 0_{l \times 1} \end{bmatrix} \\
\text{subject to } D^T_C(p)u^* = N^T_C(p)e^* \text{ and } \|e^*\|_\infty \leq K_1, \|u^*\|_\infty \leq K_2.
\end{cases}
\end{equation}

4.2 The dual of \( MD(b,p) \), denoted \( DMD(b,p) \)

For \( p \geq 2l \), a dual of \( MD(b,p) \) can be constructed in the form

\begin{equation}
DMD(b,p) : \begin{cases}
\min_{e \in \mathbb{R}^p, u \in \mathbb{R}^p} \sum_{k=1}^p K_1 |e_k| + K_2 |u_k| \\
\text{subject to } D_C(p)e + N_C(p)u = \begin{bmatrix} b_l \\ 0_{(p-l) \times 1} \end{bmatrix}.
\end{cases}
\end{equation}

The optimal values of \( DMD(b,p) \) and \( MD(b,p) \) are equal. Furthermore, if \((e,u)\) and \((e^*,u^*)\) are feasible for \( DMD(b,p) \) and \( MD(b,p) \), respectively, then a necessary and sufficient condition that they both be optimal solutions is that they be complementary. For a proof see [5].

In the next Section we give a definition of special basis vectors, which we term fixed-point basis vectors. Associated with every fixed-point basis vector, denoted \( \tilde{B} \), there is a so-called fixed-point initial condition, denoted \( \tilde{b} \). Fixed-point initial conditions are important because basic feasible solutions constructed from the fixed-point basis vector \( \tilde{B} \) are optimal for the problem \( P(\tilde{b}) \). This is the content of Theorem (5) in Section (4.3). A proof is given in [4].

4.3 Fixed point bases

**Definition 1** A vector \( \tilde{B} = [i_1, i_2, \ldots, i_p] \), whose elements are \( p \) integers chosen from \( 1, 2, \ldots, 2p \), is said to be a \( p \)-dimensional fixed-point basis vector for the problem \( P(\cdot) \) if the following 3 conditions are satisfied:

1. \( Z(\tilde{B}) := \begin{bmatrix} D_{(1:p,1:p)} & N_{(1:p,1:p)} \end{bmatrix} \tilde{B} \) is non-singular;

2. there is an eigenvector, \( \tilde{b} \), of \( G(\tilde{B}) := \begin{bmatrix} I_p - Y(\tilde{B})[Z(\tilde{B})]^{-1} \end{bmatrix}_{(1:l,1:l)} \) with corresponding simple eigenvalue \( \lambda \in [0,1) \); and

3. \( Y(\tilde{B}) \) is an optimal basis for \( DMD(\tilde{b},p) \).
Definition 2 The $l \times 1$ vector $\tilde{b}$ in Definition 1, associated with $\tilde{B}$, is termed a fixed-point initial condition of order $p$ for the program $\mathcal{P}(\cdot)$.

Definition 3 Suppose that $\tilde{b}$ is a fixed-point initial condition of order $p$ for $\mathcal{P}(\cdot)$. An optimal solution for $MD(\tilde{b}, p)$ will be termed a fixed-point dual vector corresponding to $\tilde{b}$, and will be denoted $(\tilde{e}^*, \tilde{u}^*)$. Thus $(\tilde{e}^*, \tilde{u}^*) \in \arg \max MD(\tilde{b}, p)$.

In the following Definition $\rho(G(\tilde{B}))$ denotes the spectral radius of $G(\tilde{B})$. The eigenvalue $\lambda$ is the Perron-Frobenius eigenvalue of $G(\tilde{B})$.

Definition 4 If a fixed point $\tilde{b}$ of order $p$ satisfies the additional property that $\rho(G(\tilde{B})) = \lambda$, where $\lambda \in (0, 1)$ is the eigenvalue of $G(\tilde{B})$ associated with $\tilde{b}$, then $\tilde{b}$ is said to be a stable fixed point of period $p$ for the program $\mathcal{P}$.

The following Theorem is proved in [1]. It shows that for every fixed-point initial condition $\tilde{b}$ there is an optimal solution for $\mathcal{P}(\tilde{b})$ which satisfies a $p$'th order recurrence relation, where $p$ is the order of the fixed point.

Theorem 5 Suppose $\tilde{b}$ is a fixed-point initial condition of order $p$ for the program $\mathcal{P}(\cdot)$, with corresponding fixed-point basis $\tilde{B}$, and corresponding eigenvalue $\lambda \in [0, 1)$. The optimizing solution vector for $D(\tilde{b})$ is $(\tilde{e}^*_{\text{ext}}, \tilde{u}^*_{\text{ext}})$, the infinite periodic extension of the fixed-point dual vector corresponding to $\tilde{b}$. The optimal values for the programs $\mathcal{P}(\tilde{b})$ and $D(\tilde{b})$ are equal to $[\tilde{e}^*_1, \ldots, \tilde{e}^*_p, \tilde{u}^*_1, \ldots, \tilde{u}^*_p]S^{-1}\tilde{b}$, which is also the optimal value for the program $MD(\tilde{b}, p)$. Denote by $(e^{\text{bsol}}, u^{\text{bsol}})$ the basic solution, with respect to the basis $\tilde{B}$, to the equations $D(1:p; 1:p)e + N(1:p, 1:p)u = \tilde{b}$. Then an optimizing vector for $\mathcal{P}(\tilde{b})$ is given by

$$
e^{(\text{opt})} = \begin{bmatrix} e^{\text{bsol}} \\ \lambda e^{\text{bsol}} \\ \lambda^2 e^{\text{bsol}} \\ \vdots \end{bmatrix}, \quad u^{(\text{opt})} = \begin{bmatrix} u^{\text{bsol}} \\ \lambda u^{\text{bsol}} \\ \lambda^2 u^{\text{bsol}} \\ \vdots \end{bmatrix}.$$ (10)

5 Example

We illustrate the results in this paper for the problem $\mathcal{P}(\cdot)$ having the following given data:

$\hat{a} = (1 + z/2)(1 + 2z/9)(1 - z/5) = 1 + 47/90z - 1/30z^2 - 1/45z^3$
$\hat{n} = (1 - z/3)(1 - 2z/7)(1 - 2z/5) = 1 - 107/105z + 12/35z^2 - 4/105z^3$
$K_1 = 1.$
Following the example in [4] we shall initially take $K_2 = 3/2$ and show that there is a stable fixed-point basis of period 7. We will then consider the effect of changing $K_2$, and show that for $K_2 = 7/5$ there is no longer any stable fixed point having period 15 or less. There are, however, many unstable fixed points, for example about 100 having period 15 or less. The question arises: where has the stable fixed point gone? Visual inspection of a solution of length 50 shows an apparent periodicity of 30 in the optimal solution. We will show however, using theory presented in [5] and [4], that although there is indeed a fixed point of period 30, it is in fact unstable; the solution will move away from this point eventually. The possibility of a stable fixed point of period higher than 15 cannot currently be ruled out. But for some values of $K_2$ the evidence so far available is also consistent with the existence of an infinite number of repellers, and no periodic attracting fixed points. The well-known metaphor of the pin ball machine may be applicable, with the orbit of the dynamically evolving dual optimal variables being repelled in the manner in which the pins in a pin ball machine repel the motion of the ball. Such a situation is often taken as being suggestive of chaos. Obviously this is not by any means even an outline of a proof. At this stage chaos is merely a tantalising possibility.

5.1 Fixed point bases for Example

It is possible to test for fixed-point bases by testing exhaustively all possible selections of $p$ from $2p$ integers. This involves, for a given $p$ and a given candidate basis vector $B$, testing for satisfaction of Conditions 1, 2 and 3 of Definition 1. It is found that, for $K_2$ lying between $1.4182$ and $1.5040$, there are 12 fixed-point basis vectors of period 7, all except one of these being unstable.

The fixed-point basis vector with corresponding stable fixed-point initial condition is

$$\tilde{B}_1 = [1, 2, 4, 6, 8, 10, 12]$$

with

$$\tilde{b}_1 = \left[ \frac{28655197 + \sqrt{203277932802849}}{4606468}, -1, 0 \right]^T$$

Some of the details confirming this are presented next. For $d = (1, 47/90, -1/30, -1/45)$ and $n = (1, -107/105, 12/35, -4/105)$
\[
D_{(1:7,1:7)} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
47/90 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1/30 & 47/90 & 1 & 0 & 0 & 0 & 0 \\
-1/45 & -1/30 & 47/90 & 1 & 0 & 0 & 0 \\
0 & -1/45 & -1/30 & 47/90 & 1 & 0 & 0 \\
0 & 0 & -1/45 & -1/30 & 47/90 & 1 & 0 \\
0 & 0 & 0 & -1/45 & -1/30 & 47/90 & 1 \\
\end{bmatrix},
\]

and
\[
D_{C}(7) = \begin{bmatrix}
1 & 0 & 0 & 0 & -1/45 & -1/30 & 47/90 \\
47/90 & 1 & 0 & 0 & 0 & -1/45 & -1/30 \\
-1/30 & 47/90 & 1 & 0 & 0 & 0 & -1/45 \\
-1/45 & -1/30 & 47/90 & 1 & 0 & 0 & 0 \\
0 & -1/45 & -1/30 & 47/90 & 1 & 0 & 0 \\
0 & 0 & -1/45 & -1/30 & 47/90 & 1 & 0 \\
0 & 0 & 0 & -1/45 & -1/30 & 47/90 & 1 \\
\end{bmatrix}
\]

and similarly for \(N_{(1:7,1:7)}\) and \(N_{C}(7)\).

Then
\[
Z(\tilde{B}_1) := \begin{bmatrix} D_{(1:7,1:7)} & N_{(1:7,1:7)} \end{bmatrix}_{\tilde{B}_1}
= \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 \\
47/90 & 1 & 0 & 0 & -107/105 & 0 & 0 \\
-1/30 & 47/90 & 0 & 0 & 12/35 & 1 & 0 \\
-1/45 & -1/30 & 1 & 0 & -4/105 & -107/105 & 0 \\
0 & -1/45 & 47/90 & 0 & 0 & 12/35 & 1 \\
0 & 0 & -1/30 & 1 & 0 & -4/105 & -107/105 \\
0 & 0 & -1/45 & 47/90 & 0 & 0 & 12/35 \\
\end{bmatrix}
\]

and
\[
Y(\tilde{B}_1) := \begin{bmatrix} D_{C}(7) & N_{C}(7) \end{bmatrix}_{\tilde{B}_1}
= \begin{bmatrix}
1 & 0 & 0 & -1/30 & 1 & 0 & -4/105 \\
47/90 & 1 & 0 & -1/45 & -107/105 & 0 & 0 \\
-1/30 & 47/90 & 0 & 0 & 12/35 & 1 & 0 \\
-1/45 & -1/30 & 1 & 0 & -4/105 & -107/105 & 0 \\
0 & -1/45 & 47/90 & 0 & 0 & 12/35 & 1 \\
0 & 0 & -1/30 & 1 & 0 & -4/105 & -107/105 \\
0 & 0 & -1/45 & 47/90 & 0 & 0 & 12/35 \\
\end{bmatrix}
\]

and \([F(7)]_{\tilde{B}_1} := \begin{bmatrix} D_{(1:7,1:7)} & N_{(1:7,1:7)} \end{bmatrix} - \begin{bmatrix} D_{C}(7) & N_{C}(7) \end{bmatrix}\)

Then
\[ H(\tilde{B}_1) := I_7 - YZ^{-1} = [F(7)]_{\tilde{B}_1}Z^{-1} \]

so

\[ G(\tilde{B}_1) = \left[ H(\tilde{B}_1) \right]_{(1,3;1,3)} = \begin{bmatrix} 85.550.647 & -13.412.497 & 94.742.181 \\ 5604.802.627.715 & 1120.960.525.543 & 1120.960.525.543 \\ 0 & 1120.960.525.543 & 1120.960.525.543 \end{bmatrix} \]

and one of the eigenvectors of \( G(\tilde{B}_1) \) is \( \tilde{b}_1 := \begin{bmatrix} 1 \\ -1 \\ -291997251055 \\ 7472880243352 \\ -2328578260779 \\ 7472880243352 \\ -1 \\ 11039165480207 \\ 11209320365028 \end{bmatrix} \) with corresponding eigenvalue \( \lambda_1 = 11.209.960.525.543 \). Furthermore \( DMD(\tilde{b}_1, \gamma) \) has \( \tilde{B}_1 \) as an optimal basis vector.

This is readily verified using (1). Thus it has been verified that \( \tilde{B}_1 \) is a fixed-point basis vector for \( \mathcal{P}(\cdot) \).

Also \( \tilde{b}_1 \) is a stable fixed-point of period 7 for \( \mathcal{P}(\cdot) \). This follows from the fact that the magnitude \( \lambda_1 \) is greater than the magnitude of the other eigenvalues of \( G(\tilde{B}_1) \). The fixed-point dual vectors corresponding to \( \tilde{b}_1 \) are

\[ \tilde{e}^* = \begin{bmatrix} 1 \\ -1 \\ -291997251055 \\ 7472880243352 \\ -2328578260779 \\ 7472880243352 \\ -1 \\ 11039165480207 \\ 11209320365028 \end{bmatrix}, \quad \tilde{u}^* = \begin{bmatrix} 3/2 \\ 535046818800 \\ 934110030419 \\ -3/2 \\ 2517435809853 \\ 26155080851732 \\ 3/2 \\ 2495818665649 \\ 15166976335361 \end{bmatrix} \]

and the optimal cost is therefore

\[ J_D(\tilde{b}) = \begin{bmatrix} 1, -1, -291997251055 \\ 7472880243352 \end{bmatrix}, \begin{bmatrix} 3/2 \\ 535046818800 \\ 934110030419 \\ -3/2 \\ 2517435809853 \\ 26155080851732 \\ 3/2 \\ 2495818665649 \\ 15166976335361 \end{bmatrix} S^{-1} \begin{bmatrix} \tilde{b} \\ 0_{3 \times 1} \end{bmatrix} \]

\[ = \frac{36280767673344789945}{4302947963604150092} + \frac{1185853695115\sqrt{2032779328028249}}{4302947963604150092} \]

\[ \approx 12.36. \]

The optimizing vectors for \( D(\tilde{b}_1) \) are the infinite periodic extensions of \( \tilde{e}^* \) and \( \tilde{u}^* \). See Fig 1.

Referring to Theorem 5, \( (e^{bsol}, u^{bsol}) \) denotes the basic solution, with respect to the basis \( \tilde{B}_1 \), to the equations \( D(1,7,1;7)e + N(1,7,1;7)u = \begin{bmatrix} \tilde{b}_1 \\ 0_{4 \times 1} \end{bmatrix} \)

\[ \begin{bmatrix} e^{bsol} \\ u^{bsol} \end{bmatrix} = [Z^{-1}]_{(1,7;1,3)} \tilde{b}_1 \]

\[ =: \tilde{Z}\tilde{b}_1 \]
Figure 1: $K_2 = 3/2$ and $b = \tilde{b}_1$, the stable fixed point corresponding to the stable basis vector of period 7.
and so
\[ e_{bsol}^{*} = \left[ \tilde{Z} b_1 \right]_{(1:4)} \]
\[ u_{bsol}^{*} = \left[ \tilde{Z} b_1 \right]_{(5:7)} \]

Then
\[
\| e_{bsol}^{*} \|_1 = \frac{684683277296398097}{135886020794131898} + \frac{467102372979\sqrt{203277932802849}}{2581834395088506062}, \quad (12)
\]
\[
\| u_{bsol}^{*} \|_1 = \frac{307359370491765137}{135886020794131898} + \frac{162944629169\sqrt{203277932802849}}{2581834395088506062}.
\]

The non-basic variables are of course zero, so
\[
(e_{bsol}^{*}, u_{bsol}^{*}) = (e_{bsol}^{*}, u_{bsol}^{*}, 0, 0, 0, 0, 0, 0).
\]

The optimal solution, \((e^{(opt)}, u^{(opt)})\), for \(P(\tilde{b})\) is
\[
e^{(opt)} = \begin{bmatrix} \lambda_1 e_{bsol}^{*} \\ \lambda_2 e_{bsol}^{*} \\ \vdots \end{bmatrix}, \quad u^{(opt)} = \begin{bmatrix} u_{bsol}^{*} \\ \lambda_1 u_{bsol}^{*} \\ \lambda_2 u_{bsol}^{*} \\ \vdots \end{bmatrix},
\]

having optimal cost
\[
\| e^{(opt)} \|_1 + (3/2) \| u^{(opt)} \|_1 = \frac{1}{1 - \lambda_1} \left[ \| e_{bsol}^{*} \|_1 + (3/2) \| u_{bsol}^{*} \|_1 \right] = J_D(\tilde{b}),
\]
where (11) and (12) have been used.

Now consider the program \(P(\tilde{b})\) where \(\tilde{b} = (1, 0, 0)\). It can be shown that, because \(\tilde{b}_1\) is a stable fixed-point initial condition, there is an open neighbourhood surrounding \(\tilde{b}_1\) for which \((\tilde{e}^*, \tilde{u}^*) \in \text{arg max} \ D(\tilde{b})\), and furthermore that \(\tilde{b}\) belongs to this neighbourhood. In other words, the optimizing vectors for \(D(\tilde{b})\) and \(D(\tilde{b})\) are the same. The optimal value for the program \(P(\tilde{b})\) is

\[
J_D(\tilde{b}) = \begin{bmatrix} 1, -1, -2919970251055/7472880243352, 3/2, 535046818800/934110030419, -3/2, \end{bmatrix} S^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\]
\[
= \frac{1185853695115}{934110030419}
\]

If the initial condition \(b\) moves sufficiently far away from \(\tilde{b}_1\), then \((\tilde{e}^*, \tilde{u}^*)\) will no longer be optimal for \(D(b)\). Nevertheless, after an initial aperiodic transient, the optimizing vectors for \(D(b)\) will eventually be identical with \((\tilde{e}^*, \tilde{u}^*)\). See Fig. 2. This is again a consequence of the stability of \(\tilde{b}_1\).
Figure 2: The initial condition is $b = (1, 1, 1)$ and $K_2 = 3/2$. There is a stable fixed point initial condition of period 7. After an initial aperiodic transient, the dual optimal variables are indeed periodic with period 7.
Figure 3: \( b = (1, 0, 0) \) and \( K_2 = 7/5 \). The initial condition is near an unstable fixed point of period 30. Although the optimal solution is initially periodic with period 30, it cannot remain so because the eigenvalue associated with the basis vector of period 30 is not a Perron-Frobenius eigenvalue.

5.2 Apparent disappearance of the stable fixed point

We now consider the effect of putting \( K_2 \) equal to a value outside the interval \([1.4182, 1.5040]\). Consider the problem \( \mathcal{P}(\cdot) \) defined by keeping \( \hat{d}, \hat{n} \) and \( K_1 \) the same as in the Example, but changing \( K_2 \) to 7/5. An exhaustive search has failed to find any stable fixed-point initial conditions for this problem. There are, however, plenty of unstable fixed points of all periods so far tested, which is up to about 15. The total number of fixed points so far found is more than one hundred. Whether the total number is finite or not is an open question.

A typical plot is show in Fig 3, for which \( b = (1, 0, 0) \). Although \( b \) is not apparently a fixed-point initial condition, it is close to an unstable fixed-point initial condition of period 30.

It can be verified that the basis of period 30 implied by the the first 30 values of the response in Fig 3, denoted \( \mathcal{B}_{30} \), is in fact a fixed-point basis satisfying the conditions of Definition 1. Thus there is an eigenvector of
$G(B_{30})$ which is an unstable fixed point. The eigenvalue corresponding to this fixed-point initial condition is (approximately) $1.04 \times 10^{-21}$. The other two eigenvalues (corresponding to the other eigenvectors of $G(B_{30})$, neither of which are fixed points) are zero and $-4.44 \times 10^{-20}$. Thus, unless the initial condition lies exactly in the subspace spanned by the eigenvectors corresponding to the eigenvalues zero and $1.04 \times 10^{-21}$ (and for this Example they do not lie exactly in this subspace), evolution of $e$ and $u$ according to $B_{30}$ implies that the response must eventually align itself with the eigenvector associated with the eigenvalue $-4.44 \times 10^{-20}$. But the optimal solution cannot align itself with this eigenvector because it is not a fixed-point initial condition. If the plot in Fig 3 were to be continued sufficiently far, the periodicity of $B_{30}$ would necessarily be lost. For the time being the ultimate alignment of $b(M)(e^{P(b)}, u^{P(b)})$ for this Example remains unknown.

6 Conclusions

For a specific $l_1$-norm minimization problem, having cubic polynomials as given problem data, a weighting on the cost function has been found for which the optimal solution displays features suggestive of chaos. For initial conditions $b$ in the neighbourhood of a stable fixed-point initial condition $\tilde{b}$ the mapping from $b$ to $\text{argmin } P(b)$ is linear; it is given explicitly by the results described in this paper. For other initial conditions the mapping from $b$ to $\text{argmin } P(b)$ is non-linear. Very little is known about this non-linear map. Characterizing it in special cases is a topic of current research.

References


